Logarithmic forms of algebraic varieties

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In the paper [2], "On logarithmic Kodaira dimension of algebraic varieties", the author introduced the concepts of logarithmic plurigenera $\bar{P}_m$ and of logarithmic Kodaira dimension $\bar{\kappa}$, which plays the important role in the study of algebraic varieties up to proper birational equivalence. Abelian varieties and Albanese maps are very useful in the theory of birational geometry of complete algebraic varieties.

Here we shall introduce the analogs of abelian varieties and Albanese maps into the classification theory of algebraic varieties up to proper birational equivalence. For the purpose, we study logarithmic 1-forms by applying Deligne's theorem concerning Hodge structure of algebraic varieties [1]. We consider logarithmic forms along singular divisors, which are logarithmic forms in the sense of [1] or [2]. As for 1-forms, both definitions coincide (Theorem 1). In the proof, the homomorphism $d\log: E_0 \rightarrow T_1(V)/T_1(\bar{V})$ is introduced and it may be regarded as the inverse of the residue map: Res of A. Weil. Note that by Theorem 1, we know that logarithmic 1-forms are exactly the same as rational 1-forms of the third kind in the sense of J. P. Serre [3].

Quasi-abelian varieties were defined in [2], which are extensions of abelian varieties by algebraic tori. These are occasionally called semi-abelian varieties (by Mumford). However, in proper birational geometry of quasi-projective varieties, these play the same role as abelian varieties do in birational geometry. Further, the *quasi-Albanese map* is defined and the universal property of it is proved in Proposition 4. As a simple application of the theory of quasi-Albanese maps we give another proof of the unit theorem by Ax and Lichtenbaum in [4].

Such a universal property was formulated and the existence of the universal object was proved for the wider classes of commutative algebraic groups of certain ty peby Serre [3]. His method is purely algebraic. But our theory of quasi-Albanese maps are geometric and may be useful for the classification theory of algebraic varieties in proper birational geometry. Actually, we can develop the classi fication theory of quasi-projective planes, i.e., $\boldsymbol{P}^2-$divisor, by using $\bar{\kappa}$ and $\bar{q}$, in other words, by logarithmic pluricanonical maps and quasi-Albanese maps. The classification theory of quasi-projective planes is much simpler than Enriques' classification theory of projective surfaces but is very similar to the classical one.

For that purpose we need to analyze subvarieties of a quasi-abelian variety. In particular, we have to transplant Ueno's theorems concerning abelian varieties into the field of our proper birational geometry. In fact, we formulate and prove counterparts of Ueno's theorems for closed subvarieties of a quasi-abelian variety (Theorems 4, 5, 6) and for open subvarieties of a quasi-abelian variety (Theorem 7 and Proposition 11). As an application, we get the following result: Let $V=\boldsymbol{P}^n-$a union of hyperplanes. Then $V=A^{\alpha}\times G_m^{\beta}\times V_1$, V_1 being of hyperbolic type. In this example, we have a very simple classification theory, which is, at the same time, very similar to the Enriques-Kodaira classification theory of surfaces.

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1. Let k be an algebraically closed field of characteristic zero, occasionally $k=\boldsymbol{C}$. We shall work in the category of schemes over k. Let $\bar{V}$ be a complete non-singular algebraic variety and $\bar{D}=\sum D_j$ an effective divisor with multiplicity 1 at each irreducible component (reduced divisor).

We define the *sheaf* $\Omega^q(\log D)$ *of germs of logarithmic q-forms along* $\bar{D}$ as follows:

(i) $\Omega^q(\log \bar{D})$ is the subsheaf of $\Omega^q_{\bar{V}}(*)$, the sheaf of rational q forms of $\bar{V}$,

(ii) $\Omega^q(\log \bar{D})|V=\Omega^q_V$, V being $\bar{V}-\bar{D}$,

(iii) for any $p\in\bar{D}$, let $(z_1, \cdots, z_n)$ be a system of regular parameters of V at p and let $f_1\cdots f_m=0$ define D around p. Then

$$\omega=\sum a_I(z)\frac{df_{i(1)}}{f_{i(1)}}\wedge\cdots\wedge\frac{df_{i(r)}}{f_{i(r)}}\wedge dz_{i(r+1)}\wedge\cdots\wedge dz_{i(q)},\ a_I(z)\in\mathcal{O}_p\,.$$

Define

$$T_{m_1,\ldots,m_n}(\bar{V},\bar{D})=H^0(\bar{V},(\Omega^1\log\bar{D})^{\otimes m_1}\otimes\cdots\otimes(\Omega^n\log\bar{D})^{\otimes m_n})\,.$$

Here by $(\bar{V},\bar{D})$ we denote the pair of $\bar{V}$ and $\bar{D}$. Then for any morphism $\bar{f}:\bar{V}_1\to\bar{V}_2$, we have a linear map

$$f^*: T_{m_1,\ldots,m_n}(\bar{V}_2,\bar{D}_2)\to T_{m_1,\ldots,m_n,0,\ldots,0}(\bar{V}_1,\bar{f}^{-1}(\bar{D}_2))\,,^{1)}$$

1) For an effective divisor $\bar{D}_2$, we have $\bar{f}^*(\bar{D}_2)=\sum r_jC_j$, $r_j>0$, C_j being prime divisors. Define $\bar{f}^{-1}(\bar{D}_2)=\sum C_j$.

when $n=\dim \bar{V}_2$. If $V_2=\bar{V}_2-\bar{D}_2$, $V_1=\bar{V}_1-\bar{D}_1 \underset{i}{\hookrightarrow} \bar{V}_1-\bar{f}^{-1}\bar{D}_2=\bar{f}^{-1}(V_2)$, and $f=\bar{f}^{-1}|V_1$, then

$$\begin{tikzcd}
f^*: T_{m_1,\dots,m_n}(\bar{V}_2, \bar{D}_2) \arrow[r] \arrow[dr, "\bar{f}^*"'] & T_{m_1,\dots,m_n,0,\dots,0}(\bar{V}_1, \bar{D}_1) \\
& T_{m_1,\dots,m_n,0,\dots,0}(\bar{V}_1, \bar{f}^{-1}\bar{D}_2) . \arrow[u, "i^*"']
\end{tikzcd}$$

Note that f is proper if and only if $\bar{D}_1=\bar{f}^{-1}\bar{D}_2$. Moreover, assume that $V_1=V_2$. Then

$$\dim T_{m_1,\dots,m_n}(\bar{V}_1, \bar{D}_1) \geqq \dim T_{m_1,\dots,m_n}(\bar{V}_2, \bar{D}_2) .$$

In [2], we proved that if $\bar{D}_1$ is a divisor of normal crossing type, $\dim T_{m_1,\dots,m_n}(\bar{V}_1, \bar{D}_1)$ is determined only by $V_1=\bar{V}_1-\bar{D}_1$. Hence we write

$$T_{m_1,\dots,m_n}(V_1)=T_{m_1,\dots,m_n}(\bar{V}_1, \bar{D}_1) .$$

Thus letting $\bar{V}$ be a smooth compactification of $V_1=V$ with boundary $\bar{D}$, we have

$$\bar{P}_{m_1\dots,m_n}(V)=\dim T_{m_1,\dots,m_n}(V) \geqq \dim T_{m_1,\dots,m_n}(\bar{V}, \bar{D}) .$$

PROBLEM: $\bar{P}_{m_1,\dots,m_n}(V)=\dim T_{m_1,\dots,m_n}(\bar{V}, \bar{D})$?

2. Write $T_i(V)=T_{0,\dots,\overset{i}{1},\dots,0}(V)$ and $T_i(\bar{V}, \bar{D})=T_{0,\dots,\overset{i}{1},\dots,0}(\bar{V}, \bar{D})$.

THEOREM 1. $T_1(V)=T_1(\bar{V}, \bar{D})$.

PROOF. Let $\{V_\alpha=\operatorname{Spec} A_\alpha\}$ be a covering of affine open subsets of $\bar{V}$ and for each $\bar{D}_j$, fix a regular function $F_{\alpha,j}\in A_\alpha$ which defines $D_j\cap V_\alpha$. Then $\{F_{\alpha,j}/F_{\beta,j}=e_{\alpha\beta,j}\}\in H^1(V, \mathcal{O}^*)$, which we write $\delta(\bar{D}_j)$. Thus we have a homomorphism:

$$\delta: E=\sum Z\bar{D}_j \to H^1(V, \mathcal{O}^*) .$$

Consider the following diagram.

$$\begin{tikzcd}
& H^1(V, \mathcal{O}) \arrow[d, "j"] \arrow[dr, "0"] & \\
E=\sum Z\bar{D}_j \arrow[r, "\delta"] & H^1(\bar{V}, \mathcal{O}^*) \arrow[d, "\vartheta"] \arrow[r, phantom, "\circlearrowleft"] & H^1(\bar{V}, \Omega^1) \arrow[d, phantom, "\cap"] \\
& H^2(\bar{V}, Z) \arrow[r] & H^2(\bar{V}, C)
\end{tikzcd}$$

Put $\operatorname{Ker}(\vartheta\delta)=E_0\subset E$. Take $\Delta\in E_0$. On each V_α, Δ is defined by

$$F_\alpha=c_\alpha \prod F_{\alpha,j}^{n_j}, \quad n_j\in Z, \quad c_\alpha\in A_\alpha^* .$$

$\vartheta\delta(\Delta)=0$ implies the existence of $\{a_{\alpha\beta}\}\in H^1(\bar{V}, \mathcal{O})$ such that

$$F_\alpha=(\exp 2\pi\sqrt{-1}a_{\alpha\beta})F_\beta \quad \text{on} \quad V_\alpha\cap V_\beta .$$

From this follows

$$\frac{dF_\alpha}{F_\alpha}=2\pi\sqrt{-1}da_{\alpha\beta}+\frac{dF_\beta}{F_\beta} .$$

On the other hand $o(a_{\alpha\beta})=0$ implies $da_{\alpha\beta}=(\omega_\alpha-\omega_\beta)/2\pi\sqrt{-1}$, where ω_α is a holomorphic 1-form on V_α. Hence

$$\frac{dF_\alpha}{F_\alpha}-\omega_\alpha=\frac{dF_\beta}{F_\beta}-\omega_\beta .$$

Thus we get a logarithmic form $\{dF_\alpha/F_\alpha-\omega_\alpha\}\in T_1(\bar{V}, \bar{D})$. However $\{\omega_\alpha\}$ is unique up to an addition of logarithmic 1-form. Hence, if we fix $F_{\alpha,1}, \cdots, F_{\alpha,r}$, we obtain a homomorphism $d\log$:

$$E_0\ni\Delta=\{(c_\alpha F_{\alpha,1}^{n_1}\cdots F_{\alpha,r}^{n_r})\}\mapsto\left\{\sum n_j\frac{dF_{\alpha,j}}{F_{\alpha,j}}-\omega_\alpha\right\}\in T_1(\bar{V}, \bar{D})/T_1(\bar{V}) .$$

If $d\log\Delta=0$, then dF_α/F_α is holomorphic. Hence $F_\alpha\in A_\alpha^*$, and so $\Delta=0$ in E_0. Put $\rho=\operatorname{rank}E_0$. Then we have a base $\{\Delta_1, \cdots, \Delta_\rho\}$. By $\hat{F}_{\alpha,j}$ we denote the F_α for Δ_j. We get $\boldsymbol{Q}$-linearly independent $d\log\Delta_1, \cdots, d\log\Delta_\rho$. We prove that these are $\boldsymbol{C}$-linearly independent by the following

LEMMA 1. *Let* Spec A *be a* $\boldsymbol{C}$*-algebraic non-singular variety and let* $F_1, \cdots, F_r\in A$ *and* $\hat{F}_i=\Pi F_j^{n_{ij}}$. *Assume that* $d\hat{F}_1/\hat{F}_1, \cdots, d\hat{F}_\rho/\hat{F}_\rho$ *are* $\boldsymbol{Q}$*-linearly independent modulo holomorphic* 1*-forms. Then these are* $\boldsymbol{C}$*-linearly independent modulo holomorphic* 1*-forms.*

PROOF. Assume that

$\lambda_1d\hat{F}_1/\hat{F}_1+\cdots+\lambda_\rho d\hat{F}_\rho/\hat{F}_\rho=$holom., where $\lambda_1, \cdots, \lambda_\rho\in\boldsymbol{C}$. Let $\hat{\rho}=\dim_{\boldsymbol{Q}}(\sum\boldsymbol{Q}\lambda_i)$. We choose a base $\{\mu_1, \cdots, \mu_{\hat{\rho}}\}$ of $\sum\boldsymbol{Q}\lambda_i$. Hence

$$\lambda_i=\sum_{j=1}^{\hat{\rho}}\nu_{ij}\mu_j , \qquad \nu_{ij}\in\boldsymbol{Q} .$$

Thus

$$\sum\lambda_i d\hat{F}_i/\hat{F}_i=\sum(\sum\nu_{ij}d\hat{F}_i/\hat{F}_i)\mu_j .$$

Put $\tilde{F}_j=\Pi\hat{F}_i^{\nu_{ij}c}$, where $c\neq0$ is an integer such that $c\nu_{ji}\in\boldsymbol{Z}$. Hence

$$\mu_1d\tilde{F}_1/\tilde{F}_1+\cdots+\mu_{\hat{\rho}}d\tilde{F}_{\hat{\rho}}/\tilde{F}_{\hat{\rho}}=\text{holomorphic}.$$

Then if $d\tilde{F}_1/\tilde{F}_1$ is not holomorphic, we have F_1 such that

$$\tilde{F}_1=c_1F_1^{e_1}\cdots, \qquad e_1\neq 0 .$$

We choose a point $p\in (V(F_1)-\bigcup_{j\geqq 2} V(F_j))\cap \mathrm{Reg}\,(V(F_1))$. Then we have a system of regular parameters $(F_1, \zeta_2, \cdots, \zeta_n)$ at p. Write $\tilde{F}_j=F_1^{e_j}\cdot G_j$, in which G_j is non-vanishing at p. Thus

$$\sum \mu_j d\tilde{F}_j/\tilde{F}_j=(\sum \mu_j e_j)dF_1/F_1+\text{holom.}=\text{holom.}$$

Hence $\sum \mu_j e_j=0$. This contradicts the choice of $\mu_1, \cdots, \mu_\rho$. Therefore we obtain

$$\rho=\mathrm{rank}\, E_0\leqq \dim T_1(\bar{V}, \bar{D})/T_1(\bar{V}) .$$

By duality and the fundamental exact sequence, writing $i:D\hookrightarrow \bar{V}$, we get

$$H_1(V)=H_1(\bar{V}-\bar{D})\simeq H^{2n-1}(\bar{V}, \bar{D}) ,$$

and

$$H^{2n-2}(\bar{V})\xrightarrow{i^*}H^{2n-2}(\bar{D})\to H^{2n-1}(\bar{V}, \bar{D})\to H^{2n-1}(\bar{V})\xrightarrow{i^*}H^{2n-1}(\bar{D})=0 .$$

Hence $b_1(V)=b_1(\bar{V})+r-\dim \mathrm{Im}\, i^*$. By Poincaré's duality we have the dual ${}^t i^*$ of i^*:

$${}^t i^*: H^0(\bar{D})=\sum Z\bar{D}_j\to H^2(\bar{V}, Z) .$$

Clearly, ${}^t i^*(\Delta)$ is the cohomology class of Δ. Hence,

$${}^t i^*=\vartheta\delta .$$

And so $\rho=\dim \mathrm{Ker}\, {}^t i^*=r-\dim \mathrm{Im}\, i^*$. Hence

$$b_1(V)-b_1(\bar{V})=\rho .$$

PROPOSITION 1 (Deligne).

$$\bar{q}V-q\bar{V}=b_1(V)-b_1(\bar{V}) .$$

This follows from his theorem, the degeneracy of the spectral sequence of the cohomology of logarithmic forms ([1], (3.2.13), (ii)). Thus

$$\rho=\bar{q}V-q\bar{V}=\dim\,(T_1(V)/T_1(\bar{V})) .$$

Consequently, $\dim T_1(V)=\dim T_1(\bar{V}, \bar{D})$. Hence $T_1(\bar{V}, \bar{D})=T_1(V)$. Thus we complete the proof of Theorem 1.

Example 1. Let $\bar{S}$ be an algebraic $K3$ surface and $\bar{D}$ a purely 1-dimensional curve on it. Assume $\bar{\kappa}(\bar{S}-\bar{D})=0$. Then $\bar{D}=\sum D_j$ satisfies 1) $D_j^2=-2$ and 2) $((D_i, D_j))$

is negative definite (see Example 4 of [2]). Hence we have a Dynkin diagram: $A_n, D_n, \cdots$. Anyway 2) implies that $D_1, \cdots, D_r$ are linearly independent in $H^2(\bar{S}, \boldsymbol{Z})$. Thus $\bar{q}(\bar{S}-\bar{D})=0$. Note that these $\bar{S}-\bar{D}$ may be regarded as some versions of $K3$ surfaces. We have a sublattice $\sum \boldsymbol{Z}D_j \subset H^2(\bar{S}, \boldsymbol{Z})$. By the theory of elementary divisors, we have a base $\{\tau_1, \cdots, \tau_{22}\}$ of $H^2(\bar{S}, \boldsymbol{Z})$ such that $D_1=d_1\tau_1, \cdots, D_r=d_r\tau_r$. Then $D_j^2=-2=d_j^2\tau_j^2$. Hence $d_j=1$. Moreover, the intersection form of $H^2(\bar{S}, \boldsymbol{Z})$ is of the form $E_8 \oplus E_8 \oplus U \oplus U \oplus U$, where $U=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and E_8 denotes a matrix corresponding to the Dynkin diagram E_8. Hence we get $r \leqq 16$.

3. In [1], Deligne proves that logarithmic q-forms are d-closed. Hence fixing a point $o \in V$, we have a multivalued function $\int_0^p \omega$, for $\omega \in T_1(V)$.

Choose a base $\{\omega_1, \cdots, \omega_q\}$ of $T_1(\bar{V})$ and a base $\{\omega_1, \cdots, \omega_q, \varphi_1, \cdots, \varphi_t\}$ of $T_1(V)$, where $q=q(\bar{V})$ and $t=q(\bar{V})-q(V)$. Let $\{\xi_1, \cdots, \xi_{2q}\}$ be a base of the free part of $H_1(\bar{V}, \boldsymbol{Z})$ and consider the exact sequence:

$$H_1(V, \boldsymbol{Z}) \xrightarrow{i_*} H_1(\bar{V}, \boldsymbol{Z}) \to H_1(\bar{V}, V; \boldsymbol{Z})$$
$$\qquad\qquad\qquad\qquad || $$
$$H^{2n-1}(\bar{D}; \boldsymbol{Z})=0, \quad i \text{ being the immersion: } V \hookrightarrow \bar{V}.$$

Hence we choose $\eta_1, \cdots, \eta_t \in \operatorname{Ker} i_*$ such that $\{\xi_1, \cdots, \xi_{2q}, \eta_1, \cdots, \eta_t\}$ forms a base of the free part of $H_1(V, \boldsymbol{Z})$. Put vectors in $\boldsymbol{C}^q$:

$$A_1=\left(\int_{\xi_1}\omega_1, \cdots, \int_{\xi_1}\omega_q, \cdots, \int_{\xi_1}\varphi_t\right)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A_{2q}=\left(\int_{\xi_{2q}}\omega_1, \cdots, \int_{\xi_{2q}}\omega_q, \cdots, \int_{\xi_{2q}}\varphi_t\right)$$

$$B_1=\left(\int_{\eta_1}\omega_1, \cdots, \int_{\eta_1}\omega_q, \cdots, \int_{\eta_1}\varphi_t\right)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$B_t=\left(\int_{\eta_t}\omega_1, \cdots, \int_{\eta_t}\omega_q, \cdots, \int_{\eta_t}\varphi_t\right).$$

Note that

$$\int_{\eta_i}\omega_j=\int_{\eta_i}i^*\omega_j=\int_{i_*\eta_i}\omega_j=0 .$$

PROPOSITION 2. *These vectors $A_1, \cdots, A_{2q}; B_1, \cdots, B_t$ are $\boldsymbol{R}-\boldsymbol{C}$ linearly independent.*

PROOF. Assume

$$\sum a_i A_i + \sum b_j B_j = 0 \quad \text{where} \quad a_i \in \boldsymbol{R},\ b_j \in \boldsymbol{C}.$$

From the note above, we see that $a_1, \cdots, a_{2q}$ are all 0, because the former parts of A_i are $\boldsymbol{R}$-linearly independent. Put

$$\hat{B}_j = \left(\int_{\eta_j} \varphi_1, \cdots, \int_{\eta_j} \varphi_t \right).$$

Then we have only to prove that $\hat{B}_1, \cdots, \hat{B}_t$ are $\boldsymbol{C}$-linearly independent. This is equivalent to saying that $\det({}^t B_1, \cdots, {}^t B_t) \neq 0$. Thus by the following lemma we complete the proof.

LEMMA 2. *Let $\varphi \in T_1(V)$ and assume $\int_\eta \varphi = 0$ for any $\eta \in$ the free part of $H_1(V, \boldsymbol{Z})$. Then $\varphi \in T_1(\bar{V})$.*

PROOF. If $\varphi \notin T_1(\bar{V})$, φ has a pole along some $\bar{D}_j$. Let p be a general point of $\bar{D}_j$. Then we have a system of local coordinates $(z_1, z_2, \cdots, z_n)$ around p such that $z_1 = 0$ defines $\bar{D}_j$. Hence

$$\varphi = \alpha(z) \frac{dz_1}{z_1} + \text{holomorphic 1-form}.$$

Here we may assume $\alpha(z) = \alpha(z_2, \cdots, z_n)$. Since $d\varphi = 0$, we get

$$d\varphi = d\alpha \wedge \frac{dz_1}{z_1} + d\,(\text{holom}) = 0.$$

Hence $d\alpha = 0$. This implies α is a constant. Consider a 1-circle γ_j around $\bar{D}_j$ at p. Then

$$0 = \int_{\gamma_j} \varphi = \alpha \int_{\circlearrowleft} \frac{dz_1}{z_1} = \alpha 2\pi\sqrt{-1}.$$

Hence $\alpha = 0$. This implies that φ is holomorphic at p.

In view of Lemma 2, we can choose $\varphi_1, \cdots, \varphi_t$ such that

$$\int_{\eta_j} \varphi_i = 2\pi\sqrt{-1}\, \delta_{i,j}.$$

Therefore let $\tilde{L} = \sum \boldsymbol{Z} A_i + \sum \boldsymbol{Z} B_j$ and $L = \sum \boldsymbol{Z} A_i$ modulo $\boldsymbol{C}^t$ and $L_0 = \sum \boldsymbol{Z} \hat{B}_j$. We get

$$0 \to \boldsymbol{C}^t/L_0 \to \boldsymbol{C}^{q+t}/\tilde{L} \to \boldsymbol{C}^q/L \to 0 .$$

Here $T=\boldsymbol{C}^t/L_0$ is an algebraic torus G_m^t and $\mathcal{A}_{\bar{V}}=\boldsymbol{C}^q/L$ is the Albanese variety of $\bar{V}$. Hence we may call

$$\tilde{\mathcal{A}}_V=\boldsymbol{C}^{q+t}/\tilde{L} \text{ the } \textit{quasi-Albanese variety} \text{ of } V .$$

4. For a point $p \in V$, we define

$$\alpha_V(p)=\left(\int_0^p \omega_1, \cdots, \int_0^p \omega_q, \int_0^p \varphi_1, \cdots, \int_0^p \varphi_t\right) \in \tilde{\mathcal{A}}_V .$$

This point does not depend on the choice of the path from 0 to p. Thus we get the *quasi-Albanese map*:

$$\alpha_V : V \to \tilde{\mathcal{A}}_V .$$

PROPOSITION 3. *α_V is a rational map from $\bar{V}$.*

PROOF. May assume $\bar{D}$ to be a divisor of simple normal crossing type. Let $p \in \bar{D}$ and choose a system of regular parameters $(z_1, \cdots, z_n)$ such that $z_1 \cdots z_r=0$ defines $\bar{D}$ around p. As in the proof of Proposition 2,

$$\varphi_j=\sum_{i=1}^r \alpha_{ij} dz_i/z_i+\tilde{\varphi}_j ,$$

where $\alpha_{ij} \in \boldsymbol{C}$ and the $\tilde{\varphi}_j$ are holomorphic 1-forms around p. Let δ_j be a 1-circle around D_j near p. Then $i_*\delta_j=0$. Hence,

$$\delta_j=\sum m_{ij}\gamma_i+ \text{a torsion} , \qquad m_{ij} \in \boldsymbol{Z} .$$

From this follows

$$\alpha_{ij}=\frac{1}{2\pi\sqrt{-1}}\int_{\delta_j} \varphi_i=\frac{1}{2\pi\sqrt{-1}}\sum_l m_{lj}\int_{\gamma_l} \varphi_i=m_{ij} .$$

For a point $p' \in V$ near p,

$$\exp\int_0^{p'} \varphi_i=\exp \sum m_{il} \log z_l \cdot \exp\int_0^{p'} \tilde{\varphi}_i=\Pi\, z m_{il} \exp\int_0^{p'} \tilde{\varphi}_i .$$

This implies that α_V is a rational map from $\bar{V}$. Thus we have proved that $\tilde{\mathcal{A}}_V$ is a quasi-abelian variety and α_V is a morphism in the category of $\boldsymbol{C}$-schemes. Clearly,

$$\alpha_V^* : T_1(\tilde{\mathcal{A}}_V) \rightleftarrows T_1(V) \quad \text{and} \quad (\alpha_V)_* : H_1(V, \boldsymbol{Z}) \twoheadrightarrow H_1(\mathcal{A}_V, \boldsymbol{Z})$$
$$\text{and} \quad \operatorname{Ker}(\alpha_V)_*=H_1(V, \boldsymbol{Z})_{\text{tor}} .$$

5. Let $f: V_1 \to V_2$ be a morphism. Then f induces $f_*: \tilde{\mathcal{A}}_{V_1} \to \tilde{\mathcal{A}}_{V_2}$ which satisfies $f_* \cdot \alpha_{V_2} = \alpha_{V_1} \cdot f$. Let $\mu: V' \to V$ be a proper birational morphism. Then $\mu^*: T_1(V) \cong T_1(V')$ and $H_1(V, Z)_0 \cong H_1(V', Z)_0$. Hence $\mu_*: \tilde{\mathcal{A}}_V \cong \tilde{\mathcal{A}}_{V'}$. Thus any strictly rational map $f: V_1 \to V_2$ induces a morphism

$$f_*: \tilde{\mathcal{A}}_{V_1} \to \tilde{\mathcal{A}}_{V_2} .$$

In case V is singular, we take a non-singular model (V^*, μ) of V, that is, V^* being non-singular, $\mu: V^* \to V$ being a proper birational morphism. Define the quasi-Albanese variety of V by $\tilde{\mathcal{A}}_V = \tilde{\mathcal{A}}_{V^*}$. Hence we have a strictly rational map $\alpha_V = \alpha_{V^*} \cdot \mu^{-1}: V \to V^* \to \tilde{\mathcal{A}}_V$. Summarizing the argument above, we obtain

PROPOSITION 4. *For any algebraic variety V, the quasi-Albanese variety $\tilde{\mathcal{A}}_V$ of V and the quasi-Albanese map $\alpha_V: V \to \tilde{\mathcal{A}}_V$ are defined. α_V is strictly rational and is defined at non-singular points of V. Any strictly rational map $f: V_1 \to V_2$ induces the morphism $f_*: \tilde{\mathcal{A}}_{V_1} \to \tilde{\mathcal{A}}_{V_2}$ which satisfies the following commutative diagram:*

$$\begin{array}{ccc} V_1 & \xrightarrow{f} & V_2 \\ {\scriptstyle \alpha_{V_1}}\downarrow & & \downarrow{\scriptstyle \alpha_{V_2}} \\ \tilde{\mathcal{A}}_{V_1} & \xrightarrow[f_*]{} & \tilde{\mathcal{A}}_{V_2} \end{array} .$$

Moreover, the morphism f_* satisfying $f_* \alpha_{V_1} = \alpha_{V_2} \cdot f$ is unique.

From this we get the universal property of the quasi-Albanese map: Let $\varphi: V \to \tilde{\mathcal{A}}$ be a strictly rational map from V into a quasi-abelian variety $\tilde{\mathcal{A}}$. Then there exists a morphism $\varphi_1: \tilde{\mathcal{A}}_V \to \tilde{\mathcal{A}}$ such that $\varphi_1 \cdot \alpha_V = \varphi$. Such φ_1 is unique and φ_1 is a translation of a homomorphism as an algebraic group. Hence we obtain a group homomorphism:

$$\gamma_V : \mathrm{P\,Bir}\,(V) \to \mathrm{Aut}\,(\tilde{\mathcal{A}}_V) .$$

We make the following

CONJECTURE: If $\bar{\kappa} V \geqq 0$, then $\mathrm{Ker}\,(\gamma_V)^0$ is an algebraic torus.

6.

PROPOSITION 5. *Let B be the closure of $\alpha_V(V)$ in $\tilde{\mathcal{A}}_V$. Then $\bar{q}B = \bar{q}V$ and $(\alpha_V)_*: \tilde{\mathcal{A}}_V \cong \tilde{\mathcal{A}}_B$. Moreover, $\alpha_V(V)$ generates $\tilde{\mathcal{A}}_V$ as an algebraic group.*

PROOF. Decompose α_V into a composition of $\alpha: V \to B$ and $j: B \to \tilde{\mathcal{A}}_V$. Then

$$T_1(V) \xleftarrow{\alpha^*} T_1(B) \xleftarrow{j^*} T_1(\tilde{\mathcal{A}}_V)$$

and $\alpha^* \circ j^* = \alpha_V^* = \mathrm{id}$. α is dominant and so α^* is injective. Hence α^* is bijective. Thus $\bar{q}V = \bar{q}B$. Moreover $(\alpha_V)_* = j_* \cdot \alpha_*$ is the isomorphism: $\tilde{\mathcal{A}}_V \to \tilde{\mathcal{A}}_B \to \tilde{\mathcal{A}}_V$. Hence α_* is injective. Thanks to $\dim \tilde{\mathcal{A}}_V = \bar{q}V = \bar{q}B = \dim \tilde{\mathcal{A}}_B$, α_* turns out to be isomorphic. Hence j_* is also isomorphic.

COROLLARY 1. *If V is non-singular and* $\dim B=1$, *then B is non-singular and a general fiber of α_V is irreducible.*

PROOF. Since $\dim B=1$, B coincides with $\alpha_V(V)$. Consider the normalization of $\alpha_V : V \to B$ and denote it by $\alpha_V^\star : V \to B^\star$. Assume $\dim V=1$. Then by $\bar{g}V = \bar{g}(B^\star)$ we see that $\alpha_V^\star$ is the isomorphism or $\bar{q}V=1$ and $\alpha_V^\star$ is étale by Theorems 1 and 2 in [2]. In the latter case, $V=G_m$ or an elliptic curve and so $\alpha_V = \alpha_V^\star = \mathrm{id}$. On the other hand, if $\bar{g}V \geqq 2$, $|\bar{K}+\bar{D}|$ has no base points. Hence B is non-singular and so $V=B$. Then assume $\dim V \geqq 2$. $\tilde{\mathcal{A}}_{B^\star} \rightrightarrows \tilde{\mathcal{A}}_V$ and so the normalization $\lambda : B^\star \to B$ induces $\lambda_* : \tilde{\mathcal{A}}_{B^\star} \rightrightarrows \tilde{\mathcal{A}}_V$. Since $B^\star \subset \tilde{\mathcal{A}}_{B^\star} \to \tilde{\mathcal{A}}_V$ and $B \subset \tilde{\mathcal{A}}_V$, $\lambda_*|B^\star : B^\star \rightrightarrows B$. Hence B is non-singular.

Let $\phi \cdot \beta : V \to B^\sharp \to B$ be the Stein factorization of $\alpha_V : V \to B$, in which $B^\sharp$ is the normalization of B in $k(V)$. Then $\beta : V \to B^\sharp \hookrightarrow \tilde{\mathcal{A}}_{B^\sharp}$ is written as $\beta_1 \cdot \alpha_V$, where $\beta_1 : \tilde{\mathcal{A}}_V = \tilde{\mathcal{A}}_B \to \tilde{\mathcal{A}}_{B^\sharp}$. $\phi : B^\sharp \to B$ induces $\phi_* : \tilde{\mathcal{A}}_{B^\sharp} \to \tilde{\mathcal{A}}_V$ and $\phi_*|B^\sharp = \phi$. Hence $\phi_* \cdot \beta = \alpha_V$ yields $\phi_* \cdot \beta_1 \cdot \alpha_V = \alpha_V$. Thus by the universality, we obtain $\phi_* \cdot \beta_1 = \mathrm{id}$. Hence ϕ_* is surjective and β_1 is injective. Note that $\bar{q}B^\sharp = \bar{q}B$ by the inequalities: $\bar{q}V \geqq \bar{q}B^\sharp \geqq \bar{q}B = \bar{q}V$. Hence β_1 is isomorphic and ϕ_* is so too. Q.E.D.

PROPOSITION 6. *Let V_1 and V_2 be algebraic varieties. Then*

$$T_i(V_1 \times V_2) = T_i(V_1) \oplus T_{i-1}(V) \otimes T_1(V_2) \oplus \cdots \oplus T_i(V_2) .$$

Hence

$$\bar{q}_i(V_1 \times V_2) = \bar{q}_i(V_1) + \bar{q}_{i-1}(V) \cdot \bar{q}_1(V_2) + \cdots + \bar{q}_i(V_2) .$$

Moreover, $\bar{P}_m(V_1 \times V_2) = \bar{P}_m(V_1)\bar{P}_m(V_2)$. *Thus* $\bar{q}(V_1 \times V_2) = \bar{q}(V_1) + \bar{q}(V_2)$ *and* $\bar{\kappa}(V_1 \times V_2) = \bar{\kappa}(V_1) + \bar{\kappa}(V_2)$.

PROOF. We may assume V_1 and V_2 to be non-singular. By $\bar{V}_i$ we denote compactifications of V_i with smooth boundaries $\bar{D}_i$. Then $\bar{V}_1 \times \bar{V}_2$ turns out to be the compactification of $V_1 \times V_2$ with smooth boundary $\bar{D} = \bar{D}_1 \times \bar{V}_2 + \bar{V}_1 \times \bar{D}_2$. It is easy to check that

$$\Omega^i \log \bar{D} = \Omega^i \log \bar{D}_1 \otimes_k \mathcal{O}_{\bar{V}_2} \oplus \Omega^{i-1} \log \bar{D}_1 \otimes_k \Omega^1 \log \bar{D}_2 \oplus \cdots \oplus \mathcal{O}_{\bar{V}_1} \otimes_k \Omega^i \log \bar{D}_2 .$$

Since $\bar{V}_1$ and $\bar{V}_2$ are complete, we have

$$H^0(\bar{V}_1\times\bar{V}_2,\ \Omega^{i-j}\log\bar{D}_1\otimes_k\Omega^j\log\bar{D}_2)=H^0(\bar{V}_1,\ \Omega^{i-j}\log\bar{D}_1)\otimes H^0(\bar{V}_2,\ \Omega^j\log\bar{D}_2)\ .$$

Q.E.D.

By $p_1: V_1\times V_2\to V_1$, $p_2: V_1\times V_2\to V_2$ we denote the projections. $\varphi=(p_{1*},\ p_{2*}):\widetilde{\mathcal{A}}_{V_1\times V_2}\to\widetilde{\mathcal{A}}_{V_1}\times\widetilde{\mathcal{A}}_{V_2}$ induces

$$T_1(\widetilde{\mathcal{A}}_{V_1}\times\widetilde{\mathcal{A}}_{V_2})=T_1(\widetilde{\mathcal{A}}_{V_1})\oplus T_1(\widetilde{\mathcal{A}}_{V_2})=T_1(V_1)\oplus T_1(V_2)\leftrightarrows T_1(\widetilde{\mathcal{A}}_{V_1\times V_2})\leftrightarrows T_1(V_1\times V_2)$$

and

$$\begin{aligned}H_1(\widetilde{\mathcal{A}}_{V_1\times V_2},\ \boldsymbol{Z})&=H_1(V_1\times V_2,\ \boldsymbol{Z})_0\leftrightarrows H_1(V_1,\ \boldsymbol{Z})_0\oplus H_1(V_2,\ \boldsymbol{Z})_0\\&=H_1(\widetilde{\mathcal{A}}_{V_1},\ \boldsymbol{Z})\oplus H_1(\widetilde{\mathcal{A}}_{V_2},\ \boldsymbol{Z})=H_1(\widetilde{\mathcal{A}}_{V_1}\times\widetilde{\mathcal{A}}_{V_2},\ \boldsymbol{Z})\ .\end{aligned}$$

Hence φ is isomorphic.

By this we prove

PROPOSITION 7. *Let V_1 and V_2 be algebraic varieties and $\phi: V_1\times V_2\to\widetilde{\mathcal{A}}$ a morphism into a quasi-abelian variety $\widetilde{\mathcal{A}}$. Then there exist morphisms $\phi_1: V_1\to\widetilde{\mathcal{A}}$ and $\phi_2: V_2\to\widetilde{\mathcal{A}}$ such that $\phi=\phi_1+\phi_2$, that is, $\phi(x,\ y)=\phi_1(x)+\phi_2(y)$, where $x\in V_1$, $y\in V_2$. Such a pair $(\phi_1,\ \phi_2)$ is unique up to translation. This implies that another pair $(\phi_1',\ \phi_2')$ is written as $\phi_1'=\phi_1+c$ and $\phi_2'=\phi_2-c$, where $c\in k$.*

PROOF. $\phi_*:\widetilde{\mathcal{A}}_{V_1\times V_2}=\widetilde{\mathcal{A}}_{V_1}\times\widetilde{\mathcal{A}}_{V_2}\to\widetilde{\mathcal{A}}$ is written easily as $\varphi_1+\varphi_2$, where $\varphi_1:\widetilde{\mathcal{A}}_{V_1}\to\widetilde{\mathcal{A}}$ and $\varphi_2:\widetilde{\mathcal{A}}_{V_2}\to\widetilde{\mathcal{A}}$ are linear maps between the universal covering manifolds. Define $\phi_1=\varphi_1\cdot\alpha_{V_1}$ and $\phi_2=\varphi_2\cdot\alpha_{V_2}$. Then $(\phi_i)_*=\varphi_i$. Note that $(\varphi_1,\ \varphi_2)$ is unique up to translation. We complete the proof if we show that these ϕ_1 and ϕ_2 are really morphisms. Fix $y\in\operatorname{Reg}(V_2)$. Then $\phi_1(x)+\phi_2(y)=\phi(x,\ y)$ for $x\in\operatorname{Reg}(V_1)$. Thus a rational map $\phi_1+\phi_2(y)$ is the morphism $\phi(\ ,\ y)$. Hence ϕ_1 is the morphism. And so is ϕ_2.

THEOREM 2. *Let G be a connected algebraic group and $\widetilde{\mathcal{A}}$ a quasi-abelian variety regarded as an algebraic group. Then any strictly rational map $\varphi: G\to\widetilde{\mathcal{A}}$ is a translation of a homomorphism.*

PROOF. Let $0\in\widetilde{\mathcal{A}}$ be the neutral element of $\widetilde{\mathcal{A}}$ and assume $\varphi(1)=0$. Consider $\Psi: G\times G\to\widetilde{\mathcal{A}}$ that is defined by $\Psi(x,\ y)=\varphi(xy)$. Then by Proposition 7, we obtain $\phi_1: G\to\widetilde{\mathcal{A}}$ and $\phi_2: G\to\mathcal{A}$ such that $\phi_1(1)=0$ and $\phi_1(x)+\phi_2(y)=\Psi(x,\ y)$ where $(x,\ y)\in G\times G$. Hence $\phi_1(1)+\phi_2(y)=\Psi(1,\ y)=\varphi(y)$. This implies $\phi_2=\phi_1=\varphi$.

COROLLARY TO PROPOSITION 7 AND THEOREM 2 (Unit Theorem by Ax and Lichtenbaum). *Let k be a field of characteristic zero and A, B k-domains such* *that $Q(A)/k$ and $Q(B)/k$ are algebraically closed extensions. Consider the group $U(A)$ of units of A and write $U^*(A)=U(A)/k^*$, that is a free abelian group. Then*

$$U^*(A\otimes_k B)=U^*(A)\times U^*(B) .$$

Moreover, for connected affine algebraic groups, the only invertible regular functions are scalar multiples of multiplicative characters.

PROOF. Let A be a k-algebra. Then Hom (Spec A, G_m)$=\mathrm{Hom}_k(k[T, T^{-1}], A)=U(A)$. Therefore if k is algebraically closed and if A and B are finitely generated over k, we get the result from Proposition 7. In the general case in which k is not algebraically closed, by elementary theory of Galois cohomology, we complete the proof.

7. Let V be an algebraic variety. Then by the universality of the quasi-Albanese map, we obtain

PROPOSITION 8.

$$U(\Gamma(V, \mathcal{O}_V))=\Gamma(V, \mathcal{O}_V^*)=\mathrm{Hom}\,(V, G_m) \cong \mathrm{Hom}\,(\mathcal{A}_V, G_m)=U(\Gamma(\widetilde{\mathcal{A}}_V, \mathcal{O}_{\tilde{\mathcal{A}}_V})) .$$

Denote by $\mathrm{Hom}_{\mathrm{gr}}(\widetilde{\mathcal{A}}_V, G_m)$ the group of rational homomorphisms: $\mathcal{A}_V \to G_m$. Then we have by Proposition 8

$$\mathrm{Hom}_{\mathrm{gr}}(\widetilde{\mathcal{A}}_V, G_m) \cong U^*(\Gamma(V, \mathcal{O}_V)) .$$

Since $\mathrm{Hom}_{\mathrm{gr}}(\widetilde{\mathcal{A}}_V, G_m)$ is a free abelian group $\boldsymbol{Z}^{\bar{\rho}}$, we indicate the $\boldsymbol{Z}$-base of it by $m_1, \cdots, m_{\bar{\rho}}$. Then

$$\tau_V=(m_1, \cdots, m_{\bar{\rho}}) : V \to G_m\times \cdots \times G_m=\mathcal{T}_V .$$

We say that $\mathcal{T}_V$ is the *universal torus* of V and τ_V is the *universal torus map*. Then $\mathcal{T}_{\tilde{\mathcal{A}}_V}=\mathcal{T}_V$ and $\tau_V=\tau_{\tilde{\mathcal{A}}_V}\cdot\alpha_V$; $\tau_{\tilde{\mathcal{A}}_V}: \widetilde{\mathcal{A}}_V \to \mathcal{T}_V$ is the epimorphism. Hence $\widetilde{\mathcal{A}}_V \cong \widetilde{\mathcal{A}}^\circ \times \mathcal{T}_V$ where $\widetilde{\mathcal{A}}^\circ$ is a quasi-abelian variety such that $U(\Gamma(\widetilde{\mathcal{A}}^\circ, \mathcal{O}))=k^*$. Let $\varphi: V \to T$ be a morphism from V into an algebraic torus T. Then there is a homomorphism $\varphi_1: \widetilde{\mathcal{A}}_V \to T$ such that $\varphi=\varphi_1\cdot\alpha_V+c$. Since $\widetilde{\mathcal{A}}_V=\widetilde{\mathcal{A}}^\circ\times\mathcal{T}_V$ and $\mathrm{Hom}_{\mathrm{gr}}(\widetilde{\mathcal{A}}^\circ, G_m)=0$, we have the homomorphism $\varphi_2: \mathcal{T}_V \to T$ such that $\varphi_1(x, y)=\varphi_2(y)$. This establishes the universality of $(\mathcal{T}_V, \tau_V)$. $\mathcal{T}_{V_1\times V_2} \cong \mathcal{T}_{V_1}\times\mathcal{T}_{V_2}$ is equivalent to the unit theorem. We indicate by $\bar{\rho}(V)$ the dimension of $\mathcal{T}_V$. Then $\bar{q}(V)-q(V) \geqq \bar{\rho}(V)$. $\bar{q}(V)-q(V)=\bar{\rho}(V)$ if and only if $\widetilde{\mathcal{A}}_V=\mathcal{A}^\circ\times\mathcal{T}_V$, where $\mathcal{A}^\circ$ is an abelian variety.

8. Let $\bar{V}$ be a complete non-singular variety with $q(\bar{V})=0$ and $\bar{D}=\sum_{j=1}^{r} D_j$ a reduced divisor on $\bar{V}$. Write $V=\bar{V}-\bar{D}$ and assume $\dim \alpha_V(V)=1$. Then $\bar{q}(V)\geqq 1$. Hence we have $\bar{D}'=\sum_{j=1}^{s} D_j \leqq \bar{D}$ such that $\bar{q}(\bar{V}-\bar{D}')=1$. Let $V'=\bar{V}-\bar{D}'\supset V$. The quasi-Albanese map $\alpha_{V'}: V'\to \widetilde{\mathcal{A}}_{V'}=G_m$ is surjective. Write $j: V \hookrightarrow V'$ and $B=\alpha_V(V)$. j induces $j_*: B\to G_m$ and $k(V')=k(V)=k(\bar{V}) \hookleftarrow k(B) \hookleftarrow k(G_m)$. Since $k(V')/k(G_m)$ is the algebraically closed extension, $k(B)=k(G_m)$. B is normal. Hence $B=G_m-\{p_2, \cdots, p_{\bar{q}}\}$. Let V^* be the fiber product of V' with V over G_m. Then $V^* \hookrightarrow V'$ is the open immersion satisfying $V\subset V^* \subset V'$ and $\bar{q}V=\bar{q}V^*$. If $\bar{V}=\boldsymbol{P}^n$, we have $V=V^*$. Let D_j be defined by the homogeneous polynomial F_j of degree d_j. Then, letting $\delta_0=\text{g.c.d.}\ (d_0, d_1)$, $d_0'=d_0/\delta_0$, $d_1'=d_1/\delta_0$, we get a morphism $\alpha: \boldsymbol{P}^n-D_0\cup D_1\to G_m$ sending $p\mapsto F_1^{d_0'}(p)/F_0^{d_1'}(p)$, which is the quasi-Albanese map of $\boldsymbol{P}^n-D_0\cup D_1$ by the following observation: Let $V=\boldsymbol{P}^n-D$, $D=D_0+\cdots+D_r$ where the D_j are defined by irreducible homogeneous polynomials F_j of degree d_j. Then

$$\Gamma(V, \mathcal{O}^*)=\{\alpha \Pi F_j^{a_j};\ \textstyle\sum d_j a_j=0,\ \alpha\in k^*\}\ .$$

Denoting by $m_1=(a_{1,0}, \cdots, a_{1,r}), \cdots, m_{r-1}=(a_{r-1,0}, \cdots, a_{r-1,r}), m_r=(a_{r,0}, \cdots, a_{r,r})\in \boldsymbol{Z}^r$ the $\boldsymbol{Z}$-base of Ker $(\boldsymbol{Z}^{r+1}\xrightarrow{\varphi}\boldsymbol{Z};\ \varphi(a_0, \cdots, a_r)=\sum a_j d_j)$, we have

$$\begin{array}{ccc} V & \longrightarrow & G_m^r=\mathcal{T}_V \\ \cup\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\shortmid \\ p & \longmapsto & (\phi_1(p), \cdots, \phi_r(p))\ , \end{array}$$

where $\phi_i=F_0^{a_{i,0}}\cdots F_r^{a_{i,r}}$.

PROPOSITION 9. *Let* $V=\boldsymbol{P}^n-(D_0+\cdots+D_r)$ *such that* $\dim \alpha_V(V)=1$. *Then, letting* $\delta_j=\text{g.c.d.}\ (d_0, d_j)$, $d_j''=d_j/\delta_j$, $d_0''=d_0/\delta_j$, *we have* $d_1'=d_j''$ *and*

$$F_j^{d_0''}=\alpha_j F_0^{d_1'}+\beta_j F_1^{d_0'},\quad \textit{where}\quad \alpha_j,\ \beta_j\in k\ .$$

PROOF. Let $V'=\boldsymbol{P}^n-(D_0+D_1)$, $V''=\boldsymbol{P}^n-(D_0+D_j)$. Then

$$k(\alpha_{V'}(V'))=k(F_1^{d_0'}/F_0^{d_1'})=k(\alpha_V(V))=k(\alpha_{V''}(V''))=k(F_j^{d_0''}/F_0^{d_j''})\ .$$

Hence, there are $\alpha_j, \beta_j, \gamma_j, \delta_j\in k$ such that

$$\frac{F_j^{d_0''}}{F_0^{d_j''}}=\frac{\alpha_j F_1^{d_0'}+\beta_j F_0^{d_1'}}{\gamma_j F_1^{d_0'}+\delta_j F_0^{d_1'}}\ .$$

Since F_0, F_1, F_j are distinct irreducible polynomials, we conclude that $\gamma_j=0$ and $d_1'=d_j''$.

Note that $d_0/\text{g.c.d.}\ (d_0, d_j)=d_1/\text{g.c.d.}\ (d_1, d_j)$ and $d_0/\text{g.c.d.}\ (\text{d}_0, d_1)=d_j/\text{g.c.d.}\ (d_1, d_j)$.

Hence, there are mutually coprime numbers a, b, c and an integer e such that $d_0=abe$, $d_1=bce$, $d_j=ace$.

9. Let V_1 be a non-singular algebraic variety and F_1 a Zariski closed subset of codim $\geqq 2$. Then it is easy to see that

$$\bar{P}_{m_1,\dots,m_n}(V_1-F)=\bar{P}_{m_1,\dots,m_n}(V_1) .$$

But if V_1 is singular, this does not hold any more. For example, there exists a surface S such that $\bar{\kappa}(S-\{p_1, \cdots, p_r\})=0$, but $\bar{\kappa}S=-\infty$. However, we can prove

THEOREM 3. $\bar{q}(V_1-F_1)=\bar{q}(V_1)$.

PROOF. Let $\bar{V}_1$ be a completion of V_1 and $\bar{F}_1$ the closure of F_1 in $\bar{V}_1$. By $\bar{\mu}: \bar{V}\to \bar{V}_1$ we denote the resolution of the singularity of $\bar{V}_1$ such that $\overline{\bar{\mu}^{-1}(\bar{F}_1)}=\bar{E}=\sum_{j=1}^{e}\bar{E}_j$ and $\bar{\mu}^{-1}(\bar{V}_1-V_1)=\bar{D}=\sum_{j=1}^{r}\bar{D}_j$ are divisors of simple normal crossing type, where $V=\bar{\mu}^{-1}(V_1)$ and $\mu=\bar{\mu}|V$. We prove that $\bar{E}_1, \cdots, \bar{E}_e$ are $\boldsymbol{Q}$-cohomologically independent modulo $\bar{D}_1, \cdots, \bar{D}_r$ by induction on $n=\dim V_1\geqq 2$. May assume $\bar{V}_1$ to be normal. When $n=2$, $((\bar{E}_i, \bar{E}_j))$ is a negative definite matrix by Mumford's theorem and $(\bar{E}_i, \bar{D}_j)=0$. Hence we finish the proof in this case. In the n-dimensional case ($n\geqq 3$), we first assume

$$\sum n_i\bar{E}_i+\sum m_j\bar{D}_j=0 \quad \text{in} \quad H^2(\bar{V}, \boldsymbol{Q}) .$$

Define the subset $J\subset\{1, \cdots, e\}$ by

$$i\in J \Longleftrightarrow \bar{\mu}(\bar{E}_i)=\text{point}.$$

Let $\bar{W}_1$ be a prime divisor of $\bar{V}_1$ such that $\operatorname{codim}_{\bar{W}_1}(\bar{W}_1\cap\bar{F}_1)\geqq 2$. By $\bar{V}_2$ we denote the proper image of $\bar{W}_1$ by $\bar{\mu}^{-1}$. Let $\bar{\mu}_2: \bar{V}_2^*\to\bar{V}_2$ be the resolution of the singularity of $\bar{V}_2$. Consider the relation

$$\sum n_i\bar{\mu}_2^*(\bar{E}_i|V_2)+\sum m_j\bar{\mu}_2^*(\bar{D}_j|\bar{V}_2)=0 \quad \text{in} \quad H^2(\bar{V}_2^*, \boldsymbol{Q}) .$$

By induction hypothesis, if $\bar{E}_i|\bar{V}_2\neq 0$, then $n_i=0$. Choosing $\bar{W}_1$ properly, we conclude that $n_i=0$ if $i\notin J$. In fact, let p be a general point of F_1 and take an affine neighborhood $V(p)$ of $\bar{V}_1$ with center p. Choose an irreducible local divisor $H\subset V(p)$ which intersect properly with F_1 at p. Then the closure $\bar{H}$ of H in $\bar{V}_1$ can be used as $\bar{W}_1$. Hence we have only to prove under the condition that F is finite.

Let $\bar{V}_3$ be a general hyperplane section of $\bar{V}$ and define $\bar{W}_3'=\bar{\mu}(\bar{V}_2)$, which is a prime divisor. Then $\bar{\mu}_3=\bar{\mu}|\bar{V}_3$ is birational, which satisfies i) $\bar{\mu}_3(\bar{D}_j|\bar{V}_3)$ are divisors and ii) $\bar{\mu}_3(\bar{E}_i|\bar{W}_3')$ is a finite set. Hence

$$\sum n_i\bar{E}_i|\bar{V}_3+\sum m_j\bar{D}_j|\bar{V}_3=0 \quad \text{in} \quad H^2(\bar{V}_3, \boldsymbol{Q})$$

implies $n_i=0$ by induction hypothesis. Thus by Theorem 1, $\bar{q}(V_1-F)=\bar{q}(V_1)$.

COROLLARY. *Let V be an algebraic variety and F a closed subset of* codim $\geqq 2$. *Suppose that $V-F$ is isomorphic to a quasi-abelian variety $\tilde{\mathcal{A}}$. Then $F=\varnothing$.*

PROOF. By the theorem above, $\bar{q}(V)=\bar{q}(V-F)=\bar{q}(\tilde{\mathcal{A}})=n$, $n=\dim V$. Hence the inclusion $j: V-F\hookrightarrow V$ induces $j^*: T_1(V)\rightleftarrows T_1(V-F)$ and $j_*: \tilde{\mathcal{A}}_{V-F}\rightleftarrows\tilde{\mathcal{A}}_V$. Hence, $\alpha_V: V\to\tilde{\mathcal{A}}_V\rightleftarrows\tilde{\mathcal{A}}$ induces $\alpha_{\tilde{\mathcal{A}}}$ on $V-F\rightleftarrows\tilde{\mathcal{A}}$ by the commutativity of the diagram in Proposition 4. By the following lemma by Kawamata, we complete the proof.

LEMMA 4. *Let V be an algebraic variety and F a closed subset of V. Suppose that a morphism $f: V\to V-F$ induces the isomorphism $f|V-F$ on $V-F$. Then $F=\varnothing$.*

PROOF. Take a closed point $p\in F$ and put $p'=f(p)$. There is $p_1\in V-F$ such that $f(p_1)=p'$. p and p_1 dominate p' and so $p=p_1$ by the separability of V.

Q.E.D.

This corollary was proved in (2) under the additional assumption that $\tilde{\mathcal{A}}\hookrightarrow V$ is $\tilde{\mathcal{A}}$-equivariant.

10. Let $X=\tilde{\mathcal{A}}$ be a quasi-abelian variety, that is a $G_m^t(=T)$-bundle over the abelian variety $\mathcal{A}$. Let $T\subset\boldsymbol{P}^t$ be the natural imbedding, which is T-equivariant. Hence, we have the $\boldsymbol{P}^t$-bundle over $\mathcal{A}$, whose structure group is T. Thus we obtain the projective non-singular variety $\bar{X}$ which is the $\boldsymbol{P}^t$-bundle over $\mathcal{A}$. Denote $\bar{X}-X$ by $\bar{D}$, which is of normal crossing type. $\bar{X}$ is the compactification of X with the smooth boundary $\bar{D}$. We are able to verify that $\Omega^1\log\bar{D}\rightleftarrows\mathcal{O}_{\bar{X}}^n$, n being $\dim X$. Therefore, $\bar{K}+\bar{D}\sim 0$, $\bar{q}_i(X)=\binom{n}{i}$, $\bar{P}_m(X)=1$, $\bar{p}_g(X)=1$ and finally $\bar{\kappa}X=0$.

Actually, consider the universal covering map $\pi: U=\boldsymbol{C}^m\to\mathcal{A}$. Then the fiber product of U with $\tilde{\mathcal{A}}=X$ over $\mathcal{A}$ is $T\times U$ and $U\times_{\mathcal{A}}\bar{X}=\boldsymbol{P}^t\times U$. Let ω be a logarithmic 1-form of $\bar{X}$ along $\bar{D}$. Then by pulling back, we get the logarithmic 1-form $\tilde{\omega}$ on $\boldsymbol{P}^t\times U$ which is invariant under translations in U and is logarithmic along $(\boldsymbol{P}^t-T)\times U$. Denote by $z_1, \cdots, z_m$ and $w_1, \cdots, w_t$ the coordinate of U and of $\boldsymbol{A}^t=\boldsymbol{P}^t-$a hyperplane, respectively. Then

$$\tilde{\omega}=\sum a_i\frac{dw_i}{w_i}+\sum b_j dz_j\,, \quad \text{where} \quad a_i,\ b_j\in\boldsymbol{C}\,.$$

Let $z_{m+1}=\log w_1, \cdots, z_n=\log w_t$, which are affine coordinates of the universal covering manifold $\tilde{T}=C^t$ of T. Hence

$$\omega=\sum_{i=1+m}^{n} a_i dz_i+\sum_{j=1}^{m} b_j dz_j$$

and

$$\Omega_{\bar{X}}^1 \log(\bar{D})=\sum_{i=1}^{n} \mathcal{O}_{\bar{X}} dz_i \simeq \bigoplus^{n} \mathcal{O}_{\bar{X}} .$$

We wish to generalize Ueno's theorems concerning subvarieties of an abelian variety ([5], Theorem 10.3).

THEOREM 4. *Let W be a closed subvariety of the quasi-abelian variety $\tilde{\mathcal{A}}$. Put $r=\dim W$. Then $\bar{q}_i(W) \geqq \binom{r}{i}$, $\bar{p}_g(W) \geqq 1$, $\bar{P}_m(W) \geqq 1$, and $\bar{\kappa}(W) \geqq 0$. Moreover, the following conditions are equivalent to each other*:

a) *W is a quasi-abelian variety,*

b) *$\bar{q}_i(W)=\binom{r}{i}$, for some $i \in [1, r]$,*

c) *$\bar{p}_g(W)=1$,*

d) *$\bar{P}_m(W)=1$, for some $m \geqq 1$,*

e) *$\bar{\kappa}(W)=0$.*

PROOF. Clearly, a) implies b), c), d) and e). e) implies d), which yields c). Let W be a closed subvariety of dimension r. Choose a general point $p \in W$, around which we take a system of local parameters $(\zeta_1, \cdots, \zeta_r, \zeta_{r+1}, \cdots, \zeta_n)$ such that $\zeta_{r+1}=\cdots=\zeta_n=0$ define W. Let $\pi: C^m \rightarrow \mathcal{A}$ be the universal covering map. Choose $p_u \in \pi^{-1}(p)$ and assume $z_1(p_u)=\cdots=z_n(p_u)=0$. Hence $(z_1, \cdots, z_n)$ can be regarded as the system of local analytic parameters around p. By performing a suitable linear transformation we have $\zeta_j=z_j-\varphi_j(z_1, \cdots, z_n)$, where $\varphi_j(0)=0$ and $\partial\varphi_j/\partial z_l(0)=0$ for any j, $l \in [1, n]$. The dz_j are logarithmic 1-forms and hence $dz_j|W$ are elements of $T_1(W)$. Since $d\zeta_1, \cdots, d\zeta_r$ are linearly independent holomorphic 1-forms of W around p, $dz_1|W, \cdots, dz_r|W$ are also linearly independent. By

$$dz_j|W-\sum_{l=1}^{n} \frac{\partial\varphi_j}{\partial z_l}|W \cdot dz_l|W=0 \quad \text{for} \quad j \in [r+1, n],$$

we get

$$\sum_{l=r+1}^{n}\left(\delta_{j,l}-\frac{\partial\varphi_j}{\partial z_l}|W\right) dz_l|W=-\sum_{i=1}^{r} \frac{\partial\varphi_j}{\partial z_i}|W \cdot dz_i|W .$$

Thus for $l \in [r+1, n]$, we have

$$dz_l|W=\sum_{i=1}^{r} A_{li}(\zeta_1, \cdots, \zeta_r) \cdot dz_i|W ,$$

where the A_{li} are holomorphic functions such that $A_{li}(0)=0$, $dz_1|W, \cdots, dz_r|W\in T_1(W)$, which are linearly independent. Hence $\bar{q}_1(W)\geqq r$. Similarly,

$$T_2(W)\ni(dz_1\wedge dz_2)|W, \cdots, (dz_{r-1}\wedge dz_r)|W,$$

which are linearly independent. Hence $\bar{q}_i(W)\geqq\binom{r}{i}$.

Now, assume W satisfies the condition c). Then

$$(dz_2\wedge\cdots\wedge dz_r\wedge dz_l)|W=\alpha_{1,l}(dz_1\wedge\cdots\wedge dz_r)|W\in T_r(W)=k(dz_1\wedge\cdots\wedge dz_r|W),$$

where $\alpha_{1,l}\in k$. On the other hand,

$$(dz_2\wedge\cdots\wedge dz_r\wedge dz_l)|W=\pm A_{l,1}(dz_1\wedge\cdots\wedge dz_r)|W.$$

Hence, $\pm A_{l,1}=\alpha_{1,l}$. From this, $A_{l,1}=0$ follows, because $A_{l,1}(0)=0$. In the same way, we get $A_{l,j}=0$. Thus $dz_{r+1}|W=\cdots=dz_n|W=0$. This means that

$$\pi^{-1}(W)\subset\{z_{r+1}=\cdots=z_n=0\} \quad \text{near} \quad p_u.$$

$\{z_{r+1}=\cdots=z_n=0\}$ is of dimension r and is irreducible. Hence $\pi^{-1}(W)=\{z_{r+1}=\cdots=z_n=0\}$, and so W is a quasi-abelian variety whose uniformizing parameters are $z_1, \cdots, z_r$. Similarly, we can show that $\bar{q}_i(W)=\binom{r}{i}$ yields a).

COROLLARY. *Let W be an r-dimensional closed subvariety of an algebraic torus $T=G_m^n$. Assume $\bar{\kappa}W=0$. Then $W\simeq G_m^r$ and $T\rightrightarrows W\times G_m^{n-r}$.*

PROOF. By the proof of Theorem 4, W is defined by the following equations

$$\prod_{i=1}^{n} w_i^{m_{j,i}}=1 \quad \text{where} \quad 1\leqq j\leqq n-r.$$

The elementary divisors of the matrix $(m_{j,i})$ are denoted by $d_1, \cdots, d_{n-r}$. Then changing the coordinates, we have $w_j^{d_j}=1$. Since W is irreducible, $d_j=\pm1$. Q.E.D.

THEOREM 5. *Let W be an r-dimensional closed subvariety of $\tilde{\mathcal{A}}$. If $\bar{\kappa}=\bar{\kappa}(W)>0$, then there exists an étale covering $\pi:\tilde{\mathcal{A}}^*\to\tilde{\mathcal{A}}$ such that $\pi^{-1}(W)=\tilde{\mathcal{A}}_1\times W_1$, where $W_1\subset\tilde{\mathcal{A}}_2$ that is a quasi-abelian variety of dimension $n-\bar{\kappa}$.*

Proof is easy and so omitted.

THEOREM 6. *Let $\tilde{\mathcal{A}}$ be a quasi-abelian variety and $\varphi:\tilde{\mathcal{A}}\to W$ a dominant strictly rational map. Then $\bar{\kappa}W\leqq0$.*

PROOF. Let $\bar{X}$ be the compactification of $X=\tilde{\mathcal{A}}$ defined in the beginning of §10. Let $\bar{W}^*$ be a compactification of a non-singular model of W with smooth boundary $\bar{\Delta}$. φ determines the rational map $\bar{\varphi}:\bar{X}\to\bar{W}^*$. Performing monoidal transformations we have $\bar{\mu}:\bar{X}^*\to\bar{X}$ such that $\bar{\psi}=\bar{\varphi}\cdot\bar{\mu}^{-1}$ is a morphism. Let $\omega\in$

$T_{0,\dots,0,m}(W)$, which determines $\varphi^*(\omega)\in T_{0,\dots,0,m,0,\dots,0}(\bar{X})$. Then using the notation in [2], we can write

$$\varphi^*\omega=a_I\left(\frac{dw_{i,1}}{w_{i,1}}\wedge\cdots\right)^{\otimes m}+\cdots, \quad \text{where} \quad a_I\in k .$$

On the other hand,

$$\omega=b_\alpha\left(\frac{d\zeta_1^\alpha}{\zeta_1^\alpha}\wedge\cdots\right)^{\otimes m}\in H^0(m(\bar{K}+\bar{\varDelta})) .$$

Hence, letting $(\omega)=L$, we have

$$\mathrm{Supp}\,(\bar{\varPhi}^*(L))\subset R_{\bar{\beta}}.$$

Therefore,

$$1\leq \bar{P}_{m\nu}(W)=l_{\bar{W}}(\nu L)\leqq l_{X^*}(\bar{\varPhi}^*(\nu L))\leqq l(\nu m_1R_{\bar{\beta}})=1 \quad \text{for some} \quad m_1 .$$

Next we consider open subvarieties of a quasi-abelian variety.

PROPOSITION 10. *Let D be a reduced divisor of an algebraic torus G_m^n. Let $V=G_m^n-D$. Then the following conditions are equivalent to each other:*

a) *$D=\varnothing$, in other words, $V=G_m^n$,*

b) *$\bar{q}_i(V)=\binom{n}{i}$, for some $i\in[1, n]$,*

c) *$\bar{p}_g(V)=1$,*

d) *$\bar{P}_m(V)=1$, for some $m\geqq 1$,*

e) *$\bar{\kappa}V=0$.*

PROOF. a) implies b), c), d) and e). e) implies d) and d) yields c). Hence it suffices to prove that b) implies a) and that c) implies a). Let $G_m^n=\mathrm{Spec}\,k[x_1,\cdots,x_n,1/x_1,\cdots,1/x_n]$. Then $dx_1/x_1,\cdots,dx_n/x_n\in T_1(G_m^n)$. By $g\in k[x_1,\cdots,x_n,1/x_1,\cdots,1/x_n]$ we denote the defining equation of D. Then $dg/g\in T_1(V)$, and

$$\frac{dg}{g}\wedge\frac{dx_1}{x_1}\wedge\overset{\overset{i}{\frown}}{\cdots}\wedge\frac{dx_n}{x_n}=\lambda_i\cdot\frac{dx_1}{x_1}\wedge\cdots\wedge\frac{dx_n}{x_n}\in T_n(V) ,$$

λ_i being elements of k. Hence

$$x_i\frac{\partial g}{\partial x_i}=\pm\lambda_i g \quad \text{for} \quad i=1, 2, \cdots, n .$$

Since $g=\sum a_J x_1^{j_1}\cdots x_n^{j_n}$, we have

$$x_i\frac{\partial g}{\partial x_i}=\sum a_J j_i x_1^{j_1}\cdots x_n^{j_n}=\pm\lambda_i\sum a_J x_1^{j_1}\cdots x_n^{j_n} .$$

Accordingly, $a_J(j_i \mp \lambda_i)=0$ for any $J=(j_1, \cdots, j_n)$. Hence if $j_i \neq \pm \lambda_i$, then $a_J=0$. In other words, if $a_J \neq 0$, then $j_1 = \pm \lambda_1$, $j_2 = \pm \lambda_2$, $\cdots$, $j_n = \pm \lambda_n$. Hence g turns out to be a unit and so $D(g)=\varnothing$. Similarly, the condition b) implies that $D(g)=\varnothing$.

Q.E.D.

THEOREM 7. *Let $\bar{\kappa}=\bar{\kappa}(G_m^n - D) \geqq 0$. Then there exists a reduced divisor $D_1 \subset G_m^{\bar{\kappa}}$ such that $V=G_m^{n-\bar{\kappa}} \times V_1$, where $V_1 = G_m^{n-\bar{\kappa}} - D_1$. Hence V_1 is of hyperbolic type.*

PROOF. By the fundamental theorem of logarithmic Kodaira dimension ([2] Theorem 5), we have a proper birational morphism $\mu: V^* \to V$ and a surjective morphism $f: V^* \to W$ in which V^* is non-singular, W is an irreducible constructible set of dimension $\bar{\kappa}$, any general fiber $V_w^* = f^{-1}(w)$ is a non-singular variety with $\bar{\kappa}(V_w^*)=0$. Let $V_w = \mu(V_w^*)$, which is a closed subvariety of dimension $n-\bar{\kappa}$. Since w is a general point, $\mu^{-1}(V_w)$ is an irreducible variety of dimension $n-\bar{\kappa}$. Then $V_w^* \subset \mu^{-1}(V_w)$ leads to the fact that $V_w^* = \mu^{-1}(V_w) = V_w \times_V V^*$. Hence, $\mu | V_w^*: V_w^* \to V_w$ is proper birational. Thus $\bar{\kappa}(V_w) = \bar{\kappa}(V_w^*) = 0$. Denoting by $\hat{V}_w$ the closure of V_w in G_m^n, we have $0 \leqq \bar{\kappa}(\hat{V}_w) \leqq \bar{\kappa}(V_w) = 0$, and so $\bar{\kappa}(\hat{V}_w)=0$. Hence by Theorem 4 we have $\hat{V}_w = G_m^{n-\bar{\kappa}}$. Furthermore, $\bar{\kappa}(G_m^{n-\bar{\kappa}} - G_m^{n-\bar{\kappa}} \cap D) = \bar{\kappa}(V_w) = 0$ implies $G_m^{n-\bar{\kappa}} \cap D = \varnothing$ by Proposition 10. Thus $V_w = \hat{V}_w = G_m^{n-\bar{\kappa}}$ and $G_m^n = G_m^{n-\bar{\kappa}} \times G_m^{\bar{\kappa}}$ by Corollary to Theorem 4. Hence we have a reduced divisor $D_1 \subset G_m^{\bar{\kappa}}$ such that $V = G_m^{n-\bar{\kappa}} \times (G_m^{\bar{\kappa}} - D_1)$ by the following easy lemma.

LEMMA 5. *Let V and W be algebraic varieties and D an irreducible divisor of $V \times W$ such that $D | V \times w = V \times w$ for any general point $w \in W$. Then $D = V \times D_1$, D_1 being an irreducible divisor of W.*

COROLLARY. *Let $\bar{D}$ be a union of hyperplanes L_j $(0 \leqq j \leqq t)$ in $\boldsymbol{P}^n$. Then $V = \boldsymbol{A}^\alpha \times G_m^\beta \times V_1$, where V_1 is of hyperbolic type which can be realized as a complement of a union of hyperplanes in $\boldsymbol{P}^{n-\alpha-\beta}$.*

PROOF. Consider the affine space $\boldsymbol{A}^{n+1}$ whose n-dimensional linear subspaces determine the points of $\boldsymbol{P}^n$. Let $k\boldsymbol{a}_j$ be the 1-dimensional vector space $\subset \boldsymbol{A}^{n+1}$ corresponding dually to L_j. Let $1+r = \dim(k\boldsymbol{a}_0 + \cdots + k\boldsymbol{a}_t)$. Then choosing a suitable homogeneous coordinates of $\boldsymbol{P}^n$, we have the equations:

$$
\begin{aligned}
X_0 &= 0 \quad \text{of} \quad L_0\,, \\
X_1 &= 0 \quad \text{of} \quad L_1\,, \\
&\cdots\cdots\cdots\cdots \\
X_r &= 0 \quad \text{of} \quad L_r\,,
\end{aligned}
$$

$$\sum_{j=0}^{r} a_{ij}X_j=0 \qquad \text{of} \quad L_i \quad (1\leqq i\leqq t)\,.$$

Therefore, $V=A^{n-k}\times V'$,

$$V'=\operatorname{Spec} k[x_1, \cdots, x_r, 1/x_1, \cdots, 1/x_r, 1/\sum a_{r+1,j}x_j, \cdots, 1/\sum a_{t,j}x_j]\subset G_m^r\,.$$

Hence by Theorem 7, $V'=G_m^{n-\bar\kappa}\times V_1$, where $\bar\kappa=\bar\kappa(V')$. Q.E.D.

PROPOSITION 11. *Let D be a divisor of an abelian variety $\mathcal{A}$ and $V=\mathcal{A}-D$. Then the following conditions are equivalent to each other.*

a) $V=\mathcal{A}$,

b) $\bar\kappa V=0$.

If $\bar\kappa=\bar\kappa(\mathcal{A}-D)>0$, then there is an unramified covering map $\pi:\mathcal{A}'\to\mathcal{A}$ such that $\mathcal{A}'=\mathcal{A}_1\times\mathcal{A}_2$ and $D=D_1\times\mathcal{A}_2$, D_1 being a reduced divisor $\subset\mathcal{A}_1$.

PROOF. By $\kappa(\mathcal{A})=0$, $\bar\kappa(\mathcal{A}-D)=\kappa(D, \mathcal{A})$ (Example 4, [2]). If $D\neq\emptyset$, then $\kappa(D, \mathcal{A})>0$ and $|2D|$ has no base points. Thus we have the morphism $f=\Phi_{mD}$: $\mathcal{A}\to B$ for sufficiently large m. Then B is of dimension $\bar\kappa=\kappa(D, \mathcal{A})$ and a general fiber $\mathcal{A}_b=f^{-1}(b)$ is an abelian subvariety $\mathcal{A}_2$. Hence $D=f^{-1}(D_1)$, $D_1\subset\mathcal{A}_1$. Q.E.D.

REMARK. T. Fujita proved that the similar statement as Proposition 11 holds for a quasi-abelian variety. Note that Proposition 10 cannot be generalized for a quasi-abelian variety. For instance, let V=an elliptic curve$-$one point. Then $\bar p_g V=1$ but $\bar\kappa V=1$.

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