1. Thermodynamical Origin of the Earth's Core. II.

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1. A review of the results of the last investigation.

In the previous paper\(^1\) we solved a problem of polytropic gas within the core. The density distribution in the rocky shell was assumed to be uniform in one case and to be that of Jeffreys in another. The polytropic conditions used in that paper were \(n=1\) and 5. Although the temperature in the core became of the order of \(10^5\) K, it varied proportionally with the atomic weights of the gases in the core. With the theory of nuclear transformation of elements in mind, should the earth's core be gaseous, it is likely that the elements composing that gas are of relatively low atomic numbers.

In the present investigation the polytropic conditions will be extended to a range so wide as \(n=5, 3, 2, 1, 0.5, 0.3\), from the result of which it will be shown that even if the earth's core be gaseous, the gases may be of metallic elements of relatively high atomic numbers, say, iron or nickel, but that the indices for the polytropy will be as low as \(n=0.3 \sim 0.5\).

2. Formulae for the polytropic gases in the core.

Let \(\rho, \phi, T, G (=6.66.10^{-8} \text{ C.G.S.}), \mathfrak{M} (=8.26.10^7)\) be the density, the gravitational potential, the temperature, the gravitational constant, and the universal gas constant, respectively. Let also \(\kappa, \gamma (=1+1/n)\) be the constants of polytropy showing the relation \(P=\kappa \rho \gamma\), where \(P\) is the gas pressure. Then, using the relations

\[
\rho = \left(\frac{\phi}{(n+1)\kappa}\right)^{\kappa}, \quad P = \frac{\rho \phi}{n+1}, \quad T = \frac{\beta \rho \phi}{(n+1) \mathfrak{M}},
\]

(1), (2), (3)

and writing

\[1) \text{ K. Sezawa and K. Kanai, Bull. Earthq. Res. Inst., 18 (1940), 359\sim 369.}\]
we get the equation
\[
\frac{d^2u}{dz^2} + \frac{2}{z} \frac{du}{dz} + u' = 0.
\]  
(7)

To solve this equation we write
\[
u = 1 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots \ldots \ldots
\]  
(8)

From (7), (8) it is possible to determine the constants \(a_1, a_2, \ldots\). The expressions of \(u\) thus determined for the different \(n\)'s are

\[
\begin{align*}
n = 3; \quad & u = 1 - 0.16667 z^3 + 2.5 \times 10^{-2} z^4 \\
& - 3.7698 \times 10^{-5} z^6 + 5.6860 \times 10^{-7} z^8 \\
& - 8.5763 \times 10^{-9} z^{10} + 1.2936 \times 10^{-11} z^{12} + \ldots \ldots,
\end{align*}
\]

\[
\begin{align*}
n = 2; \quad & u = 1 - \frac{1}{6} z^2 + \frac{1}{60} z^4 - \frac{11}{7560} z^6 \\
& + \frac{1}{8505} z^8 - \frac{97}{10692000} z^{10} + \ldots \ldots,
\end{align*}
\]

\[
\begin{align*}
n = 0.5; \quad & u = 1 - 0.16667 z^2 + 4.1667 \times 10^{-2} z^4 \\
& + 3.3069 \times 10^{-5} z^6 + 1.3779 \times 10^{-6} z^8 \\
& + 7.7662 \times 10^{-9} z^{10} + 5.1978 \times 10^{-11} z^{12} + \ldots \ldots,
\end{align*}
\]

\[
\begin{align*}
n = 0.3; \quad & u = 1 - 0.16667 z^3 + 2.5 \times 10^{-3} z^4 \\
& + 5.1587 \times 10^{-6} z^6 + 2.3957 \times 10^{-9} z^8 \\
& + 1.5205 \times 10^{-11} z^{10} + 1.1477 \times 10^{-13} z^{12} + \ldots \ldots.
\end{align*}
\]  
(9)

The solutions for the cases \(n = 1, 5\) can be obtained exactly as shown in the previous paper.

Let \(\rho_2\), the density in the rocky shell, be uniform and let also \(c, a\) be respectively the radius of the core and that of the earth's surface. Then, since the mass within the radius \(r\) in the shell is

\[
M_r = \frac{4\pi}{3} \rho_2 (r^3 - c^3) + M_c
\]

\[
= \frac{4\pi}{3} \rho_2 (r^3 - c^3) + \int_0^r 4\pi \rho_2 \rho^2 dr,
\]  
(10)

and since the pressure \(P_2\) in the shell is related to \(M_r\) with the formula

\[
\text{Fig. 1.}
\]
\[ \frac{dP_2}{dr} = -\frac{\rho_2 GM_e}{r^2} = -\frac{4\pi G}{3} \rho_2^2 \left( \frac{r^2}{2} + \frac{c^2}{r} \right) - \frac{\rho_2 G}{r} M_e, \]

(11)

the same pressure is expressed by

\[ P_2 = -\frac{4\pi G}{3} \rho_2^2 \left( \frac{r^2}{2} + \frac{c^2}{r} \right) + \frac{\rho_2 GM_e}{r} + B. \]

(12)

The constant \( B \) can be determined from the condition \( P=0 \) at the free surface \( r=a \), from which it follows that

\[ P_2 = -\frac{4\pi G}{3} \rho_2^2 \left( \frac{1}{2} (r^2-a^2) + c^2 \left( \frac{1}{r} - \frac{1}{a} \right) \right) + \frac{\rho_2 GM_e}{r} \left( \frac{1}{r} - \frac{1}{a} \right). \]

(13)

The conditions at the boundary \( r=c \) are that \( P=P_2 \) and that \( M_e \), the mass within the core, shall be given, namely, \( 1.915.10^{23} \) gm.mass. These conditions are expressed by

\[ \frac{\rho \phi}{n+1} = -\frac{4\pi G}{3} \rho_2^2 \left( \frac{1}{2} (c^2-a^2) + c^2 \left( \frac{1}{c} - \frac{1}{a} \right) \right) + \frac{\rho_2 GM_e}{c} \left( \frac{1}{c} - \frac{1}{a} \right), \]

(14)

\[ M_e = \int_0^c 4\pi \rho r^2 c \left[ \int_0^c 4\pi \left( \frac{\phi}{(n+1)x} \right) r^2 dr \right] \]

(15)

where \( \phi_c \) is the value of \( \phi \) at \( r=c \). Using the relations in (4), (5), (6), the above two conditions are replaced by

\[ \frac{\alpha^2 \phi^{n+1}_c u^{n+1}_c}{n+1} = 1.040.10^2, \quad \phi_0 \frac{\alpha^2}{\alpha} \int_0^{r_c} u^2 z^2 dz = 1.275.10^3, \]

(16), (17)

where \( z_c \) is the value of \( z \) at \( r=c \). There are four unknown constants \( \alpha, \phi_0, u_c, z_c \) in (16), (17). Since however the relations (6), (9) can be used as auxiliary conditions, all these unknown constants can be determined.

In the actual calculation, after drawing the curve that represents the relation in (9), we take a set of \( u_c \) and \( z_c \) for trial. Using, then, the curve just mentioned, equations (16), (17) will give the values of \( \alpha \) and \( \phi_0 \). If all the quantities thus determined satisfy the relation in (6), namely, \( z_c = \alpha c \phi_0^{(n-1)} \), the solutions are true, but if not, we shall again assume another set of \( u_c \) and \( z_c \) and repeat the above calculation until all the quantities satisfy relation (6). In this way the problem can be solved.
3. Distributions of density, pressure, and wave velocity within the core.

With the aid of the constants determined by the method shown in the preceding section, it is now possible to calculate the distributions of density, pressure, and wave velocity within the core. The formulae for the distributions of these quantities are (1), (2), and \( V = \sqrt{\gamma P / \rho} = \sqrt{\varphi / n} \). The results are shown in Figs. 2, 3, 4.

It will be seen from these figures that whereas the distributions of density and pressure within the core tend to be uniform with decrease in \( n \), the distribution of wave velocity becomes rather uniform with increasing \( n \). This arises from the condition that the increase in pressure with depth is not small for any \( n \).

The distribution of velocity of longitudinal seismic waves within the core, as given by Gutenberg and others, is also plotted by broken line in Fig. 4. This shows that the polytropic gaseous state, \( n = 0.3 \), within the core is quite similar to that of the actual core, at least, with
respect to the distribution of the longitudinal waves. In such a case, the distribution of density is almost uniform. On the other hand, the same distribution within the liquid core that was determined by Jeffreys\(^2\), is rather similar to the case \( n = 1 \) in the polytropic state. But, Bullen\(^3\) has stated quite recently that, in the usual calculation, the moment of inertia of the earth is likely to be so large that the assumption of uniform distribution of density within the core is still insufficient to explain that large moment of inertia. This shows that the small \( n \) of the polytropy does not seriously contradict the probable condition of density distribution. Jeans\(^4\) showed, although in the case of air, that the value of \( n \) decreases enormously with pressure and increases very little with temperature, so that the smallness of \( n \) in the case of core gas is almost certain. Jeans's results are shown in the above table.

<table>
<thead>
<tr>
<th></th>
<th>( p = 1 ) atm.</th>
<th>( p = 25 ) atm.</th>
<th>( p = 100 ) atm.</th>
<th>( p = 200 ) atm.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta = -79.3^\circ )C</td>
<td>( n = 2.47 )</td>
<td>( n = 1.755 )</td>
<td>( n = 0.826 )</td>
<td>( n = 0.752 )</td>
</tr>
<tr>
<td>( \theta = 0^\circ )</td>
<td>( n = 2.48 )</td>
<td>( n = 2.13 )</td>
<td>( n = 1.52 )</td>
<td>( n = 1.18 )</td>
</tr>
</tbody>
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4. Distribution of temperature within the core.

In the previous paper, we introduced the idea of nuclear transmuta-

\[ T(\ln K) \]

\[ r(\ln \text{km}) \]

Fig. 5. Temperature distribution within the core for different values of \( n \), it being assumed that \( \mu = 8.5 \).

\[ T(\ln K) \]

\[ r(\ln \text{km}) \]

Fig. 6. Temperature distribution within the core for different values of \( n \), it being assumed that \( \mu = 32 \).

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3) K. E. Bullen, *Nature*, 145 (1940), 599.
weight as \( \mu = 8.5 \) was considered. Although, in the present paper, we do not restrict ourselves to the condition of nuclear transmutation, we shall again take the case as \( \mu = 8.5 \) together with that of \( \mu = 32 \), the latter case probably representing the mean atomic weight of all the elements in the earth. The results of calculation are shown in Figs. 5, 6. The temperature at any point, from the nature of things, is proportional to the atomic weight of the polytropic gas. The variation of temperature with depth tends to diminish with increasing values of \( n \). At all events, the temperature within the core in the present cases is more than \( 10^4 \) °K.

5. Condensation of a polytropic gas ball.

The distributions of temperature and the condensing points of the elements in a polytropic gas ball corresponding to the primitive earth were investigated by Mercier\(^5\) and also by us\(^6\). Since the temperature of the gas ball outside a certain radius is less than the condensing point (corresponding to partial pressure) of some element in the ball, the same element resting outside that radius ought to condense. The element thus condensed, however, cannot fall below the radius in question in the gaseous ball. If every element in the gas ball condenses, then the remaining gaseous part will tend to be sealed in the central spherical region of that ball. According to our present idea, if the core of the earth were gaseous, the same core would have been formed in this way. Lynch\(^7\) recently suggested that the earth's core is a solid solution, that is a gas, probably hydrogen, heavily occluded by a metal. The polytropic condition of the core in our case may be assumed to be an idealized state of that suggested by Lynch.

The distribution of temperature of the polytropic gas ball for two cases \( n = 1, 0.5 \) are shown in Figs. 7, 8. The distributions of the condensing points for iron and silicon within that ball for the same cases of \( n \) are indicated by broken lines in these figures. The mass of the gas within any radius in the ball is also shown by chain line in these figures, the mass corresponding to that of the core of the present earth being represented by \( M_c \).

With increasing condensation of the various elements, the boundary

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between the condensed outer shell and the sealed inner gas tends to decrease in radial distance from the centre of the earth. This results from the increasing pressure within the shell and the core with increase in the mass of the condensed part. The temperature of the gaseous part also increases with increasing pressure. It will be seen from Figs. 7, 8 that, even if the condensed element were iron or silicon, the mass of the gaseous part that cannot condense would, although incidentally, fairly resemble mass \( M_s \) of the core of the present earth, at any rate for \( n = 1 \) or 0·5. If we specially plot the mass of the gaseous part (not condensed) on the basis of \( n \) (Fig. 9), it can be ascertained that the mass of the core in the case of iron being condensed, approaches in quantity more that of the core of the present earth. Thus, although it does not at once follow that the core of the earth is ferrous, it is possible to conclude that the shell of the earth tends to be ferrous with depth. The intersection of the line for Fe and that for \( M_s \) in Fig. 9 is near abscissa \( n = 0·3 \). In Section 3, it was shown that the distribution of wave velocity for the polytropic gas condition \( n = 0·3 \) quite resembles that of the velocity of longitudinal waves in the core of the actual earth, whence it holds that if the core of the earth were gaseous, the polytropic condition of the gas would be \( n = 0·3 \).

6. **General summary and concluding remarks.**

On the assumption that the earth's core is gaseous, the polytropic
condition of the same gaseous part was investigated mathematically. From the conditions of density, pressure, and longitudinal waves in the core, it is likely that the polytropic index of the gas is nearly \( n = 0.3 \). On the other hand, the condensation of the primitive gaseous ball of the earth was considered separately, from which it was also ascertained that the polytropic index of the ball should also be \( n = 0.3 \). It was concluded that in the beginning stage of the earth, the gaseous part that was not condensed was sealed near the central region of the ball and pressed from outside, the temperature of the gaseous part having increased with increasing pressure. However, this condition may be an idealized case of the state shown by Lynch, that is, the state of a gas heavily occluded by a metal.

In conclusion, we wish to express our thanks to the officials for Scientific Research in the Ministry of Education for financial aid (Funds for Scientific Research) granted us for a series of investigations, of which this study is a part.

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1. 地球核の熱力学的成因について (第2報)

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この前の報告でも述べたように，地球核が火山体であると仮定した場合に於ける火山のポリトロープの状態の数理的研究を積して，密度，圧力，緩波の速度の関係からポリトロープの指数は \( n = 0.3 \) 位が適當であることがわかった。一方に於ては，地球が火山状の母体から液化した状態を考へれば亦ポリトロープの指数が \( n = 0.3 \) 位である方がよい。結論として，この母体が冷却したときに外殻に液化した部分が出来，巻転の部分は中央に封じ込まれながら周辺から圧力を加へられたとすればよいのである。勿論，この巻転の部分は金属と火山の周溶体になった状態の理想的場合を考えてよいのである。巻転の砂線はこの圧力によって次第に上昇した事になるのである。