*On the asymptotic formula for the Green operators of elliptic operators on compact manifolds**

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Introduction

Minakshisundaram and Å. Pleijel [9] studied the asymptotic behaviour of the Green kernel of the Laplace-Beltrami operator on compact manifolds. There they treated the Green kernel directly.

The aim of this note is to present, using L^2 theory, an indirect method to obtain the asymptotic behaviour of Green kernels. To do this we use a kind of pseudo-differential operators treated in [4]. One of the main results which is a generalization of results in [9] is the following: Let X be a complex l-dimensional vector bundle over a compact oriented C^∞ manifold M without boundary and let P be an elliptic pseudo-differential operator (see [8]) of even order $2m$ operating on the sections of X whose principal symbol we denote by $p_0(x, \xi)$ with $x \in M$ and cotangent vector $\xi \neq 0$ of M at x. If $p_0(x, \xi)$ satisfies the condition H_θ in §2, the value $E(x, x, \sigma)$ of the kernel $E(x, y, \sigma)$ of Green operator $(P+\sigma^{2m}+\sigma_0)^{-1}$ in the diagonal set in $M \times M$ admits the asymptotic expansion

$$E(x, x, \sigma) \sim \sum_j E_j(x)\sigma^{s_j} \qquad s_j \to -\infty .$$

Where $E_j(x)$ are calculable by calculus of symbol at x of the operator P.

Results in this note are previously announced in [5].

§1. β-pseudo-differential operators

Let M be a σ-compact oriented differentiable n-manifold. We denote by X and Y differentiable complex vector bundles on M of fibre dimension l_1 and l_2. The space of C^∞ sections of X with compact support is denoted by $\mathcal{D}(M, X)$ and the space of C^∞ sections of X is denoted by $\mathcal{E}(M, X)$.

Let ω: $M \times \boldsymbol{R}^1 \to M$ be the projection, then we denote the induced bundle of X by $\omega^{-1}X$. Then $\omega^{-1}X$ is isomorphic to $X \otimes 1_{\boldsymbol{R}^1}$, where $1_{\boldsymbol{R}^1}$ is the trivial line bundle on $\boldsymbol{R}^1$.

DEFINITION. A continuous linear mapping P from $\mathcal{D}(M, X)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^1)$ into $\mathcal{E}(M, Y)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^1)$ is called a β-pseudo-differential operator of order z_0, if there is

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a sequence of real numbers $s_0 > s_1 > \cdots \to -\infty$, such that for any f in $\mathscr{D}(M, X)$ and compact set $\mathscr{K}$ in $\mathscr{E}(M)$ consisting of real elements g with $dg \neq 0$ on supp f and for any non negative integer N, $e^{-i\lambda(g\rho+s\sigma)}P(fe^{i\lambda(g\rho+s\sigma)})$ is the pull back of a section $p(f, g, \rho, \sigma, \lambda)$ in $\mathscr{E}(M, Y)$ and with some $p_j(f, g\rho, \sigma) \in \mathscr{E}(M, Y)$, $j = 0, 1, 2, \cdots$,

$$\lambda^{-s_N}(p(f, g\rho, \sigma, \lambda) - \sum_{j=0}^{N-1} p_j(f, g\rho, \sigma)\lambda^{s_j})$$

is bounded in $\mathscr{E}(M, Y)\hat{\otimes}\mathscr{E}(S_1)$, where S_1 is the annulus $S_1 = \{(\rho, \sigma) \in \boldsymbol{R}^2;\ 1/2 \leq \rho^2 + \sigma^2 \leq 2\}$ in $\boldsymbol{R}^2$.

Let $\{\varphi_j\}$ be a smooth partition of unity. Then P is a β-pseudo-differential operator if and only if each of $P_{jk} = \varphi_j P \varphi_k$ is also a β-pseudo-differential operator. Since any vector bundle is locally trivial, we may assume that P_{jk} is represented by an $l_1 \times l_2$ matrix each component of which is an operator operating on sections of trivial line bundle. Thus all the results established in [4] are available in our case. As to notations for various distribution spaces we follow the usual ones in L. Schwartz [10] or A. Grothendieck [7].

§ 2. Existence of Green operator

In the following we assume that M is a compact oriented manifold with smooth measure $d\mu$ and that $X = Y$. Further we assume the vector bundle X has a fixed hermitian metric $(\ |\ )$.

Let P be an elliptic pseudo-differential operator of even order $2m$ from $\mathscr{D}(M, X)$ into itself having the following properties;

(H_Θ) for any x in M, u in the fibre X_x over x and for non zero cotangent vector ξ the complex number $(P(x, \xi)u|u)$ lies in the shadowed sector in the Fig. 1, with $0 < \Theta < \pi$. Here $P(x, \xi)$ is the principal symbol of the operator P.

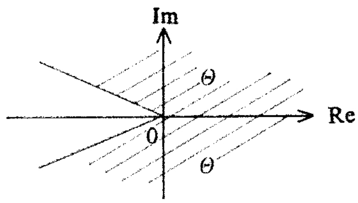

Fig. 1.

We denote by $L^2(M, X)$ the space of L^2 sections of X with the inner product $(u, v) = \int_M (u|v)d\mu$, $u, v \in L^2(M, X)$. Similarly, for real s, $H^s(M, X)$ represents the

Sobolev space of sections of X. We put $N=M\times\boldsymbol{R}^1$ in the following. First note that

$$Q=P+D_s^{2m}, \quad D_s=\frac{1}{i}\frac{\partial}{\partial s} \tag{1}$$

is an elliptic β-pseudo-differential operator on $\mathscr{D}(M, X)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$. Theorem 28 in [4] implies that there is a β-pseudo-differential operator F of order $-2m$ on $\mathscr{D}(M, X)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$ such that

$$Q\circ F=I+Q_{-\infty} \tag{2}$$

$$F\circ Q=I+Q'_{-\infty}, \tag{3}$$

where $Q_{-\infty}$ and $Q'_{-\infty}$ are β-pseudo-differential operators on $\mathscr{D}(M, X)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$ of order $-\infty$.

THEOREM 1. *For any $u\in\mathscr{D}(M, X)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$ we have the a priori estimate: For any real r*

$$||u||_{H^{r+2m}(M\times\boldsymbol{R}^1;\,\omega^{-1}X)}\leq C(||Qu||_{H^r(M\times\boldsymbol{R}^1;\,\omega^{-1}X)}+||u||_{H^r(M\times\boldsymbol{R}^1;\,\omega^{-1}X)}) \tag{4}$$

with some constant C depending on r.

PROOF. By (2) we have

$$u=FQu-Q'_{-\infty}u$$

therefore,

$$||u||_{H^{r+2m}(N;\,\omega^{-1}X)}\leq||FQu||_{H^{r+2m}(N;\,\omega^{-1}X)}+||Q'_{-\infty}u||_{H^{r+2m}(N;\,\omega^{-1}X)}. \tag{5}$$

Since order of F is $-2m$ and $Q'_{-\infty}$ is of order $-\infty$, Theorem 29 in [4] implies

$$\begin{aligned}&||FQu||_{H^{r+2m}(N;\,\omega^{-1}X)}\leq C||Qu||_{H^r(N;\,\omega^{-1}X)},\\ &||Q'_{-\infty}u||_{H^{r+2m}(N;\,\omega^{-1}X)}\leq C||u||_{H^r(N;\,\omega^{-1}X)}.\end{aligned} \tag{6}$$

Combining these with (5), we have the desired estimate (4).

THEOREM 2. *There exist constants $C>0$ and $\tau_0>0$ such that for any a in $[0, 2m]$, $\tau>\tau_0$ and any $\varphi\in\mathscr{D}(M, X)$, we have*

$$\tau^{2m-a}||\varphi||_{H^a(M,X)}\leq C||(P+\tau^{2m})\varphi||_{L^2(M,X)}. \tag{7}$$

The following proof is a modification of that of S. Agmon [1], [2], [3].

PROOF. Let $\phi\in\mathscr{D}(\boldsymbol{R}^1)$ with $\phi\equiv1$ in $|s|\leq1$ and $\phi_R(s)=\phi\left(\frac{s}{R}\right)$, then we have

$$\hat{\phi}_R(\sigma)=\int_{\boldsymbol{R}^1}e^{-i\sigma s}\phi\left(\frac{s}{R}\right)ds=R\hat{\phi}(R\sigma)$$

$$||\phi_R||^2_{H^k(R^1)} = R^2\int_{R^1} |\hat{\phi}(R\sigma)|^2(1+\sigma^{2k})d\sigma$$

$$= R^2\int_{R^1} |\hat{\phi}(\sigma)|^2\left(1+\left(\frac{\sigma}{R}\right)^{2k}\right)\frac{d\sigma}{R}$$

$$= R^1(||\phi||^2_0 + R^{-2k}||\phi||^2_k)$$

and

$$2\pi\widehat{(\phi_R e^{is\tau})}(\sigma) = \hat{\phi}_R * \delta_\tau = R\hat{\phi}(R(\sigma-\tau)) .$$

Therefore, for any non negative integer k,

$$(2\pi)^2||\phi_R e^{is\tau}||^2_{H^k(R^1)} = R^2\int_{R^1} |\hat{\phi}(R(\sigma-\tau)|^2(1+\sigma^{2k})d\sigma$$

$$= R\int_{R^1} |\hat{\phi}(\sigma)|^2\left(1+\left|\frac{\sigma+R\sigma}{R}\right|^{2k}\right)d\sigma .$$

With some constant $C>0$, we have

$$C^{-1}||\phi_R e^{is\tau}||^2_{H^k(R^1)} \leq (R^{-2k+1}||\phi||^2_{H^k(R^1)} + R^1||\phi||^2_{L^2(R^1)}(1+\tau^{2k}))$$

$$\leq C||\phi_R e^{is\tau}||^2_{H^k(R^1)} .$$

Thus

(8) $$C^{-1}||\phi_R e^{is\tau}||_{H^k(R^1)} \leq R^{1/2}(R^{-k}||\phi||_{H^k(R^1)} + ||\phi||_{L^2(R^1)}(|\tau|^k+1))$$

$$\leq C||\phi_R e^{is\tau}||_{H^k(R^1)} .$$

Now apply the estimate (4) with $r=0$ to $u=\varphi\otimes\phi_R e^{is\tau}$, $\forall\varphi\in\mathscr{D}(M, X)$, then we have

(9) $$R^{1/2}\Big[\sum_{k=0}^{2m}(R^{-k}||\phi||_{H^k(R^1)} + (1+|\tau|^k)||\phi||_{L^2(R^1)})\Big]||\varphi||_{H^{2m-k}(M,X)}$$

$$\leq C(||Qu||_{L^2(N;\,\omega^{-1}X)} + ||u||_{L^2(N;\,\omega^{-1}X)}) .$$

On the other hand, there holds

$$||Qu - ((P+\tau^{2m})\varphi)\otimes\phi_R e^{is\tau}||_{L^2(N,\omega^{-1}X)}$$

$$\leq \sum_{k=1}^{2m} ||D^k_s\phi_R\cdot D^{2m-k}_s e^{is\tau}||_{L^2(R^1)}||\varphi||_{L^2(M,X)}$$

$$\leq\Big[\sum_{k=1}^{2m}||\phi_R||_{H^k(R^1)}\tau^{2m-k}\Big]||\varphi||_{L^2(M,X)}$$

$$\leq R^{1/2}\sum_{k=1}^{2m}(||\phi||_{L^2(R^1)} + R^{-k}||\phi||_{H^k(R^1)})\tau^{2m-k}||\varphi||_{L^2(M,X)} .$$

Combining this with (9), we have

$$R^{1/2}\Big[\sum_{k=0}^{2m}(R^{-k}\|\phi\|_{H^k(R^1)}+(1+|\tau|^k)\|\phi\|_{L^2(R^1)})\|\varphi\|_{H^{2m-k}(M,X)}$$
$$\leq C(R^{1/2}\|\phi\|_{L^2(R^1)}\|(P+\tau^{2m})\varphi\|_{L^2(M,X)}+R^{1/2}\|\varphi\|_{L^2(M,X)}\|\phi\|_{L^2(R^1)}\Big]$$
$$+R^{1/2}\sum_{k=1}^{2m}(\|\phi\|_{L^2(R^1)}+R^{-k}\|\phi\|_{H^k(R^1)})\tau^{2m-k}\|\varphi\|_{L^2(M,X)}) . \tag{10}$$

Letting $R\to\infty$, we obtain from (10)

$$\sum_{k=0}^{2m}(1+|\tau|)^k\|\varphi\|_{H^{2m-k}(M,X)}\leq C(\|(P+\tau^{2m})\varphi\|_{L^2(M,X)}+\|\varphi\|_{L^2(M,X)}$$
$$+\sum_{k=1}^{2m}\tau^{2m-k}\|\varphi\|_{L^2(M,X)}) . \tag{11}$$

If τ is so large that $2C<\tau$, then we have from (11)

$$\sum_{k=0}^{2m}|\tau|^k\|\varphi\|_{2m-k}\leq C\|(P+\tau^{2m})\varphi\|_0 .$$

Therefore, for $a=0, 1, 2, \cdots, 2m$, we have proved (7). By the theory of interpolation, (7) holds for every a in $[0, 2m]$.

COROLLARY. *If P^* is the adjoint operator of P, for any $\tau>\tau_0$, a in $[0, 2m]$ and any section $\varphi\in\mathscr{D}(M, X)$, we also have*

$$\tau^{2m-a}\|\varphi\|_{H^a(M,X)}\leq C\|(P^*+\tau^{2m})\varphi\|_{H^0(M,X)} . \tag{12}$$

PROOF. Since P^* is also elliptic of order $2m$, and its principal symbol satisfies the hypothesis (H_Θ). Therefore we can replace P in (7) with P^*.

When a linear mapping Φ from $H^a(M, X)$ to $H^b(M, X)$ is continuous, we shall denote its norm as ${}_b\|\Phi\|_a$ for the time being.

PROPOSITION 3. *If P is an elliptic pseudo-differential operator of order $2m$ whose principal symol satisfies the hypothesis (H_Θ), then there is a constant $\tau_0>0$, such that for any real τ the resolvents $(P+\tau^{2m}+\tau_0)^{-1}$, and $(P^*+\tau^{2m}+\tau_0)^{-1}$ exist and satisfy*

$$_a\|(P+\tau^{2m}+\tau_0)^{-1}\|_0\leq C \tag{13}$$

$$_a\|(P^*+\tau^{2m}+\tau_0)^{-1}\|_0\leq C \tag{14}$$

$$_0\|(P+\tau^{2m}+\tau_0)^{-1}\|_0\leq C(1+|\tau|^{2m})^{-1} \tag{15}$$

$$_0\|(P^*+\tau^{2m}+\tau_0)^{-1}\|_0\leq C(1+|\tau|^{2m})^{-1} . \tag{16}$$

PROOF. We have only to show that $(P+\tau^{2m}+\tau_0)^{-1}$ and $(P^*+\tau^{2m}+\tau_0)^{-1}$ exist.

Since $(P+\tau^{2m}+\tau_0)$ and $(P^*+\tau^{2m}+\tau_0)$ have closed range, it is sufficient to prove that they are 1 to 1. This follows from (7) and (12).

PROPOSITION 4. *If P is the same as in Proposition 3, then there exist constants $\theta_0>0$, $\tau_0>0$, such that, for any τ and $|\theta|\leq\theta_0$ there hold the following estimates: for any $0\leq a\leq 2m$*

$$ {}_a||(P+\tau^{2m}e^{i\theta}+\tau_0)^{-1}||_0\leq C(1+|\tau|)^{a-2m} \tag{17} $$

$$ {}_a||(P^*+\tau^{2m}e^{i\theta}+\tau_0)^{-1}||_0\leq C(1+|\tau|)^{a-2m} . \tag{18} $$

PROOF. Making use of the first resolvent equation together with (15) or (16), we can prove that $(P+\tau^{2m}e^{i\theta}+\tau_0)^{-1}$ and $(P^*+\tau^{2m}e^{i\theta}+\tau_0)^{-1}$ exist and satisfy (17) and (18), with $a=0$. The first resolvent equation reads

$$ (P+\tau^{2m}e^{i\theta}+\tau_0)^{-1}=(P+\tau_0)^{-1}+\tau^{2m}e^{i\theta}(P+\tau_0)^{-1}(P+\tau^{2m}e^{i\theta}+\tau_0)^{-1} . $$

Thus we have

$$ {}_{2m}||(P+\tau^{2m}e^{i\theta}+\tau_0)^{-1}||_0\leq C_{2m}||(P+\tau_0)^{-1}||_0+\tau^{2m}(1+|\tau|^{2m})^{-1}{}_{2m}||(P+\tau_0)^{-1}||_0 . \tag{19} $$

This proves (17) with $a=2m$.

By interpolation, we have (17) for general $a\in[0, 2m]$. Similar argument proves (18).

THEOREM 5. *If P is an elliptic pseudo-differential operator of order $2m$ whose principal symbol satisfies the condition (H_Θ), then for any Z in the complex domain $\Sigma(\delta_1, \theta_1)$ in the figure 2, there exist resolvents $(P+Z^{2m}+\tau_0)^{-1}$ and $(P^*+Z^{2m}+\tau_0)^{-1}$ satisfying, with any $a\in[0, 2m]$,*

$$ {}_a||(P+Z^{2m}+\tau_0)^{-1}||_0\leq C(1+|Z|)^{a-2m} \tag{20} $$

$$ {}_a||(P^*+Z^{2m}+\tau_0)^{-1}||_0\leq C(1+|Z|)^{a-2m} . \tag{21} $$

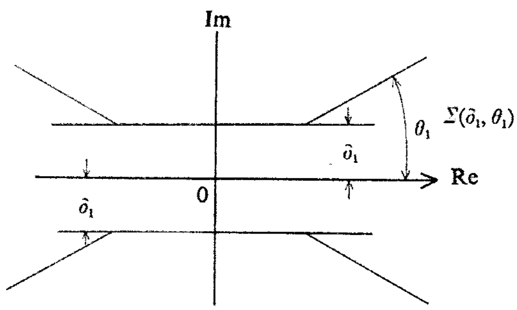

Fig. 2.

§ 3. Green operators and pseudo-differential operators

Let Z be in the complex domain $\Sigma(\delta_1, \theta_1)$ in Theorem 5. Then the linear mapping $E^{(k)}(Z): \mathscr{D}(M, X) \to \mathscr{D}(M, X)$

$$E^{(k)}(Z)\varphi = D_Z^k (P+Z^{2m}+\tau_0)^{-1}\varphi\,, \qquad k=0, 1, 2, \cdots, \tag{22}$$

is continuous for fixed Z and $E^{(k)}(Z)$ belong to

$$L(L^2(M, X),\ L^2(M, X))\hat{\otimes}\mathscr{H}(\Sigma(\delta_1, \theta_1))$$

where $\mathscr{H}(\Sigma(\delta_1, \theta_1))$ is the space of holomorphic functions in $\Sigma(\delta_1, \theta_1)$.

LEMMA 6. *For any k, there is a constant $C>0$ such that, for any real $g\in\mathscr{D}(M)$, any $\varphi\in\mathscr{D}(M, X)$, any $Z\in\Sigma(\delta_2, \theta_2)$ with $\delta_2<\delta_1$, $\theta_2<\theta_1$ and any $\lambda\in \boldsymbol{R}^1$, we have*

$$\|e^{-i\lambda g}E^{(k)}(Z)(\varphi e^{i\lambda g})\|_{L^2(M,X)} \leq C\|\varphi\|_{L^2(M,X)}(1+|Z|)^{-2m}\,. \tag{23}$$

PROOF. Consider the following bilinear form B_Z on $\mathscr{D}(M, X)\times\mathscr{D}(M, X)$

$$\begin{aligned} B_Z(\varphi, \psi) &= (e^{-i\lambda g}E^{(0)}(Z)(\varphi e^{i\lambda g}),\ \psi) \\ &= (E^{(0)}(Z)(\varphi e^{i\lambda g}),\ e^{i\lambda g}\psi)\,. \end{aligned} \tag{24}$$

For fixed φ and ψ, $B_Z(\varphi, \psi)$ is holomorphic in $\Sigma(\delta_1, \theta_1)$ and satisfies

$$|B_Z(\varphi, \psi)| \leq C(1+|Z|)^{-2m}\|\varphi\|_{L^2(M,X)}\cdot\|\psi\|_{L^2(M,X)}\,. \tag{25}$$

By Cauchy integral formula, we have for $Z\in\Sigma(\delta_2, \theta_2)$ with $\delta_2<\delta_1$, $\theta_2<\theta_1$,

$$|D_Z^k B_Z(\varphi, \psi)| \leq C(1+|Z|)^{-2m}\|\varphi\|_{L^2(M,X)}\cdot\|\psi\|_{L^2(M,X)}\,. \tag{26}$$

From this, (23) follows, because

$$D_Z^k B_Z(\varphi, \psi) = (e^{-i\lambda g}E^{(k)}(Z)(\varphi e^{i\lambda g}),\ \psi)\,. \tag{27}$$

LEMMA 7. *Let R_1, R_2 be pseudo-differential operators of order $-k$, $k\geq 0$, on $\mathscr{D}(M, X)$ and let φ_1, φ_2 be in $\mathscr{D}(M)$ whose supports are contained in a coordinate neighbourhood U (not necessarily connected). Then for any linear function $x\cdot\xi$ of the coordinate function $x_1, \cdots, x_n$ in U, there exists a constant $C>0$, such that, for any a, $b\in[0, k]$, $\xi\in\boldsymbol{R}^n$, $Z\in\Sigma(\delta_1, \theta_1)$, $u\in\mathscr{D}(M, X)$, we have*

$$\begin{aligned} &\|e^{-ix\cdot\xi}\varphi_1 R_1\cdot E^{(0)}(Z)R_2(\varphi_2 e^{ix\cdot\xi}u)\|_{H^a(M,X)} \\ &\leq C(1+|\xi|)^{a-b}(1+|Z|)^{-2m}\|u\|_{H^b(M,X)} \end{aligned} \tag{28}$$

PROOF. Using (53) in [4], we have

$$\|e^{-ix\cdot\xi}\varphi_1 R_1 E^{(0)}(Z)R_2(\varphi_2 e^{ix\cdot\xi}u)\|_{H^a(M,X)}$$

$$\leq (1+|\xi|)^a \| \varphi_1 R_1 E^{(0)}(Z) R_2(\varphi_2 e^{ix\cdot\xi} u) \|_{H^a(M,X)}$$

$$\leq C(1+|\xi|)^a \| E^{(0)}(Z) R_2(\varphi_2 e^{ix\cdot\xi} u) \|_{L^2(M,X)}$$

$$\leq C(1+|\xi|)^a (1+|Z|)^{-2m} \| R_2(\varphi_2 e^{ix\cdot\xi} u) \|_{L^2(M,X)}$$

$$\leq C(1+|\xi|)^a (1+|Z|)^{-2m} \| e^{-ix\cdot\xi} R_2(\varphi_2 e^{ix\cdot\xi} u) \|_{L^2(M,X)}$$

$$\leq C(1+|\xi|)^{a-b} (1+|Z|)^{-2m} \| u \|_{H^b(M,X)} .$$

THEOREM 8. *The operator $E^{(k)}(Z)$, $k=0, 1, 2, \cdots$ is a pseudo-differential operator, in the sense of Hörmander* [6], *of order $-2m$. The asymptotic expansion of $e^{-i\lambda g}E^{(k)}(Z)(fe^{i\lambda g})$ is locally uniform in $Z\in\Sigma(\delta_2, \theta_2)$. That is, there is a sequence $s_0=-2m>s_1>s_2>\cdots\to-\infty$ of reals such that for all $f\in\mathscr{D}(M, X)$ and compact set $\mathscr{K}$ of real functions $g\in\mathscr{D}(M)$ with $dg\neq 0$ on* supp f, *and for any integer $N>0$,*

$$\lambda^{-s_N}\Big(e^{-i\lambda g}E^{(k)}(Z)(fe^{i\lambda g})-\sum_0^{N-1} e_j^{(k)}(f, g)\lambda^{s_j}\Big) \tag{29}$$

remains bounded in $\mathscr{D}(M, X)$ uniformly in $\lambda\geq 1$, $g\in\mathscr{K}$ and Z belonging to a compact set in $\Sigma(\delta_2, \theta_2)$.

PROOF. $E^{(0)}(Z)$ satisfies

$$(P+Z^{2m}+\tau_0)E^{(0)}(Z)=I \tag{30}$$

$$E^{(0)}(Z)(P+Z^{2m}+\tau_0)=I . \tag{31}$$

On the other hand we can choose a pseudo-differential operator E_0 of order $-2m$ on $\mathscr{D}(M, X)$ satisfying

$$(P+\tau_0)E_0=I+P_{-\infty} , \tag{32}$$

and

$$E_0(P+\tau_0)=I+P'_{-\infty} , \tag{33}$$

where $P_{-\infty}$ and $P'_{-\infty}$ are pseudo-differential operators of order $-\infty$. Applying E_0 from the left to (30), we obtain

$$E^{(0)}(Z)=E_0-(Z^{2m}E_0+P'_{-\infty})E^{(0)}(Z) . \tag{34}$$

Similarly,

$$E^{(0)}(Z)=E_0-E^{(0)}(Z)(Z^{2m}E_0+P_{-\infty}) .$$

So that we have

$$E^{(0)}(Z)=E_0-Z^{2m}E_0^2-P'_{-\infty}E_0+(Z^{2m}E_0+P'_{-\infty})E^{(0)}(Z)(Z^{2m}E_0+P'_{-\infty}) . \tag{35}$$

Replacing $E^{(0)}(Z)$ in the right side of (35) by the whole sum in the right side of (35), we have

$$E^{(0)}(Z)=\sum\tilde{A}_j Z^j+(Z^{2m}E_0+P'_{-\infty})^2 E^{(0)}(Z)(Z^{2m}E_0+P_{-\infty})^2 .$$

Where $\tilde{A}_j$ is a pseudo-differential operator of order $-2m-j$. Repeating this process k times, we obtain

$$E^{(0)}(Z)=\sum_{j=0}^{2(k+1)m} A_j Z^j+(Z^{2m}E_0+P'_{-\infty})^k E^{(0)}(Z)(Z^{2m}E_0+P_{-\infty})^k \tag{36}$$

where A_j is a pseudo-differential operator of order $-2m-j$ which is independent of Z.

Let φ_1, φ_2 by any functions in $\mathscr{D}(M)$, whose supports are both contained in a coordinate neighbourhood U (not necessarily connected). Then for any integer $k\geq 0$, $Z\in\Sigma(\delta_2,\ \theta_2)$ with $\delta_2<\delta_0$, $\theta_2<\theta_0$, and any $a\geq 0$, there exists a constant C, such that for any u in $\mathscr{D}(M,\ X)$ and any linear function $x\cdot\xi$ of coordinate functions $(x_1,\ \cdots,\ x_n)$ in U, we have

$$||e^{-ix\cdot\xi}\varphi_1 E^{(k)}(Z)\varphi_2(ue^{ix\cdot\xi})||_{H^a(M,X)}\leq C(1+|Z|)^{2a}||u||_{H^a(M,X)} . \tag{37}$$

In fact, it follows from (53) in [4],

$$||e^{-ix\cdot\xi}\varphi_1 A_j(\varphi_2 e^{ix\cdot\xi}u)||_{H^a(M,X)}\leq C||u||_{H^a(M,X)} . \tag{38}$$

Applying Lemma 7 to $(Z^{2m}E_0+P'_{-\infty})^k E^{(0)}(Z)(Z^{2m}E_0+P_{-\infty})^k$ we have

$$\begin{aligned}&||e^{-ix\cdot\xi}\varphi_1(Z^{2m}E_0+P'_{-\infty})^k E^{(0)}(Z)(Z^{2m}E_0+P_{-\infty})^k(\varphi_2 e^{ix\cdot\xi}u)||_{H^{2mk}(M,X)}\\ &\leq C(1+|Z|)^{4mk}||u||_{H^{2mk}(M,X)},\ Z\in\Sigma(\delta_1,\ \theta_1) .\end{aligned} \tag{39}$$

(36), (38) and (39) give that if $Z\in\Sigma(\delta_1,\ \theta_2)$,

$$||e^{-ix\cdot\xi}\varphi_1 E^{(0)}(Z)(\varphi_2 ue^{ix\cdot\xi})||_{H^{2mk}(M,X)}\leq C(1+|Z|)^{4mk}||u||_{H^{2mk}(M,X)} .$$

Interpolating these, we have for $Z\in\Sigma(\delta_1,\ \theta_1)$

$$||e^{-ix\cdot\xi}\varphi_1 E^{(0)}(Z)(\varphi_2 ue^{ix\cdot\xi})||_{H^a(M,X)}\leq C(1+|Z|)^{2a}||u||_{H^a(M,X)} . \tag{40}$$

Consider the following bilinear form $B_Z(u,\ v)$ on $H^a(M,\ X)\times H^{-a}(M,\ X)$ defined by

$$B_Z(u,\ v)=\langle e^{-ix\cdot\xi}\varphi_1 E^{(0)}(Z)\varphi_2 ue^{ix\cdot\xi},\ v\rangle . \tag{41}$$

Then for fixed u, v, $B_Z(u,\ v)$ is holomorphic in $\Sigma(\delta_1,\ \theta_1)$ and satisfies the estimate

$$|B_Z(u,\ v)|\leq C(1+|Z|)^{2a}||u||_{H^a(M,X)}\cdot||v||_{H^{-a}(M,X)} . \tag{42}$$

Hence for $Z\in\Sigma(\delta_2,\ \theta_2)$ with $\delta_2<\delta_1$, $\theta_2<\theta_1$, by Cauchy's integral formula with another constant C,

$$(43) \qquad |D_Z^k(B_Z(u, v))| \leq C(1+|Z|)^{2a} \|u\|_{H^a(M,X)} \|v\|_{H^{-a}(M,X)} .$$

Taking account of the relation

$$(44) \qquad D_Z^k B_Z(u, v) = \langle e^{-ix\cdot\xi}\varphi_1 E^{(k)}(Z)\varphi_2 u e^{ix\cdot\xi}, v\rangle ,$$

(37) follows from (43).

Since $\mathscr{D}(M, X) = \mathscr{E}(M, X) = \bigcap_{a\geq 0} H^a(M, X)$, (37) implies the linear mapping from $\mathscr{E}(M, X)$ to $\mathscr{E}(M, X)$

$$u \to e^{-ix\cdot\xi}\varphi_1 E^{(k)}(Z)\varphi_2 e^{ix\cdot\xi} u$$

are equi-continuous.

Now we can prove that $E^{(0)}(Z)$, $Z \in \Sigma(\delta_2, \theta_2)$, are pseudo-differential operators.

Consider a finite smooth partition of unity $\sum_{j\in J} \Psi_j^2 = 1$, $\Psi_j \in C_0^\infty(M)$. We may assume that for any three j, k, $l \in J$, there exists a coordinate neighbourhood U (not necessarily connected) containing the union $\operatorname{supp}\Psi_j \cup \operatorname{supp}\Psi_k \cup \operatorname{supp}\Psi_l$. Taking (35) into account, we have only to prove that $E^{(0)}(Z)A$ is a pseudo-differential operator whenever A is a given pseudo-differential operator. To this aim it is sufficient to show that for any j, k, $l \in J$,

$$(45) \qquad R_{j,k,l} = \Psi_j^2 E^{(0)}(Z) \Psi_k^2 A \Psi_l^2$$

is a pseudo-differential operator.

Let $x\cdot\xi$ be any linear function of coordinate functions $(x_1, \cdots, x_n)$ in U, then for any $u \in \mathscr{D}(M, X)$,

$$(46) \qquad e^{-ix\cdot\xi} R_{j,k,l} e^{ix\cdot\xi} u = e^{-ix\cdot\xi} \Psi_j^2 E^{(0)}(Z) \Psi_k^2 A(\Psi_l^2 u e^{ix\cdot\xi})$$
$$= e^{-ix\cdot\xi} \Psi_j^2 E^{(0)}(Z) \Psi_k e^{ix\cdot\xi} e^{-ix\cdot\xi} \Psi_k A(\Psi_l^2 e^{ix\cdot\xi} u) .$$

Since $\Psi_k A \Psi_l^2$ is a pseudo-differential operator, there exists an asymptotic expansion

$$(47) \qquad e^{-ix\cdot\xi} \Psi_k A(\Psi_l^2 e^{ix\cdot\xi} u) \sim \sum_m a_m(u, x, \xi)$$

in $\mathscr{E}(M, X)$ topology. That is, there exists a sequence $s_0 > s_1 > \cdots \to -\infty$ of reals such that

$$(48) \qquad \{e^{-ix\cdot\xi} \Psi_k A(\Psi_l^2 e^{i\lambda x\cdot\xi} u) - \sum_{m=0}^{N-1} a_m(u; x, \xi)\lambda^{s_j}\} (|\lambda|+1)^{-s_N}$$

remains in a bounded set B_N in $\mathscr{D}(M, X)$.

Setting

$$(49) \qquad b_m(u; x, \lambda\xi, Z) = e^{i\lambda x\cdot\xi} \Psi_j^2 E^{(0)}(Z) e^{i\lambda x\cdot\xi} (\Psi_k a_m(u; x, \xi)\lambda^{s_m})$$

we know that

$$\{e^{-i\lambda x\cdot\xi}R_{j,k,l}e^{i\lambda x\cdot\xi}u-\sum_{m=0}^{N-1}b_m(u;\ x,\ \lambda,\ Z)\}(|\lambda|+1)^{-s_N}$$

are in the set

$$e^{-i\lambda x\cdot\xi}\Psi_j^2E^{(0)}(Z)(e^{i\lambda x\cdot\xi}\Psi_kB_N)\ .$$

Because the mappings

$$u\to e^{-i\lambda x\cdot\xi}\Psi_j^2E^{(0)}(Z)e^{i\lambda x\cdot\xi}\Psi_ku$$

are equi-continuous from $\mathscr{D}(M, X)$ to $\mathscr{D}(M, X)$, $e^{-i\lambda x\cdot\xi}\Psi_j^2E^{(0)}(Z)(e^{i\lambda x\cdot\xi}\Psi_kB_N)$ remains in a bounded set $\tilde{B}_N$ in $\mathscr{D}(M, X)$ when $\lambda\to\infty$, $Z\in\Sigma(\delta_2,\ \theta_2)$ $|\xi|=1$. From this it follows that $b_m(u;\ x,\ \xi,\ Z)$ are positively homogeneous of degree s_j in ξ. In fact, we have

$$b_m(u,\ x,\ \xi,\ Z)=\lim_{\lambda\to\infty}\lambda^{-s_m}(e^{-i\lambda x\cdot\xi}R_{j,k,l}e^{i\lambda x\cdot\xi}u-\sum_{l=0}^{m-1}b_l(u;\ x,\ \lambda\xi,\ Z))\ .$$

Thus we have proved that $E^{(0)}(Z)$, $Z\in\Sigma(\delta_2,\ \theta_2)$ with $\delta_2<\delta_1$, $\theta_2<\theta_1$ is a pseudo-differential operator. Local uniformity in Z of the asymptotic expansion is also obvious.

Finally by induction in k we shall prove that $E^{(k)}(Z)$, $k=0, 1, 2, \cdots$, $Z\in\Sigma(\delta_2,\ \theta_2)$ are all pseudo-differential operators. This has been already proved for $k=0$. Differentiating

$$E^{(0)}(Z)\varphi=(P+Z^{2m}+\tau_0)^{-1} \tag{50}$$

in Z, and using (22) we have

$$E^{(1)}(Z)=-(2m)Z^{2m-1}(E^{(0)}(Z))^2\ . \tag{51}$$

From (51) it follows that $E^{(1)}(Z)$ is a pseudo-differential operator. Repeating this process, we can prove that $E^{(k)}(Z)$ $Z\in\Sigma(\delta_1,\ \theta_1)$ are pseudo-differential operators. This completes our proof.

REMARK. As to the estimate (37), a sharper form will be proved later.

Let Z be a point in an section $\Sigma(\theta_1)$ in the complex plane defined in the Figure 3.

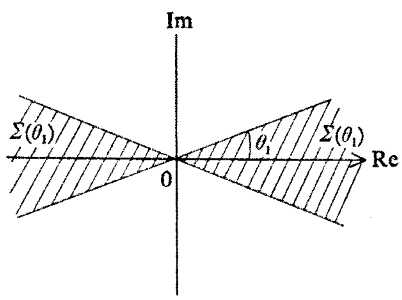

Fig. 3.

Then, for $\sigma \in \boldsymbol{R}^1$, we have, from (37)

$$\|E^{(k)}(\sigma Z)u\|_{L^2(M,X)} \leq C\|u\|_{L^2(M,X)} . \tag{52}$$

Therefore the bilinear mapping

$$L^2(M, X) \times \mathscr{S}'(\boldsymbol{R}^1) \to L^2(M, X)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$$

$$(\varphi, \phi) \to (2\pi)^{-1}\int_{R^1} e^{is\sigma}\hat{\phi}(\sigma)(E^{(k)}(\sigma Z)\varphi)d\sigma$$

is continuous, because this is separately continuous and $L^2(M, X)$ and $\mathscr{S}'(\boldsymbol{R}^1)$ are barelled $(\mathscr{D}\mathscr{F})$ spaces (cf. Cor. 1, Th. 2, § 3 Chap. IV [6]). Therefore there is a continuous linear mapping

$$G_Z^{(k)}:\ L^2(M, X)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1) \to L^2(M, X)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$$

$$\varphi\otimes\phi \to (2\pi)^{-1}\int \hat{\phi}(\sigma)e^{i\sigma s}(E^{(k)}(\sigma Z)\varphi)d\sigma .$$

THEOREM 9. *$G_Z^{(0)}$ satisfies*

$$(P+Z^{2m}D_s^{2m}+\tau_0)G_Z^{(0)}=I \tag{53}$$

$$G_Z^{(0)}(P+Z^{2m}D_s^{2m}+\tau_0)=I \tag{54}$$

on $\mathscr{D}(M, X)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$.

PROOF. Both the mappings $(P+Z^{2m}D_s^{2m}+\tau_0)G_Z^{(0)}$ and $G_Z^{(0)}(P+Z^{2m}D_s^{2m}+\tau_0)$ are continuous mappings from $\mathscr{D}(M, X)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$ to $\mathscr{D}'(M\times\boldsymbol{R}^1;\ \omega^{-1}X)$. Therefore, we have only to prove (53) and (54) for functions of the form $\varphi\otimes\phi$, $\varphi \in \mathscr{D}(M, X)$, $\varphi \in \mathscr{D}(\boldsymbol{R}^1)$. In this case, the integral

$$G_Z^{(0)}(\varphi\otimes\phi)=(2\pi)^{-1}\int_{R^1} \hat{\phi}(\sigma)e^{i\sigma s}(E^{(0)}(\sigma Z)\varphi)d\sigma \tag{55}$$

converges in $L^2(M, X)\hat{\otimes}\mathscr{E}(\boldsymbol{R}^1)$, (see (20)). Therefore, applying $G_Z^{(0)}$ to (55), we have

$$\begin{aligned} &(P+Z^{2m}D_s^{2m}+\tau_0)G_Z^{(0)}(\varphi\otimes\phi) \\ &=(2\pi)^{-1}\int_{R^1} \hat{\phi}(\sigma)e^{i\sigma s}(P+Z^{2m}\sigma^{2m}+\tau_0)(E^{(0)}(\sigma Z)\varphi)d\sigma \\ &=(2\pi)^{-1}\int_{R^1} \hat{\phi}(\sigma)\otimes\varphi e^{i\sigma s}d\sigma \\ &=\varphi\otimes\phi. \end{aligned} \tag{56}$$

Similarly,

$$(57)\qquad G_Z^{(0)}(P+Z^{2m}D_s^{2m}+\tau_0)(\varphi\otimes\phi)$$

$$=(2\pi)^{-1}\int_{R^1}\widehat{Z^{2m}D_s^{2m}}\phi(\sigma)e^{is\sigma}(E^{(0)}(\sigma Z)\varphi)d\sigma$$

$$+(2\pi)^{-1}\int_{R^1}e^{is\sigma}\hat{\phi}(\sigma)E^{(0)}(\sigma Z)((P+\tau_0)\varphi))d\sigma$$

$$=(2\pi)^{-1}\int_{R^1}e^{is\sigma}\hat{\phi}(\sigma)(E^{(0)}(\sigma Z)(P+\tau_0+Z^{2m}\sigma^{2m})\varphi)d\sigma$$

$$=\varphi\otimes\phi\ .$$

PROPOSITION 10. *$G_Z^{(0)}$ continuously maps $\mathscr{D}(M,\ X)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$ into*

$$\mathscr{D}(M,\ X)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)\ .$$

PROOF. Theorem 28 in [4] reads that there exists a β-pseudo-differential operator F_Z of order $-2m$ satisfying

$$(58)\qquad F_Z(P+Z^{2m}D_s^{2m}+\tau_0)=I+F_{Z,-\infty}$$

where $F_{Z,-\infty}$ is a β-pseudo-differential operator of order $-\infty$. Applying F_Z to (53), from the left, we have

$$(59)\qquad G_Z^{(0)}=F_Z-F_{Z,-\infty}\cdot G_Z^{(0)}\ .$$

Obviously $G_Z^{(0)}$ maps $\mathscr{D}(M,\ X)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$ into $\mathscr{D}'(M,\ X)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$ continuously. It follows from Theorem 27 in [4] $F_{Z,-\infty}G_Z^{(0)}$ maps $\mathscr{D}(M,\ X)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$ into $\mathscr{D}(M,\ X)\hat{\otimes}\mathscr{O}_M(\boldsymbol{R}^1)$ continuously. This proves our Proposition 10.

PROPOSITION 11. *For any $\varphi\in\mathscr{D}(M,\ X)$,*

$$(60)\qquad e^{-is\tau}G_Z^{(k)}(\varphi e^{is\tau}))=E^{(k)}(Z\tau)\varphi\ ,\qquad k=0,\ 1,\ 2,\ \cdots$$

$$(61)\qquad D_Z^k(e^{-is\tau}G_Z^{(0)}(\varphi e^{is\tau}))=\tau^k e^{-is\tau}G_Z^{(k)}(\varphi e^{is\tau})\ ,\qquad k=0,\ 1,\ 2,\ \cdots\ .$$

PROOF. For any k, from the definition of $G_Z^{(k)}$,

$$e^{-is\tau}G_Z^{(k)}(\varphi e^{is\tau})=(2\pi)^{-1}e^{-is\tau}\int_{R^1}(2\pi)\delta(\sigma-\tau)e^{i\sigma s}(E^{(k)}(\sigma Z)\varphi)d\sigma$$

$$=E^{(k)}(\tau Z)\varphi\ .$$

(61) follows directly from (60).

LEMMA 12. *Let f be in $\mathscr{D}(M,X)$, and g be a real element in $\mathscr{D}(M,X)$ then there is a constant $C\gtrless 0$ such that, for any $u\in\mathscr{D}(M,\ X)$*

$$(62)\qquad ||e^{-i(g+s\tau)}G_Z^{(0)}(e^{i(g+s\tau)}u)||_{L^2(M,X)}\leq C(1+|Z\tau|)^{-2m}||u||_{L^2(M,X)}\ .$$

PROOF. By Proposition 11

$$\|e^{-i(g+s\tau)}G_Z^{(0)}(e^{i(g+s\tau)}u)\|_{L^2(M,X)}$$
$$\leq \|E^{(0)}(Z\tau)(e^{ig}u)\|_{L^2(M,X)}$$
$$\leq C(1+|Z\tau|)^{-2m}\|u\|_{L^2(M,X)}$$

where we have used (23).

LEMMA 13. *Assume that R_1 and R_2 are β-pseudo-differential operators of order $-k\leq 0$, and that φ_1 and φ_2 are in $\mathscr{D}(M)$ whose supports are both contained in a coordinate neighbourhood U (not necessarily connected). Then for any real linear function $x\cdot\xi$ of the coordinate functions $(x_1, \cdots, x_n)$ in U, there is a constant $C>0$ such that for any $u\in\mathscr{D}(M, X)$, $Z\in\Sigma(\delta_2, \theta_2)$ and $0\leq a\leq k$, we have*

$$\|e^{-i(x\cdot\xi+s\tau)}\varphi_1R_1G_Z^{(0)}R_2(\varphi_2ue^{i(x\cdot\xi+s\tau)})\|_{H^a(M,X)} \tag{63}$$
$$\leq C(1+|Z\tau|)^{-2m}\|u\|_{H^a(M,X)}\ .$$

PROOF. Using (102) and (100) in [4], we have

$$\|e^{-i(x\cdot\xi+s\tau)}\varphi_1R_1G_Z^{(0)}R_2(\varphi_2ue^{i(x\cdot\xi+s\tau)})\|_{H^a(M,X)} \tag{64}$$
$$\leq C(1+|\xi|)^a\|e^{-is\tau}\varphi_1R_1G_Z^{(0)}R_2(\varphi_2ue^{i(x\cdot\xi+s\sigma)})\|_{H^a(M,X)}$$
$$\leq C(1+|\xi|)^a(1+|\tau|)^{a-k}\|e^{-is\tau}G_Z^{(0)}R_2(\varphi_2ue^{i(x\cdot\xi+s\tau)})\|_{L^2(M,X)}$$
$$\leq C(1+|\xi|)^a(1+|\tau|)^{a-k}(1+|\tau Z|)^{-2m}\|e^{-is\tau}R_2(\varphi_2ue^{i(x\cdot\xi+s\tau)})\|_{L^2(M,X)}$$
$$\leq C(1+|\xi|)^a(1+|\tau|)^{a-k}(1+|\tau Z|)^{-2m}\|e^{-i(x\cdot\xi+s\tau)}R_2(\varphi_2ue^{i(x\cdot\xi+s\tau)})\|_{L^2(M,X)}$$
$$\leq C(1+|\xi|)^a(1+|\tau|)^{a-k}(1+|\tau Z|)^{-2m}(1+|\xi|+|\tau|)^{-a}\|u\|_{H^a(M,X)}$$
$$\leq C(1+|\tau Z|)^{-2m}\|u\|_{H^a(M,X)}\ .$$

PROPOSITION 14. *Under the same hypothesis as in Lemma 13, for any $a\geq 0$, there exists a constant $C>0$, such that for any $u\in\mathscr{D}(M, X)$ and any $Z\in\Sigma(\theta_2)$ with $\theta_2<\theta_1$ and any k, we have*

$$\|e^{-i(x\cdot\xi+s\sigma)}\varphi_1G_Z^{(k)}(\varphi_2e^{i(x\cdot\xi+s\sigma)}u)\|_{H^a(M,X)}\leq C(1+|\sigma Z|)^{-k}\|u\|_{H^a(M,X)} \tag{65}$$

PROOF. $G_Z^{(0)}$ satisfies

$$(P+Z^{2m}D_s^{2m}+\tau_0)G_Z^{(0)}=I \tag{66}$$

$$G_Z^{(0)}(P+Z^{2m}D_s^{2m}+\tau_0)=I\ . \tag{67}$$

There exists a β-pseudo-differential operator F_Z of order $-2m$ which satisfies

$$F_Z(P+Z^{2m}D_s^{2m}+\tau_0)=I+Q'_{Z,-\infty} \tag{68}$$

$$(P+Z^{2m}D_s^{2m}+\tau_0)F_Z=I+Q_{Z,-\infty} \tag{69}$$

with β-pseudo-differential operators $Q_{Z,-\infty}$ and $Q'_{Z,-\infty}$ both of order $-\infty$.

By the same process as we used in the proof of Theorem 8, we obtain

$$G_Z^{(0)} = F_Z - G_Z^{(0)} Q_{Z,-\infty} \tag{70}$$

and

$$G_Z^{(0)} = F_Z - Q'_{Z,-\infty} F_Z + Q'_{Z,-\infty} G_Z^{(0)} Q_{Z,-\infty} \tag{71}$$

$F_Z - Q'_{Z,-\infty} F_Z$ is a β-pseudo-differential operator of order $-2m$. Therefore, if fixing Z in $\Sigma'(\theta_1)$, we have from Theorem 30 in [4]

$$||e^{-i(x\cdot\xi+s\sigma)}\varphi_1(F_Z - Q'_{Z,-\infty}F_Z)\varphi_2 e^{i(x\cdot\xi+s\sigma)}u||_{H^a(M,X)} \leq C||u||_{H^a(M,X)} . \tag{72}$$

On the other hand, applying Lemma 13, we have

$$||e^{-i(x\cdot\xi+s\sigma)}\varphi_1 Q'_{Z,-\infty} G_Z^{(0)} Q_{Z,-\infty}\varphi_2 e^{i(x\cdot\xi+s\sigma)}u||_{H^a(M,X)} \leq C(1+|Z\tau|)^{-2m}||u||_{H^a(M,X)} . \tag{73}$$

Combining (71) with (72) and (73), we have

$$||e^{-i(x\cdot\xi+s\sigma)}\varphi_1 G_Z^{(0)}\varphi_2 e^{i(x\cdot\xi+s\sigma)}u||_{H^a(M,X)} \leq C||u||_{H^a(M,X)} . \tag{74}$$

The constant C is independent of (ξ, σ) and u, but may depend on Z. We must show that the constant C in (74) is independent of Z, if Z remain in the set $\Sigma'(\theta_2)$, $\theta_2 < \theta_1$.

Note that

$$e^{-is\sigma}\varphi_1 G_{\tau Z}^{(0)}(\varphi_2 e^{is\sigma}u) = e^{-is\tau\sigma}\varphi_1 G_Z^{(0)}(\varphi_2 e^{is\tau\sigma}u) . \tag{75}$$

Really, by (60), we have

$$\begin{aligned} &e^{-is\sigma}\varphi_1 G_{Z\tau}^{(0)}(\varphi_2 e^{is\sigma}u) \\ &= \varphi_1 E^{(0)}(Z\tau\sigma)(\varphi_2 u) \\ &= e^{-is\tau\sigma} G_Z^{(0)}(\varphi_2 u e^{is\sigma\tau}) . \end{aligned} \tag{76}$$

It follows from (75) and (74) that we can choose a common constant $0 < C < \infty$ in (74) when Z runs on four rays $Z = re^{\pm i\theta_2}$ and $Z = re^{i(\pi\pm\theta_2)}$, $r \geq 0$.

Consider the bilinear form on $\mathscr{D}(M, X) \times \mathscr{D}(M, X)$

$$B_Z(\xi, \sigma, u, v) = (e^{-i(x\cdot\xi+s\sigma)}\varphi_1 G_Z^{(0)}\varphi_2 e^{i(x\cdot\xi+s\sigma)}u, v), \; Z \in \Sigma'(\theta_1) .$$

This is a holomorphic function if we fix ξ, σ and u, v. On the other hand, from (60), (41) and (42), this satisfies if $Z \in \Sigma'(\theta_2)$,

$$\begin{aligned} |B_Z(\xi, \sigma, u, v)| &= |\langle e^{-ix\cdot\xi}\varphi_1 E^{(0)}(Z\sigma)(\varphi_2 e^{ix\cdot\xi}u), v\rangle| \\ &= |B_{Z\sigma}(u, v)| \\ &\leq C(1+|Z\sigma|)^{2a}||u||_{H^a(M,X)}||v||_{H^{-a}(M,X')} . \end{aligned} \tag{77}$$

What we have proved means that with a constant C,

$$(78)\qquad |B_Z(\xi,\ \sigma,\ u,\ v)|\leq C\|u\|_{H^a(M,X)}\|v\|_{H^{-a}(M,X')}\ ,$$

when Z runs on the four rays: $Z=re^{\pm i\theta_2}$, $Z=re^{-i(\pi\pm\theta_2)}$.

It follows from Phragmén-Lindelöf theorem that there is a constant C independent of Z, ξ, r, such that

$$(79)\qquad |B_Z(\xi,\ \sigma,\ u,\ v)|\leq C\|u\|_{H^a(M,X)}\|v\|_{H^{-a}(M,X')}\ .$$

This implies that

$$(80)\qquad \|e^{-i(x\cdot\xi+s\sigma)}\varphi_1 G_Z^{(0)}\varphi_2 e^{i(x\cdot\xi+s\sigma)}u\|_{H^a(M,X)}\leq C\|u\|_{H^a(M,X)}$$

where constant C is independent of $(\xi,\ \sigma)\in \boldsymbol{R}^{n+1}$ and $Z\in\Sigma'(\theta_2)$.

Finally we shall prove (65) for $k=1,\ 2,\ \cdots$.

From (60) we have

$$(81)\qquad \begin{aligned} &e^{-i(x\cdot\xi+s\sigma)}\varphi_1 G_Z^{(k)}(\varphi_2 e^{i(x\cdot\xi+s\sigma)}u)\\ &=e^{-ix\cdot\xi}\varphi_1 E^{(k)}(Z\sigma)(\varphi_2 e^{ix\cdot\xi}u)\ .\end{aligned}$$

So that from (37), we have

$$(82)\qquad \|e^{-i(x\cdot\xi+s\sigma)}\varphi_1 G_Z^{(k)}(\varphi_2 e^{i(x\cdot\xi+s\sigma)}u)\|_{H^a(M,X)}\leq C(1+|Z\sigma|)^{2a}\|u\|_{H^a(M,X)}$$

Consider the following bilinear form

$$\begin{aligned} \mathrm{B}_Z^{(k)}(\xi,\ \sigma,\ u,\ v)&=(e^{-i(x\cdot\xi+s\sigma)}\varphi_1 G_Z^{(k)}(\varphi_2 e^{i(x\cdot\xi+s\sigma)}u),\ v)\\ &\qquad u\in\mathscr{D}(M,\ X),\ v\in\mathscr{D}(M,\ X)\ .\end{aligned}$$

Then from (61)

$$(83)\qquad \begin{aligned} \sigma^k B_Z^{(k)}(\xi,\ \sigma,\ u,\ v)&=D_Z^k(e^{-i(x\cdot\xi+s\sigma)}\varphi_1 G_Z^{(0)}(\varphi_2 e^{i(x\cdot\xi+s\sigma)}u),\ v)\\ &=D_Z^k B_Z(\xi,\ \sigma,\ u,\ v),\qquad Z\in\Sigma'(\theta_1)\ .\end{aligned}$$

Since (79) holds we obtain by Cauchy's integral formula,

$$(84)\qquad |\sigma^k B_Z^{(k)}(\xi,\ \sigma,\ u,\ v)|\leq C|Z|^{-k}\|u\|_{H^a(M,X)}\|v\|_{H^{-a}(M,X')}$$

for any $Z\in\Sigma'(\theta_2)$, $\theta_2<\theta_1$. Thus (84) gives

$$(85)\qquad \|e^{-i(x\cdot\xi+s\sigma)}\varphi_1 G_Z^{(k)}\varphi_2 e^{i(x\cdot\xi+s\sigma)}u\|_{H^a(M,X)}\leq C|\sigma Z|^{-k}\|u\|_{H^a(M,X)}\ ,$$

Combining this with (82), we have

$$(86)\qquad \|e^{-i(x\cdot\xi+s\sigma)}\varphi_1 G_Z^{(k)}\varphi_2 e^{i(x\cdot\xi+s\sigma)}u\|_{H^a(M,X)}\leq C(1+|\sigma Z|)^{-k}\|u\|_{H^a(M,X)}$$

for $Z\in\Sigma'(\theta_2)$ with $\theta_2<\theta_1$.

Now we can prove the fundamental theorem.

THEOREM 15. *For any fixed* $Z\in\Sigma(\theta_2)$ *with* $\theta_2<\theta_1$, *the operator* $G_Z^{(0)}$ *is a* β*-pseudo-differential operator of order* $-2m$.

PROOF. There exists a β-pseudo-differential operators F_Z of order $-2m$ which satisfies

$$(P+Z^{2m}D_s^{2m}+\tau_0)F_Z=I+Q_{-\infty} \tag{87}$$

where $Q_{-\infty}$ is a β-pseudo-differential operator of order $-\infty$. Applying F_Z from the right to (67), we have

$$G_Z^{(0)}=F_Z-G_Z^{(0)}\cdot Q_{-\infty}\,. \tag{88}$$

Let $\sum_{j\in J}\Psi_j^2\equiv 1$ be a finite smooth partition of unity. We may assume that for any three Ψ_j, Ψ_k, Ψ_l, there exists a coordinate neighbourhood U (not necessarily connected) containing the union $\operatorname{supp}\Psi_j\cup\operatorname{supp}\Psi_k\cup\operatorname{supp}\Psi_l$.

We wish to study the asymptotic behaviour of

$$Q_{j,k,l}=e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_j^2G_Z^{(0)}\Psi_k^2Q_{-\infty}\Psi_l^2(ue^{i\lambda(x\cdot\xi+s\sigma)})\,,\qquad 1/2\le|\xi|^2+|\sigma|^2\le 2 \tag{89}$$

where $x\cdot\xi$ is a real linear function $x_1\xi_1+\cdots+x_n\xi_n$ of coordinate functions $x=(x_1,\cdots,x_n)$ in U. Note that

$$\begin{aligned}&e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_j^2G_Z^{(0)}\Psi_k^2Q_{-\infty}\Psi_l^2(ue^{i\lambda(x\cdot\xi+s\sigma)})\\&=e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_j^2G_Z^{(0)}\Psi_ke^{i\lambda(x\cdot\xi+s\sigma)}e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_kQ_{-\infty}\Psi_l^2(ue^{i\lambda(x\cdot\xi+s\sigma)})\,.\end{aligned} \tag{90}$$

Since $Q_{-\infty}$ is a β-pseudo-differential operator of order $-\infty$, for any $N>0$,

$$\lambda^N[e^{-i(x\cdot\xi+s\sigma)\lambda}\Psi_kQ_{-\infty}\Psi_l^2(ue^{i\lambda(x\cdot\xi+s\sigma)})]$$

remains in a bounded set A_N in $\mathscr{D}(M,X)\hat{\otimes}\mathscr{E}(S)$, where

$$S=\{(\xi,\sigma)\in\boldsymbol{R}^{n+1},\ 1/2\le|\xi|^2+|\sigma|^2\le 2\}\,.$$

On the other hand, Proposition 14 means that the mappings

$$u\to e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_j^2G_Z^{(0)}\Psi_ke^{i\lambda(x\cdot\xi+s\sigma)}u$$

are equi-continuous from $\mathscr{D}(M,X)$ to $\mathscr{D}(M,X)$. Therefore,

$$\lambda^N(e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_j^2G_Z^{(0)}\Psi_k^2Q_{-\infty}\Psi_l^2e^{i\lambda(x\cdot\xi+s\sigma)}u) \tag{91}$$

remains bounded in $\mathscr{D}(M,X)$ if $\lambda\to\infty$ and $(\xi,\sigma)\in S$, that is (91) remains bounded in $\mathscr{D}(M,X)\hat{\otimes}\mathscr{C}(S)$, when $\lambda\to\infty$.

To complete the proof, we must prove that (91) is bounded in $\mathscr{D}(M,X)\hat{\otimes}\mathscr{E}(S)$. Differentiating (91) by ξ_p, we have

$$(92)\qquad \begin{aligned} &\lambda^N D_{\xi_p}(e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_j^2 G_Z^{(0)}\Psi_k^2 Q_{-\infty}\Psi_l^2 e^{i\lambda(x\cdot\xi+s\sigma)}u) \\ &=\lambda^{N+1}[-x_p e^{-i(x\cdot\xi+s\sigma)}\Psi_j^2 G_Z^{(0)}\Psi_k^2 Q_{-\infty}\Psi_l^2(ue^{i\lambda(x\cdot\xi+s\sigma)}) \\ &\quad +e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_j^2 G_Z^{(0)}\Psi_k^2 Q_{-\infty}\Psi_l^2(x_p u e^{i\lambda(x\cdot\xi+s\sigma)})]\,. \end{aligned}$$

Since the mapping $v\to -x_p v$ is a continuous mapping on $\mathscr{D}(M, X)\hat{\otimes}\mathscr{E}(S)$,

$$\lambda^{N+1}(x_p e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_j^2 G_Z^{(0)}\Psi_k^2 Q_{-\infty}\Psi_l^2(ue^{i\lambda(x\cdot\xi+s\sigma)})$$

is bounded in $\mathscr{D}(M, X)\hat{\otimes}\mathscr{E}(S)$. Since $x_p u\in\mathscr{D}(M)$,

$$\lambda^{N+1}(e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_j^2 G_Z^{(0)}\Psi_k^2 Q_{-\infty}\Psi_l^2(ue^{i\lambda(x\cdot\xi+s\sigma)}))$$

is also bounded in $\mathscr{D}(M, X)\hat{\otimes}\mathscr{E}(S)$. Combining these, we prove that (92) is bounded in $\mathscr{D}(M, X)\hat{\otimes}\mathscr{E}(S)$.

Next, differentiation of (91) by σ gives

$$(93)\qquad \begin{aligned} &\lambda^N D_\sigma[e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_j^2 G_Z^{(0)}\Psi_k^2 Q_{-\infty}(\Psi_l^2 e^{i\lambda(x\cdot\xi+s\sigma)}u)] \\ &=\lambda^N D_\sigma[e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_j^2 G_Z^{(0)}\Psi_k e^{i\lambda(x\cdot\xi+s\sigma)}e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_k Q_{-\infty}(\Psi_l^2 e^{i\lambda(x\cdot\xi+s\sigma)}u)] \\ &=\lambda^N D_\sigma[e^{-i\lambda x\cdot\xi}\Psi_j^2 E^{(0)}(\lambda Z\sigma)\Psi_k e^{i\lambda x\cdot\xi}e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_k Q_{-\infty}(\Psi_l^2 e^{i\lambda(x\cdot\xi+s\sigma)}u)] \\ &=\lambda^N[e^{-i\lambda x\cdot\xi}(\lambda Z)\Psi_j^2 E^{(1)}(\lambda Z\sigma)(\Psi_k e^{i\lambda x\cdot\xi}e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_k Q_{-\infty}\Psi_l^2 e^{i\lambda(x\cdot\xi+s\sigma)}u) \\ &\quad +e^{-\lambda(x\cdot\xi+s\sigma)}\Psi_j^2 G_Z^{(0)}\Psi_k e^{i\lambda(x\xi\cdot+s\sigma)}D_\sigma(e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_k Q_{-\infty}(\Psi_l^2 e^{i\lambda(x\cdot\xi+s\sigma)}u)] \\ &=\lambda^N[\lambda Z e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_j^2 G_Z^{(1)}(\Psi_k e^{i\lambda(x\cdot\xi+s\sigma)}e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_k Q_{-\infty}\Psi_l^2 e^{i\lambda(x\cdot\xi+s\sigma)}u)] \\ &\quad +\lambda^N[e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_j^2 G_Z^{(0)}\Psi_k e^{i\lambda(x\cdot\xi+s\sigma)}D_\sigma[e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_k Q_{-\infty}\Psi_l^2(e^{i\lambda(x\cdot\xi+s\sigma)}u)]]\,. \end{aligned}$$

Here we used (60).

Since $\lambda^{N+1}e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_k Q_{-\infty}\Psi_l^2 e^{i\lambda(x\cdot\xi+s\sigma)}u$ are bounded in $\mathscr{D}(M, X)\hat{\otimes}\mathscr{E}(S)$, and the mappings $v\to e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_j^2 G_Z^{(1)}(\Psi_k e^{i\lambda(x\cdot\xi+s\sigma)}v)$ are equicontinuous on $\mathscr{D}(M, X)$ (Proposition 14),

$$\lambda^{N+1}Z e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_j^2 G_Z^{(1)}(\Psi_k e^{i\lambda(x\cdot\xi+s\sigma)}e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_k Q_{-\infty}\Psi_l^2 e^{i\lambda(x\cdot\xi+s\sigma)}u)$$

remain bounded in $\mathscr{D}(M, X)\hat{\otimes}\mathscr{E}(S)$.

$\lambda^N D_\sigma(e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_k Q_{-\infty}\Psi_l^2 e^{i\lambda(x\cdot\xi+s\sigma)}u)$ are bounded in $\mathscr{D}(M, X)\hat{\otimes}\mathscr{E}(S)$, because $Q_{-\infty}$ is a β-pseudo-differential operator of order $-\infty$. The mapping

$$v\to e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_j^2 G_Z^{(0)}\Psi_k e^{i\lambda(x\cdot\xi+s\sigma)}v$$

being equicontinuous on $\mathscr{D}(M, X)$ (Proposition 14),

$$\lambda^N[e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_j^2 G_Z^{(0)}\Psi_k e^{i\lambda(x\cdot\xi+s\sigma)}D_\sigma e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_k Q_{-\infty}(\Psi_l^2 e^{i\lambda(x\cdot\xi+s\sigma)}u)]$$

remain bounded in $\mathscr{D}(M, X)\hat{\otimes}\mathscr{E}(S)$. Combining these with (93)

$$\lambda^N D_\sigma(e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_j^2 G_Z^{(0)}\Psi_k^2 Q_{-\infty}(\Psi_l^2 e^{i\lambda(x\cdot\xi+s\sigma)}u))$$

are bounded in $\mathscr{D}(M, X)\hat{\otimes}\mathscr{E}(S)$. Therefore

$$\lambda^N e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_j^2 G_Z^{(0)}\Psi_k^2 Q_{-\infty}\Psi_l^2 e^{i\lambda(x\cdot\xi+s\sigma)}u$$

are bounded in $\mathscr{D}(M, X)\hat{\otimes}\mathscr{E}^1(S)$.

Repeating these processes, we can prove that

$$\lambda^N e^{-i\lambda(x\cdot\xi+s\sigma)}\Psi_j^2 G_Z^{(0)}\Psi_k^2 Q_{-\infty}\Psi_l^2 e^{i\lambda(x\cdot\xi+s\sigma)}u$$

are bounded in $\mathscr{D}(M, X)\hat{\otimes}\mathscr{E}(S)$.

This completes the proof.

§ 3. Asymptotic behaviour of kernels

Let $G_Z^{(k)}$ and $E^{(k)}(Z)$ be as in § 2. First we shall collect estimates obtained.

In the following three theorems, we assume that φ_1 and φ_2 are fixed functions in $\mathscr{D}(M)$ whose supports are both contained in a coordinate neighbourhood U (not necessarily connected). Further we assume that $x\cdot\xi$ is a linear function of coordinates $x_1, \cdots, x_n$ on U.

First from Theorem 29 in [4]

THEOREM 16. *For any $a\in \boldsymbol{R}^1$, there is a constant C such that for any $b\in[-2m, 2m]$, $Z\in\Sigma'(\theta_2)$ and any $u\in\mathscr{D}(M, X)$, we have*

$$(95)\qquad \|e^{-i(x\cdot\xi+s\sigma)}\varphi_1 G_Z^{(0)}(\varphi_2 e^{i(x\cdot\xi+s\sigma)}u)\|_{H^{a+b}(M,X)} \leq C(1+|\xi|+|\sigma|)^b\|u\|_{H^a(M,X)}\,.$$

COROLLARY. *For any $u\in \boldsymbol{R}$, there is a constant C such that for any $b\in[-2m, 2m]$, $Z\in\Sigma(\delta_2, \theta_2)$ with $\delta_2<\delta_1$, $\theta_2<\theta_1$, and any $u\in\mathscr{D}(M, X)$, we have*

$$(96)\qquad \|e^{-ix\cdot\xi}\varphi_1 E^{(k)}(Z)(\varphi_2 e^{ix\cdot\xi}u)\|_{H^{a+b}(M,X)}\leq C(1+|\xi|)^b(1+|Z|)^{-k}\|u\|_{H^a(M,X)}\,.$$

This is a sharper form of (37). However we omit the proof.

Finally, from Theorem 31 in [4], we have

THEOREM 18. *There exists a constant $C>0$ such that for any $b\in[0, 2m]$, $Z\in\Sigma'(\theta_2)$ with $\theta_2<\theta_1$ and for any $u\in\mathscr{D}(M, X)$, we have*

$$(97)\qquad \|e^{-is\sigma}G_Z^{(0)}(e^{is\sigma}u)\|_{H^{a+b}(M,X)}\leq C(1+|\sigma|)^{b-2m}\|u\|_{H^a(M,X)}\,.$$

COROLLARY. *For any $a\in\boldsymbol{R}$, there exists a constant $C>0$, such that, for any $b\in[0, 2m]$, $Z\in\Sigma(\delta_2, \theta_2)$ with $\delta_2<\delta_1$, $\theta_2<\theta_1$, and any $u\in\mathscr{D}(M)$, we have*

$$(98)\qquad \|E^{(0)}(Z)u\|_{H^{a+b}(M,X)}\leq C(1+|Z|)^{b-2m}\|u\|_{H^a(M,X)}\,.$$

PROOF. By (60) and (97) we have for any $Z\in\Sigma'(\theta_2)$

$$(99)\qquad \|E^{(0)}(Z\sigma)u\|_{H^{a+b}(M,X)}\leq C(1+|\sigma|)^{b-2m}\|u\|_{H^a(M,X)}\,.$$

Replace σ by $|Z|$ and Z by $Z/|Z|$, then we have (98) for z in $\Sigma'(\theta_1)$. This and (37) give (98).

THEOREM 19 (Approximation in uniform norm). *For any Z in $\Sigma'(\theta_2)$, let $G_{Z,l}$ be any β-pseudo-differential operator such that $G_Z^{(0)}-G_{Z,l}$ is of order less than $-l$. We define the operator $E_l(Z)$ by*

$$E_l(Z)\varphi=e^{-is|Z|}G_{\frac{Z}{|Z|},l}(e^{is|Z|}\varphi) . \tag{100}$$

Then

$$||E^{(0)}(Z)\varphi-E_l(Z)\varphi||_{H^{a+b}(M,X)}\leq C(1+|Z|)^{b-l}||u||_{H^a(M,X)} , \tag{101}$$

for any $0\leq b\leq l$, and $\varphi\in\mathcal{D}(M)$.

PROOF. From Theorem 31 in [4] with $Q=G_Z^{(0)}-G_Z$, we have (101).

COROLLARY. *For any $Z\in\Sigma'(\theta_2)$ let F_Z be an arbitrary β-pseudo-differential operator such that the symbol of either $F_Z(P+Z^{2m}D_s^{2m}+\tau_0)$ or $(P+Z^{2m}D_s^{2m}+\tau_0)F_Z$ is identically I on $M\times\boldsymbol{R}^1$. We define the operator $\tilde{E}(Z)$ by*

$$\tilde{E}(Z)\varphi=e^{-is|Z|}F_{\frac{Z}{|Z|}}(e^{is|Z|}\varphi) , \qquad \forall\varphi\in\mathcal{D}(M, X) .$$

Then, for any $l\geq 0$, $b\geq 0$, and $a\in\boldsymbol{R}$ there is a constant $C>0$ such that for any $\varphi\in\mathcal{D}(M)$, we have

$$||E^{(0)}(Z)-\tilde{E}(Z)\varphi||_{H^{a+b}(M,X)}\leq C(1+|Z|)^{-l}||\varphi||_{H^a(M,X)} . \tag{102}$$

REMARK. We saw in Theorem 28 in [4] that we could find F_Z through a calculus of symbols.

Now let us consider the kernel of the Green operator $E(Z)$.

THEOREM 20. *Let A be a β-pseudo-differential operator of order $s_0<-n$ then the operator $B(\sigma)$ defined by*

$$B(\sigma)\varphi=e^{-is\sigma}Ae^{is\sigma}\varphi \qquad \varphi\in\mathcal{D}(M, X) \tag{103}$$

has a kernel $B(x, y, \sigma)$ in $\mathcal{E}(M\times M, \mathrm{Hom}(X, X))\hat{\otimes}\mathcal{O}_M(\boldsymbol{R}^1)$ with the estimate: For any $\forall\varepsilon>0$, there is a constant $C_\varepsilon>0$,

$$|B(x, y, \sigma)|\leq C_\varepsilon(1+|\sigma|)^{s_0+n+\varepsilon} . \tag{104}$$

PROOF. Let $0<2\gamma<-s_0-n$ and let φ_1 and φ_2 be arbitrary functions in $\mathcal{D}(M)$. Then we have from Theorem 31, in [4]

$$|\langle B(\sigma)\varphi_1, \varphi_2\rangle|\leq C(1+|\sigma|)^{s_0+n+2\gamma}||\varphi_1||_{H^{-(n/2)-\gamma}(M,X)}||\varphi_2||_{H^{-(n/2)-\gamma}(M,X)} . \tag{105}$$

Thus this bilinear form $\varphi_1, \varphi_2\to\langle B(\sigma)\varphi_1, \varphi_2\rangle$ has the unique continuous extension on $H^{-(n/2)-\gamma}(M, X)\times H^{-(n/2)-\gamma}(M, X')$, which we denote again with the symbol $\langle B(\sigma)*, *\rangle$.

By Sobolev's embedding theorem, the embedding mapping ι: $H^{(n/2)+\gamma}(M, X) \to \mathscr{C}(M, X)$ is completely continuous. Therefore its adjoint

$$\iota^*: \mathscr{C}'(M, X') \to H^{(-n/2)-\gamma}(M, X') \tag{106}$$

is also completely continuous, see [11]. We denote with δ_x, δ_y the Dirac measures concentrated on arbitrary points x, y on M. For any σ, let us define the section $B(x, y, \sigma)$ of the bundle $\mathrm{Hom}(X, X) = X \otimes X'$ on $M \times M$ by

$$(B(x, y, \sigma)u, v) = (B(\sigma)u\delta_y, v\delta_x) \tag{107}$$

$u \in X_x$, $v \in X_x$, where X_x is the fibre of X over the point x in M. In fact if $x_n \to x$, $y_n \to y$ in M, then $\delta_{x_n} \to \delta_x$, $\delta_{y_n} \to \delta_y$ in $\sigma(\mathscr{C}'(M), \mathscr{C}(M))$ topology. ι^* being complete continuous $\delta_{x_n} \to \delta_x$, $\delta_{y_n} \to \delta_y$ strongly in $H^{(-n/2)-\gamma}(M)$. This implies $B(x, y, \sigma)$ is continuous in (x, y). We must prove that $B(x, y, \sigma)$ is the kernel of the operator $B(\sigma)$. To prove this, it is sufficient to show that

$$\langle B(\sigma)\varphi, \psi\rangle = \iint_{M \times M} B(x, y, \sigma)\, \varphi(x)\, \psi(y)\, \mu(dx)\, \mu(dy)\ . \tag{108}$$

for any $\varphi \in \mathscr{D}(M, X)$, $\psi \in \mathscr{D}(M, X')$, whose supports are both contained in a coordinate neighbourhood U (not necessarily connected) of M. By coordinate functions $x_1, \cdots, x_n$ in U, we can identify U with an open set Ω in $\boldsymbol{R}^n$. Let $\rho(x)$ be the density of measure $\mu(dx)$, that is $\rho(x)dx_1 \cdots dx_n = \mu(dx)$.

Divide U into a net $\mathscr{N}_k$ of cubes with sides $1/k$. Let $\{x_{j,k}\}_{j=1}^{\infty}$ be the set of vertices of the net $\mathscr{N}_k$. By the theory of Riemannian integral, the measure

$$\nu_k = \sum_{j=1}^{\infty} \frac{1}{k^n}\ \varphi(x_{j,k})\, \rho(x_{j,k})\, \delta_{x_{j,k}}$$

$$\left(\nu_k' = \sum_{j=1}^{\infty} \frac{1}{k^n}\ \psi(x_{j,k})\, \rho(x_{j,k})\, \delta_{x_{j,k}}, \quad \text{respectively}\right)$$

tends to $\varphi(x)\rho(x)dx$, $(\psi(x)\rho(x)dx$, respectively) in $\mathscr{C}(M)'$, when $k \to \infty$. Therefore we have

$$\begin{aligned}
&\iint_{M\,M} B(x, y, \sigma)\, \varphi(x)\, \psi(y)\, \mu(dx)\, \mu(dy) \\
&= \lim_{k\to\infty} \frac{1}{k^{2n}} \sum_{j=1}^{\infty} B(x_{j,k}, y_{j,k}, \sigma)\varphi(x_{j,k})\rho(x_{j,k})\psi(y_{j,k})\rho(y_{j,k}) \\
&= \lim_{k\to\infty} \frac{1}{k^{2n}} \sum_{j=1}^{\infty} \langle B(\sigma)\delta_{y_{j,k}}, \delta_{x_{j,k}}\rangle\varphi(x_{j,k})\rho(x_{j,k})\psi(y_{j,k})\rho(y_{j,k}) \\
&= \lim_{k\to\infty} \langle B(\sigma)\nu_k', \nu_k\rangle \\
&= \langle B(\sigma)\psi, \varphi\rangle\ .
\end{aligned}$$

This proves (108).

The estimate (104) follows from (105) and (107).

Finally we shall prove that $B(x, y, \sigma)$ belongs to $\mathscr{E}(M\times M, \mathrm{Hom}(X, X))\hat{\otimes}\mathscr{O}_M(\boldsymbol{R}^1)$. For this purpose, it is sufficient to show that for any measure ν in $\mathscr{E}'(M\times M; X\otimes X)$,

$$\phi_\nu(\sigma)=\iint_{M\,M} B(x, y, \sigma)\,\nu(dx, dy) \tag{109}$$

belongs to $\mathscr{O}_M(\boldsymbol{R}^1)$. (cf. A. Grothendieck [7], Theorem 13, n^0 3 Chap. II.)

Let φ_1 and φ_2 be arbitrary functions in $\mathscr{D}(M)$ whose supports are both contained in a coordinate neighbourhood U (not necessarily connected). Then we can identify $X|U$ with $U\times \boldsymbol{C}^l$. For any linear function $x\cdot\xi=x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n$ of coordinates $x_1, \cdots, x_n$ valid in U, we define $b(x, \xi, \sigma)$ by

$$b(x, \xi, \sigma)=e^{-i(x\cdot\xi+s\sigma)}\varphi_1 B(\varphi_2 e^{i(x\cdot\xi+s\sigma)})\ . \tag{110}$$

This is an $l\times l$ matrix each component of which is a function in $\mathscr{D}(U)\hat{\otimes}\mathscr{O}_M$ satisfying

$$|D_x^\alpha D_\xi^\beta D_\sigma^\gamma b(x, \xi, \sigma)|\leq C(1+|\xi|+|\sigma|)^{s_0-|\beta|-|\gamma|} \tag{111}$$

(cf. (10) in [4]). Then from Lemma 5 in [4], for any $u\in\mathscr{D}(U, X)$

$$\begin{aligned}B(\sigma)u(x)&=e^{-is\sigma}\varphi_1 B(\varphi_2 e^{is\sigma}u)\\&=(2\pi)^{-n}\int_{R^n}\hat{u}(\xi)b(x, \xi, \sigma)e^{ix\cdot\xi}d\xi\end{aligned}$$

Therefore by direct calculation, for any $u\in\mathscr{D}(U, X)$, $v\in\mathscr{D}(U, X')$,

$$\begin{aligned}\langle B(x, y, \sigma)u(y), v(x)\rangle&=\langle (B(\sigma)u\delta_y, v\delta_x\rangle\\&=(2\pi)^{-n}\frac{1}{\rho(y)}\int_{R^n}\langle b(x, \xi, \sigma)e^{i(x-y)\cdot\xi}u(y), v(x)\rangle d\xi\ .\end{aligned} \tag{112}$$

Hence

$$D_\sigma^\gamma B(x, y, \sigma)=(2\pi)^{-n}\frac{1}{\rho(y)}\int_{R^n}D_\sigma^\gamma b(x, \xi, \sigma)e^{i(x-y)\cdot\xi}d\xi \qquad \gamma=1, 2, 3, \cdots, \tag{113}$$

which is continuous in $(x, y, \sigma)\in M\times M\times\boldsymbol{R}^1$ by (10) in [4]. Moreover there holds the estimate,

$$\begin{aligned}|D_\sigma^\gamma B(x, y, \sigma)|&\leq C\int_{R^n}(1+|\xi|+|\sigma|)^{s_0-\gamma}d\xi\\&\leq C(1+|\sigma|)^{s_0-\gamma+n}\ .\end{aligned} \tag{114}$$

Therefore, for any measure ν in $\mathscr{E}'(M\times M,\ X\otimes X')$, $D_\sigma^\gamma\phi_\nu(\sigma)$ is continuous, and satisfies the estimate

(115) $$|D_\sigma^\gamma\phi_\nu(\sigma)|\leq C||\nu||(1+|\sigma|)^{s_0-\gamma+n}\ ,\quad \gamma=0,\ 1,\ 2,\ \cdots$$

where C is a constant independent of ν and σ, $||\nu||$ is the total variation of ν. Therefore $\phi_\nu\in\mathscr{O}_M(\boldsymbol{R}^1)$. This completes the proof.

We have proved

COROLLARY. *Let φ_1 and φ_2 be functions in $\mathscr{D}(M)$ with the following properties both* $\operatorname{supp}\varphi_1$ *and* $\operatorname{supp}\varphi_2$ *are contained in a coordinate neighbourhood U (not necessarily connected) and $\varphi_i\equiv 1$ in some neighbourhood of x_i, in M, $i=1,2$. Then expressing by coordinate, we have*

(116) $$B(x_1,\ x_2,\ \sigma)=(2\pi)^{-n}\rho(x_2)^{-1}\int_{R^n}b(x_1,\ \xi,\ \sigma)e^{i(x_1-x_2)\cdot\xi}d\xi$$

where

$$b(x,\ \xi,\ \sigma)=\varphi_1e^{-i(x\cdot\xi+s\sigma)}B(\varphi_2e^{i(x\cdot\xi+s\sigma)})\ .$$

THEOREM 21. *If $2m-n>\gamma>0$, then the Green operator $(P+Z^{2m}+\tau_0)^{-1}$ has the kernel $E(x_1,x_2,Z)$ belonging to $\mathscr{E}(M\times M,X\otimes X')\hat{\otimes}\mathscr{H}(\Sigma(\delta_2,\theta_2))$ with the estimate, for $(x_1,x_2,Z)\in M\times M\times\Sigma(\delta_2,\theta_2)$,*

(117) $$|\ E(x_1,\ x_2,\ Z)\ |\leq C_\gamma(1+|\ Z\ |)^{-\gamma}\ .$$

$E(x_1,x_2,Z)$ is given by (116) with $B=G_Z^{(0)}$.

PROOF. We shall show that $E\in\mathscr{E}(M\times M;\ X\otimes X')\hat{\otimes}\mathscr{H}(\Sigma(\delta_2,\theta_2))$. For this purpose, it is sufficient to show

$$\Phi_\nu(Z)=\iint_{M\times M}E(x_1,\ x_2,\ Z)\nu(dx_1dx_2)$$

is holomorphic in $\Sigma(\delta_2,\theta)$ for any measure ν in $\mathscr{E}'(M\times M,\ X'\otimes X)$. Obviously if ν is in $\mathscr{D}(M,X')\otimes\mathscr{D}(M,X)$, then $\Phi_\nu(Z)$ is holomorphic in $\Sigma(\delta_2,\theta_2)$ and satisfies

(118) $$|\ \Phi_\nu(Z)\ |\leq C_\gamma(1+|\ Z\ |)^{-\gamma}||\ \nu\ ||\ ,$$

where $||\ \nu\ ||$ is the total variation of ν. If ν is an arbitrary measure in $\mathscr{E}'(M\times M,X\otimes X')$, we choose a sequence $\{\nu_k\}\subset\mathscr{D}(M,X)\otimes\mathscr{D}(M,X')$ which tends to ν in measure. We may assume total variations of ν_k, $k=1,2,3,\cdots$ are uniformly bounded. Therefore $\Phi_{\nu_k}(Z)$ tends pointwise to $\Phi_\nu(Z)$ and uniformly bounded. Since $\Phi_{\nu_k}(Z)$ are holomorphic in $\Sigma(\delta_2,\theta_2)$, by Montel's theorem, $\Phi_\nu(Z)$ is also holomorphic and satisfies the estimate (118). This proves Theorem 21.

From Theorem 21, follows

THEOREM 22. *We assume that* $2m-n>\gamma>0$. *For any* $Z\in\Sigma'(\theta_2)$, *let* F_Z *be any* β*-pseudo-differential operator such that the symbol of either* $F_Z(P+Z^{2m}D_s^{2m}+\tau_0)$ *or* $(P+Z^{2m}D_s^{2m}+\tau_0)F_Z$ *is identically I. We define the operator* $\tilde{E}(Z)$ *by*

$$\tilde{E}_Z(\sigma)u=e^{-is\sigma}F_Z(e^{is\sigma}u) \qquad \sigma\in\boldsymbol{R}^1,\ u\in\mathscr{D}(M,X)\ . \tag{119}$$

Then, $\tilde{E}_Z(\sigma)$ *has a kernel* $\tilde{E}(x,y,\sigma)$ *which belongs to* $\mathscr{E}(M\times M, X\otimes X')\hat{\otimes}\mathscr{O}_M(\boldsymbol{R}^1)$ *with*

$$|\tilde{E}(x,y,\sigma)|\le C(1+|Z\sigma|)^{-\gamma}\ . \tag{120}$$

Moreover for any $k>0$, *there is a constant* C *such that*

$$|E(x,y,Z\sigma)-\tilde{E}_Z(x,y,\sigma)|\le C(1+|Z\sigma|)^{-k}\ . \tag{121}$$

$\tilde{E}_Z(x,y,\sigma)$ *is given by the corollary to Theorem* 21.

PROOF. Since the β-pseudo-differential operator $G_Z^{(0)}-F_Z$ is of order $-\infty$, Theorem 22 follows from Theorem 20.

Now our main theorem follows from Theorem 22.

THEOREM 23. *We assume* $2m-n>\gamma>0$. *We denote the Green kernel of the elliptic operator* $(P+Z^{2m}+\tau_0)$ *by* $E(x,y,Z)$, *where* Z *is in* $\Sigma'(\theta_2)$, $\theta_2<\theta_1$. *Let* F_Z *be any* β*-pseudo-differential operator such that the symbol of either* $F_Z(P+Z^{2m}D_s^{2m}+\tau_0)$ *or* $(P+Z^{2m}D_s^{2m}+\tau_0)F_Z$ *is* 1 *and let* φ *be in* $\mathscr{D}(M)$ *which is equal to* 1 *in some neighbourhood of* x *and whose support is contained in a coordinate neighbourhood* U. *With a real linear function* $x\cdot\xi=x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n$ *of coordinate functions* $x_1,\cdots,x_n$ *in* U, *we denote the asymptotic expansion of* $e^{-i(x\cdot\xi+s\sigma)}\varphi F_Z\varphi e^{i(x\cdot\xi+s\sigma)}$ *by*

$$e^{-i(x\cdot\xi+s\sigma)}\varphi F_Z\varphi e^{i(x\cdot\xi+s\sigma)}\sim\sum_{j=0}^{\infty}f_{j,Z}(x,\xi,\sigma)\ . \tag{122}$$

Then, the asymptotic expansion of $E(x,x,Z\sigma)$, $Z\in\Sigma'(\theta_2)$, *in* σ, *is given by*

$$E(x,x,Z\sigma)\sim\sum_{j=0}^{\infty}\sigma^{s_j+n}(2\pi)^{-n}\frac{1}{\rho(x)}\int_{\boldsymbol{R}^n}f_{j,Z}(x,\xi,1)d\xi\ , \tag{123}$$

where s_j *is the homogeneous degree of* $f_{j,Z}$ *in* ξ,σ.

PROOF. First we note that the kernel $\tilde{E}_Z(x,y,\sigma)$ of the mapping $u\to e^{-is\sigma}F_Z(e^{is\sigma}u)$ exists in $\mathscr{E}(M\times M, X\otimes X)\hat{\otimes}\mathscr{O}_M(\boldsymbol{R}^1)$ and for any $k>0$ there exists a contant $C>0$ such that

$$|E(x,y,Z\sigma)-\tilde{E}_Z(x,y,\sigma)|\le C(1+|Z\sigma|)^{-k} \tag{124}$$

(Theorem 23).

On the other hand $\tilde{E}_Z(x,x,\sigma)$ is given by

$$\tilde{E}_Z(x, x, \sigma)=(2\pi)^{-n}\rho(x)^{-1}\int_{R^n} f_Z(x, \xi, \sigma)d\xi \ , \tag{125}$$

where $f_Z(x, \xi, \sigma)=e^{-i(x\cdot\xi+s\sigma)}\varphi F_Z\varphi e^{i(x\cdot\xi+s\sigma)}$. Since, if $|\xi|+|\sigma|$ is sufficiently large,

$$\left|(f_Z(x, \xi, \sigma)-\sum_{j=0}^{N-1} f_{j, Z}(x, \xi, \sigma)\right|\leq C(|\xi|+|\sigma|)^{s_N} . \tag{126}$$

we have, for large σ,

$$\begin{aligned}
&\left|\tilde{E}_Z(x, x, \sigma)-(2\pi)^{-n}\rho(x)^{-1}\sum_{j=0}^{N-1}\sigma^{s_j+n}\int_{R^n} f_{j, Z}(x, \xi, 1)d\xi\right| \\
&\leq\left|\tilde{E}_Z(x, x, \sigma)-(2\pi)^{-n}\rho(x)^{-1}\sum_{j=0}^{N-1}\int_{R^n} f_{j, Z}(x, \sigma\xi, \sigma)d\sigma\right. \\
&\leq(2\pi)^{-n}\rho(x)\int_{R^n}\left(f_Z(x, \xi, \sigma)-\sum_{j=0}^{N-1} f_{j, Z}(x, \xi, \sigma)\right)d\xi \\
&\leq(2\pi)^{-n}\rho(x)\int_{R^n}(|\xi|+|\sigma|)^{s_N}d\xi \\
&\leq(2\pi)^{-n}\rho(x)(|\sigma|)^{s_N+n} .
\end{aligned} \tag{127}$$

(123) follows from (124) and (127).

COROLLARY. *Using the notations in Theorem* 23,

$$\begin{aligned}
&\mathrm{Trace}\,(P+\sigma^{2m}Z^{2m}+\tau_0)^{-1} \\
&\quad\sim\sum_{j=0}^{\infty}\sigma^{s_j+n}(2\pi)^{-n}\int_M\frac{d\mu(x)}{\rho(x)}\int_{R^n}\mathrm{tr}\, f_{j, Z}(x, \xi, 1)d\xi .
\end{aligned} \tag{128}$$

when $\sigma\to\infty$.

§ 4. Examples.

Here we shall show some examples. All the vector bundles treated in the following are trivial line bundles. First we begin with the simplest case where $n=1, m=1$.

Example 1. Let M be the unit sphere S^1 and X be the trivial line bundle on S^1. As a local parameter of S^1, we choose the arc length x. Consider the elliptic operators

$$P=-\frac{d^2}{dx^2}+b(x)\frac{d}{dx}+c(x), \qquad b, c\in\mathscr{D}(S^1), \text{ on } S^1$$

and

$$Q=-\frac{d^2}{dx^2}-\frac{d^2}{ds^2}+b(x)\frac{d}{dx}+c(x)+\tau_0, \qquad \tau_0>0 \text{ on } S^1\times R^1 .$$

Let $x_0 \in S^1$ and $x\cdot\xi$ be a linear function of the coordinate x, in some neighbourhood of x. Then for any C^∞ function φ which is equal to 1 near x_0 and has its support contained in a coordinate neighbourhood of x_0,

$$e^{-i(x\cdot\xi+s\sigma)}\varphi Q\varphi e^{i(x\cdot\xi+s\sigma)}=\xi^2+\sigma^2+ib(x)\xi+c(x)+\tau_0\ .$$

By Theorem 15, if τ_0 is sufficiently large, the inverse Q^{-1} of Q exists as a β-pseudo-differential operator. Setting

$$e^{-i(x\cdot\xi+s\sigma)}\varphi Q^{-1}\varphi e^{i(x\cdot\xi+s\sigma)}\sim\sum_{j=0}^{\infty}g_j(x,\xi,\sigma)$$

near x_0, g_j, $j=0,1,2,\cdots$, are determined by the generalized Leibniz rule:

$$g_0\cdot q_0=1 \tag{129}$$

$$g_1\cdot q_0+\frac{\partial g_0}{\partial x}\frac{\partial q_0}{\partial\xi}+g_0\cdot q_1=0 \tag{130}$$

$$g_2\cdot q_0+g_1\cdot q_1+\frac{\partial g_1}{\partial x}\frac{\partial q_0}{\partial\xi}+g_0\cdot q_2+\frac{\partial g_0}{\partial x}\frac{\partial q_1}{\partial\xi}+\frac{1}{2}\frac{\partial^2 g_0}{\partial x^2}\frac{\partial^2 q_0}{\partial\xi^2}=0 \tag{131}$$

where

$$q_0=\xi^2+\sigma^2 \tag{132}$$

$$q_1=ib(x)\xi\ , \tag{133}$$

$$q_2=c(x)+\tau_0\ . \tag{134}$$

$$\vdots$$

Therefore, we have

$$g_0=(\xi^2+\sigma^2)^{-1} \tag{135}$$

$$g_1=-ib\xi(\xi^2+\sigma^2)^{-1} \tag{136}$$

$$\begin{aligned}g_2=&(2ib'-b^2-c-\tau_0)(\xi^2+\sigma^2)^{-2}\\&+(b^2-2ib')(\xi^2+\sigma^2)^{-3}\sigma^2\end{aligned} \tag{137}$$

$$\vdots$$

Since

$$\int_{R^1}\frac{d\xi}{\xi^2+\sigma^2}=\pi\sigma^{-1},\qquad \int_{R^1}\frac{d\xi}{(\xi^2+\sigma^2)^2}=\frac{1}{2}\pi\sigma^{-3}\ ,$$
$$\int_{R^1}\frac{d\xi}{(\xi^2+\sigma^2)^3}=\frac{3}{8}\pi\sigma^{-5}\ .$$

Applying Theorem 23 to this case, we have

$$E(x_0, x_0, \sqrt{\sigma^2+\tau_0})=\frac{1}{2}\sigma^{-1}+\left\{\frac{1}{16}(2ib'(x_0)-b(x_0)^2)-\frac{1}{4}(c(x_0)+\tau_0)\right\}\sigma^{-3}+0(\sigma^{-5}) \tag{138}$$

where $E(x, y, \sigma)$ is the kernel of the Green operator $(P+\sigma^2)^{-1}$. Replacing σ^2 with $\sigma^2-\tau_0$ in (138), we obtain

THEOREM 24.

$$E(x_0, x_0, \sigma)\sim\frac{1}{2}\sigma^{-1}+\left\{\frac{1}{16}(2ib'(x_0)-b(x_0)^2)-\frac{1}{4}c(x_0)\right\}\sigma^{-3}+0(\sigma^{-5}) \tag{139}$$

and

$$\text{trace}\,(P+\sigma^2)^{-1}\sim\pi\sigma^{-1}+\sigma^{-3}\int_0^{2\pi}\left\{\frac{1}{16}(2ib'(x)-b(x))-\frac{1}{4}c(x)\right\}dx+0(\sigma^{-5})\,. \tag{140}$$

REMARK. The second term of (139) and (140) does not always vanish.

As special cases, the following two cases among others are interesting.

(i) $b\equiv c\equiv 0$.

In this case, we have from the generalized Leibniz rule,

$$g_0=(\xi^2+\sigma^2)^{-2},\ g_1=0,\ g_2=-\frac{\tau}{(\xi^2+\sigma^2)^2} \tag{141}$$

$$0=g_3=g_4=\cdots=g_{n-1}=g_n\cdots\,.$$

Therefore,

$$E(x_0, x_0, \sqrt{\sigma^2+\tau_0})\sim\frac{1}{2}\sigma^{-1}-\frac{\tau_0}{4}\sigma^{-3}+0(\sigma^{-N})$$

for any $N>0$.

Thus for any $N>0$,

$$\lim_{\sigma\to\infty}\sigma^N\left(E(x_0, x_0, \sigma)-\frac{1}{2}\sigma^{-1}\right)=0\,. \tag{142}$$

$$\lim_{\sigma\to\infty}(\sigma^N\,\text{trace}\,(P+\sigma^{2m})^{-1}-\pi\sigma^{-1})=0\,. \tag{143}$$

(142) and (143) can be verified by calculating $E(x, x, \sigma)$ exactly. In fact, we have

$$\begin{aligned} E(x, x, \sigma)&=\frac{1}{2}\sigma^{-1}\coth 2\sigma \\ &=\frac{1}{2}\sigma^{-1}(1-2e^{-2\sigma}+\cdots)\,. \end{aligned} \tag{144}$$

(ii) $b\equiv 0,\ c=-2q\cos 2x$.

In this case, P is called the Mathieu's operator. (139) and (140) give,

(145) $$E(x_0, x_0, \sigma) \sim \frac{1}{2}\sigma^{-1} + \frac{1}{2}(q\cos 2x_0)\sigma^{-3} + 0(\sigma^{-5})$$

(146) $$\text{trace}\,(P+\sigma^2)^{-1} \sim \pi\sigma^{-1} + 0(\sigma^{-5})\,.$$

Now we examine higher order term. (135), (136), (137) give

(147) $$g_0 = (\xi^2+\sigma^2)^{-1},\ g_1 = 0,\ g_2 = (2q\cos 2x - \tau_0)(\xi^2+\sigma^2)^{-2}\,.$$

And the generalized Leibniz rule gives

$$g_3\cdot q_0 + g_2\cdot q_1 + \frac{\partial g_2}{\partial x}\frac{\partial q_0}{\partial \xi} + g_0\cdot q_3 + \frac{\partial g_0}{\partial x}\frac{\partial q_2}{\partial \xi} + \frac{1}{2}\frac{\partial^2 g_0}{\partial x^2}\frac{\partial^2 q_1}{\partial \xi^2} + \frac{1}{3!}\frac{\partial^3 g_0}{\partial x^3}\frac{\partial^3 q_0}{\partial \xi^3} = 0.$$

Therefore

(148) $$\begin{aligned} g_3 &= -q_0^{-2}\left(\frac{\partial g_2}{\partial x}\frac{\partial q_0}{\partial \xi}\right) \\ &= 4q(\xi^2+\sigma^2)^{-3}(\sin 2x)2\xi\,. \end{aligned}$$

As to g_4, we have

(149) $$g_4\cdot q_0 + \frac{\partial g_2}{\partial x}\frac{\partial q_0}{\partial \xi} + g_2\cdot q_2 + \frac{1}{2}\frac{\partial^2 g_2}{\partial x_2}\frac{\partial^2 q_0}{\partial \xi^2} = 0$$

replacing g_2, g_3 with (147), (148) we have

(150) $$\begin{aligned} g_4 = &\{(2q\cos 2x - \tau_0)^2 - 24q\cos 2x\}(\xi^2+\sigma^2)^{-3} \\ &+ 32q\cos 2x\sigma^2(\xi^2+\sigma^2)^{-4}\,. \end{aligned}$$

Since

$$\int_{R^1}\frac{d\xi}{(\xi^2+\sigma^2)^4} = \frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\pi\sigma^{-7},\quad \int_{R^1}\frac{d\xi}{(\xi^2+\sigma^2)^3} = \frac{3}{4\cdot 2}\pi\sigma^{-5}\,,$$

integrating (150) in ξ over $\boldsymbol{R}^1$, we have

$$\frac{1}{(2\pi)}\int_{R^1} g_4(x, \xi, \sigma)d\xi = \left\{\frac{3}{4}\cos^2 2x + \frac{q}{2}\cos 2x - \frac{3}{4}\tau_0 q\cos 2x + \tau_0^2\frac{3}{16}\right\}\sigma^{-5}\,.$$

Therefore,

$$\begin{aligned} E(x_0, x_0, \sqrt{\sigma^2+\tau_0}) \sim &\frac{1}{2}\sigma^{-1} + \left(\frac{1}{2}q\cos 2x - \frac{1}{4}\tau_0\right)\sigma^{-3} \\ &+ \left(\frac{3}{4}\cos^2 2x + \frac{1}{2}q\cos 2x - \frac{3}{4}\tau_0 q\cos 2x + \frac{3}{16}\tau_0^2\right)\sigma^{-5} \\ &+ 0(\sigma^{-7})\,. \end{aligned}$$

Replacing σ^2 with $\sigma^2-\tau_0$, and integrating in x, we have

THEOREM 25. *For Mathieu's operator P, we have the asymptotic expansion,*

(147) $$E(x, x, \sigma)\sim\frac{1}{2}\sigma^{-1}+\frac{1}{2}q\cos 2x\sigma^{-3}+\left(\frac{3}{4}\cos^2 2x+\frac{1}{2}q\cos 2x\right)\sigma^{-5}+0(\sigma^{-7}) .$$

(148) $$\text{trace } (P+\sigma^2)^{-1}\sim\pi\sigma^{-1}+\frac{3}{4}\pi\sigma^{-5}+0(\sigma^{-7}) .$$

Example 2. Let M be a compact orientable compact Riemannian manifold of dimension n and let X be the trivial line bundle over M. In the following we put $P=\Delta^m$, where Δ is the Laplace-Beltrami operator associated with the Riemannian structure of M. First we note that if $2m>n$, the value $E(x, x, \sigma)$ of the Green kernel of $(\Delta^m+\sigma^{2m})^{-1}$ on the diagonal set of $M\times M$ is obviously independent of local coordinates.

A simple but interesting consequence of Theorem 23 is the following generalization of Example 1 (i).

THEOREM 26. *If the metric of M is locally Euclidean near x in M, for any $N>0$, we have*

(149) $$\lim_{\sigma\to\infty}\sigma^N\left\{E(x, x, \sigma)-(2\pi)^{-n}\omega_{n-1}(2m)^{-1}\frac{\pi}{\sin\frac{n\pi}{2m}}\sigma^{n-2m}\right\}=0 .$$

If M is locally Euclidean,

(150) $$\lim_{\sigma\to\infty}\sigma^N\left\{\text{trace } (\Delta^m+\sigma^{2m})^{-1}-(2\pi)^{-n}\omega_{n-1}(2m)^{-1}\frac{\pi}{\sin\frac{n\pi}{2m}}v(M)\sigma^{n-2m}\right\}=0 .$$

Where, ω_{n-1} is the volume of the unit $n-1$ sphere and $v(M)$ is the volume of M.

PROOF. We have only to prove (149), because (150) follows from (149).

For any $\varphi\in\mathscr{D}(M)$ with $\varphi=1$ near x and with its support in a coordinate neighbourhood, setting,

$$e^{-i(x\cdot\xi+s\sigma)}\varphi Q\varphi e^{i(x\cdot\xi+s\sigma)}=q(x, \xi, \sigma)=|\xi|^{2m}+\sigma^{2m}+\tau_0 ,$$

where $x\cdot\xi$ is a linear function $x_1\xi_1+\cdots+x_n\xi_n$ of the coordinate functions $x_1, \cdots, x_n$. Therefore, we have

$$g(x, \xi, \sigma)=(|\xi|^{2m}+\sigma^{2m}+\tau_0)^{-1} .$$

Replacing σ^{2m} with $\sigma^{2m}-\tau_0$ and integrating in ξ over $\boldsymbol{R}^n$, we obtain

$$E(x, x, \sigma)\sim(2\pi)^{-n}\int_{\boldsymbol{R}^n}(|\xi|^{2m}+\sigma^{2m})^{-1}d\xi$$

$$=(2\pi)^{-n}(2m)^{-1}\omega_{n-1}B\left(\frac{n}{2m}, 1-\frac{n}{2m}\right)\sigma^{n-2m} .$$

This completes the proof.

It is interesting but very complicated to calculate the asymptotic expansion of $E(x, x, \sigma)$. In the following, we shall confine ourselves to the case that M is an orientable compact hypersurface embedded in $\boldsymbol{R}^{n+1}$ with the metric induced by the embedding and we shall give a geometric interpretation of the second term of the expansion.

Let M be an orientable smooth compact hypersurface in $\boldsymbol{R}^{n+1}$ and let ω be a unit tangent vector to M at x in M. Then the curvature of the geodesic tangent to ω at x is a symmetric quadratic form $\mathfrak{a}(\omega)$. The eigenvalues $p_1(x), p_2(x), \cdots, p_n(x)$ of the matrix associated with $\mathfrak{a}(\omega)$ are called principal curvature of M at x. The j-th mean curvature $K_j(x)$ of M at x is the j-th elementary symmetric function of n-principal curvatures divided by $\binom{n}{j}$, that is

$$\binom{n}{j} K_j(x) = \sum_{k_1 \cdots k_j} p_{k_1}(x) \cdots p_{k_j}(x) . \tag{151}$$

Our results are the following

THEOREM 27. *There exist constants A and B depending only on n and m such that for any smooth compact oriented hypersurface M in $\boldsymbol{R}^{n+1}$, the value of the Green kernel of $P = \Delta^{2m}$, $2m > n$, on the diagonal set in $M \times M$ admits the asymptotic expansion,*

$$\begin{aligned} E(x, x, \sigma) \sim & (2\pi)^{-n}(2m)^{-1} \frac{\pi}{\sin \dfrac{n\pi}{2m}} \sigma^{n-2m} \\ & + (AK_1(x)^2 + BK_2(x))\sigma^{n-2m-2} \\ & + 0(\sigma^{n-2m-4}) . \end{aligned} \tag{151}$$

PROOF. Choosing a suitable orthonormal frame in $\boldsymbol{R}^{n+1}$, we may assume that M is given by $x_{n+1} = 1/2(p_1 x_1^2 + \cdots + p_n x_n^2) + O_3$, $p_j = p_j(0)$, where and hereafter we denote by O_k those functions u of $x_1, \cdots, x_n$, which satisfy

$$|(|x_1| + \cdots + |x_n|)^k u(x_1 \cdots x_n)| < M \qquad \text{as } x_j \longrightarrow 0 .$$

Thus the metric on M is of the form

$$ds^2 = \sum_{k,l}^{n} (\delta_{kl} - a_{kl}(p_1 x_1 \cdots p_n x_n) + O_3) dx_k dx_l$$

here $a_{ij}(\gamma_1, \cdots, \gamma_n)$ is a quadratic form of $\gamma_1 \cdots \gamma_n$. So that the Laplace-Beltrami operator is

$$-\Delta f=\sum_{i,j}(\delta_{ij}-a_{ij}(p_1x_1,\cdots,p_nx_n)+O_3)\frac{\partial^2 f}{\partial x_i\partial x_j}$$
$$+\sum_{i,j}(b_{ij}(p_1,\cdots,p_n)x_i+O_2)\frac{\partial f}{\partial x_j},$$

where $c_{ij}(p_1,\cdots,p_n)$ is a homogeneous polynomial in $p_1,\cdots,p_n$ of degree 2. From this, we have

$$\Delta^2 f=\sum_{i,j,k}(\delta_{ij}-2a_{ij}(p_1x_1,\cdots,p_nx_n)+O_3)\frac{\partial^4 f}{\partial x_i\partial x_j\partial x_k^2}$$
$$+\sum_{i,j,k,l}(c_{ijkl}(p_1,\cdots,p_n)x_l+O_2)\frac{\partial^3 f}{\partial x_i\partial x_j\partial x_k}$$
$$+\sum_{i,j,k}(d_{ij}(p_1,\cdots,p_n)+O_1)\frac{\partial^2 f}{\partial x_i\partial x_j}$$
$$+\text{lower order derivatives of } f,$$

where $c_{ijkl}(\gamma_1,\cdots,\gamma_n)$, and $d_{ij}(\gamma_1,\cdots,\gamma_n)$ are homogeneous polynomials in $\gamma_1,\cdots,\gamma_n$ of degree 2.

If $x\cdot\xi=x_1\cdot\xi_1+\cdots+x_n\cdot\xi_n$ is a linear function of x, and if $\varphi\in\mathscr{D}(M)$, $\varphi\equiv 1$ near the origin and has its support in a small neighbourhood of the origin, we have the following asymptotic expansion near the origin

$$e^{-i(x\cdot\xi+s\sigma)}\varphi(\Delta^{2m}+D_s^{2m}+\tau_0)\varphi e^{i(x\cdot\xi+s\sigma)}$$
$$=\sum_{i,j,k}(\delta_{ij}-2a_{ij}+O_3)\xi_i\xi_j\xi_k{}^2|\xi|^{2m-4}+\sigma^{2m}$$
$$+\binom{m}{2}\sum_{i,j,k,l}(c_{ijkl}(p_1,\cdots,p_n)x_i+O_2)\xi_j\xi_k\xi_l|\xi|^{2m-4}$$
$$+\binom{m}{2}\sum_{i,j}(d_{ij}(p_1,\cdots,p_n)+O_1)\xi_i\xi_j|\xi|^{2m-4}$$
$$+\text{lower order term in } \xi,\sigma.$$

Therefore, setting

$$(153)\qquad\begin{cases} q_0(x,\xi,\sigma)=\sum_{i,j,k}(\delta_{ij}-2a_{ij}(p_1x_1,\cdots,p_nx_n)+O_3)\xi_i\xi_j\xi_k^2|\xi|^{2m-4} \\ q_1(x,\xi,\sigma)=\binom{m}{2}\sum_{i,j,k,l}(c_{ijkl}(p_1,\cdots,p_n)x_i+O_2)\xi_j\xi_k\xi_l|\xi|^{2m-4} \\ q_2(x,\xi,\sigma)=\binom{m}{2}\sum_{i,j,k}(d_{ij}(p_1,\cdots,p_n)+O_1)\xi_i\xi_j|\xi|^{2m-4} \end{cases}$$

we have, by the generalized Leibniz rule,

$$(154)\begin{cases} g_0\cdot q_1=1 \\ g_1\cdot q_0+g_0\cdot q_1+\sum_i \frac{\partial g_0}{\partial x_i}\frac{\partial q_0}{\partial \xi_i}=0 \\ g_2\cdot q_0+g_1\cdot q_1+\sum_i \frac{\partial g_1}{\partial x_i}\frac{\partial q_0}{\partial \xi_i}+g_0\cdot q_2+\sum_i \frac{\partial g_0}{\partial x_i}\frac{\partial q_1}{\partial \xi_i}+\frac{1}{2}\sum_{i,j}\frac{\partial^2 g_0}{\partial x_i\partial x_j}\frac{\partial^2 q_0}{\partial \xi_i\partial \xi_j}=0\,. \end{cases}$$

From (154)

$$g_0=q_0^{-1}$$
$$q_1=-\left(q_0^{-2}q_1-q_0^{-3}\sum_i \frac{\partial q_0}{\partial x_i}\frac{\partial q_0}{\partial \xi_i}\right).$$

Considering $q_0(0,\xi,\sigma)=|\xi|^{2m}+\sigma^{2m}$ and $q_1(0,\xi,\sigma)=0$, we have

$$g_1(0,\xi,\sigma)=0\,,$$

and

$$\left.\frac{\partial g_1}{\partial x_j}\right|_{x=0}=-\left(q_0^{-2}\frac{\partial q_1}{\partial x_j}-q_0^{-3}\sum_{i,j}\frac{\partial^2 q_0}{\partial x_i\partial x_j}\frac{\partial q_0}{\partial \xi_i}\right)\Bigg|_{x=0}$$
$$\left.\frac{\partial^2 g_0}{\partial x_i\partial x_j}\right|_{x=0}=-q_0^{-2}\left.\frac{\partial^2 q_0}{\partial x_i\partial x_j}\right|_{x=0}.$$

So that

$$(155)\qquad g_2\Big|_{x=0}=-q_0^{-1}\left(\sum_i\frac{\partial q_0}{\partial \xi_i}\left(q_0^{-3}\sum_{i,j}\frac{\partial^2 q_0}{\partial x_i\partial x_j}\frac{\partial q_0}{\partial \xi_j}-q_0^{-2}\frac{\partial q_1}{\partial x_i}\right)\right)\Bigg|_{x=0}$$
$$-q_0^{-1}q_0^{-1}q_2\Big|_{x=0}$$
$$-q_0^{-1}\left(\frac{1}{2}\sum_{i,j}\frac{\partial^2 q_0}{\partial \xi_i\partial \xi_j}(-q_0^{-2})\frac{\partial^2 q_0}{\partial x_i\partial x_j}\right)\Bigg|_{x=0}.$$

After replacing q_0, q_1, q_2 in (155) with (153), integrate in ξ over $\boldsymbol{R}^n$, then we have the result.

From Theorem 27 follows

THEOREM 28. *Under the same condition as in Theorem 27*

$$(156)\qquad \operatorname{trace}(\Delta^m+\sigma^{2m})^{-1}\sim(2\pi)^{-n}(2m)^{-1}\frac{\pi}{\sin\frac{n\pi}{2m}}v(M)\sigma^{n-2m}$$
$$+\left(A\int_M K_1(x)^2dx+B\int K_2(x)dx\right)\sigma^{n-2m-2}$$
$$+0(\sigma^{n-2m-2})\,.$$

A, B are constants depending only on n and m. $v(M)$ is the volume of M.

REMARK. It is possible but rather complicated to determine the constants A and B above. We omit to do this here. But in the case of $n=2$ $m=2$, after a somewhat long calculation we can prove

THEOREM 29. *If M is an oriented compact C^∞ surface in $\boldsymbol{R}^3$,*

$$E(x, x, \sigma) \sim \frac{1}{16\pi}\sigma^{-2} + \left\{21\pi^{-1}H(x)^2 - \frac{10}{3}\pi^{-1}K(x)\right\}\sigma^{-4} + 0(\sigma^{-6}), \tag{157}$$

where $H(x)$ (resp. $K(x)$) the mean (resp. Gaussian) curvature of M at x.

THEOREM 30. *If M is an oriented compact C^∞ surface in $\boldsymbol{R}^3$,*

$$\begin{aligned} \text{trace}\,(\Delta^2 + \sigma^4) \sim & \frac{1}{16\pi} v(M)\sigma^{-2} \\ & + \left\{21\pi^{-1}\int_M H(x)^2 dx - \frac{20}{3}\pi^{-1}v(M)\chi(M)\right\}\sigma^{-4} \\ & + 0(\sigma^{-6}), \end{aligned} \tag{158}$$

where $\chi(M)$ is the Euler-Poincaré number of M.

To obtain (158), we have only to integrate (157) over M and apply the Gauss-Bonnet formula.

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