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New Beam and Plate Bending Elements in Finite Element Analysis

有限要素法における新しい梁及び平板曲げ要素

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Summary

New beam and plate bending finite elements are proposed in this paper. A new beam elements consists of rigid bars connected by a rotational spring, while a plate element consists of rigid plates with a rotational spring. Size of stiffness matrices of these bending elements are 1/2 of a well-known beam bending element, and 1/3 of a triangular plate bending element respectively and therefore to obtain solutions of the same accuracy at least matrix condensation to 2/3 of the original matrices of three dimensional structures can be made so that considerble reduction of computing time can be expected in finite element analysis of structures with minor penality in increase of mesh division. The auther believes that the use of these new elements will especially make the nonlinear analysis of complex structures much more practical.

1. Derivation of new bending finite elements

It is well known that the bending stiffness of a framed structure is much smaller than the direct stiffness of the same structure, and in case of plate and shell structures the membrane stiffness of a given structure is much larger than the bending stiffness. And therefore bending analysis of these structure can be made under the assumption of infinitely large direct stiffness so long as the deformation is considered small.

Basing on such consideration two rigid bars connected by one rotational spring, two rigid triangular plates connected with one rotational spring are proposed for new beam and plate bending elements.

(α) a new beam bending element

Deformation of two rigid bars connected by one spring is considered (Fig. 1).

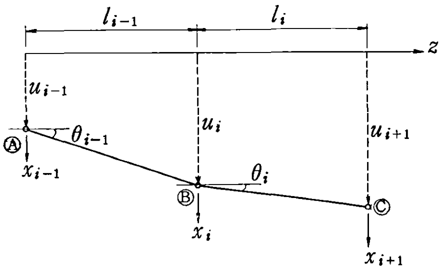

Fig. 1. A new beam bending element

Denoting the displacements of the point A, B and C by u_{i-1}; u_i, u_{i+1}; k spring constant, the following strain energy expression can be derived easily

$$V=(u_{i-1},u_i,u_{i+1})=\frac{k}{2}\left\{\left(\frac{u_{i+1}-u_i}{l_i}\right)^2-\left(\frac{u_i-u_{i-1}}{l_{i-1}}\right)^2\right\}\cdots\cdots(1)$$

It is not difficult to derive the following stiffness matrix of a given system by applying Castigliano's Theorem;

$$\begin{Bmatrix}X_{i-1}\\X_i\\X_{i+1}\end{Bmatrix}=k\begin{bmatrix}\frac{1}{l_{i-1}^2} & & SYM\\ -\frac{1}{l_{i-1}}\left(\frac{1}{l_i}+\frac{1}{l_{i-1}}\right) & \left(\frac{1}{l_{i-1}}+\frac{1}{l_i}\right)^2 & \\ \frac{1}{l_{i-1}+l_i} & -\frac{1}{l_i}\left(\frac{1}{l_i}+\frac{1}{l_{i-1}}\right) & \frac{1}{l_i^2}\end{bmatrix}\begin{Bmatrix}u_{i-1}\\u_i\\u_{i+1}\end{Bmatrix}\cdots\cdots\cdots(2)$$

$$M_i=k\left\lfloor\frac{1}{l_{i-1}},\ -\left(\frac{1}{l_{i-1}}+\frac{1}{l_i}\right),\ \frac{1}{l_i}\right\rfloor\begin{Bmatrix}u_{i-1}\\u_i\\u_{i+1}\end{Bmatrix}\cdots\cdots\cdots(3)$$

(β) a new plate bending element

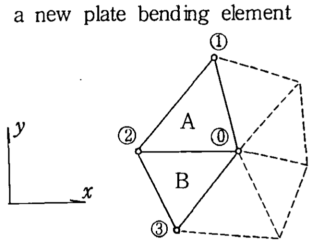

Fig. 2. A new plate bending element

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Consider assemblage of rigid plate elements shown in Fig. 2. Triangular plates Δ⓪ ① ② and Δ⓪ ② ③, are connected by a spring whose constant is k_{AB}. Before loading, these plates were on the x,y plane and under a given loading they are displaced to the position whose equation is given by the following equation;

$$l_A x + m_A y + n_A z = p_A \qquad \cdots\cdots(4)$$

Lateral displacement of the point ⓪ (x_0, y_0), ① (x_1, y_1), ② (x_2, y_2) are denonted by w_0, w_1, and w_2 respectively. Consequently the following three equation can be obtained:

$$\begin{aligned} l_A x_0 + m_A y_0 + n_A w_0 &= p_A \\ l_A x_1 + m_A y_1 + n_A w_1 &= p_A \qquad \cdots\cdots(5) \\ l_A x_2 + m_A y_2 + n_A w_2 &= p_A \end{aligned}$$

From these equation (l_A, m_A, n_A) and p_A can be expressed as follows:

$$\begin{aligned} l_A &= \pm \frac{\Delta_{11}}{\sqrt{\Delta_{11}^2 + \Delta_{12}^2 + \Delta_{10}^2}} \\ m_A &= \pm \frac{\Delta_{12}}{\sqrt{\Delta_{11}^2 + \Delta_{12}^2 + \Delta_{10}^2}} \\ n_A &= \pm \frac{\Delta_{10}}{\sqrt{\Delta_{11}^2 + \Delta_{12}^2 + \Delta_{10}^2}} \qquad \cdots\cdots(6) \\ p_A &= \pm \frac{x_0\Delta_{11} + y_0\Delta_{12} + w_0\Delta_{10}}{\sqrt{\Delta_{11}^2 + \Delta_{12}^2 + \Delta_{10}^2}} \end{aligned}$$

where

$$\Delta_{10} = \begin{vmatrix} x_{01} & y_{01} \\ x_{12} & y_{12} \end{vmatrix}, \quad \Delta_{11} = -\begin{vmatrix} w_{01} & y_{01} \\ w_{12} & y_{12} \end{vmatrix},$$

$$\Delta_{12} = -\begin{vmatrix} x_{01} & w_{01} \\ x_{12} & w_{12} \end{vmatrix} \qquad \cdots\cdots(7)$$

$x_{ij} = x_i - x_j$, $y_{ij} = y_i - y_j$, and

$$w_{ij} = w_i - w_j$$

The similar expressions for Δ⓪ ② ③ can drived. In the deformed state Δ⓪ ① ② and Δ⓪ ② ③ are inclined each other through the rotation angle θ_{AB} and the following relation can be easily obtained:

$$\cos\theta_{AB} = l_A l_B + m_A m_B + n_A n_B \qquad \cdots\cdots(8)$$

When θ_{AB} is small such that

$$\cos\theta_{AB} = 1 - \frac{\theta_{AB}^2}{2} \qquad \cdots\cdots(9)$$

then eq. (8) can be simplified with the aid of eqs.(6) and (9) as follows:

$$\frac{\theta_{AB}^2}{2} = 1 - (l_A l_B + m_A m_B + n_A n_B) \text{ or} \qquad \cdots\cdots(10)$$

$$\frac{\theta_{AB}^2}{2} = -\frac{1}{\Delta_{10}\Delta_{20}}(\Delta_{11}\Delta_{21} + \Delta_{12}\Delta_{22}) + \frac{1}{2\Delta_{10}{}^2}(\Delta_{11}^2 + \Delta_{12}^2) + \frac{1}{2\Delta_{20}{}^2}(\Delta_{21}^2 + \Delta_{22}^2)$$

Consequently the strain energy stored in the connection spring will be given as follows:

$$V(w) = \frac{1}{2}k_{AB}\theta_{AB}^2 = -\frac{1}{\Delta_{10}\Delta_{20}}(\Delta_{11}\Delta_{21} + \Delta_{12}\Delta_{22}) + \frac{1}{2\Delta_{10}{}^2}(\Delta_{11}^2 + \Delta_{12}^2) + \frac{1}{2\Delta_{20}{}^2}(\Delta_{21}^2 + \Delta_{22}^2)$$

where $w^T = \lfloor w_0, w_1, w_2, w_3 \rfloor \qquad \cdots\cdots(11)$

Applying Castingliano's theorm, the following reaction force vecter R can be derived

$$R = \frac{\partial V}{\partial w} = Kw \qquad \cdots\cdots(12)$$

where K is the stiffness matrix to be obtained. The final from of the stiffness matrix is shown in the following Table. 1

	w_0	w_1	w_2	w_3
Z_0	$-2(x_{12}x_{23} + y_{12}y_{23}) + \frac{\Delta_{02}}{\Delta_{01}}(x_{02}^2 + y_{12}^2) + \frac{\Delta_{01}}{\Delta_{02}}(x_{23}^2 + y_{23}^2)$	$(x_{02}x_{23} + y_{02}y_{23}) - \frac{\Delta_{02}}{\Delta_{01}}(x_{02}x_{12} + y_{02}y_{12})$	$(x_{12}x_{03} - x_{01}x_{23} + y_{12}y_{03} - y_{01}y_{23}) + \frac{\Delta_{02}}{\Delta_{01}}(x_{02}x_{12} + y_{01}y_{12}) + \frac{\Delta_{01}}{\Delta_{02}}(x_{03}x_{23} + y_{03}y_{23})$	$-(x_{02}x_{12} + y_{02}y_{12}) + \frac{\Delta_{01}}{\Delta_{02}}(x_{02}x_{23} + y_{02}y_{23})$
Z_1	$(x_{02}x_{03} + y_{02}y_{03}) - \frac{\Delta_{02}}{\Delta_{01}}(x_{02}x_{12} + y_{02}y_{12})$	$\frac{\Delta_{02}}{\Delta_{01}}(x_{02}^2 + y_{02}^2)$	$-(x_{02}x_{03} + y_{02}y_{03}) - \frac{\Delta_{02}}{\Delta_{01}}(x_{01}x_{02} + y_{01}y_{02})$	$(x_{02}^2 + y_{02}^2)$
Z_2	$(x_{12}x_{03} - x_{01}x_{23} + y_{12}y_{03} - y_{01}y_{23}) + \frac{\Delta_{02}}{\Delta_{01}}(x_{01}x_{12} + y_{01}y_{12}) - \frac{\Delta_{01}}{\Delta_{02}}(x_{03}x_{23} + y_{03}y_{23})$	$-(x_{02}x_{03} + y_{02}y_{03}) - \frac{\Delta_{02}}{\Delta_{01}}(x_{01}x_{02} + y_{01}y_{02})$	$2(x_{01}x_{02} + y_{01}y_{02}) - \frac{\Delta_{02}}{\Delta_{01}}(x_{01}^2 + y_{01}^2) + \frac{\Delta_{01}}{\Delta_{02}}(x_{03}^2 + y_{03}^2)$	$-(x_{01}x_{02} + y_{01}y_{02}) - \frac{\Delta_{01}}{\Delta_{02}}(x_{02}x_{03} + y_{02}y_{03})$
Z_3	$-(x_{02}x_{12} + y_{02}y_{12}) + \frac{\Delta_{01}}{\Delta_{02}}(x_{02}x_{03} + y_{02}y_{03})$	$(x_{02}^2 + y_{02}^2)$	$-(x_{01}x_{02} + y_{01}y_{02}) - \frac{\Delta_{01}}{\Delta_{02}}(x_{02}x_{03} + y_{02}y_{03})$	$\frac{\Delta_{01}}{\Delta_{02}}(x_{02}^2 + y_{02}^2)$

$\times \dfrac{k_{AB}}{\Delta_{10}\Delta_{20}}$

Table 1. Stiffness matrix of a new plate bending element

(γ) method of determination of the spring constants

The spring constants k in these matrices can be computed theoretically by useing the second order polynomial approximation.

In case of a beam element (Fig. 1), for example the curvature r can be expressed by

$$r = \frac{2}{l_i + l_{i-1}} (\theta_i - \theta_{i-1}) \qquad \cdots\cdots\cdots(13)$$

From the moment-curvature relation the following equation is derived:

$$M = k\ (\theta_i - \theta_{i-1}) = EI\,r \qquad \cdots\cdots\cdots(14)$$

Substituting eq.(13) into eq.(14) the following relation is easily obtained

$$k = \frac{2EI}{l_i + l_{i-1}} \qquad \cdots\cdots\cdots(15)$$

It is not difficult to derive the following formula for the spring constant of a given plate element as shown in Fig. 3

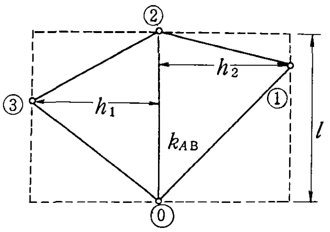


Fig. 3. Determination of the spring constant in the plate bending element

$$k = \frac{2Dl}{h_1 + h_2} \qquad \cdots\cdots\cdots(16)$$

(δ) Convergency test of beam and plate bending solutions

Fig. 4 shows the result of limit analaysis of a beam clamped at both ends under a single concentrated load.

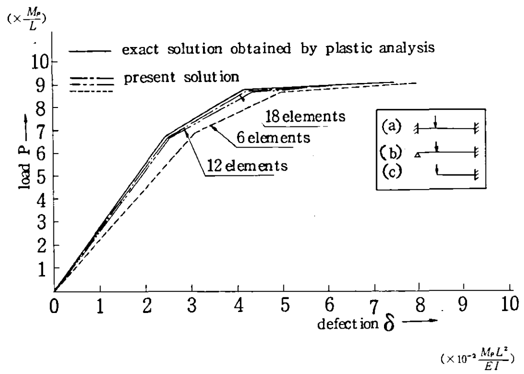


Fig. 4. Collapse load analysis of a clamped beam under a concentrated load

Fig. 5 shows the result of convergency test of a rectangular plate bending solutions under a concentrated as well as uniformly distributed load.

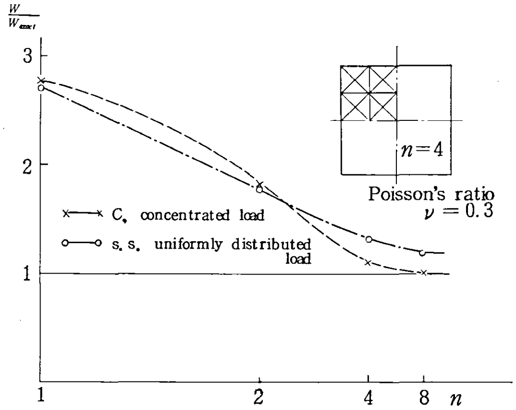


Fig. 5. Result of convergency tests

2. Some numerical examples on the collapse load analysis of rectangular plates

To show validity of a newly proposed plate bending element, a series of collapse load analysis of rectangular plates was made under the assumption of a concentrated as well as uniformly distributed load with unusual boundary condition. Two typical examples of analysis are shown in Fig. 6 and Fig. 7. Agreement between the present calculation and experiment made by other investigators (1) was founed to be extremely good.

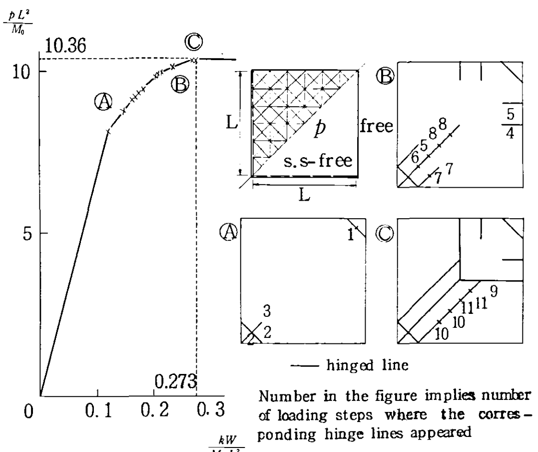
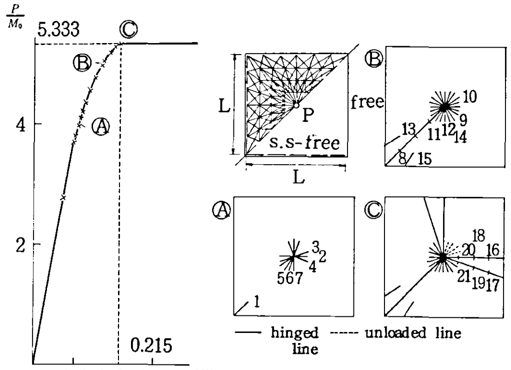


Fig. 6. Collapse load analysis of a rectangular plate under a uniformly distributed load



Number in the figure implies number of loading steps where the corresponding hinge lines appeared

Fig. 7. Collapse load analysis of a rectangular plate under centrally concentrated load

3. Conclution

The auther believes that the use of these new elements will especially make the nonlinear analysis of complex structures much more practical. Program development of dynamic collapse load analysis of plate structures including stability is now under way. The authors would like to express their thanks to Messrs. Y. Fujitani, T. Okamoto and other colleague for their valuable discussion and help of numerical calculation.

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References

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2) M. A. Save and C. E. Massonet, "Plastic Analysis and Design of Plates, Shells and Disks", North-Holland Publishing Company, Amsterdam, London, 1972.

正誤表

（7月号）

頁	段	行	種別	正	誤
309	右	↑1~5	本文	文字の出力の際はPASCAL 内部コード→EBCDIC変換を行う(CHR サブルーチン). 文字の入力の場合は逆にEBCDIC,→PASCAL内部コードの変換が必要になる.（ORDサブルーチン）整数や実数は直接内部2進表示と変換されるので、これらのルーチンは要らない.	出力に際してはすべて文字として出力されるので, ……………… 文字入力の場合だけ EBCDIC→ PASCAL 内部コードの変換が必要となる.（ORD サブルーチン）
333	左	↑	(2) 式	$3\hbar^2$	$3h^2$

（8月号）

頁	段	行	種別	正	誤
355	左	↑9	(1) 式	$\varDelta$	$\triangle$
〃	〃	↑8	本文	〃	〃
〃	〃	↑6	(2) 式	〃	〃
〃	右	↑20	本文	〃	〃
〃	〃	↑19	(4) 式	〃	〃
356	左	↑2	本文	技　官	教　官
357			見出し	and	AND
376	右	↑2	本文	東京製綱株式会社	東京製鋼株式会社
377	〃	↑16	式(2)の第1式	$+a\ \cos\theta)B$	$+a\ \sin\theta)B$
〃	〃	↑14	式(2)の第2式	$+s^2\sin 2\theta)A$	$+s^2\sin\theta)A$
〃	〃	↑12	本文中の式	$=\left(\frac{V_s}{V_p}\right)^2$	$=\frac{V_s^2}{V_p}$
〃	〃	↑3	〃	$=\frac{\cos\nu}{s}>$	$=\frac{\cos\theta}{s}>$
378	左	↑9	式 (4)	$=\frac{1}{\sqrt{1+\left(\frac{2\pi}{\alpha\lambda}\right)^2}}$	$=\frac{1}{1+\sqrt{\frac{2\pi^2}{\alpha\lambda}}}$
〃	右	↓16	本文中の式	$=2\pi\frac{\ell}{\lambda}$	$=2\pi\cdot\cdot$ -