Theory of Hyperfunctions, II.

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Suggested by the theory of distributions of L. Schwartz, we have introduced another generalized concept of functions, that of hyperfunctions, which includes that of distributions provided that the underlying manifold is analytic.

In our previous paper [9], we have expounded the theory of hyperfunctions of one variable, but we have hitherto given only a rough outline of the theory in case of more than one variables in [8]. The purpose of the present paper is to give a full account of the general formulation of our theory. (The proof will be published in a subsequent paper.) All definitions and theorems are so formulated that this paper can be understood without reference to our former papers.

We have announced in [9] that we shall develop further our theory in case of one variable in a forthcoming paper. We have however noticed in the meantime that we had better include the "further theory" we intended to develop in a "forthcoming paper" in the general theory, so that we have changed our program.

As we have sketched in [8], our generalization of the function concept is carried out in utilizing the (relative) cohomology theory with sheaf-coefficients. We can define each hyperfunction on a given m-dimensional real analytic manifold M which can be 'analytically prolonged' to a paracompact m-dimensional complex analytic manifold X, as a relative m-cohomology class of X mod$(X-M)$ with coefficients in the sheaf of analytic functions (or more generally, in a locally free analytic sheaf). By the excision theorem for relative cohomology moduli, our definition does not essentially depend on the choice of the 'prolongation X'. In case M is an oriented manifold, our hyperfunction may be (at least locally) regarded as a sum of 'boundary values' of some holomorphic functions in the prolonged complex manifold. The concept of hyperfunctions may also be considered as a special case of another new concept: that of 'analytic distributions' which is defined on any (locally) closed subset of a complex analytic manifold. The relation between these concepts can be schematized as follows:

$$\begin{tikzcd}
\text{Analytic distributions} & \text{Hyperfunctions} \arrow[l, Rightarrow] \\
\begin{array}{c}\text{Cousin's additive distributions}\\ \text{(modulo an ideal)}\end{array} \arrow[u, Rightarrow] \arrow[r, dashed, -] & \text{Dirac functions} \arrow[u, Rightarrow]
\end{tikzcd}$$

It is shown that each hyperfunction on M can be identified with a cross-section of an analytic sheaf over M (the 'localizability' of hyperfunctions), that each hyperfunction defined on a subdomain of M is always extended to a hyperfunction on M (the 'completeness' of hyperfunctions), and that the module of holomorphic functions (of some type $\boldsymbol{B}$) on M and the module of hyperfunctions (of the type complementary to $\boldsymbol{B}$) with compact carrier on M, constitute a couple of mutually dual topological vector spaces in a natural manner provided that the manifold M is paracompact (the 'duality theorem', which is closely related to the Serre's duality theorem for complex analytic manifolds).

The present paper consists of three Chapters. In Chapter I (§1–§3), we summarize briefly the results of [9] and show that they can be neatly expressed in the language of the cohomology theory of sheaves. (The knowledge on sheaf theory is not assumed beforehand.) This will also serve as a motivation to the formulation of the general theory in terms of sheaf theory (see especially §3). Chapter II (§4–§5) concerns the cohomology theory of sheaves in general. In §4, we summarize the well-known results on cohomology moduli with sheaf coefficients after Cartan [1], and extend them to the relative case. In §5, we introduce all sheaf-theoretical notions used in our theory, in particular those of associated relative cohomology moduli, of sheaves of distributions, and of pure codimensionality. After these preparations, it will be easy to formulate our theory systematically in Chapter III (§6–§10); we shall extend all the results of Chapter I in case of dimension 1 to the case of arbitrary dimensions in this last and main Chapter of the present paper. (For the sake of simplicity, we confine our considerations to *paracompact* (complex and real) analytic manifolds throughout this Chapter.) Further details as well as various applications of the theory will be given in our subsequent papers.

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Chapter I. A Summary of the Case of One Variable.

§ 1. Definitions.

Throughout this paper, $\boldsymbol{C}$, $\boldsymbol{R}$ denote the complex plane and the real axis, respectively. For any open set $D \subset \boldsymbol{C}$, $\mathfrak{A}(D)$ will denote the ring of all holomorphic functions in D.

1.1. Let S be any locally closed subset of $\boldsymbol{R}$ (S will keep this meaning throughout this paragraph). We denote with $\mathfrak{D}(S)$ the family of all the *complex neighborhoods* of S, i.e. the open sets of $\boldsymbol{C}$ containing S as a closed subset. We denote with $\mathfrak{A}(S)$ and with $\widetilde{\mathfrak{A}}(S)$ the inductive limit of $\{\mathfrak{A}(D)\,;\ D \in \mathfrak{D}(S)\}$ and that of $\{\mathfrak{A}(D-S)\,;\ D \in \mathfrak{D}(S)\}$ by the canonical homomorphisms, respectively. $\widetilde{\mathfrak{A}}(S)$ is regarded as an extension ring of $\mathfrak{A}(S)$ in a natural manner, whence *an* $\mathfrak{A}(S)$*-module* $\mathfrak{B}(S)$ *is defined by*

$$\mathfrak{B}(S) = \widetilde{\mathfrak{A}}(S) \bmod \mathfrak{A}(S). \tag{1.1.1}$$

Each element of $\mathfrak{B}(S)$ is called a hyperfunction on S. By the definition, each hyperfunction $g\in\mathfrak{B}(S)$ is represented by an element of $\widetilde{\mathfrak{A}}(S)$ and hence by an element $\varphi\in\mathfrak{A}(D-S)$ with some $D\in\mathfrak{D}(S)$. We call such φ a *defining function* of g, and write

$$g=[\varphi, D]=[\varphi] \quad \text{or} \quad g(x)=[\varphi(z)]_{z=x}.$$

Setting $I=\boldsymbol{R}\frown D$ (a *real neighborhood* of S), and regarding $g(x)$ as a hyperfunction on I in a natural manner, the above expression is written in the form*⁾

$$g(x)=\varphi(x+i0)-\varphi(x-i0) \tag{1.1.2}$$

where the 'boundary values' $\varphi(x\pm i0)\in\mathfrak{B}(I)$ are defined by

$$\varphi(x+i0)=[\varepsilon(z)\varphi(z)]_{z=x},\ \varphi(x-i0)=-[\bar{\varepsilon}(z)\varphi(z)]_{z=x}$$

with

$$\varepsilon(z)=\begin{cases}1\\0\end{cases} \qquad \bar{\varepsilon}(z)=\varepsilon(-z)=\begin{cases}0 & (\Im z>0)\\1 & (\Im z<0).\end{cases}$$

Furthermore, defining $1\in\mathfrak{B}(I)$ by $1=[\varepsilon(z)]_{z=x}=-[\bar{\varepsilon}(z)]_{z=x}$ and identifying each $f\in\mathfrak{A}(I)$ with $f\cdot 1\in\mathfrak{B}(I)$, we can consider $\mathfrak{A}(I)$ as canonically embedded in $\mathfrak{B}(I)$:

$$\mathfrak{A}(I)\subset\mathfrak{B}(I). \tag{1.1.3}$$

We shall list now operations on hyperfunctions ((i)~(vi)).

(i) We have, for any $f_\nu\in\mathfrak{A}(S)$ and $g_\nu=[\varphi_\nu]\in\mathfrak{B}(S)$ $(\nu=1, 2,\cdots, n)$,

$$\sum_{\nu=1}^{n} f_\nu(x)g_\nu(x)=[\sum_{\nu=1}^{n} f_\nu(z)\varphi_\nu(z)]_{z=x}.$$

(ii) For $g(x)=[\varphi(z)]_{z=x}\in\mathfrak{B}(S)$, we define

$$\frac{d}{dx}g(x)=\left[\frac{d}{dz}\varphi(z)\right]_{z=x},$$

whence we have, for any linear differential operator $L_x=f_0(x)\dfrac{d^n}{dx^n}+\cdots+f_n(x)$ with $f_\nu(x)\in\mathfrak{A}(S)$,

$$L_x g(x)=[L_z\varphi(z)]_{z=x}.$$

(iii) For $g(x)=[\varphi(z)]_{z=x}\in\mathfrak{B}(S)$, we define

$$\bar{g}(x)=-[\bar{\varphi}(z)]_{z=x}, \qquad (\bar{\varphi}(z)=\overline{\varphi(\bar{z})}),$$

whence the real part $\Re g(x)$ and the imaginary part $\Im g(x)$ of $g(x)$ are defined in an obvious manner; a hyperfunction $g(x)$ is called *real-valued* if $\Im g(x)=0$.

(iv) For any open subset S' of S, we have a canonical homomorphism

$$\mathfrak{B}(S)\to\mathfrak{B}(S')$$

in which the image $g|S'$ of $g=[\varphi, D]\in\mathfrak{B}(S)$ (the *restriction* of g onto S') is given by

*⁾ The symbol $[\varphi(z)]_{z=x}$ may now be regarded as an abridgement of the symbol $[\varphi(z)]_{z=x-i0}^{z=x+i0}$.

$$g|S'=[\varphi, D']$$

with $D'\in\mathfrak{D}(S')$, $D'\subset D$. The *carrier* of g is the smallest of closed F such that $g|(S-F)=0$, while the *carrier of singularity* of g is the smallest of closed F such that $g|(S-F)$ is holomorphic.

(v) Let $x=\xi(x')$ be a holomorphic function on I' with the value in I such that $d\xi(x')/dx'\neq 0$ on I', I and I' being open sets of $\boldsymbol{R}$. For any $g(x)=[\varphi(z)]_{z=x}$ $\in\mathfrak{B}(I)$, we define

$$g(\xi(x'))=\pm[\varphi(\xi(z'))]_{z'=x'}\in\mathfrak{B}(I'),$$

where $\pm$ denotes the sign of $d\xi(x')/dx'$.

(vi) Let $g=[\varphi, D]$ be a hyperfunction on a compact set $K\subset\boldsymbol{R}$. We define the *definite integral* of $g(x)$ by

$$\int_K g(x)dx=-\oint_\Gamma \varphi(z)dz$$

where Γ denotes a rectifiable path in D going around K in the positive sense.*) For each $g\in\mathfrak{B}(K)$, we have $g=[\varphi_0, \boldsymbol{C}]$ where $\varphi_0\in\mathfrak{A}(\boldsymbol{C}-K)$, the *standard defining function* of g, is defined by

$$\varphi_0(z)=\frac{1}{2\pi i}\int_K \frac{1}{x-z}g(x)dx \qquad (z\in\boldsymbol{C}-K).$$

1.2. For example, a hyperfunction $\delta(x)$ on $\{0\}$, the Dirac function, is defined by

$$\delta(x)=-\frac{1}{2\pi i}\left[\frac{1}{z}\right]_{z=x}=-\frac{1}{2\pi i}\left(\frac{1}{x+i0}-\frac{1}{x-i0}\right),$$

for which we have

$$x\cdot\delta(x)=0,\quad x\cdot\delta^{(n)}(x)=-n\delta^{(n-1)}(x),\quad \left(\delta^{(n)}(x)=\frac{d^n}{dx^n}\delta(x)\right),$$

whence it follows that the $\boldsymbol{C}$-module spanned by $\delta(x), \delta'(x),\cdots,\delta^{(n)}(x)$ constitutes an $\mathfrak{A}(\{0\})$-submodule of $\mathfrak{B}(\{0\})$. $\delta(x)$ and hence the derivatives $\delta'(x), \delta''(x),\cdots$ are all real-valued. We have, for any $f(x)\in\mathfrak{A}(\{0\})$,

$$\int_{\{0\}} f(x)\delta(x)dx=f(0), \qquad \int_{\{0\}} f(x)\delta^{(n)}(x)dx=(-1)^n f^{(n)}(0).$$

§2. Localizability and Completeness of Hyperfunctions.

Now we shall give main properties of hyperfunctions.

2.1. *Completeness.* First of all, the existence of the standard defining function for a hyperfunction on a *compact* set in $\boldsymbol{R}$ stated in 1.1. (vi) is generalized for

*) It is also easily shown that if $g(x)$ depends on some 'holomorphic parameters', then the definite integral of $g(x)$ is a holomorphic function of these parameters.

an arbitrary hyperfunction in the following manner.

Let S be a locally closed set in $\boldsymbol{R}$. For any $g\in\mathfrak{B}(S)$ and for any $D_0\in\mathfrak{D}(S)$, there exists $\varphi_0\in\mathfrak{A}(D_0)$ such that $g=[\varphi_0, D_0]$. This means that the exact sequence

$$0\longrightarrow\mathfrak{A}(D_0)\longrightarrow\mathfrak{A}(D_0-S)\longrightarrow\mathfrak{B}(S)$$

with canonical homomorphisms is now completed to the exact sequence

$$0\longrightarrow\mathfrak{A}(D_0)\longrightarrow\mathfrak{A}(D_0-S)\longrightarrow\mathfrak{B}(S)\longrightarrow 0, \tag{2.1.1}$$

or equivalently, that we can define $\mathfrak{B}(S)$ by

$$\mathfrak{B}(S)\simeq\mathfrak{A}(D_0-S)\bmod\mathfrak{A}(D_0)$$

by means of any fixed $D_0\in\mathfrak{D}(S)$, instead of using the inductive limit as in (1.1.1).

From (2.1.1) follows

(i) that for any closed set F of S, *hyperfunctions on F* are identified with *hyperfunctions on S whose carriers are contained in F*, or equivalently, we have an exact sequence

$$0\longrightarrow\mathfrak{B}(F)\longrightarrow\mathfrak{B}(S)\longrightarrow\mathfrak{B}(S-F)$$

with canonical homomorphisms, and

(ii) that for any open set S' of S, each *hyperfuction on S'* is extended to a *hyperfunction on S*, i.e. the homomorphism by restriction: $\mathfrak{B}(S)\xrightarrow{\text{rest}}\mathfrak{B}(S')$ is *surjective* (Completeness theorem).

The results (i) and (ii) are subsumed in a single exact sequence:

$$0\longrightarrow\mathfrak{B}(F)\longrightarrow\mathfrak{B}(S)\xrightarrow{\text{rest.}}\mathfrak{B}(S-F)\longrightarrow 0. \tag{2.1.2}$$

2.2. *Composition and decomposition.* On the other hand, let $\{F_n;\ n=1, 2, \cdots\}$ be a locally finite family of closed sets of S, and let $g_n(x)\in\mathfrak{B}(F_n)$, $n=1, 2, \cdots$, be given. Then it is proved that there exists $g(x)\in\mathfrak{B}(S)$ such that, for any open subset S' of S which intersects with only a finite number of F_n, we have

$$g|S'=\sum_n g_n|S'$$

where the sum in the right-hand side runs over those n for which S' intersects with F_n. Clearly such $g(x)$ is unique. We therefore call $g(x)$ *the sum of* $\{g_n(x); n=1, 2,\cdots\}$ and denote

$$g(x)=\sum_{n=1}^{\infty} g_n(x). \tag{2.2.1}$$

(2.2.1) implies

$$f(x)g(x)=\sum_{n=1}^{\infty} f(x)g_n(x)\qquad (f(x)\in\mathfrak{A}(S)),$$

$$\bar{g}(x)=\sum_{n=1}^{\infty}\bar{g}_n(x),\qquad g^{(\nu)}(x)=\sum_{n=1}^{\infty} g_n^{(\nu)}(x).$$

For example, let $\xi(x)$ be a real-valued holomorphic function on an open set $I \subset \boldsymbol{R}$ whose zeros $x=a_1, a_2, \cdots (\in I)$ are all simple. Then we can define $\delta(\xi(x)) \in \mathfrak{B}(I)$ by **1.1.** (v), and obtain

$$\delta(\xi(x)) = \sum_n |\xi'(a_n)|^{-1} \cdot \delta(x-a_n),$$

$$f(x)\delta(\xi(x)) = \sum_n f(a_n)|\xi'(a_n)|^{-1} \cdot \delta(x-a_n).$$

Now it follows from the results of 2. 1. that, *for any* $g(x) \in \mathfrak{B}(S)$ *and any locally finite closed covering* $\{F_n\,;\, n=1, 2, \cdots\}$ *of* S, *there exists a decomposition* $g(x) = \sum_{n=1}^{\infty} g_n(x)$ *with* $g_n(x) \in \mathfrak{B}(F_n)$ (Decomposition theorem).

2.3. *Localization theorem.* Furthermore, let $\{S_\alpha\,;\, \alpha \in N\}$ be an open covering of S, and let $g_\alpha(x) \in \mathfrak{B}(S_\alpha)$. We shall say that $\{(S_\alpha, g_\alpha)\,;\, \alpha \in N\}$ constitutes a *localized hyperfunction* if every pair g_α, g_β has a common restriction on $S_\alpha \frown S_\beta$: $g_\alpha | S_\alpha \frown S_\beta = g_\beta | S_\alpha \frown S_\beta$.

Then, it follows from the above results that, *for any localized hyperfunction* $\{(S_\alpha, g_\alpha)\,;\, \alpha \in N\}$, *there exists a hyperfunction* $g \in \mathfrak{B}(S)$ *such that* $g_\alpha = g | S_\alpha$ *for every* $\alpha \in N$. In short, *every localized hyperfunction is equivalent to one* (*and only one*) *hyperfunction* (Localization theorem).*)

The definition (2.2.1) of the sum of $\{g_n\}$, $g_n \in \mathfrak{B}(F_n)$, can be now paraphrased as follows. Let $\{S_\alpha\,;\, \alpha \in N\}$ be an open covering of S such that each S_α intersects with only a finite number of F_n, and define $h_\alpha \in \mathfrak{B}(S_\alpha)$ by $h_\alpha = \sum g_n | S_\alpha$. Then $\{(h_\alpha, S_\alpha)\,;\, \alpha \in N\}$ constitutes a localized hyperfunction, and hence defines a hyperfunction $g \in \mathfrak{B}(S)$. This g, which does not depend on the choice of $\{S_\alpha\}$, will be called the sum of $\{g_n\,;\, n=1, 2, \cdots\}$.

Another application of the localization theorem concerns the product of hyperfunctions ([7], §3). Let $g_1(x), \cdots, g_n(x)$ be hyperfunctions on S such that their carriers of singularity are mutually disjoint. Then we can define their product $g(x) = g_1(x) \cdots g_n(x) \in \mathfrak{B}(S)$ as follows. By the assumption, we can choose an open covering $\{S_\alpha\,;\, \alpha \in N\}$ of S so that $g_1(x), \cdots, g_n(x)$ are all holomorphic on S_α with possibly one exception. Hence an $h_\alpha(x) \in \mathfrak{B}(S_\alpha)$ is defined by $h_\alpha = \prod_{\nu=1}^{n} g_\nu | S_\alpha$. Then $\{(h_\alpha, S_\alpha)\,;\, \alpha \in N\}$ constitutes a localized hyperfunction, and hence defines a hyperfunction $g(x) \in \mathfrak{B}(S)$. This g, which does not depend on the choice of $\{S_\alpha\}$, will be called the product of $g_1(x), \cdots, g_n(x)$.

Furthermore, let $\varphi_\nu(z)$ be a defining function of $g_\nu(z)$ ($\nu=1, \cdots, n$). Then, on

*) $\{(S_\alpha, g | S_\alpha)\,;\, \alpha \in N\}$ in Proposition 24, 1 on p. 180, [9], is the misprint of $\{(S_\alpha, g_\alpha)\,;\, \alpha \in N\}$.

each S_α, the 'boundary values' $\varphi_\nu(x\pm i0)$ are also holomorphic for every $\nu=1,\cdots,n$ with possibly one exception; hence we can expand the product $g(x)=\prod_{\nu=1}^{n}(\varphi_\nu(x+i0)-\varphi_\nu(x-i0))$ as follows:

$$g(x)=\sum_{(\pm_1,\cdots,\pm_n)}\pm_1\cdots\pm_n\varphi_1(x\pm_1 i0)\cdots\varphi_n(x\pm_n i0) \tag{2.3.1}$$

where right-hand side stands for the sum running over all 2^n combinations of signs.

2.4. *Localizability from the view-point of sheaf theory.* The above results can be now neatly reformulated in the language of sheaf theory. We shall remind here some concepts of the sheaf theory which will be used in the following.

Let X be any topological space. We shall denote with $\mathfrak{L}(X)$ the totally of open sets of X. If for each $D\in\mathfrak{L}(X)$ there corresponds a module $\mathfrak{F}(D)$, and if for each D, $D'\in\mathfrak{L}(X)$, $D\supset D'$, there corresponds a homomorphism $\rho_{D'D}$: $\mathfrak{F}(D)\longrightarrow\mathfrak{F}(D')$ such that $\rho_{DD}=$identity and $\rho_{D''D'}\circ\rho_{D'D}=\rho_{D''D}$ $(D\supset D'\supset D'')$, then we say that the family of moduli $\{\mathfrak{F}(D);\ D\in\mathfrak{L}(X)\}$, together with the family of homomorphisms $\{\rho_{D'D}\}$, constitutes a *pre-sheaf (of moduli) over X*. (For each $\varphi\in\mathfrak{F}(D)$, the image $\rho_{D'D}(\varphi)\in\mathfrak{F}(D')$ is usually called the *restriction* of φ onto D'). If, in addition, each $\mathfrak{F}(D)$ is a ring, and each $\rho_{D'D}$ is a ring-homomorphism, then we are dealing with a *pre-sheaf of rings.*

Let $\mathfrak{U}=\{U_\alpha;\ \alpha\in N\}$ be an open covering of some $D\in\mathfrak{L}(X)$. By a *0-cochain of* $\mathfrak{U}$ *with coefficients in* $\mathfrak{F}$ is meant an element of the product module $\prod_{\alpha\in N}\mathfrak{F}(U_\alpha)$, i.e. a 'vector' $\varphi=(\varphi_\alpha)_{\alpha\in N}$ with components $\varphi_\alpha\in\mathfrak{F}(U_\alpha)$. A 0-cochain $\varphi=(\varphi_\alpha)_{\alpha\in N}$ is called a *0-cocycle* if each pair φ_α, φ_β of the components have a common restriction on $U_\alpha\frown U_\beta$. A pre-sheaf $\mathfrak{F}$ is called a *sheaf* if for any $D\in\mathfrak{L}(X)$, any open covering $\mathfrak{U}=\{U_\alpha;\ \alpha\in N\}$ of D, and any 0-cocycle φ of $\mathfrak{U}$, there exists one and only one $\psi\in\mathfrak{F}(D)$ such that each component φ_α of φ is the restriction of ψ onto U_α. For any pre-sheaf $\mathfrak{F}$, the totality of 0-cocycles of $\mathfrak{U}$ constitutes a submodule $Z^0(\mathfrak{U},\mathfrak{F})$ of $\prod_{\alpha\in N}\mathfrak{A}(U_\alpha)$. The *section module* $\Gamma(D,\mathfrak{F})$ *of* $\mathfrak{F}$ *over* D is the module naturally obtained as the inductive limit of $Z^0(\mathfrak{U},\mathfrak{F})$ by refining the covering $\mathfrak{U}$ of D. Each element of $\Gamma(D,\mathfrak{F})$ is called a cross-section of $\mathfrak{F}$ over D. By the definition, we see that $\{\Gamma(D,\mathfrak{F}),\ D\in\mathfrak{L}(X)\}$, together with the canonical homomorphisms, constitutes a sheaf over X (i.e. the *sheaf determined by the pre-sheaf* $\mathfrak{F}$), and that there are canonical homomorphisms $\mathfrak{F}(D)\longrightarrow\Gamma(D,\mathfrak{F})$ $(D\in\mathfrak{L}(X))$ which are all bijective if (and only if) $\mathfrak{F}$ is a sheaf. That is, each $\varphi\in\mathfrak{F}(D)$ can be identified with each cross-section of $\mathfrak{F}$ over D if and only if $\mathfrak{F}$ is a sheaf.

In this terminology, the family of rings $\{\mathfrak{A}(S');\ S'\in\mathfrak{L}(S)\}$, the families of moduli $\{\tilde{\mathfrak{A}}(S');\ S'\in\mathfrak{L}(S)\}$, $\{\mathfrak{B}(S');\ S'\in\mathfrak{L}(S)\}$, together with canonical homomorphisms all constitute pre-sheaves over S. It is clear that the preceding two pre-sheaves are actually sheaves, while the localization theorem asserts that *the last one, the pre-sheaf of hyperfunctions on S, is also a sheaf.*[*)]

Remember that our sheaf $\mathfrak{B}_S$ of hyperfunctions depends on S. If we want to describe the module $\mathfrak{B}(S)$ through a single sheaf $\mathfrak{B}_R \equiv \mathfrak{B}$, we need generalize the above considerations to the relative case as below.

2.5. *Relative case.* Let X, $\mathfrak{F}$ be of the same meaning as above, and suppose that we have an open covering $\mathfrak{U}=\{U_\alpha\,;\ \alpha\in N\}$ of $D\in\mathfrak{L}(X)$ such that a sub-family $\mathfrak{U}'$ of $\mathfrak{U}$ (i.e. $\mathfrak{U}'=\{U_\alpha\,;\ \alpha\in N'\}$ with $N'\subset N$) is an open covering of some $D'\in\mathfrak{L}(X)$, $D'\subset D$. (We call the couple $(\mathfrak{U}, \mathfrak{U}')$ an open covering of (D, D')). Each 0-cochain (or 0-cocycle) $\varphi=(\varphi_\alpha)_{\alpha\in N}$ of $\mathfrak{U}$ with coefficients in $\mathfrak{F}$ will be called a (*relative*) *0-cochain* (or *0-cocycle*) *of* $\mathfrak{U} \bmod \mathfrak{U}'$ if $\varphi_\alpha=0$ for every $\alpha\in N'$. The totality of 0-cocycles of $\mathfrak{U} \bmod \mathfrak{U}'$ constitutes a submodule $Z^0(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{F})$ of $Z^0(\mathfrak{U}, \mathfrak{F})$. We shall denote with $\Gamma(D \bmod D', \mathfrak{F})$ the submodule of $\Gamma(D, \mathfrak{F})$ naturally obtained as the inductive limit of $Z^0(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{F})$ by refining the covering $(\mathfrak{U}, \mathfrak{U}')$ of (D, D'). By the definition, $Z^0(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{F})$ is the kernel of the canonical homomorphism: $Z^0(\mathfrak{U}, \mathfrak{F})\to Z^0(\mathfrak{U}', \mathfrak{F})$; hence $\Gamma(D \bmod D', \mathfrak{F})$ is the kernel of the canonical homomorphism: $\Gamma(D, \mathfrak{F})\to\Gamma(D', \mathfrak{F})$, or equivalently, the diagram

$$0 \longrightarrow \Gamma(D \bmod D', \mathfrak{F}) \longrightarrow \Gamma(D, \mathfrak{F}) \longrightarrow \Gamma(D', \mathfrak{F}) \tag{2.5.1}$$

is exact. If (D_j, D'_j), $D_j\supset D'_j$, $j=1, 2$, are two couples of open sets of X such that $D_1\supset D_2$, $D'_1\supset D'_2$, we have a canonical homomorphism (the homomorphism by re-

[*)] If we use Cartan-Serre's theory on coherent analytic sheaves (Cartan [2], [3]), we can derive the localization theorem and the completeness theorem as follows ([7]). Let X be a non-compact Riemann surface, S a nowhere dense closed set in X. For any $D\in\mathfrak{L}(X)$, we denote with $\mathfrak{A}(D)$ the ring of all holomorphic functions on D, and with $\mathfrak{B}(D)$ the quotient module $\mathfrak{A}(D-S) \bmod \mathfrak{A}(D)$ (regarding $\mathfrak{A}(D)$ as a submodule of $\mathfrak{A}(D-S)$). Let $\mathfrak{A}$, $\widetilde{\mathfrak{A}}$, $\mathfrak{B}_S$ denote sheaves over X determined by the pre-sheaves $\{\mathfrak{A}(D)\,;\ D\in\mathfrak{L}(X)\}$, $\{\mathfrak{A}(D-S)\,;\ D\in\mathfrak{L}(X)\}$, $\{\mathfrak{B}(D)\,;\ D\in\mathfrak{L}(X)\}$ respectively. The sheaf $\mathfrak{B}_S$ has the stalk 0 except over S, and reduces to the sheaf of hyperfunctions when S is a locally closed set in $\boldsymbol{R}$ and X is a complex neighborhood of S.

Clearly we have an exact sequence of sheaves

$$0 \longrightarrow \mathfrak{A} \longrightarrow \widetilde{\mathfrak{A}} \longrightarrow \mathfrak{B}_S \longrightarrow 0.$$

As X is a Stein manifold of dimension 1, and $\mathfrak{A}$ is a coherent analytic sheaf, the cohomology moduli $H^n(X, \mathfrak{A})$ of X with coefficients in $\mathfrak{A}$ vanish for all $n\geq 1$ by the theorem of Cartan-Serre. Therefore, the above exact sequence gives rise to the exact sequence of section modules immediately:

$$0 \longrightarrow \mathfrak{A}(X) \longrightarrow \mathfrak{A}(X-S) \longrightarrow \Gamma(X, \mathfrak{B}_S) \longrightarrow 0$$

In case where S is a locally closed set in $\boldsymbol{R}$, this result shows that any localized hyperfunction on S has a defining function $\in\mathfrak{A}(X-S)$. This includes the localization theorem and the completeness theorem.

striction): $\Gamma(D_1 \bmod D_1', \mathfrak{F}) \to \Gamma(D_2 \bmod D_2', \mathfrak{F})$. In particular, for any $D_1, D_2 \in \mathfrak{L}(X)$, it is easily verified that the homomorphism

$$\Gamma(D_1 \cup D_2 \bmod D_2, \mathfrak{F}) \to \Gamma(D_1 \bmod D_1 \cap D_2, \mathfrak{F})$$

is bijective (the excision theorem); in other words, (identifying moduli which are mutually canonically isomorphic) the relative section-module $\Gamma(D \bmod D', \mathfrak{F})$ essentially depends only on the difference $S = D - D'$ (which is a locally closed set of X).—This is why we use the notation '$D \bmod D'$' and the adjective 'relative'. To emphasize this fact, we shall also write $\Gamma_{\mathrm{rel}}(S, \mathfrak{F})$ instead of $\Gamma(D \bmod D', \mathfrak{F})$; thus, for *any* locally closed $S \subset X$, we have $\Gamma_{\mathrm{rel}}(S, \mathfrak{F}) = \Gamma(D \bmod (D-S), \mathfrak{F})$ where D denotes any of open set $\in \mathfrak{L}(X)$ containing S as a closed subset, or equivalently, (2.5.1) is rewritten as follows:

$$(2.5.1)' \qquad 0 \longrightarrow \Gamma_{\mathrm{rel}}(S, \mathfrak{F}) \longrightarrow \Gamma(D, \mathfrak{F}) \longrightarrow \Gamma(D-S, \mathfrak{F}) \quad \text{(exact)}.$$

For any $D \in \mathfrak{L}(X)$, we have clearly $\Gamma_{\mathrm{rel}}(D, \mathfrak{F}) = \Gamma(D \bmod \phi, \mathfrak{F}) = \Gamma(D, \mathfrak{F})$ (ϕ=empty set). On the other hand, (2.5.1) is generalized as follows: Let $D, D', D'' \in \mathfrak{L}(X)$ be a triple such that $D \supset D' \supset D''$. Then we have an exact sequence by canonical homomorphisms

$$(2.5.2) \qquad 0 \longrightarrow \Gamma(D \bmod D', \mathfrak{F}) \longrightarrow \Gamma(D \bmod D'', \mathfrak{F}) \longrightarrow \Gamma(D' \bmod D'', \mathfrak{F}).$$

Again, this can be rewritten as follows:

$$(2.5.2)' \qquad 0 \longrightarrow \Gamma_{\mathrm{rel}}(F, \mathfrak{F}) \longrightarrow \Gamma_{\mathrm{rel}}(S, \mathfrak{F}) \longrightarrow \Gamma_{\mathrm{rel}}(S-F, \mathfrak{F})$$

where S and F denote any locally closed set of X and any closed subset of S, respectively.

Employing these notations, it will now be obvious that the module $\mathfrak{B}(S)$ of hyperfunctions over a locally closed subset $S \subset \boldsymbol{R}$ is expressed as follows by means of the sheaf $\mathfrak{B} = \mathfrak{B}_{\boldsymbol{R}}$:

$$\mathfrak{B}(S) = \Gamma_{\mathrm{rel}}(S, \mathfrak{B}) = \Gamma(I \bmod (I-S), \mathfrak{B})$$

where I denotes *any* real neighborhood of S.

Now, a sheaf $\mathfrak{F}$ over X will be called *complete* (or *hyperfine*) if for any $D, D' \in \mathfrak{L}(X)$, $D \supset D'$, the canonical homomorphism $\mathfrak{F}(D) \to \mathfrak{F}(D')$ is *surjective*. This is equivalent to say that we can add '$\to 0$' to the exact sequences (2.5.1) and (2.5.1)', or further, to the exact sequences (2.5.2) and (2.5.2)'. Furthermore, it is proved that a sheaf $\mathfrak{F}$ is complete if (and only if) it is locally complete, i.e. for any $p \in X$ there exists an open neighborhood U of p such that the restriction $\mathfrak{F}|U$ of $\mathfrak{F}$ onto U (=the sheaf over U defined by $\{\mathfrak{F}(D); D \in \mathfrak{L}(U)\}$) is complete.

Using this terminology, the completeness theorem for hyperfunctions, or equivalently the formula (2.1.2), is nothing else but the assertion that the sheaf

$\mathfrak{B}$ over $\boldsymbol{R}$ which we have described above, is a hyperfine sheaf.

On the other hand, for any sheaf $\mathfrak{F}$ over X and any locally finite closed covering $\{F_\nu\,;\,\nu\in N\}$ of X, we can define a canonical homomorphism $\prod_{\nu\in N}\mathfrak{F}(F_\nu)\to\mathfrak{F}(X)$ in an analogous way as we have defined the sum of hyperfunctions in (2.2.1). It is proved that this homomorphism is surjective if $\mathfrak{F}$ is a complete sheaf; in other words, the decomposition theorem is a consequence of the completeness theorem. (This result is proved in the same way as the decomposition theorem for hyperfunctions ([9], Proposition 23.2) by employing Zorn's lemma.)

We shall call a sheaf $\mathfrak{F}$ *fine* if for any *closed* set $F\subset X$ the canonical homomorphism $\rho_{FX}:\mathfrak{F}(X)\to\mathfrak{F}(F)$ is *surjective*. Then, $\mathfrak{F}$ is fine if and only if it is *locally fine*, i.e. if and only if each $p\in X$ has a neighborhood $U\ni p$ such that $\mathfrak{F}|U$ is a fine sheaf over U. By the definition, every hyperfine sheaf is fine. If a sheaf is fine, then for any $D\in\mathfrak{L}(X)$ and for any locally finite open covering $\{U_\alpha\,;\,\alpha\in N\}$ of D, the canonical homomorphism $\prod_{\alpha\in N}\Gamma_{\mathrm{rel}}(\overline{U}_\alpha,\mathfrak{F})\to\Gamma(D,\mathfrak{F})$ is *surjective*, and conversely. In case the module $\mathfrak{F}(D)$ is *completely reducible* for every $D\in\mathfrak{L}(X)$ (which is the case if $\mathfrak{F}$ is the sheaf of *moduli over a certain field*)*), the condition that F is fine is equivalent to the condition that for any closed set $F\subset X$ there exists a homomorphism $\gamma_F:\mathfrak{F}(F)\to\mathfrak{F}(X)$ such that $\rho_{FX}\circ\varepsilon_F=$identity mapping of $\mathfrak{F}(F)$, or further, to the condition that for $D\in\mathfrak{L}(X)$ and for any locally finite open covering $\{U_\alpha\,;\,\alpha\in N\}$ of D, there exists a system of homomorphisms γ_α: $\Gamma(X,\mathfrak{F})\to\Gamma_{\mathrm{rel}}(\overline{U}_\alpha,\mathfrak{F})$ such that, denoting with ι_α the canonical homomorphism $\Gamma_{\mathrm{rel}}(\overline{U}_\alpha,\mathfrak{F})\to\Gamma(X,\mathfrak{F})$, we have $\sum_{\alpha\in N}\iota_\alpha\circ\gamma_\alpha=1$; hence in this case our definition of fine sheaves coincides to that of H. Cartan [1].**)

§3. Local defining functions.

3.1. Let $\{(S_\alpha, g_\alpha)\,;\,\alpha\in N\}$ be a localized hyperfunction on S, D a complex neighborhood of S. For each $\alpha\in N$, we have

$$g_\alpha=[\varphi_\alpha,\ U_\alpha] \tag{3.1.1}$$

with some $\varphi_\alpha\in\mathfrak{A}(U_\alpha-S)$ and some $U_\alpha\in\mathfrak{D}(S)$, where, by replacing U_α by $U_\alpha\frown D$ if necessary, we may assume $U_\alpha\subset D$ from the beginning (Fig. 1). By the assumption, there is a $\varphi_{\alpha\beta}\in\mathfrak{A}(U_\alpha\frown U_\beta)$ such that

$$\varphi_\alpha(z)-\varphi_\beta(z)=\varphi_{\alpha\beta}(z)\ \text{ on }\ U_\alpha\frown U_\beta-S \tag{3.1.2}$$

*) Actually, we consider in Chapter III only analytic sheaves, hence only sheaves of moduli over the complex number field $\boldsymbol{C}$.

**) In R. Godement [5], hyperfine sheaves and fine sheaves are called faisceaux flasques and faisceaux mous, respectively.

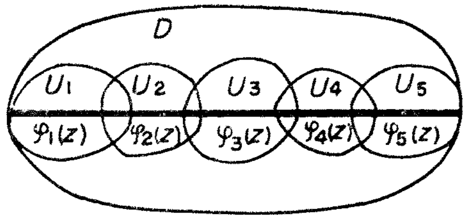

Fig. 1

for any $\alpha, \beta \in N$. Conversely, if we have a complex neighborhood D of S, a family $\{U_\alpha\,;\ \alpha\in N\}$ of open subsets of D such that $\bigcup_{\alpha\in N} U_\alpha \supset S$, and a family $\{\varphi_\alpha\,;\ \alpha\in N\}$ of holomorphic functions $\varphi_\alpha\in\mathfrak{A}(U-S)$ satisfying the condition (3.1.2) with some $\varphi_{\alpha\beta}\in\mathfrak{A}(U_\alpha\frown U_\beta)$, then a localized hyperfunction $\{(S_\alpha, g_\alpha)\,;\ \alpha\in N\}$ on S is determined by (3.1.1), and hence a hyperfunction $g(x)$ on S. We call $\{(\varphi_\alpha, U_\alpha);\ \alpha\in N\}$ a *system of local defining functions* of $g(x)$, and denote

$$g=[(\varphi_\alpha, U_\alpha)\,;\ \alpha\in N]=[\varphi_\alpha\,;\ \alpha\in N],\ \text{or} \tag{3.1.3}$$
$$g(x)=[(\varphi_\alpha(z), U_\alpha)\,;\ \alpha\in N]_{z\to x}=[\varphi_\alpha(z)\,;\ \alpha\in N]_{z\to x}.$$

In particular, we denote

$$g=[(\varphi_1, U_1),\cdots,(\varphi_n, U_n)]=[\varphi_1,\cdots,\varphi_n]$$

if the index set N is the finite set $\{1, 2,\cdots, n\}$.

Now take any open covering of $D-S$. We shall denote it with $\mathfrak{U}'=\{U_\kappa\,;\ \kappa\in N'\}$ regarding N' as a set of indices disjoint to N, and define an open covering of D by $\mathfrak{U}=\{U_\alpha\,;\ \alpha\in N\smile N'\}$ $(\supset\mathfrak{U}')$. Then, setting $\varphi_\kappa=0$ for every $\kappa\in N'$, we can define $\varphi_{\alpha\beta}\in\mathfrak{A}(U_\alpha\frown U_\beta)$ such that (3.2) holds even when $\alpha, \beta\in N\smile N'$. Clearly they satisfy

$$\varphi_{\alpha\alpha}(z)=0,\qquad \varphi_{\alpha\beta}(z)=-\varphi_{\beta\alpha}(z), \tag{3.1.4}$$
$$\varphi_{\alpha\beta}(z)-\varphi_{\alpha\gamma}(z)+\varphi_{\beta\gamma}(z)=0\ \text{on}\ U_\alpha\frown U_\beta\frown U_\gamma\ (\alpha, \beta\in N\smile N')$$

and

$$\varphi_{\kappa\lambda}(z)=0\ \text{if}\ \kappa, \lambda\in N'. \tag{3.1.5}$$

The identities (3.1.4) *mean that* $\varphi=(\varphi_{\alpha\beta})_{\alpha,\beta\in N\smile N'}$ *constitutes a* 1-*cocycle of* $\mathfrak{U}$ *with coefficients in* $\mathfrak{A}$, where $\mathfrak{A}$ denotes the *sheaf of holomorphic functions* defined by $\{\mathfrak{A}(U)\,;\ U\in\mathfrak{L}(C)\}$ (together with the homomorphisms by restriction), while *the additional identities* (3.1.5) *show that the* 1-*cocycle* φ *is a* (*relative*) 1-*cocycle of* $\mathfrak{U}\bmod\mathfrak{U}'$. Conversely, if $\mathfrak{U}$, $\mathfrak{U}'$ are as described above (i.e. $\mathfrak{U}'$ is an open covering of $D-S$ and $\mathfrak{U}$ is an open covering of D containing $\mathfrak{U}'$ as a subfamily—in short: $(\mathfrak{U}, \mathfrak{U}')$ is an open covering of $D\bmod(D-S)$), and if $\varphi=(\varphi_{\alpha\beta})_{\alpha,\beta\in N\smile N'}$ is any relative 1-cocycle of $\mathfrak{U}\bmod\mathfrak{U}'$ with coefficients in $\mathfrak{A}$, then a system of local defining functions $\{(\varphi_\alpha, U_\alpha)\,;\ \alpha\in N\}$, $\varphi_\alpha\in\mathfrak{A}(U_\alpha-S)$, is determined by the identities

$$\varphi_\alpha(z)=\varphi_{\alpha\kappa}(z) \text{ on } U_\alpha\frown U_\kappa \quad (\alpha\in N,\ \kappa\in N'),$$

and hence, a hyperfunction $g(x)\in\mathfrak{B}(S)$. It is easy to see from the definition that two of such 1-cocycles of $\mathfrak{U} \bmod \mathfrak{U}'$, say $\varphi=(\varphi_{\alpha\beta})_{\alpha,\beta\in N-N'}$ and $\varphi'=(\varphi'_{\alpha\beta})_{\alpha,\beta\in N-N'}$, determine one and the same hyperfunction $g\in\mathfrak{B}(S)$ if and only if there are a (*relative*) 0-*cochain* $\psi=(\psi_\alpha)_{\alpha\in N-N'}$ of $\mathfrak{U} \bmod \mathfrak{U}'$ with coefficients in $\mathfrak{A}$ (i.e. a 0-cochain of $\mathfrak{U}$ in which φ_ν vanishes for every $\nu\in N'$) such that

$$\varphi_{\alpha\beta}(z)-\varphi'_{\alpha\beta}(z)=\psi_\alpha(z)-\psi_\beta(z) \text{ on } U_\alpha\frown U_\beta \tag{3.1.6}$$

for every $\alpha,\beta\in N\smile N'$, i.e. if and only if $\varphi-\varphi'$ is a (*relative*) 1-*coboundary of* $\mathfrak{U} \bmod \mathfrak{U}'$. In other words, *it is each* (*relative*) *cohomology class of* $\mathfrak{U} \bmod \mathfrak{U}'$ *with coefficients in* $\mathfrak{A}$ *that corresponds in a* 1-1 *manner to each hyperfunction on* S. Roughly speaking, a 'localized hyperfunction on S' as defined in the preceding paragraph is nothing but a 'cohomology class of $\mathfrak{U} \bmod \mathfrak{U}'$ with coefficients in $\mathfrak{A}$'. The module consisting of all these 1-cohomology classes of $\mathfrak{U} \bmod \mathfrak{U}'$ is the (*relative*) 1-*cohomology module* $H^1(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{A})$. The assertion that this $H^1(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{A})$ is canonically isomorphic to $\mathfrak{B}(S)$ is another expression of the localization theorem.

If we take a refinement $(\mathfrak{V}, \mathfrak{V}')$ of $(\mathfrak{U}, \mathfrak{U}')$, we have a canonical homomorphism: $H^1(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{A})\to H^1(\mathfrak{V} \bmod \mathfrak{V}', \mathfrak{A})$ which however is *bijective* in the present case. Consequently, we can replace $H^1(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{A})$ by the inductive limit thereof by refining $(\mathfrak{U}, \mathfrak{U}')$, i.e. by the 1-cohomology module $H^1(D \bmod (D-S), \mathfrak{A})$ of $D \bmod (D-S)$. We have thus

$$\mathfrak{B}(S)\simeq H^1(D \bmod (D-S), \mathfrak{A}) \qquad \text{canonically}; \tag{3.1.7}$$

or equivalently, *we can define each hyperfunction* $g\in\mathfrak{B}(S)$ *as a* 1-*cohomology class of* $D \bmod (D-S)$ *with coefficients in* $\mathfrak{A}$. Through handling hyperfunctions in one variable from the *local* standpoint, we are naturally lead to conceive our theory in the framework of *the relative cohomology theory of sheaves.*

Now, as we have announced in the foreword, and will expound fully in Chapter III, a hyperfunction *in several variables* are defined analogously to (3.1.7) as follows: Suppose that M is an m-dimensional real analytic manifold, and is 'analytically prolonged' to a paracompact m-dimensional complex analytic manifold X. Then, a hyperfunction on M is an m-cohomology class of $X \bmod (X-M)$ with coefficients (in the sheaf) of holomorphic functions. Or equivalently, the module $\mathfrak{B}(M)$ of hyperfunctions on M is defined by

$$\mathfrak{B}(M)=H^m(X \bmod (X-M), \mathfrak{A})$$

where $\mathfrak{A}$ denotes the sheaf of holomorphic functions on M. Moreover, it is proved that we have

$$H^n(X \bmod (X-M), \mathfrak{A})=0 \qquad \text{if } n \neq m;$$

whence it is derived that the localization theorem and the completeness theorem is valid also for hyperfunctions in several variables. As we shall develop in Chapter III, all the results described in this Chapter for hyperfunctions in one variable are then generalized in a natural manner to the case of arbitrary dimensions.

3.2. *Representation of a definite integral by local defining functions.* Now if $\{(\varphi_\alpha, U_\alpha); \alpha \in N\}$ is a system of local defining functions of a *perfect* hyperfunction $g \in \mathfrak{B}(K)$, K denoting a compact set in $\boldsymbol{R}$, then we can calculate the definite integral of $g(x)$ directly from these $\varphi_\alpha(z) \in \mathfrak{A}(U_\alpha - K)$.

In this case, we can choose from $\{U_\alpha; \alpha \in N\}$ a finite subfamily, say $\{U_\nu; \nu=1, \cdots, n\}$, such that $\bigcup_{\nu=1,\cdots,n} U_\nu \supset S$ (and hence, $\{(\varphi_\nu, U_\nu); \nu=1, \cdots, n\}$ already is a system of local defining functions of g). Set $U_0 = D-K$, where D denote a complex neighborhood of K containing $\bigcup_{\nu=1,\cdots,n} U_\nu$. Then, setting $\mathfrak{U}=\{U_\nu; \nu=0, 1, \cdots, n\}$ and $\mathfrak{U}'=\{U_0\}$, an open covering $(\mathfrak{U}, \mathfrak{U}')$ of $D \bmod (D-K)$ is defined. Let $\varphi=(\varphi_{\mu\nu})_{\mu,\nu=0,1,\ldots,n}$ be a 1-cocycle of $\mathfrak{U} \bmod \mathfrak{U}'$ corresponding to the given system $\{(\varphi_\nu, U_\nu); \nu=1, \cdots, n\}$ of local defining functions of $g(x)$. Furthermore, let $\Gamma_{\mu\nu}$, $\mu, \nu=0, 1, \cdots, n$ be a differentiable singular 1-chain (i. e. a sum of a finite number of oriented differentiable arcs) in $U_\mu \frown U_\nu$ subject to the relation $\Gamma_{\nu\nu}=0$, $\Gamma_{\nu\mu}=-\Gamma_{\mu\nu}$, such that $\Gamma_\nu = \Gamma_{0\nu} + \cdots + \Gamma_{n\nu}$, a singular 1-chain in U_ν, satisfies the following condition (Fig. 2):

(i) for $\nu=1, \cdots, n$, Γ_ν is a bounding cycle in U_ν,

(ii) Γ_0 is a cycle in U_0 going around K in a positive sense.

Then we have

$$\int_K g(x)dx = -\sum_{\mu>\nu} \int_{\Gamma_{\mu\nu}} \varphi_{\mu\nu}(z)dz. \tag{3.2.1}$$

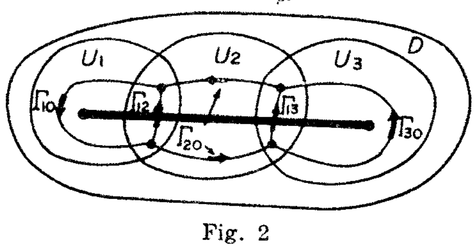

Fig. 2

The formula (3.2.1) is verified as follows.

Take a defining function (e.g. the standard defining function) $\varphi^0(z) \in \mathfrak{A}(U_0)$ of $g(x)$, and set

$$\varphi_{\mu\nu}^0=\begin{cases}\varphi^0|U_\mu\frown U_0 & (\mu>0,\ \nu=0)\\ -\varphi^0|U_0\frown U_\nu & (\mu=0,\ \nu>0)\\ 0 & (\text{otherwise}).\end{cases}$$

It is clear that $(\varphi_{\mu\nu}^0)_{\mu,\nu=0,1,\cdots,n}$ constitute a 1-cocycle of $\mathfrak{U}$ mod $\mathfrak{U}'$ cohomologous to $\varphi=(\varphi_{\mu\nu})_{\mu,\nu=0,1,\cdots,n}$; hence we have

$$\varphi_{\mu\nu}(z)=\varphi_{\mu\nu}^0(z)+\psi_\mu(z)-\psi_\nu(z) \text{ on } U_\mu\frown U_\nu$$

with some 0-cochain $(\psi_\nu)_{\nu=0,1,\cdots,n}$ of $\mathfrak{U}$ mod $\mathfrak{U}'$ (i.e. some $\psi_\nu\in\mathfrak{A}(U_\nu)$, $\alpha=1,\cdots,n$ and $\psi_0=0$). Therefore, the right-hand side of (3.2.1) is devided into a sum I_1+I_2, where I_1 and I_2 stand for

$$I_1=-\sum_{\mu>\nu}\int_{\Gamma_{\mu\nu}}\varphi_{\mu\nu}^0(z)dz, \qquad I_2=-\sum_{\mu>\nu}\int_{\Gamma_{\mu\nu}}(\psi_\mu(z)-\psi_\nu(z))dz$$

respectively. Now we have

$$I_1=-\sum_{\mu=1}^n\int_{\Gamma_{\mu 0}}\varphi^0(z)dz=-\oint_{\Gamma_0}\varphi^0(z)dz=\int_K g(x)dx,$$

and

$$I_2=\sum_{\mu,\nu}\int_{\Gamma_{\mu\nu}}\psi_\nu(z)dz=\sum_{\nu=1}^n\oint_{\Gamma_\nu}\psi_\nu(z)dz=\sum_{\nu=1}^n 0=0$$

(by the integral theorem of Cauchy), and hence the formula (3.2.1).

Our formula (3.2.1) now provides us a practical means to calculate the standard defining function $\varphi_0(z)=\frac{1}{2\pi i}\int_K\frac{g(x)}{x-z}dx\in\mathfrak{A}(C-K)$ from a *given* system of local defining functions; namely we have

$$\varphi_0(z)=-\frac{1}{2\pi i}\sum_{\mu>\nu}\int_{\Gamma_{\mu\nu}}\frac{\varphi_{\mu\nu}(\zeta)}{\zeta-z}d\zeta$$

for $z\in\boldsymbol{C}-D$.

Chapter II. Preparations from the Cohomology Theory of Sheaves.

This chapter is devoted to the sheaf-theoretical preparations we need in our theory. Only a brief account of results is given. Further details will be given in our forthcoming paper. As to the general theory of sheaves, we refer to H. Cartan [1], R. Godement [5]. When we fix a topological space X, we can define for sheaves over X such notions as subsheaf, residue class sheaf, homomorphism between sheaves, kernel and cokernel of a homomorphism, etc. in a natural manner. The totality of sheaves over X constitutes a category, and even an exact category in the sense of Cartan-Eilenberg [4], Appendix. As to the algebraic aspects of homology and cohomology theory, we refer to [4].

§4. Cohomology Theory of Sheaves.

4.1. *Cohomology moduli with sheaf-coefficients.*

Let X denote a paracompact T_2-space. For any sheaf $\mathfrak{F}$ over X, the *cohomology moduli of X with coefficients in $\mathfrak{F}$*

$$H^n(X, \mathfrak{F}), \quad n=0, 1, 2, \cdots$$

are defined (Cartan [1]). They have following properties.

(i) $$H^0(X, \mathfrak{F})=\Gamma(X, \mathfrak{F}),$$

where $\Gamma(X, \mathfrak{F})$ denotes the *section module* of $\mathfrak{F}$ over X.

(ii) If $\mathfrak{F}$ is *fine* (and hence, if $\mathfrak{F}$ is *hyperfine*)

$$H^n(X, \mathfrak{F})=0 \quad \text{for} \quad n \geqq 1.$$

(iii) If $$0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{H} \longrightarrow 0$$

is an *exact sequence* of sheaves over X, then we have an exact sequence of cohomology moduli

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathfrak{F}) &\longrightarrow H^0(X, \mathfrak{G}) \longrightarrow H^0(X, \mathfrak{H}) \\ \xrightarrow{\delta^*} H^1(X, \mathfrak{F}) &\longrightarrow H^1(X, \mathfrak{G}) \longrightarrow H^1(X, \mathfrak{H}) \\ \xrightarrow{\delta^*} H^2(X, \mathfrak{F}) &\longrightarrow H^2(X, \mathfrak{G}) \longrightarrow H^2(X, \mathfrak{H}) \\ \xrightarrow{\delta^*} \cdots & \end{aligned}$$

where δ^* denote *connecting homomorphisms.*

(iv) If $$0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{X}^0 \xrightarrow{d} \mathfrak{X}^1 \xrightarrow{d} \cdots$$

is a *fine resolution* of $\mathfrak{F}$, i.e. if it is an exact sequence of sheaves, each $\mathfrak{X}^n$ being fine, then $H^n(X, \mathfrak{F})$ is canonically isomorphic to the n-th cohomology module of the chain complex

$$\Gamma(X, \mathfrak{X}^0) \xrightarrow{d} \Gamma(X, \mathfrak{X}^1) \xrightarrow{d} \cdots.$$

(v) If $\mathfrak{U}=\{U_\alpha\,;\ \alpha \in N\}$ is an open covering of X such that

$$H^n(S_{\alpha_0} \cap \cdots \cap S_{\alpha_m}, \mathfrak{F})=0 \quad (n \geqq 1)$$

holds for any $m \geqq 0$ and $(\alpha_0, \cdots, \alpha_m) \in N^{m+1}$, then $H^n(X, \mathfrak{F})$ is canonically isomorphic to the n-th cohomology module of the chain complex

$$C^0(\mathfrak{U}, \mathfrak{F}) \xrightarrow{d} C^1(\mathfrak{U}, \mathfrak{F}) \xrightarrow{d} \cdots$$

where $C^n(\mathfrak{U}, \mathfrak{F})$, $n=0, 1, 2, \cdots$, denote the cochain moduli of Čech for the covering $\mathfrak{U}$ and δ denotes the coboundary operator.

Let us add two remarks:

a) Roughly speaking, the properties (i) to (iii) characterize the symbols $H^n(X, \mathfrak{F})$ completely. Namely, if $\{\tilde{H}^n;\ n=0, 1, 2, \cdots\}$ is a *connected sequence of covariant functors* (in the sense of Cartan-Eilenberg [1], Chap. III, §4) from the category of sheaves over X to the category of moduli such that (i), (ii) and (iii) hold, when H^n is replaced by $\tilde{H}^n$, then we have $\tilde{H}^n \simeq H^n$ canonically.

b) The sole property (iv) also suffices to determine our cohomology moduli, because it is shown that each sheaf has a fine (even a hyperfine) resolution.

4.2. *Relative case.* We shall next show how the above properties of cohomology moduli $H^n(X, \mathfrak{F})$ are extended to the relative cohomology moduli.

Let X' be open subset of X which also is assumed to be paracompact. To each sheaf $\mathfrak{F}$ over X, there correspond *relative cohomology moduli of* $X \bmod X'$ *with coefficients in* $\mathfrak{F}$

$$H^n(X \bmod X', \mathfrak{F}), \qquad n=0, 1, 2, \cdots.$$

They have following properties.

(i) $$H^0(X \bmod X', \mathfrak{F}) = \Gamma(X \bmod X', \mathfrak{F}),$$

where $\Gamma(X \bmod X', \mathfrak{F})$ denotes kernel of the natural homomorphism by restriction $\Gamma(X, \mathfrak{F}) \to \Gamma(X', \mathfrak{F})$, i.e. the module consisting of sections of $\mathfrak{F}$ whose carriers lie in $X - X'$.

(ii) If $\mathfrak{F}$ is *hyperfine*

$$H^n(X \bmod X', \mathfrak{F}) = 0 \qquad \text{for } n \geqq 1.$$

(If the condition '$\mathfrak{F}$ is hyperfine' is replaced by '$\mathfrak{F}$ is fine', the result is not necessarily true.)

(iii) If $$0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{H} \longrightarrow 0$$

is an *exact sequence* of sheaves over X, then we have an exact sequence of moduli

$$\begin{aligned} 0 &\longrightarrow H^0(X \bmod X', \mathfrak{F}) \longrightarrow H^0(X \bmod X', \mathfrak{G}) \longrightarrow H^0(X \bmod X', \mathfrak{H}) \\ &\xrightarrow{\delta^*} H^1(X \bmod X', \mathfrak{F}) \longrightarrow H^1(X \bmod X', \mathfrak{G}) \longrightarrow H^1(X \bmod X', \mathfrak{H}) \\ &\xrightarrow{\delta^*} H^2(X \bmod X', \mathfrak{F}) \longrightarrow H^2(X \bmod X', \mathfrak{G}) \longrightarrow H^2(X \bmod X', \mathfrak{H}) \\ &\xrightarrow{\delta^*} \cdots. \end{aligned}$$

(iv) If

$$0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{X}^0 \xrightarrow{d} \mathfrak{X}^1 \xrightarrow{d} \cdots$$

is a *hyperfine resolution* of $\mathfrak{F}$, i.e. if it is an exact sequence of sheaves, each $\mathfrak{X}^n$ being hyperfine, then $H^n(X \bmod X', \mathfrak{F})$ is canonically isomorphic to the n-th cohomology module of the chain complex

$$\Gamma(X \bmod X', \mathfrak{X}^0) \xrightarrow{d} \Gamma(X \bmod X', \mathfrak{X}^1) \xrightarrow{d} \cdots.$$

(v) If $\mathfrak{U}=\{U_\alpha;\ \alpha\in N\}$ is of the same meaning as in 4.1. (v), and if $\mathfrak{U}'=\{U_\alpha;\ \alpha\in N'\}$ with $N'\subset N$, a sub-family of $\mathfrak{U}$, is a covering of X', then $H^n(X \bmod X', \mathfrak{F})$ is canonically isomorphic to the n-th cohomology module of the chain complex

$$C^0(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{F}) \xrightarrow{\delta} C^1(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{F}) \xrightarrow{\delta} \cdots$$

where $C^n(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{F})$, $n=0, 1, 2, \cdots$, denote the *relative cochain moduli* of Čech for the couple of coverings $(\mathfrak{U}, \mathfrak{U}')$, i.e. they are defined by exact sequences

$$0 \longrightarrow C^n(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{F}) \longrightarrow C^n(\mathfrak{U}, \mathfrak{F}) \longrightarrow C^n(\mathfrak{U}', \mathfrak{F}) \longrightarrow 0.$$

(vi) For $X'=\phi$ (=empty set) we have

$$H^n(X \bmod X', \mathfrak{F}) = H^n(X, \mathfrak{F}).$$

(vii) We have an exact sequence of moduli

$$\begin{aligned} 0 &\longrightarrow H^0(X \bmod X', \mathfrak{F}) \longrightarrow H^0(X, \mathfrak{F}) \longrightarrow H^0(X', \mathfrak{F}) \\ &\xrightarrow{\delta^*} H^1(X \bmod X', \mathfrak{F}) \longrightarrow H^1(X, \mathfrak{F}) \longrightarrow H^1(X', \mathfrak{F}) \\ &\xrightarrow{\delta^*} \cdots. \end{aligned}$$

More generally, for any descending triple $X\supset X'\supset X''$ of paracompact open sets, we have an exact sequence of moduli

$$\begin{aligned} 0 &\longrightarrow H^0(X \bmod X', \mathfrak{F}) \longrightarrow H^0(X \bmod X'', \mathfrak{F}) \longrightarrow H^0(X' \bmod X'', \mathfrak{F}) \\ &\xrightarrow{\delta^*} H^1(X \bmod X', \mathfrak{F}) \longrightarrow H^1(X \bmod X'', \mathfrak{F}) \longrightarrow H^1(X' \bmod X'', \mathfrak{F}) \\ &\xrightarrow{\delta^*} \cdots. \end{aligned}$$

(viii) If X_1 and X_2 are both paracompact open sets, we have canonical isomorphisms

$$H^n(X_1\smile X_2 \bmod X_2, \mathfrak{F}) \simeq H^n(X_1 \bmod X_1\frown X_2, \mathfrak{F}).$$

Again, let us add some remarks:

a) The properties (i) to (iii) completely characterize the symbols $H^n(X \bmod X', \mathfrak{F})$ in the same sense as the corresponding properties on $H^n(X, \mathfrak{F})$ characterize $H^n(X, \mathfrak{F})$.

b) The sole property (iv) also suffices to determine our relative cohomology moduli because each sheaf has a hyperfine resolution.

c) The properties (vi) to (viii) concern the functors $H^n(*, \mathfrak{F})$ from the category of topological spaces to that of moduli with fixed $\mathfrak{F}$. They constitute the main part of the axioms of Eilenberg-Steenrod for cohomology moduli if our $H^n(*, \mathfrak{F})$ are replaced by the cohomology moduli with constant coefficients (i.e. if the sheaf $\mathfrak{F}$ reduces to a *simple sheaf* in the sense of [1]).

§5. The Sheaf of Distributions.

5.1. *Associated cohomology moduli and the sheaf* $\mathrm{Dist}^n(S, \mathfrak{F})$ *of distributions.* Let X and X' be of the same meaning as in 2,2, and set $F=X-X'$. F is a closed set of X, and hence, paracompact. Let $\mathfrak{L}(X)$ denote the totality of open sets of X, and D, D' the variable elements of $\mathfrak{L}(X)$ with $D'\subset D$. The correspondence $D\rightarrow\Gamma(D\bmod(D-F), \mathfrak{F})$*), together with the canonical homomorphisms induced by restriction $\rho_{D'D}$: $\Gamma(D\bmod(D-F), \mathfrak{F})\rightarrow\Gamma(D'\bmod(D'-F), \mathfrak{F})$, defines a pre-sheaf over X. This pre-sheaf is a sheaf. Indeed, it is the greatest sub-sheaf of $\mathfrak{F}$ whose stalks vanish except over F. We denote this sheaf with $\mathrm{Dist}^0(F, \mathfrak{F})$. By the definition, we have

$$(5.1.1)\qquad \Gamma(D\bmod D\frown X', \mathfrak{F})=\Gamma(D, \mathrm{Dist}^0(F, \mathfrak{F})).$$

Now we introduce a new symbol ${}^0H^n(D\bmod D\frown X', \mathfrak{F})$**) by

$${}^0H^n(D\bmod D\frown X', \mathfrak{F})=H^n(D, \mathrm{Dist}^0(F, \mathfrak{F}))$$

which reduces to $\Gamma(D\bmod D\frown X', \mathfrak{F})$ when $n=0$.

The correspondence $D\rightarrow H^n(D\bmod(D-F), \mathfrak{F})$ for $D\in\mathfrak{L}(X)$, together with the canonical homomorphisms induced by restriction $\rho_{D'D}$: $H^n(D\bmod(D-F), \mathfrak{F})\rightarrow H^n(D'\bmod(D'-F), \mathfrak{F})$ for $D, D'\in\mathfrak{L}(X)$, $D'\subset D$, defines a pre-sheaf over X.***) The sheaf determined by this pre-sheaf will be called the *sheaf of n-distributions of* $\mathfrak{F}$ and denoted with $\mathrm{Dist}^n(F, \mathfrak{F})$. It is 'confined' to F, i.e. the stalks of $\mathrm{Dist}^n(F, \mathfrak{F})$ vanish except over F.

On the other hand, let $\mathfrak{G}^n$ denote a sheaf which is determined by the pre-sheaf defined by the correspondence $D\rightarrow H^n(D-F, \mathfrak{F})$ for $D\in\mathfrak{L}(X)$ (together with

*) Note that $D-F=D\frown X'\in\mathfrak{L}(X)$.

**) The notation $D\bmod D\frown X'$ is used to suggest the fact that the excision theorem is valid for our symbol ${}^0H^n(*, \mathfrak{F})$, i.e. that ${}^0H^n(D\bmod D\frown X', \mathfrak{F})$ depends only on the difference $D-D\frown X'$ (which is a locally closed subset of X). We shall also use other symbols such as ${}^pH^n(D\bmod D\frown X', \mathfrak{F})$ in the same sense. Generally speaking, let $\mathfrak{M}(X)$ denote the totality of locally closed subsets of X, and for S, $S'\in\mathfrak{M}(X)$, let $M(S, S')$ denote the totality of subsets S_1 of $S\frown S'$ which are open in S (i.e. $S_1\in\mathfrak{L}(S)$) and, at the same time, closed in S'. $M(S, S')$ is a subset of $\mathfrak{M}(X)$. For S, S', $S''\in\mathfrak{M}(X)$, $S_1\in M(S, S')$ and $S_2\in M(S', S'')$ imply $S_1\frown S_2\in M(S, S'')$ and hence this '$\frown$-multiplication' defines a map:

$$M(S', S'')\times M(S, S')\longrightarrow M(S, S'').$$

It is now easy to see that $\mathfrak{M}(X)$, together with the totality of $M(S, S')$ ($S, S'\in\mathfrak{M}(X)$), constitutes a category (where each $M(S, S')$ stands for the totality of 'maps' from S to S'), which also will be denoted with $\mathfrak{M}(X)$ for the sake of simplicity. The excision theorem asserts that our symbols ${}^0H^n(*, \mathfrak{F})$, ${}^pH^n(*, \mathfrak{F})$, etc. are functors defined on the category $\mathfrak{M}(X)$ (where the argument is $S=D-D\frown X'$). See the end of this subsection.

***) For $n>0$, this pre-sheaf itself is not necessarily a sheaf, contrarily to the case $n=0$ described above.

the canonical homomorphisms induced by restriction).*) In particular, $\mathfrak{G}^0$ coincides with the sheaf $\mathfrak{F}_{X'}$ (which we shall call the *confinement of* $\mathfrak{F}$ *onto* X') defined by

$$\Gamma(D, \mathfrak{F}_{X'}) = \Gamma(D \frown X', \mathfrak{F}).$$

We have now, in a natural manner,

$$0 \longrightarrow \mathrm{Dist}^0(F, \mathfrak{F}) \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{G}^0 \longrightarrow \mathrm{Dist}^1(F, \mathfrak{F}) \longrightarrow 0 \qquad \text{(exact)} \tag{5.1.2}$$

and

$$\mathfrak{G}^n \simeq \mathrm{Dist}^{n+1}(F, \mathfrak{F}) \qquad (\text{for } n \geqslant 1). \tag{5.1.3}$$

Now let

$$0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{A}^0 \longrightarrow \mathfrak{A}^1 \longrightarrow \cdots \tag{5.1.4}$$

be a hyperfine resolution of $\mathfrak{F}$, and, for each $q \geqslant 0$, let $\mathfrak{H}^q$ be defined by

$$0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{A}^0 \longrightarrow \mathfrak{A}^1 \longrightarrow \cdots \longrightarrow \mathfrak{A}^{q-1} \longrightarrow \mathfrak{H}^q \longrightarrow 0 \qquad \text{(exact)}.$$

(In particular, we set $\mathfrak{H}^0 = \mathfrak{F}$). We define new symobols ${}^qH^n(*, \mathfrak{F})$, $q \geqslant 0$, $n \geqslant 0$, which we shall call *associated relative cohomology moduli*,**) by

$${}^qH^n(D \bmod D\frown X', \mathfrak{F}) = \begin{cases} H^n(D \bmod D\frown X', \mathfrak{F}) & (n \leqslant q) \\ {}^0H^{n-q}(D \bmod D\frown X', \mathfrak{H}^q) & (n > q). \end{cases}$$

It is proved that our definition does not essentially depend on the choice of the resolution (5.1.4). We have, further, an exact sequence of moduli which we shall call *the fundamental exact sequence of* $D \bmod D\frown X'$***):

*) Our $\mathfrak{G}^n$ is to be denoted by $\mathcal{H}^n(X', \mathfrak{F})$ in notation of R. Godement [1], 4.17; the continuous map π considered there being here specialized to the injection ι: $X' \longrightarrow X$.

**) When we wish to distinguish the relative cohomology module $H^n(X \bmod X', \mathfrak{F})$ defined in 4.2 from these associated moduli, we shall use the term 'principal relative cohomology module'.

***) As we shall explain in a subsequent paper, the method sketched below is essentially equivalent to that of *spectral sequence*: (5.1.5) implies namely the existence of a spectral sequence

$$E_2^{pq} \underset{p}{\Rightarrow} H^n(D \bmod D\frown X', \mathfrak{F})$$

where the initial term E_2^{pq} is given by $H^p(D, \mathrm{Dist}^q(F, \mathfrak{F}))$. (This result is closely related to a theorem of J. Leray; see R. Godement, loc. cit., Théorème 4. 17. 1.)

More precisely: We can define still more *associated relative cohomology moduli* ${}^q_rH^n = {}^q_rH^n(D \bmod D\frown X', \mathfrak{F})$ $(q > r)$ which satisfy exact sequences

$$E_{q,r}: \quad \cdots \xrightarrow{\delta^*} {}^rH^n \longrightarrow {}^qH^n \longrightarrow {}^q_rH^n \xrightarrow{\delta^*} {}^rH^{n+1} \longrightarrow {}^qH^{n+1} \longrightarrow \cdots.$$

They satisfy in particular ${}^q_rH^n = 0$ (for $r = q$), $= H^{n-q}(D, \mathrm{Dist}^q(S, \mathfrak{F}))$ (for $r = q-1$), $= {}^qH^n$ (for $r < 0$). Generally, we have for $q \geqslant q'$, $r \geqslant r'$ a canonical homomorphism ${}^{q'}_{r'}H^n \longrightarrow {}^q_rH^n$ so that we have a commutative diagram yielding a translation between exact sequences: $E_{q',r'} \longrightarrow E_{q,r}$. We have further, for $q \geqslant r \geqslant s$, an exact sequences

$$E_{q,r,s}: \quad \cdots \xrightarrow{\delta^*} {}^r_sH^n \longrightarrow {}^q_sH^n \longrightarrow {}^q_rH^n \xrightarrow{\delta^*} {}^r_sH^{n+1} \longrightarrow {}^q_sH^{n+1} \longrightarrow {}^q_rH^{n+1} \xrightarrow{\delta^*} \cdots.$$

The general term E_r^{pq} of the spectral sequence mentioned above is given by the image of the canonical homomorphism ${}^{\;\;\;q}_{q-r+1}H^n \longrightarrow {}^{q+r-2}_{q-1}H^n$, $(n = p+q,\ r \geqslant 2)$.

(5.1.5)

$$\begin{aligned}
0 &\longrightarrow {}^{q-1}H^q(D \bmod D\frown X', \mathfrak{F}) \longrightarrow {}^qH^q(D \bmod D\frown X', \mathfrak{F}) \longrightarrow H^0(D, \mathrm{Dist}^q(F, \mathfrak{F}))\\
&\longrightarrow {}^{q-1}H^{q+1}(D \bmod D\frown X', \mathfrak{F}) \longrightarrow {}^qH^{q+1}(D \bmod D\frown X', \mathfrak{F}) \longrightarrow H^1(D, \mathrm{Dist}^q(F, \mathfrak{F}))\\
&\longrightarrow \cdots \qquad \cdots \qquad \cdots\\
&\longrightarrow {}^{q-1}H^{q+p}(D \bmod D\frown X', \mathfrak{F}) \longrightarrow {}^qH^{q+p}(D \bmod D\frown X', \mathfrak{F}) \longrightarrow H^p(D, \mathrm{Dist}^q(F, \mathfrak{F}))\\
&\longrightarrow \cdots \qquad \cdots \qquad \cdots.
\end{aligned}$$

Note that the second term ${}^qH^q(D \bmod D\frown X', \mathfrak{F})$ is the *principal* relative cohomology module of $D \bmod D\frown X'$.

For the sake of brevity, we shall introduce following convensions.

For any locally closed $S\subset X$, $\mathfrak{D}(S, X)$ will denote the totality of $D\in\mathfrak{L}(X)$ containing S as a closed subset. By the excision theorem, the relative cohomology moduli $H^n(D \bmod (D-S), \mathfrak{F})$ (where D varies in $\mathfrak{D}(S, X)$) are combined by canonical isomorphisms. Identitying the corresponding elements of these moduli, we denote these moduli simply with $H^n_{\mathrm{rel}}(S, \mathfrak{F})$. Similarly with the associated relative cohomology module ${}^qH^n_{\mathrm{rel}}(S, \mathfrak{F})$.

If F is a closed subset of S, we have natural homomorphisms

$$H^n(D \bmod (D-F), \mathfrak{F}) \longrightarrow H^n(D \bmod (D-S), \mathfrak{F}),$$
$$H^n(D \bmod (D-S), \mathfrak{F}) \longrightarrow H^n((D-F) \bmod (D-S), \mathfrak{F}),$$

i. e.

$$\text{(5.1.6)} \qquad H_{\mathrm{rel}}(F, \mathfrak{F}) \xrightarrow{\iota^*} H_{\mathrm{rel}}(S, \mathfrak{F}), \qquad H_{\mathrm{rel}}(S, \mathfrak{F}) \xrightarrow{\varepsilon^*} H_{\mathrm{rel}}(S-F, \mathfrak{F}).$$

The homomorphisms ι^* and ε^* in (5.1.6) are independent of the choice of $D\in\mathfrak{D}(S, X)$, as are easily verified, and will be called *dilution* and *restriction*, respectively. If S, S' are both locally closed in X, and if $S_1\subset S\frown S'$ is such that it is open in S and closed in S', then a homomorphism $H_{\mathrm{rel}}(S, \mathfrak{F})\to H_{\mathrm{rel}}(S', \mathfrak{F})$ is defined by composing the restriction $H_{\mathrm{rel}}(S, \mathfrak{F})\to H_{\mathrm{rel}}(S_1, \mathfrak{F})$ and the dilution $H_{\mathrm{rel}}(S_1, \mathfrak{F})\to H_{\mathrm{rel}}(S', \mathfrak{F})$.

Similarly with the associated relative cohomology moduli ${}^qH^n(*, \mathfrak{F})$.

5.2. *Pure codimensionality.* In the following, we describe some consequences of the fundamental exact sequence (5.1.5) which we shall make use in Chapter III.

In the first place, we have a canonical homomorphism

$$H^q(D\mathrm{mod}(D-F), \mathfrak{F}) \longrightarrow \Gamma(D, \mathrm{Dist}^q(F, \mathfrak{F})).$$

Furthermore, if $\mathrm{Dist}^q(F,\mathfrak{F})=0$ for $q=0, 1, \cdots, m-1$, then we have as a consequence of (5.1.5),

$$\text{(5.2.1)} \qquad H^q(D \bmod (D-F), \mathfrak{F}) = \begin{cases} 0 & (q=0, 1, \cdots, m-1)\\ \Gamma(D, \mathrm{Dist}^m(F, \mathfrak{F})) & (q=m).\end{cases}$$

This means that, for $q=0, 1, \cdots, m-1$, the pre-sheaves defined by the correspondence $D \rightarrow H^q(D \bmod (D-F), \mathfrak{F})$ (not only the sheaves $\mathrm{Dist}^q(F, \mathfrak{F})$ defined by them) vanish identically, and that the correspondence $D \rightarrow H^m(D \bmod (D-F), \mathfrak{F})$ defines (not merely a pre-sheaf but) itself a sheaf. (This is a generalization of (5.1.1)).

Definition. The closed set $F \subset X$ is called *purely m-codimensional (with respect to* $\mathfrak{F}$) if we have $\mathrm{Dist}^q(F, \mathfrak{F})=0$ for $q \neq m$.

Then, F is purely m-codimensional if and only if we have

$$H^n(D \bmod (D-F), \mathfrak{F}) = \begin{cases} 0 & (n<m) \\ H^{n-m}(D, \mathrm{Dist}^m(F, \mathfrak{F})) & (n \geqq m) \end{cases} \tag{5.2.2}$$

for all $D \in \mathfrak{L}(X)$.

Furthermore, if F is purely m-codimensional, we have, for any closed subset F' of F,

$${}^pH^n(X \bmod (X-F'), \mathfrak{F}) = \begin{cases} {}^{p-m}H^{n-m}(F \bmod (F-F'), \mathfrak{G}) & (p \geqq m,\ n \geqq m) \\ 0 & (\text{otherwise}), \end{cases} \tag{5.2.3}$$

where $\mathfrak{G}$ stands for $\mathrm{Dist}^m(F, \mathfrak{F})$ which we here regard as a sheaf over F. In particular we have

$$H^n(X \bmod (X-F'), \mathfrak{F}) = \begin{cases} H^{n-m}(F \bmod (F-F'), \mathfrak{G}) & (n \geqq m) \\ 0 & (n<m) \end{cases} \tag{5.2.4}$$

which reduces to (5.2.2) if we set $F'=F$. (5.2.4) implies

$$\mathrm{Dist}^n(F', \mathfrak{F}) = \begin{cases} \mathrm{Dist}^{n-m}(F', \mathfrak{G}) & (n \geqq m) \\ 0 & (n<m). \end{cases} \tag{5.2.5}$$

Hence

Proposition 5.2.1. *Let F be a purely m-codimensional closed subset of X with respect to $\mathfrak{F}$. The following two conditions for a closed subset F' of F is equivalent:*

(i) *F' is a purely n-codimensional subset of X with respect to $\mathfrak{F}$.*

(ii) *F' is a purely $(n-m)$-codimensional subset of F with respect to* $\mathrm{Dist}^m(F, \mathfrak{F})$.

5.3. *A generalization.* Thus far we have considered the section-module $\Gamma(*, \mathfrak{F})$ or more generally the cohomology moduli $H^n(*, \mathfrak{F})$ only in case the 'argument' is an *open set* $D \in \mathfrak{L}(X)$. We can however define these moduli equally well in case the argument D is replaced by *any subset* $E \subset X$ without any essential alteration, and have again the results analogous to those of 4.1.-5.2. The module $H^n(E, \mathfrak{F})$ thus defined, however, turns out to coincide with the inductive limit of $\{H^n(D, \mathfrak{F});\ D \in \mathfrak{L}(X),\ D \supset S\}$. Consequently, we can derive results for $H^n(E, \mathfrak{F})$ from the results for $H^n(D, \mathfrak{F})$. This principle applies equally well to the relative case; in particular, we can introduce the relative cohomology moduli $H^n(E \bmod E', \mathfrak{F})$ and ${}^qH^n(E \bmod E', \mathfrak{F})$ of $E \bmod E' (E' \subset E \subset X)$ as the inductive limits of those of $D \bmod D'$ satisfying $D, D' \in \mathfrak{L}(X)$, $D \supset E$ and $D \supset D' \supset E'$.

Further, the *confinement* $\mathfrak{F}_E$ of $\mathfrak{F}$ to E is defined in the same manner by the relation

$$\Gamma(D, \mathfrak{F}_E) = \Gamma(D \frown E, \mathfrak{F}). \tag{5.3.1}$$

By the definition, i) there is a canonical homomorphism $\varepsilon: \mathfrak{F} \to \mathfrak{F}_E$, which we shall call *confining homomorphism*; ii) $\mathfrak{F}_E$ vanishes (i.e. has stalk 0) except over F; and iii) if a sheaf $\mathfrak{G}$ over X such that $\mathfrak{G}|(X-E)=0$ and a homomorphism $f: \mathfrak{F} \to \mathfrak{G}$ are given, then there exists a unique $f': \mathfrak{F}_E \to \mathfrak{G}$ with $f = f' \circ \varepsilon$.

Using the confinement $\mathfrak{F}_E$, we can legitimately denote, for any E, E' (with $E' \subset E \subset X$) and $E_1 = E - E'$,

$$H^n(E \bmod E', \mathfrak{F}) = H^n_{\mathrm{rel}}(E_1, \mathfrak{F}_E)$$

after the conventions of 5.1. Further, $\mathrm{Dist}^n(E_1, \mathfrak{F}_E)$ are defined as the sheaves determined by the pre-sheaves $\{H^n(E \frown D \bmod E' \frown D);\ D \in \mathfrak{L}(X)\}$, or equivalently, by the relation

$$\begin{gathered} 0 \longrightarrow \mathrm{Dist}^0(E_1, \mathfrak{F}_E) \longrightarrow \mathfrak{F}_E \longrightarrow \mathfrak{G}_0 \ (=\mathfrak{F}_{E'}) \\ \longrightarrow \mathrm{Dist}^1(E_1, \mathfrak{F}_E) \longrightarrow 0 \ \text{(exact)}, \qquad \text{and} \\ \mathfrak{G}_n \simeq \mathrm{Dist}^{n+1}(E_1, \mathfrak{F}_E) \ \text{canonically, (for } n \geqslant 1), \end{gathered}$$

where $\mathfrak{G}_n$ denotes the sheaf determined by the pre-sheaf $\{H^n(E' \frown D, \mathfrak{F}_E);\ D \in \mathfrak{L}(X)\}$.

The situation becomes simpler either if E' is *closed in* E*) or if $\mathfrak{F}$ is *a hyperfine sheaf*. Namely, we have in either case

i) the confining homomorphism $\mathfrak{F}_E \to \mathfrak{F}_{E'}$ is *surjective*, i. e.

$$0 \longrightarrow \mathrm{Dist}^0(E_1, \mathfrak{F}_E) \longrightarrow \mathfrak{F}_E \longrightarrow \mathfrak{F}_{E'} \longrightarrow 0 \ \text{(exact)},$$

ii) $\mathrm{Dist}^n(E_1, \mathfrak{F}_E) = 0$ for $n \geqslant 1$,

iii) $H^n_{\mathrm{rel}}(E_1, \mathfrak{F}_E) = {}^qH^n_{\mathrm{rel}}(E_1, \mathfrak{F}_E) = H^n(E_1, \mathrm{Dist}^0(E, \mathfrak{F}_E))$. (In case $\mathfrak{F}$ is hyperfine, these moduli are all 0 for $n \geqslant 1$).

5.4. *Dimension of a sheaf.* We shall say that a sheaf $\mathfrak{F}$ over a paracompact T_2-space X is *of dimension* $\leqslant m$ (in notation: $\dim \mathfrak{F} \leqslant m$), if (and only if) there exists a hyperfine resolution of $\mathfrak{F}$ of length $m+1$:

$$\begin{gathered} 0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{X}^0 \longrightarrow \mathfrak{X}^1 \longrightarrow \cdots \longrightarrow \mathfrak{X}^m \longrightarrow 0 \\ \text{with } \mathfrak{X}^0, \mathfrak{X}^1, \cdots, \mathfrak{X}^m \text{ hyperfine.} \end{gathered} \tag{5.4.1}$$

(Some of $\mathfrak{X}^j$ may be the sheaf 0.) We have: $\dim \mathfrak{F} \leqslant -1$ if and only if $\mathfrak{F} = 0$, and $\dim \mathfrak{F} \leqslant 0$ if and only if $\mathfrak{F}$ is hyperfine.

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*) This is the case treated by Godement [5] (§2.9 and §4.10).

If

$$0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{X}^0 \longrightarrow \mathfrak{X}^1 \longrightarrow \cdots \longrightarrow \mathfrak{X}^m \longrightarrow 0 \tag{5.4.2}$$

is any exact sequence such that $\dim \mathfrak{X}^j \leqslant m-j$ for $j=0, 1, \cdots, m-1$. Then, $\mathfrak{F}$ is dimension $\leqslant m$ if and only if $\mathfrak{X}^m$ is hyperfine. In particular, if $\mathfrak{F}$ is of dimension $\leqslant m$, and $\mathfrak{X}^0, \mathfrak{X}^1, \cdots, \mathfrak{X}^{m-1}$ are hyperfine, then (5.4.2) yields a hyperfine resolution of $\mathfrak{F}$.

If $\mathfrak{F}$ is of dimension $\leqslant m$, then

(i) for any $D, D' \in \mathfrak{L}(X)$, $D \supset D'$, the canonical homomorphisms $H^m(D, \mathfrak{F}) \to H^m(D', \mathfrak{F})$, ${}^pH^m(D, \mathfrak{F}) \to {}^pH^m(D', \mathfrak{F})$ are surjective, and $H^n(D \bmod D', \mathfrak{F})$, ${}^pH^n(D \bmod D', \mathfrak{F})$ vanishes for all $n > m$.

(ii) for any locally closed $S \subset X$, the sheaf $\mathrm{Dist}^n(S, \mathfrak{F})$ is of dimension $\leqslant m-n$. In particular, $\mathrm{Dist}^m(S, \mathfrak{F})$ is hyperfine, and $\mathrm{Dist}^n(S, \mathfrak{F})$ vanishes for all $n > m$.

Conversely, $\mathfrak{F}$ is of dimension $\leqslant n$ if any one of the following conditions holds:

(i) For any $D \in \mathfrak{L}(X)$, the canonical homomorphism $H^m(X, \mathfrak{F}) \to H^m(D, \mathfrak{F})$ is surjective.

(ii) For any $D \in \mathfrak{L}(X)$, $H^{m+1}(X \bmod D, \mathfrak{F}) = 0$.

(iii) For any closed subset $F \subset X$, $\mathrm{Dist}^m(F, \mathfrak{F})$ is hyperfine.

(iv) For any closed subset $F \subset X$, $\mathrm{Dist}^{m+1}(F, \mathfrak{F}) = 0$.

Further, $\mathfrak{F}$ is of dimension $\leqslant m$ if and only if it is *locally of dimension* $\leqslant m$, i.e. if and only if, for any $p \in X$, there exists an open neighborhood U of p such that $\dim \mathfrak{F} | U \leqslant m$.

Chapter III. General Theory of Hyperfunctions.

In case X is a *manifold*, the condition that X is *paracompact* means that *each connected component of X is perfectly separable*. This implies that *any subset* of X is also paracompact. As we have mentioned in the foreword, *all (complex and real) analytic manifolds considered in this Chapter are supposed to be paracompact unless otherwise is stated.*

§6. Analytic Distributions.

6.1. *Analytic distributions.* Let X be a complex analytic manifold of (complex) dimension m. For any $D \in \mathfrak{L}(X)$, let $\mathfrak{A}(D)$ denote *the ring of all (single-valued) holomorphic functions* defined on D. The correspondence $D \to \mathfrak{A}(D)$, together with the canonical homomorphisms induced by restriction, defines (not merely a pre-sheaf but) a sheaf over X. This sheaf, *the sheaf of (germs of) holomorphic functions*, will be denoted by $\mathfrak{A}$. We have, by the definition,

$$\Gamma(D, \mathfrak{A})=\mathfrak{A}(D).$$

A sheaf of $\mathfrak{A}$-moduli is called an *analytic sheaf*. That is, an analytic sheaf $\mathfrak{F}$ is a sheaf of moduli such that each $\Gamma(D, \mathfrak{F})$ is an $\mathfrak{A}(D)$-module and each canonical homomorphism: $\Gamma(D, \mathfrak{F})\to\Gamma(D', \mathfrak{F})$, $D'\subset D$, is an $\mathfrak{A}(D)$-homomorphism, if we regard $\Gamma(D', \mathfrak{F})$ as an $\mathfrak{A}(D)$-module in natural manner. An analytic sheaf is called *locally free* if for each point $p\in X$ there are an integer $r\geqslant 0$ and a neighborhood U of p such that, when restricted to U, $\mathfrak{F}$ is $\mathfrak{A}$-isomorphic to $\mathfrak{A}+\cdots+\mathfrak{A}$ (direct sum of r copies of $\mathfrak{A}$). Clearly the integer r is uniquely determined at each $p\in X$, and assumes a constant value within each connected component of X. If it is constant throughout the whole domain X, $\mathfrak{F}$ will be called a locally free analytic sheaf *of rank* r. In this case, we can regard $\mathfrak{F}$ as a sheaf consisting of all local sections of some *analytic vector bundle* $\boldsymbol{B}$ with typical fiber $\boldsymbol{C}^r$; we shall write $\mathfrak{F}=\mathfrak{A}_B$ for it.

Let $\mathfrak{A}_B$ denote a locally free analytic sheaf as described above. For any closed subset S of X, the analytic sheaf $\mathrm{Dist}^n(S, \mathfrak{A}_B)$ is called the *sheaf of analytic n-distributions* of type $\boldsymbol{B}$ over S. Each element of $\Gamma(S, \mathrm{Dist}^n(S, \mathfrak{A}_B))$ is called an *analytic n-distribution* of type $\boldsymbol{B}$ over S.

Proposition 6.1.1. *Any locally free analytic sheaf $\mathfrak{F}$ over X is of dimension $\leqslant m$.*

This proposition is equivalent to the following

Proposition 6.1.2. *For any open set D in $\boldsymbol{C}^m$, we have*

$$H^m(D, \mathfrak{A})=0.$$

Suppose that the proposition 6.1.1 is true. Then we have a *surjective* homomorphism

$$H^m(\boldsymbol{C}^m, \mathfrak{A})\longrightarrow H^m(D, \mathfrak{A}).$$

On the other hand, we have $H^n(\boldsymbol{C}^m, \mathfrak{A})=0$ for any $n>0$, as is well known. Hence the proposition 6.1.2.

Conversely, assume the proposition 6.1.2. For any $p\in X$, we can clearly choose an open neighborhood U of p and a local coordinate $z=(z_1, \cdots, z_m)$ defined in U such that $\mathfrak{F}|U$ is isomorphic to a direct sum of some copies of $\mathfrak{A}|U$. $\mathfrak{F}|U$ is of dimension $\leqslant m$, because the condition that $H^m(U, \mathfrak{F}|U)\to H^m(U', \mathfrak{F}|U')$ is surjective for any $U'\in\mathfrak{L}(U)$ is trivially satisfied. Hence the proposition 6.1.1.

A closed subset S of X is called *purely s-codimensional* if it is purely s-codimensional with respect to the sheaf $\mathfrak{A}$. Clearly such S is always purely s-codimensional with respect to any locally free analytic sheaf.

6.2. *Pure codimensionality of subvarieties.* Let S be a closed subset of X. Let D denote any open set of the complex plane $\boldsymbol{C}$ (or more generally, any open Riemann surface), K a compact subset of D. $X\times D$ constitute a complex analytic manifold of dimension $m+1$. By 4.2. (vii), we have an exact sequence

$$\cdots\xrightarrow{\delta^*}H^n_{\mathrm{rel}}(S\times K, \mathfrak{A})\longrightarrow H^n_{\mathrm{rel}}(S\times D, \mathfrak{A})\longrightarrow H^n_{\mathrm{rel}}(S\times(D-K), \mathfrak{A})\xrightarrow{\delta^*}\cdots$$

where $\mathfrak{A}$ denotes the sheaf of holomorphic functions on $X\times D$. However, it is shown that we have a homomorphism $f\colon H^{n+1}_{\mathrm{rel}}(S\times K, \mathfrak{A})\rightarrow H^n_{\mathrm{rel}}(S\times(D-K), \mathfrak{A})$ such that $\delta^*\circ f=1$ $(n=0, 1, 2, \cdots)$; whence the exact sequence above reduces to

$$0\longrightarrow H^n_{\mathrm{rel}}(S\times D, \mathfrak{A})\longrightarrow H^n_{\mathrm{rel}}(S\times(D-K), \mathfrak{A})\xrightarrow{\delta^*}H^{n+1}_{\mathrm{rel}}(S\times K, \mathfrak{A})\longrightarrow 0 \text{ (exact).}$$

Setting in particular $K=\{p\}$ (the set consisting of a single point), we know that $S\times\{p\}$ is a purely $(s+1)$-codimensional subset of $X\times D$ if, for every open subset D' of D, $S\times D'$ is a purely s-codimensional subset of $X\times D$. Whence we obtain

Proposition 6.2.1. *Let S be a closed subset of a paracompact complex analytic manifold X of dimension m. Let $D_1, \cdots, D_k$ denote open subsets of the complex plane $\boldsymbol{C}$, and set $D=D_1\times\cdots\times D_k$ $(\subset\boldsymbol{C}^k)$. If $S\times D$ is a purely s-codimensional subset of the $(m+k)$-dimensional complex analytic manifold $X\times D$, then $S\times\{p\}$ ($\{p\}$ denoting a set consisting of a single point of D) is a purely $(s+k)$-codimensional subset of $X\times D$.*

Further, we have

Proposition 6.2.2. *Each $(m-s)$-dimensional closed analytic subvariety V of an m-dimensional paracompact complex analytic manifold X is purely s-codimensional.*

When, in particular, V is a closed submanifold of X (i.e. a closed subvariety without singularity) this proposition is derived from the preceding proposition immediately.

6.3. *Analytic distribution on a point.* Again, let X be a paracompact complex analytic manifold of dimension m. Let Ω^n $(0\leqslant n\leqslant m)$ denote *the sheaf of holomorphic exterior differential forms of order n* (in short, *holomorphic n-forms*). As is well known, Ω^n is a locally free analytic sheaf of rank $\binom{m}{n}=\dfrac{m!}{n!(m-n)!}$ and hence be expressed by $\mathfrak{A}_{T^n}$, where T^n denotes an analytic vector bundle (viz. *the vector bundle of antisymmetric covariant tangential tensors of order n*). In particular, $\Omega^0=\mathfrak{A}$. $\Omega=\Omega^0+\Omega^1+\cdots+\Omega^m$ constitutes a sheaf of $\mathfrak{A}$-algebras, i.e. the Grassmann algebra generated by Ω^1.

Suppose a point $p\in X$ is given. Choosing an open neighborhood U of p and a local coordinates $z=(z_1, \cdots, z_m)$ defined in U and satisfying $z(p)=0$, we can define a homomorphism between sheaves over U:

$$\partial:\ \Omega^n|U \longrightarrow \Omega^{n-1}|U$$

by setting

$$\partial(dz_{j_1}\wedge\cdots\wedge dz_{j_n})=\frac{1}{2\pi i}\sum_{\nu=1}^{n}(-1)^\nu z_{j_\nu}\cdot dz_{j_1}\wedge\cdots\wedge\overset{\text{(excluded)}}{\widehat{dz_{j_\nu}}}\wedge\cdots\wedge dz_{j_n}.$$

It is clearly an $\mathfrak{A}|U$-homomorphism. Now we have an exact sequence of sheaves:

$$(6.3.1)\qquad 0\longrightarrow\Omega^m|U\xrightarrow{\partial}\cdots\xrightarrow{\partial}\Omega^1|U\xrightarrow{\partial}\Omega^0|U\longrightarrow C_p|U\longrightarrow 0\ \text{(exact)},$$

where C_p denotes a sheaf which has a stalk C on p and vanishes on $X-\{p\}$. (6.3.1) induces an iterated connecting homomorphism for cohomology moduli:

$$(6.3.2)\qquad H^0_{\mathrm{rel}}(\{p\},C_p)\longrightarrow H^m_{\mathrm{rel}}(\{p\},\Omega^m).$$

By proposition 6.2.1, the set $\{p\}$ consisting of a single point p is purely m-codimensional; whence we have $H^q_{\mathrm{rel}}(\{p\},\Omega^n)=0$ for $q=0,1,\cdots,m-1$, and $=\Gamma(\{p\},\mathrm{Dist}^m(\{p\},\Omega^n))$ for $q=m$. This fact implies that the homomorphism (6.3.2) is injective, and that the right-hand term of (6.3.2) is replaced by $\Gamma(\{p\},\mathrm{Dist}^m(\{p\},\Omega^m))$. The left-hand term of (6.3.2), on the other hand, is clearly equal to $\Gamma(\{p\},C_p)$ $(=C)$. Therefore (6.3.2) is restated as an injective homomorphism for sheaves over X as follows:

$$(6.3.3)\qquad C_p\longrightarrow\mathrm{Dist}^m(\{p\},\Omega^m)\qquad\text{(injective)}.$$

It is proved that the homomorphism (6.3.2), and hence the homomorphism (6.3.3), does not depend on the choice of U and $z=(z_1,\cdots,z_n)$.

6.4. *Analytic distributions on a submanifold.* Now let X, Ω^n be as described above, and suppose that we have a $(m-s)$-dimensional closed submanifold X_1 of X. Take an open set U of X and a local coordinate $z=(z_1,\cdots,z_m)$ defined in U such that $X_1\frown U$ coincides with the inverse image of $\{0\}^s\times C^{m-s}$ in the analytical mapping $z:U\to C^m$. Let Ξ denote the sheaf over U of subalgebras of $\Omega|U$ generated by $dz_1,\cdots,dz_s$; i.e. $\Xi=\Xi^0+\Xi^1+\cdots+\Xi^s$, $\Xi^0=\mathfrak{A}|U$, $\Xi^1=(\mathfrak{A}|U)dz_1+\cdots+(\mathfrak{A}|U)dz_s$, $\cdots$, $\Xi^s=(\mathfrak{A}|U)dz_1\wedge\cdots\wedge dz_s$. For each $n=1,2,\cdots,s$, define an $(\mathfrak{A}|U)$-homomorphism $\partial:\Xi^n\to\Xi^{n-1}$ by

$$\partial(dz_{j_1}\wedge\cdots\wedge dz_{j_n})=\frac{1}{2\pi i}\sum_{\nu=1}^{n}(-1)^\nu z_{j_\nu}\cdot dz_{j_1}\wedge\cdots\wedge\overset{\text{(excluded)}}{\widehat{dz_{j_\nu}}}\wedge\cdots\wedge dz_{j_n},\qquad(j_1,\cdots,j_n=1,\cdots,s).$$

Then we obtain an exact sequence of sheaves by $(\mathfrak{A}|U)$-homomorphisms:

$$(6.4.1)\qquad 0\longrightarrow\Xi^s\xrightarrow{\partial}\cdots\xrightarrow{\partial}\Xi^1\xrightarrow{\partial}\Xi^0\xrightarrow{\varepsilon}\mathfrak{A}_1|U\longrightarrow 0,$$

where $\mathfrak{A}_1$ denotes the sheaf over X induced by the sheaf of holomorphic functions over the $(m-s)$-dimensional complex analytic manifold X_1, and ε denotes the canonical homomorphism $\mathfrak{A}\to\mathfrak{A}_1$ induced by the injection $X_1\to X$.

(6.4.1) provides us an iterated connecting homomorphism for cohomology moduli for any $U'\in\mathfrak{L}(U)$:

$$H^0_{\mathrm{rel}}(X_1\frown U', \mathfrak{A}_1)\longrightarrow H^s_{\mathrm{rel}}(X_1\frown U', \Xi^s). \tag{6.4.2}$$

As X_1 is a purely s-codimensional subset of X in this case, we can infer from (6.4.1) that the homomorphism (6.4.2) is injective, and that the right-hand term of (6.4.2) is equal to $\Gamma(X_1\frown U', \mathrm{Dist}^s(X_1, \Xi^s))$. The left-hand term of (6.4.2) is clearly equal to $\Gamma(X_1\frown U', \mathfrak{A}_1)$ $(=\mathfrak{A}_1(X_1\frown U'))$. Therefore we have an injective homomorphism for sheaves over U:

$$\mathfrak{A}_1|U\longrightarrow \mathrm{Dist}^s(X_1, \Xi^s)|U. \tag{6.4.3}$$

On the other hand, we have an exact sequence of *locally free* analytic sheaves by canonical homomorphisms:

$$0\longrightarrow \Xi^n\longrightarrow \Omega^n\longrightarrow \Omega^n \bmod \Xi^n\longrightarrow 0,$$

whence we obtain an injective homomorphism for cohomology moduli:

$$H^s_{\mathrm{rel}}(X_1\frown U', \Xi^n)\longrightarrow H^s_{\mathrm{rel}}(X_1\frown U', \Omega^n) \quad (\text{for any } U'\in\mathfrak{L}(U)),$$

and further, an injective homomorphism for sheaves over U:

$$\mathrm{Dist}^s(X_1, \Xi^n)|U\longrightarrow \mathrm{Dist}^s(X_1, \Omega^n)|U. \tag{6.4.4}$$

Setting $n=s$ in (6.4.4) and combining it with (6.4.3), we obtain

$$\mathfrak{A}_1|U\longrightarrow \mathrm{Dist}^s(X_1, \Omega^s)|U \quad (\text{injective}). \tag{6.4.5}$$

Furthermore, it is proved that if $(U_j, z_j=(z_{j1},\cdots, z_{jm}))$, $j=1, 2$, are two couples of an open set of X and a local coordinates defined there satisfying the condition described before, then two homomorphisms corresponding to (6.4.5) coincide when they are restricted onto $U_1\frown U_2$. Consequently, we have now a canonical injective $\mathfrak{A}$-homomorphism:

$$\mathfrak{A}_1\longrightarrow \mathrm{Dist}^s(X_1, \Omega^s). \tag{6.4.6}$$

We shall remind here that $\mathfrak{A}_1$ is a *coherent analytic sheaf* over X[*]. Each cross-section of $\mathfrak{A}_1$ over X constitutes a *Cousin's additive distribution in a generalized sense* (i. e. *a Cousin's additive distribution modulo the sheaf* $\mathfrak{J}$ *of* (*prime*) *ideals of* $\mathfrak{A}$ defining the subvariety V); and the formula (6.4.6) provides

[*] The exact sequence (6.4.1) is called a *projective resolution of* $\mathfrak{A}_1|U$ *by locally free analytic sheaves*. Generally, it follows from a theorem of K. Oka that, for any coherent analytic sheaf $\mathfrak{F}$ over X and for any $p\in X$, we have, choosing a suitable open neighborhood U of p, a projective resolution

$$\cdots\xrightarrow{\partial}\mathfrak{X}_2\xrightarrow{\partial}\mathfrak{X}_1\xrightarrow{\partial}\mathfrak{X}_0\longrightarrow\mathfrak{F}|U\longrightarrow 0 \quad (\text{exact})$$

of $\mathfrak{F}|U$ consisting of locally free analytic sheaves $\{\mathfrak{X}_n;\ n=0,1,\cdots\}$ and $\mathfrak{A}$-homomorphisms ∂.

us an example that a Cousin's additive distribution may be regarded as an analytic distribution on some analytic subvariety of X. For further discussion, see a forthcoming paper of ours.

§7. Hyperfunctions.

7.1. *Analytic transformation.* For any complex analytic manifold X of dimension m, we shall denote with $\boldsymbol{T}(X)$ *the tangential vector bundle over* X. The fiber of $\boldsymbol{T}(X)$ at p (which we shall denote with $T(p)$) is *the tangent space of* M *at* p, i.e. the $\boldsymbol{C}$-module of rank m consisting of all tangential vectors. Let X, X' be paracompact complex analytic manifolds of dimension m, m' respectively, and Φ an analytic transformation from X' into X. For each $p\in X'$, Φ induces a homomorphism $d\Phi_p$: $T(p)\to T(\Phi(p))$ called the *differential of* Φ *at* p. Denoting with V_n the totality of $p\in X'$ where $d\Phi_p(T(p))$, the image of $T(p)$ by $d\Phi_p$, is of a rank $\leqslant n$, we have an ascending sequence of *closed subvarieties* of X':

$$V_0\subset V_1\subset\cdots\subset V_\mu=V_{\mu+1}=\cdots=X',\ (\mu=\min(m, m')).$$

If $V_n=X'$, for some n, $0\leqslant n\leqslant\mu$, and if V_{n-1} is a proper subvariety of X' (i.e. if, for any connected component X_α' of X', $V_{n-1}\frown X_\alpha'$ is a proper subvariety of X_α'), we shall say that Φ is an *analytic transformation* (*or analytic mapping*) *of rank* n. Each $p\in X'$ will be called *degenerate* or *non-degenerate* point of Φ according as $p\in V_{n-1}$ or not. If $V_{n-1}=\phi$ (the empty set), Φ will be called *non-degenerate on* X'. (In general cases, an analytic transformation may have different ranks on different connected components of X'.) In case $m\leqslant m'$, an analytic transformation Φ is of rank m if and only if Φ is an open mapping.

If $m'=m+r$, and if Φ: $X'\to X$ is a non-degenerate analytic transformation of rank m, the couple (X, Φ) will be called an *analytic fiber space of dimension* r *over* X.

What we have described above for complex analytic manifolds and transformations can be restated for the real analytic case without any alteration other than replacing the qualifier 'complex analytic' by 'real analytic' everywhere.

In particular, if Φ: $M'\to M$ is a non-degenerate analytic mapping of rank m between real analytic manifolds of dimension m' and m $(m'=m+r)$, (M', Φ) will be called an *analytic fiber space of dimension* r *over* M.

Suppose that we have a complex manifold X of dimension m, a real analytic manifold M of dimension n laid in X $(n\leqslant m)$, an analytic fiber space (X', Φ) of dimension r over X, and a subset M' of the inverse image $\Phi^{-1}(M)$ of M such that $(M', \Phi|M')$ is an analytic fiber space of dimension r over M. We shall then call the triple (X', M', Φ) an *analytic fiber space of dimension* r *over* $(X. M)$. The (complex and real) dimensions of X and M are given by $m'=m+r$ and $n'=n+r$, respectively. (As to the meaning of 'real manifold laid in X', see 7.2.)

7.2. *Pure codimensionality of a real analytic manifold.*

Lemma 7.2.1. *Let $D=D_1\times\cdots\times D_k$ be an open subset of C^k, denoting with $D_1,\cdots, D_k$ open sets in C (or more generally, let D be any Stein manifold of dimension k). Regarding $R^m\times D$ as a closed subset of an $(m+k)$-dimensional complex analytic manifold $C^m\times D$, we have*

$$H^n_{\mathrm{rel}}(R^m\times D, \mathfrak{A})=0 \qquad (\text{for } n\neq m). \tag{7.2.1}$$

In particular we have, setting $k=0$,

$$H^n_{\mathrm{rel}}(R^m, \mathfrak{A})=0 \qquad (\text{for } n\neq m).$$

We can define an open covering $(\mathfrak{U}, \mathfrak{U}')$ of $(C^m\times D, (C^m-R^m)\times D)$ by

$$\mathfrak{U}=\{U_j\,;\ j=0, 1, \cdots, m\},\ \ \mathfrak{U}'=\{U_j\,;\ j=1,\cdots, m\},$$
$$U_0=C^m\times D,$$
$$U_j=C^{j-1}\times(C-R)\times C^{m-j}\times D \qquad (j=1,\cdots, m).$$

This covering is a Stein covering, i.e. an open covering whose constituents are Stein manifolds, and hence, the condition of 4.2. (v) is satisfied for any locally free analytic sheaf. We have, consequently,

$$H^n_{\mathrm{rel}}(R^m\times D, \mathfrak{A})\simeq H^n(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{A}).$$

The dimension of the nerve of $\mathfrak{U}$ being m, we have $H^n_{\mathrm{rel}}(R^m\times D, \mathfrak{A})=H^n(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{A})=0$ for $n>m$ at once.

On the other hand, we have an exact sequence

$$\cdots\xrightarrow{\delta^*} H^n_{\mathrm{rel}}(R^{m+1}\times D, \mathfrak{A})\longrightarrow H^n_{\mathrm{rel}}(R^m\times C\times D, \mathfrak{A})\longrightarrow H^n_{\mathrm{rel}}(R^m\times(C-R)\times D, \mathfrak{A})\xrightarrow{\delta^*}\cdots.$$

Consequently, if the formula (7.2.1) is valid for $n=0, 1, \cdots, k$ for some m, it is also valid for $n=0, 1,\cdots, k$ with m replaced by $m+1$. Therefore we need prove (7.2.1) only for $n=m-1$, i.e. we need only prove

$$H^{m-1}_{\mathrm{rel}}(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{A})=0. \tag{7.2.2}$$

Utilizing the integral formula of Cauchy-Weil, we can actually verify the formula (7.2.2).

Consider a complex analytic manifold X of dimension m. A subset M of X will be called *a real analytic manifold of dimension n $(\leqq m)$ laid in X* if for each $p\in X$ there exist an open neighborhood U of p and a local coordinate $z=(z_1,\cdots, z_m)$ defined in U such that $M\frown U$ coincides with the inverse image of

$R^n \times \{0\}^{m-n}$ in the analytic mapping $z: U \to C^m$. (When this is the case, M is an n-dimensional closed submanifold of X regarded as a real analytic manifold of dimension $2m$.) In case of $n=m$, we may legitimately regard X as an "analytical prolongation of M".

Proposition 7.2.2. *Let X be a paracompact complex analytic manifold of dimension m. Any real analytic manifold (of any dimension $n \leqslant m$) laid in X is a purely m-codimensional subset of X.*

This proposition is contained in the following proposition as a special case of $X=X'$, $\Phi=1$.

Proposition 7.2.3. *Let (X', Φ) be an analytic fiber space (of any dimension) over a paracompact complex analytic manifold X of dimension m. Let M be a real analytic manifold (of any dimension $n \leqslant m$) laid in X, M' the inverse image of M in the analytic mapping $\Phi: X' \to X$. Then, M' is a purely m-codimensional subset of X'.*

As we can easily see from the proposition 6.2.2 and the local nature of pure codimensionality (5.2., Definition), we need prove the proposition 7.2.3 only in case $m=n$.

By the definition, it suffices for the proof of the proposition 7.2.3 to show that, for any point $p \in M'$ and any open neighborhood U of p, there exists an open neighborhood U_1 of p contained in U such that $H^n_{\text{rel}}(M' \frown U_1, \mathfrak{A})=0$ (for $n \neq m$).

We can choose an open neighborhood U_0 of $\Phi(p) \in X$, a local coordinate $z=(z_1, \cdots, z_m)$ defined in U, an open neighborhood U_1 of $p \in X'$, and a local coordinate $z'=(z'_1, \cdots, z'_{m'})$ defined in U_1, so that they satisfy following conditions: i) $U_1 \subset U$, ii) $z'_j = z_j \circ \Phi$ for $j \leqslant m$, iii) $(\Phi, z'_{m+1}, \cdots, z'_{m'}): U_1 \to X \times C^{m'-m}$ maps U_1 onto $U_0 \times D$ homeomorphically, where $D = D_{m+1} \times \cdots \times D_{m'}$ with D_j $(j=m+1, \cdots, m')$ being an open set in C, iv) $z: U_1 \to C^{m'}$ maps $M' \frown U_1$ onto $R^m \times D$ homeomorphically.*) In short, $z': U_1 \to C^{m'}$ maps $M' \frown U_1$ onto $R^m \times D$ homeomorphically. Hence, by the lemma 7.2.1, we obtain $H^n_{\text{rel}}(M' \frown U_1, \mathfrak{A})=0$ for $n \neq m$, hence the proposition 7.2.3.

7.3. *Hyperfunctions.* Consider a paracompact complex analytic manifold X of dimension m and a real analytic manifold M of dimension m laid in X. By pro-

*) More generally, suppose that z maps $M' \frown U_0$ onto a rectangular domain $I=\{(x_1, \cdots, x_m) \in R^m;\ -a_j < x_j < a_j \text{ for } j=1, \cdots, m\}$ in R^m homeomorphically $(0 < a_j \leqslant \infty)$. If $a_j < \infty$ for some j, replace x_j by $\tilde{x}_j$ defined e.g. by $\tilde{x}_j = x_j/(a_j^2 - x_j^2)$, and U_0 by a suitable open $\widetilde{U}_0$ contained in U_0 and containing $I' \frown U_0$. And then, the situation is reduced to the case $a_j = \infty$, $(j=1, \cdots, m)$.

position 7.2.1, all the sheaves $\mathrm{Dist}^n(M, \mathfrak{A})$, $n=0, 1, \cdots$, vanish except for $n=m$. Similarly for $\mathrm{Dist}^n(M, \mathfrak{A}_B)$ (where $\mathfrak{A}_B$ denotes the locally free analytic sheaf consisting of local sections of an analytic vector bundle $\boldsymbol{B}$ over X).

Definition. We call $\mathrm{Dist}^m(M, \mathfrak{A}_B)$ *the sheaf of hyperfunctions of type* $\boldsymbol{B}$ *over* M, and denote $\mathfrak{B}_B$ for it. For any locally closed subset S of M, we denote: $\mathfrak{B}_B(S)=\Gamma_{\mathrm{rel}}(S, \mathfrak{B}_B)$. Each element of $\mathfrak{B}_B(S)$, i.e. each analytic m-distribution of type $\boldsymbol{B}$ over S, will be called a *hyperfunction of type* $\boldsymbol{B}$ *over* S.

If $\mathfrak{A}_B=\mathfrak{A}$, i.e. if $\boldsymbol{B}=X\times \boldsymbol{C}$ (product bundle), we shall simply write $\mathfrak{B}$, $\mathfrak{B}(S)$ instead of $\mathfrak{B}_B$, $\mathfrak{B}_B(S)$, and omit the qualifying pharse "of type $\boldsymbol{B}$". If $\boldsymbol{B}$ is a vector bundle of differential forms, of tensors, of differential operators, etc. (of some given type, respectively), then the hyperfunctions of the corresponding type $\boldsymbol{B}$ will be called the hyperfunctions of differential forms, of tensors, of differential operators, etc.

By (5.2.2) we have

$$\mathfrak{B}_B(S)=H^m_{\mathrm{rel}}(S, \mathfrak{A}_B). \tag{7.3.1}$$

By *a complex neighborhood* (resp. *a real neighborhood*) of S is meant an open subset of X (resp. of M) containing S as a closed subset. The totality of complex neighborhood of S will be denoted by $\mathfrak{D}(S)=\mathfrak{D}(S, X)$. By the definition, (7.3.1) is rewritten as follows:

$$\mathfrak{B}_B(S)=H^m(D \bmod (D-S), \mathfrak{B}_B) \qquad \text{for any } D\in\mathfrak{D}(S);$$

i.e. a hyperfunction $g\in\mathfrak{B}_B(S)$ is nothing else but an *m-cohomology class of D* mod $(D-S)$ *with coefficients in* $\mathfrak{A}_B$. By proposition 5.2.1, $\mathfrak{B}_B$ is a hyperfine sheaf over M. Hence we have, for any locally closed $S\subset M$ and any closed subset $F\subset S$,

$$0\longrightarrow\mathfrak{B}_B(F)\longrightarrow\mathfrak{B}_B(S)\longrightarrow\mathfrak{B}_B(S-F)\longrightarrow 0 \qquad \text{(exact)}$$

(the completeness theorem)*); and for any locally finite closed covering $\{F_\nu\}$ of S, we can naturally define a *surjective* homomorphism: $\prod_\nu\mathfrak{B}_B(F_\nu)\to\mathfrak{B}_B(S)$ (the decomposition theorem). The *carrier* of $g\in\mathfrak{B}_B(I)$ (in notation: $\mathrm{car}\, g$) is defined as the complementary set $I-I'$ of the greatest open set $I'\subset I$ such that $g|I'=0$. Clearly it is a closed set in I. We have, for any closed subset $S\subset I$,

$$\mathfrak{B}_B(S)=\{g\in\mathfrak{B}_B(I)\,;\ \mathrm{car}\, g\subset S\}.$$

It is clear from the definition that $\mathfrak{B}_B$ is an analytic sheaf (i.e. a sheaf of $\mathfrak{A}$-moduli). Hence $\mathfrak{B}_B(S)$ constitutes an $\mathfrak{A}(S)$-module for any S and any type $\boldsymbol{B}$.

*) Here we derived the completeness theorem from proposition 6.1.1. Conversely, the completeness theorem implies the existence of a hyperfine resolution of $\mathfrak{A}$ of length $m+1$ as we shall show in 10.3, and hence the proposition 6.1.1.

7.4. *Derivation.* Further, consider an *analytic sheaf* $\mathfrak{T}$ *over* X consisting of all germs of *holomorphic linear differential operators.* $\mathfrak{T}$ constitutes a sheaf of $\mathfrak{A}$-algebras. On the other hand, $\mathfrak{A}$ is naturally regarded as a sheaf of $\mathfrak{T}$-left-moduli. Therefore, our sheaf $\mathfrak{B}=\mathrm{Dist}^m(M, \mathfrak{A})$ also constitutes a sheaf of $\mathfrak{T}$-left-moduli in a natural manner, and hence, denoting $\mathfrak{T}(S)=\Gamma(S, \mathfrak{T})$, the module $\mathfrak{B}(S)$ of hyperfunctions constitutes a $\mathfrak{T}(S)$-left-module. Therefore we can talk about the holomorphic linear differential equations satisfied by hyperfunctions.

There is a special class of hyperfunctions, that of *analytic hyperfunctions*, which we define as follows. A hyperfunction $g\in\mathfrak{B}(S)$ is called *analytic at* $p\in S$ if there exists a real neighborhood $I_p\ni p$ such that the $\mathfrak{T}(I_p)$-submodule $\mathfrak{T}(I_p)\cdot g|I_p$ of $\mathfrak{B}(I_p)$ generated by $g|I_p$ is finite dimensional as an $\mathfrak{A}(I_p)$-module, i. e. contains only a finite number of linearly independent elements over $\mathfrak{A}(I_p)$. g is called *an analytic hyperfunction* on S if it is analytic at every point of S, i.e. if the subsheaf $\mathfrak{T}\cdot g$ of $\mathfrak{B}$ generated by g as a sheaf of $\mathfrak{T}$-left moduli is locally finite dimensional as a sheaf of $\mathfrak{A}$-moduli. Roughly speaking, this condition means that g satisfies a sufficient number of (independent) holomorphic linear differential equations. It is proved that the carrier of singularity (as will be defined in 8.1.) of an analytic hyperfunction g on I $(\in\mathfrak{L}(M))$ constitutes a (proper) closed subvariety V of I. Each irreducible component of V will be called a *threshold* of g. Furthermore, thresholds are devided into two classes: *regular thresholds* (*degenerate* and *non-degenerate*), and *irregular thresholds.* It is proved that an analytic hyperfunction g on I whose thresholds are all non-degenerate regular thresholds is completely determined from the restriction $g|(I-V')$, with V' denoting any (proper) closed subvariety of I.

The utility of this notion of analytic hyperfunction consists in the fact that almost all of the functions of frequent use in the applied analysis can be regarded as hyperfunctions of this category. For further details, see subsequent papers of ours.

7.5. *Transformation of variables.* Consider an analytic fiber space (X', Φ) of dimension r over a complex analytic manifold X of dimension m. Let M denote a real analytic manifold of dimension m laid in X, and set $M'=\Phi^{-1}(M)$. Let further, $\boldsymbol{B}$ denote any analytic vector bundle over X, $\boldsymbol{B}'$ the analytic vector bundle over X' induced from $\boldsymbol{B}$ by the transformation Φ. By proposition 7.2.3, we have $\mathrm{Dist}^n(M', \mathfrak{A}_{B'})=0$ except for $n=m$ (where $\mathfrak{A}_{B'}$ denotes the locally free analytic sheaf over X' corresponding to $\boldsymbol{B}'$). We shall call $\mathrm{Dist}^m(M', \mathfrak{A}_{B'})$ the sheaf of hyperfunctions of type $\boldsymbol{B}'$ over M', and denote $\mathfrak{B}_{B'}$ for it. For any locally closed subset S' of M', we denote $\mathfrak{B}_{B'}(S')=\Gamma_{\mathrm{rel}}(S', \mathfrak{B}_{B'})$ $(=H^m_{\mathrm{rel}}(S', \mathfrak{A}_{B'}))$. Each element of $\mathfrak{B}_B$ will be called a *hyperfunction of type* B' *over* M'. (Roughly speaking, it stands for a hyperfunction which is 'holomorphic on each fibers', or: a hyperfunction containing r 'complex holomorphic parameters'. See further 9.4.)

Now the transformation Φ satisfies $\Phi(X'-M')\subset X-M$, and hence induces a homomorphism between cohomology moduli $\Phi^*: H^m_{\mathrm{rel}}(S, \mathfrak{A}_B)\to H^m_{\mathrm{rel}}(\Phi^{-1}(S), \mathfrak{A}_{B'})$ for any locally closed subset S of M, or equivalently

$$\Phi^*: \mathfrak{B}_B(S)\longrightarrow \mathfrak{B}_{B'}(\Phi^{-1}(S)). \tag{7.5.1}$$

In other words, a *hyperfunction on* M induces in a nature manner a *hyperfunction on* M' (i.e. a hyperfunction on M "assuming constant value on each fiber").

In case $r=0$ in particular, $\mathfrak{B}_{B'}$ reduces to a sheaf of analytic distributions of the original sense; and our homomorphism Φ^* yields a *transformation of variables* for hyperfunctions.

7.6. *Hyperfunctions as boundary values of holomorphic functions.* Let I_ν be an open set in $\boldsymbol{R}$, $D_\nu(\subset \boldsymbol{C})$ a complex neighborhood of I_ν $(\nu=1,\cdots,m)$; hence $I=I_1\times\cdots\times I_m$ and $D=D_1\times\cdots\times D_m$ denote a rectangular open set in $\boldsymbol{R}^m$ and a complex neighborhood of I, respectively. Let $(\mathfrak{U}, \mathfrak{U}')$ be an open covering of $(D, D-I)$ defined by

$$\mathfrak{U}=\{U_\nu;\ \nu=0,1,\cdots,m\}, \qquad \mathfrak{U}'=\{U_\nu;\ \nu=1,\cdots,m\}$$
$$U_0=D,\quad U_\nu=D_1\times\cdots\times D_{\nu-1}\times(D_\nu-I_\nu)\times D_{\nu+1}\times\cdots\times D_m \qquad (\nu=1,\cdots,m).$$

As this is a Stein covering, we have by 4.2.(v)

$$\mathfrak{B}(I)=H^m(D \bmod (D-I), \mathfrak{A})=H^m(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{A}). \tag{7.6.1}$$

Namely, each hyperfunction $g\in\mathfrak{B}(I)$ corresponds to a m-cohomology class of $\mathfrak{U} \bmod \mathfrak{U}'$ with coefficients in $\mathfrak{A}$, and hence, is represented by some m-cocycle φ $(\varphi_{\alpha_0\cdots\alpha_m})\in Z^m(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{A})$ of $\mathfrak{U} \bmod \mathfrak{U}'$. Clearly φ has only one independent component, for which we can choose the component $\varphi_{01\cdots m}\in\mathfrak{A}(U_1\frown\cdots\frown U_m)$. In other words, we have an isomorphism $Z^m(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{A})\simeq\mathfrak{A}(U_1\frown\cdots\frown U_m)$ by the correspondence $\varphi\leftrightarrow\varphi_{01\cdots m}$, or equivalently a surjective homomorphism

$$\theta:\ \mathfrak{A}(U_1\frown\cdots\frown U_m)\longrightarrow\mathfrak{B}(I).$$

Of course the homomorphism θ depends on the ordering of sufficies $1,\cdots,m$. If a permutation is applied to these sufficies, the resulting homomorphism will become θ or $-\theta$ according as the permutation is even or odd.

The homomorphism θ is derived also in the following way. In the first place, denote with $\mathfrak{A}_\nu$ the confinement of $\mathfrak{A}$ onto $(\boldsymbol{C}-\boldsymbol{R})^\nu\times\boldsymbol{C}^{m-\nu}$, and with $\mathfrak{G}_\nu$ and $\mathfrak{H}_{\nu-1}$ the confinements of $\mathfrak{A}_\nu$ onto $\boldsymbol{R}^\nu\times\boldsymbol{C}^{m-\nu}$ and onto $\boldsymbol{R}^{\nu-1}\times(\boldsymbol{C}-\boldsymbol{R})\times\boldsymbol{C}^{m-\nu}$, respectively ($\nu=$

$1,\cdots,m$). The sheaves $\mathfrak{G}_\nu$, $\mathfrak{H}_\nu$ may be also defined recursively as follows: $\mathfrak{G}_0=\mathfrak{A}$, $\mathfrak{G}_\nu=$confinement of $\mathfrak{H}_{\nu-1}$ onto $\boldsymbol{R}^\nu\times\boldsymbol{C}^{m-\nu}$ $(1\leqslant\nu\leqslant m)$, $\mathfrak{H}_\nu=$confinement of $\mathfrak{G}_\nu$ onto $\boldsymbol{R}^\nu\times(\boldsymbol{C}-\boldsymbol{R})\times\boldsymbol{C}^{m-\nu-1}$ $(0\leqslant\nu\leqslant m-1)$. Clearly we have exact sequences by confining homomorphisms

$$0\longrightarrow\mathfrak{G}_\nu\longrightarrow\mathfrak{H}_\nu\longrightarrow\mathfrak{G}_{\nu+1}\longrightarrow 0\qquad(\nu=0,1,\cdots,m-1)$$

and hence, an exact sequence

$$(7.6.2)\qquad 0\longrightarrow\mathfrak{A}\longrightarrow\mathfrak{H}_0\longrightarrow\mathfrak{H}_1\longrightarrow\cdots\longrightarrow\mathfrak{H}_{m-1}\longrightarrow\mathfrak{G}_m\longrightarrow 0$$

which yields an iterated connecting homomorphism

$$H^0_{\mathrm{rel}}(I,\mathfrak{G}_m)\longrightarrow H^m_{\mathrm{rel}}(I,\mathfrak{A}).$$

By the definition, we have $H^0_{\mathrm{rel}}(I,\mathfrak{G}_m)=\Gamma(I,\mathfrak{A}_m)$. Therefore, combining this homomorphism with the restriction $\Gamma(D,\mathfrak{A}_m)\to\Gamma(I,\mathfrak{A}_m)$, we have a homomorphism

$$(7.6.3)\qquad \Gamma(U_1\frown\cdots\frown U_m,\mathfrak{A})\longrightarrow\mathfrak{B}(I).$$

It is easy to show that θ coincides with the homomorphism (7.6.3). The surjectivity of the homomorphism (7.6.3) is now derived from another fact that $\boldsymbol{R}^m$ is purely $(m-\nu-1)$-codimensional with respect to $\mathfrak{H}_\nu$.

Now, for any $\psi=\psi(z_1,\cdots,z_m)\in\mathfrak{A}(U_1\frown\cdots\frown U_m)$, and for any one $(\pm_1,\cdots,\pm_m)$ of the 2^m combinations of signs $+$ and $-$, we shall define $\psi(x_1\pm_1 i0,\cdots,x_m\pm_m i0)$, the 'boundary value' of $\psi|U_1^{\pm_1}\frown\cdots\frown U_m^{\pm_m}$ (denoting $U_\nu^\pm=U_\nu\frown C^\pm$, with $\boldsymbol{C}^+=\{z\in\boldsymbol{C},\ \Im z>0\}$ and $\boldsymbol{C}^-=\{z\in\boldsymbol{C},\ \Im z<0\}$), as follows:

$$\psi(x_1\pm_1 i0,\cdots,x_m\pm_m i0)=\theta(\prod_{\nu=1}^m \pm_\nu\varepsilon(\pm_\nu z)\cdot\psi(z_1,\cdots,z_m))$$

where $\varepsilon(\pm z)$ is as defined in **1.1**. (Clearly this definition remains invariant if we perform an *even* permutation of sufficies $1,\cdots,m$.) It follows immediately from this definition that we have for any $g\in\mathfrak{B}(I)$

$$\begin{aligned}g(x_1,\cdots,x_m)&=\theta(\psi(z_1,\cdots,z_m))\\&=\sum_{(\pm_1,\cdots,\pm_m)}\pm_1\cdots\pm_m\psi(x_1\pm_1 i0,\cdots,x_m\pm_m i0)\end{aligned}$$

with some $\psi\in\mathfrak{A}(U_1\frown\cdots\frown U_m)$. (The sum extends over 2^m combinations of signs.) Thus, *any hyperfunction on I is represented as a linear combination of boundary values of holomorphic functions in 2^m "quadrants" $U_1^{\pm_1}\frown\cdots\frown U_m^{\pm_m}$.*

As is well known, a real analytic manifold M of dimension m is called *oriented* if all the couples of an open set $U\in\mathfrak{L}(X)$ and a local coordinate $x=(x_1,\cdots,x_m)$ defined in U are classified into those of positive parity and those of negative parity so that for any two: (U,x), (U',x') of such couples, the Jacobian of the transformation $x\to x'$ satisfies the following condition:

$$\left\{\begin{array}{l} \qquad\qquad\qquad \dfrac{\partial(x_1,\cdots,x_m)}{\partial(x'_1,\cdots,x'_m)}>0 \text{ or } <0 \text{ on } U\frown U' \\ \text{according as } (U,x) \text{ and } (U',x') \text{ are of the same parity or of different parities.}\end{array}\right.$$

Now let M be an oriented real analytic manifold of dimension m laid in a complex analytic manifold X of dimension m. Then, for any $p\in M$, we can choose a complex neighborhood U of p and a local parameter z defined in U and adapted to M such that (i) z maps U onto a rectangular open set of $\boldsymbol{C}^m$ and hence, $I=U\frown M$ onto a rectangular open set of $\boldsymbol{R}^m$, and (ii) $(I, z|I)$ is of positive parity. Hence, for each $g\in\mathfrak{B}(M)$ we obtain (using the coordinates $(z_1,\cdots,z_m)$ in this order) a representation of $g|I$ as a linear combination of boundary values of holomorphic functions.

7.7. *Hyperfunctions on the Cartesian product of manifolds.* Let X_1, X_2 denote complex analytic manifolds of dimension m_1, m_2 respectively, $X=X_1\times X_2$ the Cartesian product thereof, Φ_j the projection $X\to X_j$ $(j=1, 2)$. Let further M_1, M_2 denote real analytic manifolds of dimension m_1, m_2 laid in X_1, X_2 respectively, $M=M_1\times M_2$ $(\subset X)$ the Cartesian product thereof. Then, for any type $\boldsymbol{B}$ and any $D\in\mathfrak{L}(X)$, we have canonical injective homomorphisms by 8.1.1 (see 8.1)

$$(7.7.1)\qquad \begin{array}{c} H^{m_j}_{\mathrm{rel}}(M\frown D, \mathfrak{A}_{\Phi_j^{-1}(M_j),B})\longrightarrow\mathfrak{B}_B(M\frown D) \quad\text{and}\\ \mathfrak{A}_B(D)\longrightarrow\mathfrak{B}_B(\Phi_j^{-1}(M_j)\frown D).\end{array}$$

On the other hand, we have a canonical homomorphism

$$H^{m_1}(D \bmod (D-\Phi_1^{-1}(M_1)), \mathfrak{A}_B)\longrightarrow H^{m_1}(D \bmod D\frown(\Phi_2^{-1}(M_2)-M), \mathfrak{A}_B)$$

i. e. $$H^{m_1}_{\mathrm{rel}}(\Phi_1^{-1}(M_1)\frown D, \mathfrak{A}_B)\longrightarrow H^{m_1}_{\mathrm{rel}}(M\frown D, \mathfrak{A}_{\Phi_2^{-1}(M_2),B})$$

and the corresponding homomorphism with sufficies 1, 2 interchanged

$$H^{m_2}_{\mathrm{rel}}(\Phi_2^{-1}(M_2)\frown D, \mathfrak{A}_B)\longrightarrow H^{m_2}_{\mathrm{rel}}(M\frown D, \mathfrak{A}_{\Phi_1^{-1}(M_1),B}).$$

The left hand sides of the homomorphisms above are $\mathfrak{B}_B(\Phi_j^{-1}(M_j)\frown D)$, $j=1, 2$. Further, it is easy to verify that, setting $D\in\mathfrak{D}(M)$ and then going to the inductive limit, these canonical homomorphisms become injective. Combining them with (7.7.1), we have

$$(7.7.2)\qquad \begin{array}{l} \mathfrak{A}_B(M)\subset(\text{the inductive limit of } \{\mathfrak{B}_B(\Phi_1^{-1}(M_1)\frown D\,;\ D\in\mathfrak{D}(M)\})\\ \qquad\subset H^{m_2}_{\mathrm{rel}}(M, \mathfrak{A}_{\Phi_2^{-1}(M_2),B})\subset\mathfrak{B}_B(M).\end{array}$$

and the corresponding relation with sufficies 1, 2 interchanged. This result may be schematized as follows :

	Hyperfunctions on M	
⇗		⇖
Hyperfunctions on M containing real holomorphic parameters on M_1		Hyperfunctions on M containing real holomorphic parameters on M_2
⇑		⇑
Hyperfunctions on M containing complex holomorphic parameters on X_1		Hyperfunctions on M containing complex holomorphic parameters on X_2
⇖		⇗
	Holomorphic functions on M	

Now suppose for a moment that X_1, X_2 denote any paracompact T_2-spaces, $X=X_1\times X_2$ the Cartesian product thereof, and $\mathfrak{F}$ any sheaf over X. Let $\mathfrak{F}_1$, $\mathfrak{F}_2$ denote sheaves over X_1, X_2 respectively, f a bilinear mapping from $(\mathfrak{F}_1, \mathfrak{F}_2)$ to $\mathfrak{F}$ (so that f induces a bilinear mapping $\Gamma(D_1, \mathfrak{F}_1)\times\Gamma(D_2, \mathfrak{F}_2)\rightarrow\Gamma(D_1\times D_2, \mathfrak{F})$ for any $D_1\in\mathfrak{L}(X_1)$, $D_2\in\mathfrak{L}(X_2)$). Then, as is well known, f induces bilinear mappings between cohomology moduli (see e.g. §6, Chapitre II of [5])

$$f^*:\ H^{n_1}(D_1, \mathfrak{F}_1)\times H^{n_2}(D_2, \mathfrak{F}_2)\longrightarrow H^{n_1+n_2}(D_1\times D_2, \mathfrak{F})$$

for any $D_j\in\mathfrak{L}(X_j)$, $j=1, 2$.

We have further,

$$\begin{aligned}f^*:\ &H^{n_1}(D_1 \bmod D_1', \mathfrak{F}_1)\times H^{n_2}(D_2 \bmod D_2', \mathfrak{F}_2)\\ &\longrightarrow H^{n_1+n_2}(D_1\times D_2 \bmod (D_1'\times D_2)\smile(D_1\times D_2'), \mathfrak{F})\end{aligned}$$

for any D_j, $D_j'\in\mathfrak{L}(X_j)$, $D_j'\subset D_j$; or equivalently,

$$(7.7.3)\qquad f^*:\ H_{\mathrm{rel}}^{n_1}(S_1, \mathfrak{F}_1)\times H_{\mathrm{rel}}^{n_2}(S_2, \mathfrak{F}_2)\longrightarrow H_{\mathrm{rel}}{}^{n_1+n_2}(S_1\times S_2, \mathfrak{F})$$

for any locally closed $S_j\subset X_j$.

Returing to the former notations, and letting $\mathfrak{F}_j$ be a locally free analytic sheaf $\mathfrak{A}_{B_j}$, and f a bilinear mapping $(\mathfrak{A}_{B_1}, \mathfrak{A}_{B_2})\rightarrow\mathfrak{A}_B$ induced by a bilinear mapping $(\boldsymbol{B}_1, \boldsymbol{B}_2)\rightarrow\boldsymbol{B}$ for vector bundles (i. e. a homomorphism $\boldsymbol{B}_1\otimes\boldsymbol{B}_2\rightarrow\boldsymbol{B}$ between analytic vector bundles over X), we obtain from (7.7.3)

$$f^*:\ \mathfrak{B}_{B_1}(S_1)\times\mathfrak{B}_{B_2}(S_2)\longrightarrow\mathfrak{B}_B(S_1\times S_2)$$

for any locally closed $S_j\subset M_j$. On the other hand we have clearly

$$f:\ \mathfrak{A}_{B_1}(D_1)\times\mathfrak{A}_{B_2}(D_2)\longrightarrow\mathfrak{A}_L(D_2\times D_2)\qquad (D_j\in\mathfrak{L}(X_j)).$$

We have further

$$f^*:\ \mathfrak{B}_{B_1}(S_1)\times\mathfrak{A}_{B_2}(D_2)\longrightarrow\mathfrak{B}_L(S_1\times D_2)$$

$$f^*:\ \mathfrak{A}_{B_1}(D_1)\times\mathfrak{B}_{B_2}(S_2)\longrightarrow\mathfrak{B}_B(D_1\times S_2)$$

where the right hand terms stand for the moduli of hyperfunctions containing complex holomorphic parameters as we have defined before.

Setting in particular $\boldsymbol{B}_2=X_2\times\boldsymbol{C}$, we can choose as $\boldsymbol{B}$ the vector bundle induced from $\boldsymbol{B}_1$ by Φ_1, and as f the natural mapping. Setting further $S_2=M_2$, and using $1\in\mathfrak{B}_{B_2}(M_2)$, we have a canonical homomorphism

$$\mathfrak{B}_{B_1}(S_1)\longrightarrow\mathfrak{B}_B(\Phi_1^{-1}(S_1))$$

by letting each $g_1\in\mathfrak{B}_{B_1}(S_1)$ correspond to $g_1\otimes 1\in\mathfrak{B}_B(S_1\times M_2)$ (note that $\Phi_1^{-1}(S_1)=S_1\times M_2$). This coincides with the homomorphism (7.5.1).

§8. Hyperfunctions Containing Real Holomorphic Parameters.

8.1. Let X be a paracompact complex analytic manifold of dimension m, M a real analytic manifold of dimension n ($\leqslant m$) laid in X, and consider an analytic fiber space (X', M', Φ) of dimension r over (X, M). Let $\mathfrak{F}=\mathfrak{A}_B$ denote a locally free analytic sheaf over X' (with $\boldsymbol{B}$ denoting an analytic vector bundle over X), and $\mathfrak{F}_0$ the confinement of $\mathfrak{F}$ onto $\Phi^{-1}(M)$. In this case, it is proved that M' is purely r-codimensional with respect to $\mathfrak{F}_0$.

In the following, we shall restrict our considerations to the case $n=m$. (Generalization to the general case is immediate). By what was stated above, we have for any $I\in\mathfrak{L}(M')$

$$H^{r}_{\mathrm{rel}}(I, \mathfrak{F}_0)=\Gamma(I, \mathrm{Dist}^{r}(M', \mathfrak{F}_0)).$$

Now, each element $g\in H^{r}_{\mathrm{rel}}(I, \mathfrak{F}_0)$ will be legitimately regarded as a special kind of hyperfunction on I, i. e. *a hyperfunction on I holomorphic in variables over M*, as we call it. This interpretation is justified by the following

Proposition 8.1.1. *We have an injective canonical homomorphism for relative cohomology moduli*:

$$(8.1.1) \qquad H^{r}_{\mathrm{rel}}(I, \mathfrak{F}_0)\longrightarrow H^{m'}_{\mathrm{rel}}(M', \mathfrak{F}_0) \quad \text{for any} \quad I\in\mathfrak{L}(M')$$

or equivalently, an injective canonical homomorphism for sheaves:

$$(8.1.1)' \qquad \mathrm{Dist}^{r}(M', \mathfrak{F}_0)\longrightarrow \mathfrak{B}_B.$$

The homomorphism (8.1.1) or (8.1.1)′ is derived as follows.

Choose an open set U of X', and a local coordinate $z'=(z'_1, \cdots, z'_{m'})$ defined in U and adapted to M', such that $z'_j=z_j\circ\Phi$ for $j=m+1, \cdots, m'$, with some local coordinate $z=(z_{m+1}, \cdots, z_{m'})$ in M. For $\nu=0, 1, \cdots, m$, let I_ν denote the inverse image of $\boldsymbol{R}^\nu\times\boldsymbol{C}^{m'-\nu}$ in the analytic mapping z': $U\to\boldsymbol{C}^{m'}$; hence $U=I_0\supset I_1\supset\cdots\supset I_m=\Phi^{-1}(M)\frown U$. Denoting with $\overline{\boldsymbol{C}}^{+}$, $\overline{\boldsymbol{C}}^{-}$ the upper and the lower half of the complex plane $\boldsymbol{C}$ including the real axis respectively (i. e. $\overline{\boldsymbol{C}}^{+}=\{z\in\boldsymbol{C};\ \Im z\geqslant 0\}$, $\overline{\boldsymbol{C}}^{-}=\{z\in\boldsymbol{C};\ \Im z\leqslant 0\}$), we further define I_ν^{+}, I_ν^{-} to mean the inverse images of $\boldsymbol{R}^\nu\times\overline{\boldsymbol{C}}^{+}\times\boldsymbol{C}^{m'-\nu-1}$, $\boldsymbol{R}^\nu\times\overline{\boldsymbol{C}}^{-}\times\boldsymbol{C}^{m'-\nu-1}$, respectively; hence we have $I_\nu=I_\nu^{+}\smile I_\nu^{-}$, $I_\nu^{+}\frown I_\nu^{-}=I_{\nu+1}$. The confinements of $\mathfrak{F}|U$ onto I_ν, I_ν^{+}, I_ν^{-} (which are sheaves over U) will be denoted with $\mathfrak{G}_\nu$, $\mathfrak{G}_\nu^{+}$, $\mathfrak{G}_\nu^{-}$ respectively. (Hence $\mathfrak{G}_0=\mathfrak{F}|U$, $\mathfrak{G}_m=\mathfrak{F}_0|U$.) We set further, $\mathfrak{H}_\nu=\mathfrak{G}_\nu^{+}+\mathfrak{G}_\nu^{-}$. By the definition, there are confining homomorphisms

$$\varepsilon^{\pm}:\ \mathfrak{G}_\nu\longrightarrow\mathfrak{G}_\nu^{\pm}, \qquad \eta^{\pm}:\ \mathfrak{G}_\nu^{\pm}\longrightarrow\mathfrak{G}_{\nu+1}$$

and hence (combining the above homomorphisms with the natural injections resp.

projections), the homomorphisms

$$\varepsilon^{\pm}: \mathfrak{G}_\nu \longrightarrow \mathfrak{H}_\nu, \qquad \eta^{\pm}: \mathfrak{H}_\nu \longrightarrow \mathfrak{G}_{\nu+1}.$$

(Clearly these are all $\mathfrak{A}$-homomorphisms.) Then we have exact sequences

$$(8.1.2) \qquad 0 \longrightarrow \mathfrak{G}_\nu \xrightarrow{\varepsilon^+ + \varepsilon^-} \mathfrak{H}_\nu \xrightarrow{\eta^+ - \eta^-} \mathfrak{G}_{\nu+1} \longrightarrow 0, \qquad (\nu=0, 1, \cdots, m-1)$$

and hence, an exact sequence

$$(8.1.3) \qquad 0 \longrightarrow \mathfrak{F} \xrightarrow{\varepsilon} \mathfrak{H}_0 \xrightarrow{\partial} \mathfrak{H}_1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathfrak{H}_{m-1} \xrightarrow{\eta} \mathfrak{G}_m \longrightarrow 0$$

where we set: $\varepsilon=\varepsilon^+ + \varepsilon^-$, $\eta=\eta^+ - \eta^-$, $\partial=\varepsilon\circ\eta=\varepsilon^+\circ\eta^+ - \varepsilon^-\circ\eta^-$.

Consequently, for any open subset I of I_m, we have connecting homomorphisms for cohomology moduli:

$$(8.1.4) \qquad \partial^*: H^{n-1}_{\mathrm{rel}}(I, \mathfrak{G}_{\nu+1}) \longrightarrow H^n_{\mathrm{rel}}(I, \mathfrak{G}_\nu) \qquad (\nu=0, 1, \cdots, m-1)$$

and an iterated connecting homomorphism for cohomology moduli:

$$(8.1.5) \qquad H^{n-m}_{\mathrm{rel}}(I, \mathfrak{F}_0) \longrightarrow H^n_{\mathrm{rel}}(I, \mathfrak{F}).$$

By the fact stated before, I_m is $(m'-\nu)$-codimensional with respect to $\mathfrak{G}_\nu$ ($\nu \leqslant m$). It is proved, further, that I_m is $(m'-\nu-1)$-codimensional with respect to $\mathfrak{G}^{\pm}_\nu$ (and hence, with respect to $\mathfrak{H}_\nu$). This fact implies that the homomorphisms (8.1.4) are injective for $n=m-\nu'$, and hence, the homomorphism (8.1.5) is injective for $n=m'$:

$$(8.1.5)' \qquad H^r_{\mathrm{rel}}(I, \mathfrak{F}_0) \longrightarrow H^{m'}_{\mathrm{rel}}(I, \mathfrak{F}).$$

Furthermore, it is proved that the homomorphism (8.1.5)′ does not depend on the choice of the local coordinate $z=(z'_1, \cdots, z'_{m'})$; whence we obtain the required formulae (8.1.1), (8.1.1)′.

8.2. *Substitution of particular values into real holomorphic parameters.* Given an ordinary function (or more definitely, a holomorphic function) $f(x_1, \cdots, x_{m'})$ in $m'=m+r$ real variables, we can substitute special values $c_1, \cdots, c_m$ into part of variables $x_1, \cdots, x_m$ and obtain $f(c_1, \cdots, c_m, x_{m+1}, \cdots, x_{m'})$, a function in the remaining r variables $x_{m+1}, \cdots, x_{m'}$.

This process of substitution can be generalized to hyperfunctions in the following case:

Let X, M, X', M', Φ, B be of the same meaning as in 8.1. For each $p\in M$, set $M'_p=\Phi^{-1}(p)\frown M'$, and denote with $B(p)$ the restriction of B onto $\Phi^{-1}(p)$. (Clearly $\Phi^{-1}(p)$ is a closed analytic submanifold of X' of dimension r.) Denoting with $\mathfrak{F}_p$ the sheaf over X' induced from $\mathfrak{A}_{B(p)}$ by the injection: $\Phi^{-1}(p)\rightarrow X'$, we have clearly a natural (surjective) homomorphism ε_p: $\mathfrak{F}_0 \rightarrow \mathfrak{F}_p$, and hence a natural homomorphism

$$\varepsilon_p^*: H^r_{\mathrm{rel}}(I, \mathfrak{F}_0) \longrightarrow H^r_{\mathrm{rel}}(M'_p \frown I, \mathfrak{F}_p) \ (=\mathfrak{B}_{B(p)}(M'_p \frown I)) \text{ for any } I \in \mathfrak{L}(M'),$$

or equivalently, a natural homomorphism

$$\varepsilon_p^*: \ \mathrm{Dist}^r(M', \mathfrak{F}_0)|M'_p \longrightarrow \mathfrak{B}_{B(p)}. \tag{8.2.1}$$

For each $g \in H^r_{\mathrm{rel}}(I, \mathfrak{F}_0)$, the image $\varepsilon_p^*(g) \in \mathfrak{B}_{B(p)}(M'_p \frown I)$ will be called the *value* or the *specialization of* g *on the fiber* M'_p.

8.3. *Another case.* So far we have investigated mutual relations of analytic distributions, hyperfunctions and holomorphic functions under some typical situations and obtained, among others, formulae (6.4.6), (8.1.1), (8.2.1). There are, however, various generalizations of these results corresponding to the situations more complicated.

For example, let X be a paracompact complex analytic manifold of dimension m, X_1 a closed submanifold of X of dimension $m_1 = m - s$, M_1 a real analytic manifold of dimension m_1 laid in M_1. Further, let $\boldsymbol{B}$ denote any analytic vector bundle over X, and $\mathfrak{B}_1$ the restriction of $\boldsymbol{B}$ onto X_1. Then we have a commutative diagram of sheaves over M_1 consisting of four *injective* homomorphisms:

$$\begin{array}{ccc} \mathfrak{A}_{B_1}|M_1 & \xrightarrow{f} & \mathrm{Dist}^s(X_1, \mathfrak{A}_B)|M_1 \\ g\big\downarrow & & \big\downarrow g' \\ \mathrm{Dist}^{m_1}(M_1, \mathfrak{A}_{B_1}) & \xrightarrow[f']{} & \mathrm{Dist}^m(M_1, \mathfrak{A}_B), \end{array} \tag{8.3.1}$$

er equivalently, a commutative diagram of moduli for any $I_1 \in \mathfrak{L}(M_1)$:

$$\begin{array}{ccc} \mathfrak{A}_{B_1}(I_1) & \xrightarrow{f} & \Gamma(I_1, \mathrm{Dist}^s(X_1, \mathfrak{A}_B)) \\ \big\downarrow g & & \big\downarrow g' \\ \mathfrak{B}_{B_1}(I_1) & \xrightarrow[f']{} & \mathfrak{B}_B(I_1). \end{array} \tag{8.3.1$'$}$$

Of these four homomorphisms, f and g are the homomorphisms as introduced in (6.4.6) and (8.1.1), respectively, while f' and g' denote homomorphisms induced by f and g respectively, according to 5.2. The diagram (8.3.1)$'$ may be interpreted graphically as follows:

Holomorphic δ-functions on M_1	$\Rightarrow$	Analytic distributions on the germ of analytic prolongation of M_1
$\Downarrow$		$\Downarrow$
δ-functions on M_1	$\Rightarrow$	Hyperfunctions on M_1

§ 9. Hyperfunctions on a Real Analytic Manifold.

9.1. *Spaces with analytic structure.* Let S be a topological space, $\mathfrak{S}$ a sheaf over S of commutative algebras over the complex number field $\boldsymbol{C}$. If we have a complex analytic manifold X of dimension m, a homeomorphism Φ from S onto a subset of X, and an isomorphism $h: \Phi^{-1}(\mathfrak{A}_X) \xrightarrow{\sim} \mathfrak{S}$ between sheaves of algebras, then we call the triple (X, Φ, h) an *analytic prolongation* of $(S, \mathfrak{S})$ (of dimension m).

For example, if X, X' are analytic manifolds of dimension m, and Ψ is an analytic homeomorphism from X' onto an open set of X, then, denoting with h_Ψ the canonical isomorphism $\Phi^{-1}(\mathfrak{A}_X) \xrightarrow{\sim} \mathfrak{A}_{X'}$, (X, Ψ, h_Ψ) constitute an analytic prolongation of X'.

Roughly speaking, the analytic prolongation of $(S, \mathfrak{S})$ is determined uniquely in a local sense whenever it exists, provided that the analytic prolongation X is paracompact; that is, if we have analytic prolongations (X_j, Φ_j, h_j), $j=1, 2$, of $(S, \mathfrak{S})$ of dimension m such that at least one of them is paracompact, then we have still another analytic prolongation (X_3, Φ_3, h_3) of $(S, \mathfrak{S})$ of dimension m, and analytic homeomorphisms Ψ_j, $j=1, 2$, from X_3 onto open subsets of X_j such that Ψ_j induces Φ_j, h_j from Φ_3, h_3 (i. e. the analytic prolongation $(X_j, \Psi_j, h_{\Psi_j})$ of $(X_3, \mathfrak{A}_{X_3})$ induces (X_j, Φ_j, h_j) from (X_3, Φ_3, h_3)).

Now, let S and $\mathfrak{S}$ be of the same meaning as above. We say: $\mathfrak{S}$ defines an *analytic structure* of S (of dimension m), or $(S, \mathfrak{S})$ is a *space with analytic structure* (of dimension m), if for each point $p \in S$, there exists a neighborhood U of p such that $(U, \mathfrak{S}|U)$ admits an analytic prolongation of dimension m.

When this is the case, we call each cross-section $\varphi \in \Gamma(S_1, \mathfrak{S})$ of $\mathfrak{S}$ over a subset $S_1 \subset S$ *a holomorphic function on* S_1. Any $(S, \mathfrak{S})$ which admits an analytic prolongation of dimension m clearly constitutes a space with analytic structure. Conversely, if we have a space $(S, \mathfrak{S})$ with analytic structure of dimension m, then we can construct an analytic prolongation (X, Φ, h) of $(S, \mathfrak{S})$ of dimension m, and even a paracompact one, provided that S is paracompact. This being the case, we can further choose (X, Φ, h) so that X contains $\Phi(S)$ as a closed subset, if and only if S is locally compact; we shall call such an analytic prolongation (X, Φ, h) a *complex neighborhood* of S. The totality of complex neighborhoods of S (which is a *category* rather than a *set*) will be denoted with $\mathfrak{D}(S)$.

For the sake of simplicity, *we shall hereafter use the symbol* $\mathfrak{A}_S$ *in place of* $\mathfrak{S}$, to denote the analytic structure imposed on a topological space S. In the rest of this paragraph, S will always denote a paracompact space equipped with an analytic structure $\mathfrak{A}_S$ of dimension m.

9.2. *Analytic prolongation of a locally free analytic sheaf.* By a locally free analytic sheaf over S, we mean a locally free sheaf of $\mathfrak{A}_S$-moduli. If we have a locally free analytic sheaf $\mathfrak{F}$ over S, then we can choose an analytic prolongation (X, Φ, h) of S with X paracompact, a locally free analytic sheaf $\tilde{\mathfrak{F}}$ over X, and an isomorphism $f: \Phi^{-1}(\tilde{\mathfrak{F}}) \to \mathfrak{F}$ compatible with the isomorphism h between the sheaves of their operator ring. The couple $(\tilde{\mathfrak{F}}, f)$, together with the analytic prolongation (X, Φ, h) of S, will be called an analytic prolongation of $\mathfrak{F}$. Such an analytic prolongation of $\mathfrak{F}$ is determined uniquely in the sense of local isomorphism; that is, if we have analytic prolongations $(\tilde{\mathfrak{F}}_j, f_j)$, $j=1, 2$, of $\mathfrak{F}$ defined on the analytic prolongations (X_j, Φ_j, h_j), $j=1, 2$, of S, respectively, then we have still another analytic prolongation $(\tilde{\mathfrak{F}}_3, f_3)$ of $\mathfrak{F}$ defined on an analytic prolongation (X_3, Φ_3, h_3) of S, analytic homeomorphisms Ψ_j $(j=1, 2)$ from X_3 onto open sets of X_j which induce Φ_j, h_j from Φ_3, h_3, and isomorphisms $\Psi_j^{-1}(\tilde{\mathfrak{F}}_j) \to \tilde{\mathfrak{F}}_3$ compatible with the canonical isomorphisms h_{Ψ_j} between the sheaves of operator rings which induce f_j from f_3.

Let (X, Φ, h) and $(\tilde{\mathfrak{F}}, f)$ be any analytic prolongation of S (with X paracompact) and any analytic prolongation of $\mathfrak{F}$ over (X, Φ, h), respectively, and consider the relative cohomology module $H^n_{\mathrm{rel}}(\Phi(S), \tilde{\mathfrak{F}})$. It is then clear that these cohomology moduli corresponding to different choices of the prolongations (X, Φ, h) and $(\tilde{\mathfrak{F}}, f)$ are mutually combined by canonical isomorphisms; hence, identifying the corresponding elements in different moduli, we have one and the same cohomology module which we shall call the *realative cohomology module of* S and denote with $H^n_{\mathrm{rel}}(S, \mathfrak{F})$. Similarly with the associated relative cohomology moduli ${}^q H^n_{\mathrm{rel}}(S, \mathfrak{F})$, and the sheaf of distributions $\mathrm{Dist}^n(S, \mathfrak{F})$. Further, we can speak of pure codimensionality of S; i.e. S is called purely n-codimensional if and only if $\Phi(S)$ is purely n-codimensional in X.

9.3. *Real analytic manifolds.* Consider a paracompact real analytic manifold M of dimension m and the sheaf $\mathfrak{A}_M$ of holomorphic functions over M. Then $\mathfrak{A}_M$ clearly defines an analytic structure of M of dimension m; M is then purely m-codimensional as a result of proposition 7.2.1. Therefore, denoting with $\boldsymbol{B}$ an analytic vector bundle (whose fibers are $\boldsymbol{C}$-moduli) over M, and with $\mathfrak{A}_B = \mathfrak{A}_{M,B}$ the locally free analytic sheaf over M consisting of holomorphic local sections of $\boldsymbol{B}$, we can define the sheaf $\mathfrak{B}_B = \mathfrak{B}_{M,B}$ of hyperfunctions and the module $\mathfrak{B}_B(M)$ of hyperfunctions by formulae

$$\mathfrak{B}_B = \mathrm{Dist}^m(M, \mathfrak{A}_B) \quad \text{and} \quad \mathfrak{B}_B(M) = \Gamma(M, \mathfrak{B}_B) = H^m_{\mathrm{rel}}(M, \mathfrak{A}_B),$$

respectively. Clearly $\mathfrak{B}_B$ is a hyperfine sheaf.

More generally, we shall define *a real analytic manifold* (*of* (*base*) *dimension* m) *with local complex fiber space* (*of dimension* r) as follows: namely, it means a topological space M equipped with an analytic structure $\mathfrak{A}_M$ of dimension $m+r$ such that we can find for any $p \in M$ an open neighborhood $I \ni p$ and an analytical prolongation (U, z, h) of $(I, \mathfrak{A}_M|I)$ such that z maps I onto an open subset of $R^m \times C^r$. Clearly this notion reduces to that of a real analytic manifold or of a complex analytic manifold if we set $r=0$ or $m=0$. By proposition 7.2.3, such M is purely m-codimensional. Again, we can construct a locally free analytic sheaf $\mathfrak{A}_B$ over M of holomorphic local sections from an analytic vector bundle B (which has a partly complex analytic, partly real analytic structure); we shall define the sheaf $\mathfrak{B}_B = \mathfrak{B}_{M,B}$ and the module $\mathfrak{B}_B(M)$ by

$$\mathfrak{B}_B = \mathrm{Dist}^m(M, \mathfrak{A}_B) \text{ and } \mathfrak{B}_B(M) = \Gamma(M, \mathfrak{B}_B) = H^m_{\mathrm{rel}}(M, \mathfrak{A}_B),$$

respectively. In this case, the sheaf $\mathfrak{B}_B$ is of dimension $\leqslant r$. We shall call each element of $\mathfrak{B}_B(M)$ *a hyperfunction of type* B *on* M. (It stands for "a hyperfunction of m variables containing r (complex) holomorphic parameters.")

It will be now clear that the results of §7–§8 can be extended to the case of any paracompact real analytic manifolds; the generalized results, however, will not be repeated here.

9.4. *The complex conjugate hyperfunction.* For any complex analytic manifold X of dimension m (with the structure sheaf $\mathfrak{A}_X$), we can define in an obvious manner the *complex conjugate manifold* $\bar{X}$ (which is a copy of X) with the structure sheaf $\mathfrak{A}_{\bar{X}}$ (which is a copy of $\mathfrak{A}_X$) so that, denoting the natural (bijective) mapping $X \to \bar{X}$ with Θ, we have a natural semi-linear isomorphism k; $\Theta^{-1}(\mathfrak{A}_{\bar{X}}) \xrightarrow{\sim} \mathfrak{A}_X$ (i.e. an isomorphism over the automorphism of the operator ring C defined by taking the complex conjugate). Clearly the complex conjugate manifold $\bar{X}$ of X thus defined is again a complex analytic manifold of dimension m.

Consider a real analytic manifold M of dimension m. For any analytic prolongation (X, Φ, h) of M, we can define another analytic prolongation $(\bar{X}, \bar{\Phi}, \bar{h})$ as follows: $\bar{X}$ denotes the complex conjugate manifold of X, $\bar{\Phi}$ denotes the homomorphism $\Theta \circ \Phi: M \to \bar{X}$, and $\bar{h}$ denotes the isomorphism $h \circ (\Phi^{-1}(k)): \bar{\Phi}^{-1}(\mathfrak{A}_{\bar{X}}) \to \mathfrak{A}_M$ (where $\Theta: X \to \bar{X}$ and $k: \Theta^{-1}(\mathfrak{A}_{\bar{X}}) \xrightarrow{\sim} \mathfrak{A}_X$ are as defined above). We shall call this $(\bar{X}, \bar{\Phi}, \bar{h})$ the complex conjugate analytic prolongation of (X, Φ, h). The homeomorphism Θ and the isomorphism k naturally induce the isomorphism: $H^m(\bar{D} \bmod (\bar{D} - \bar{\Phi}(M)), \mathfrak{A}_{\bar{X}}) \xrightarrow{\sim} H^m(D \bmod (D - \Phi(M)), \mathfrak{A}_X)$ for any $D \in \mathfrak{L}(X)$, and $\bar{D} = \Theta(D) \in \mathfrak{L}(\bar{X})$, i.e. the automorphism: $\mathfrak{B}(I) \xrightarrow{\sim} \mathfrak{B}(I)$ for any $I \in \mathfrak{L}(M)$. Or equivalently, we have an automorphism c of the sheaf $\mathfrak{B}_M$ of hyperfunctions over M which is compatible with the automorphism of the operator-sheaf $\mathfrak{A}_M$ defined by taking complex conjugate. It is easily verified that the automorphism c is independent of the choice of the analytic prolongation (X, Φ, h). Clearly we have $c\circ c=1$. The image $c(g)$ of a hyperfunction $g\in\mathfrak{B}(S)$ (S being a locally closed set of M) will be called *the complex conjugate of* g, and denoted with $\bar{g}$.

§10. Integration.

10.1. *Integration.* If M is a paracompact, oriented (topological) manifold of dimension m, then we can define, as is well known, the homology moduli $H_n(M, \boldsymbol{C})$ and the cohomology moduli $H^n(M, \boldsymbol{C})$ of M with coefficients in $\boldsymbol{C}$, and in addition, an inner multiplication between $H_n(M, \boldsymbol{C})$ and $H^n(M, \boldsymbol{C})$, i.e. a bilinear mapping from $H_n(M, \boldsymbol{C})\times H^n(M, \boldsymbol{C})$ into $\boldsymbol{C}$ $(n=0, 1, \cdots, m)$.

If, in particular, M is a compact manifold, then we have a special element of $H_m(M, \boldsymbol{C})$ called *the fundamental cycle of* X. Accordingly, we can specify a canonical homomorphism

$$H^m(M, \boldsymbol{C})\to\boldsymbol{C}. \tag{10.1.1}$$

Let M be as described above (without assuming the compactness) and let M' be an open subset of M. Then we have the homology moduli $H_n(M \bmod M', \boldsymbol{C})$ and cohomology moduli $H^n(M \bmod M', \boldsymbol{C})$ of $M \bmod M'$ with coefficients in $\boldsymbol{C}$, and in addition, a bilinear mapping from $H_n(M \bmod M', \boldsymbol{C})\times H^n(M \bmod M', \boldsymbol{C})$ into $\boldsymbol{C}$ for each n.

If, in particular, $K=M-M'$ is compact, then we have a special element of $H_m(M \bmod M', \boldsymbol{C})$ which we shall call *the fundamental (relative) cycle of* $M \bmod M'$. Accordingly, we can specify a canonical homomorphism

$$H^m(M \bmod M', \boldsymbol{C})\longrightarrow\boldsymbol{C}. \tag{10.1.2}$$

Now let X be a paracompact complex analytic manifold of dimension m. Let $\Omega^n(=\mathfrak{A}\Gamma^n)$ denote the sheaf of holomorphic n-forms. Then we have an *exact sequence of sheaves*

$$0\longrightarrow\boldsymbol{C}\xrightarrow{\iota}\Omega^0\xrightarrow{d}\Omega^1\xrightarrow{d}\cdots\xrightarrow{d}\Omega^m\longrightarrow 0 \tag{10.1.3}$$

where $\boldsymbol{C}$ denotes the simple sheaf with constant stalk $\boldsymbol{C}$, while ι and d denote *injection* and *exterior derivation*, respectively.

Consequently, we have a iterated connecting homomorphism between cohomology moduli:

$$H^n(X, \Omega^m)\longrightarrow H^{n+m}(X, \boldsymbol{C}) \qquad (n=0, 1, \cdots, m). \tag{10.1.4}$$

If, in particular, X is compact, then we can apply the homomorphism (10.1.1) with M replaced by X regarded as a topological manifold of dimension $2m$, and obtain a homomorphism

$$(10.1.5) \qquad H^{2m}(X, \boldsymbol{C}) \longrightarrow \boldsymbol{C}.$$

Setting $m=n$ in (10.1.4) and combining it with (10.1.5), we obtain a canonical homomorphism

$$(10.1.6) \qquad H^{m}(X, \Omega^{m}) \longrightarrow \boldsymbol{C}.$$

Let X and Ω^{n} be as described above (without assuming the compactness of X), and let X' be an open subset of X. From (10.1.3), we have again an iterated connecting homomorphism between cohomology moduli:

$$(10.1.7) \qquad H^{n}(X \operatorname{mod} X', \Omega^{m}) \longrightarrow H^{n+m}(X \operatorname{mod} X', \boldsymbol{C}) \qquad (n=0, 1, \cdots, m).$$

If, moreover, $K=X-X'$ is compact, then we can apply the homomorphism (10.1.2) with M replaced by X regarded as a topological manifold of dimension $2m$, and obtain a homomorphism

$$(10.1.8) \qquad H^{2m}(X \operatorname{mod} X', \boldsymbol{C}) \longrightarrow \boldsymbol{C}.$$

Setting $m=n$ in (10.1.7) and combining it with (10.1.8), we obtain a canonical homomorphism

$$(10.1.9) \qquad H^{m}(X \operatorname{mod} X', \Omega^{m}) \longrightarrow \boldsymbol{C}.$$

Let X be a paracompact complex analytic manifold of dimension m, M a real manifold in X, K a compact subset of M. *The formula* (10.1.9) *now yields the following canonical homomorphism:*

$$(10.1.10) \qquad \mathfrak{B}_{T^m}(K) \longrightarrow \boldsymbol{C}.$$

The image of each $g \in \mathfrak{B}_{T^m}(K)$ by the canonical homomorphism (10.1.10) will be called *the (definite) integral of* g, and denoted by

$$(10.1.11) \qquad \underbrace{\int \cdots \int_K}_{m\text{-fold}} g\,.$$

(The domain K of integration may be replaced by M.)

As to a more concrete representation of the integral (10.1.11), see p. 607 of [8].

We shall further generalize the notation of integration in a subsequent paper.

10.2. *Duality theorem.* Consider a paracompact real analytic manifold M of dimension m and an analytic vector bundle $\boldsymbol{B}$ over M. We denote with $\mathfrak{B}^*_B(S)$ the $\mathfrak{A}(S)$-submodule of $\mathfrak{B}_B(S)$ consisting of $g \in \mathfrak{B}_B(S)$ whose carrier is compact.

Such $g \in \mathfrak{B}_B(S)$ will be called a *perfect hyperfunction* on S. Let $\boldsymbol{B}'$ denote the analytic vector bundle of the type complementary to $\boldsymbol{B}$, i.e. the vector bundle consisting of fibers $B'_p = \mathrm{Hom}_C(B_p, T_p^m)$ where B_p, T_p^m denote the fibers at p of $\boldsymbol{B}$ and $\boldsymbol{T}^m$ respectively. ($\boldsymbol{T}^m$ denotes the analytic vector bundle of m-forms.) By the definition, we have a bilinear map: $\mathfrak{A}_{B'}(S) \times \mathfrak{B}_B^*(S) \to \mathfrak{B}_{T^m}^*(S)$. Therefore, combining this map with the integration $B_{T^m}^*(S) \to \boldsymbol{C}$, we can define an inner product (f, g) between $f \in \mathfrak{A}_{B'}(S)$ and $g \in \mathfrak{B}_B^*(S)$. Further, it is proved that (i) $(f, g)=0$ for every $g \in \mathfrak{B}_B^*(S)$ implies $f=0$, and (ii) $(f, g)=0$ for every $f \in \mathfrak{A}_{B'}(S)$ implies $g=0$. Hence we have

Proposition 10.2.1. *$\mathfrak{A}_{B'}(S)$ and $\mathfrak{B}_B^*(S)$ constitute a pair of mutually dual vector spaces* (Duality theorem). The Mackey topology induced in $\mathfrak{A}_{B'}(S)$ by $\mathfrak{B}_B^*(S)$ coincides with the ordinary topology in $\mathfrak{A}_{B'}(S)$.

Now let S_1 denote any locally closed subset of S. Then we have obviously a canonical injective homomorphism: $\mathfrak{B}_B^*(S_1) \to \mathfrak{B}_B^*(S)$. Hence the proposition 10.2.1 yields (replacing $\boldsymbol{B}'$ by $\boldsymbol{B}$)

Corollary. *The image of the canonical homomorphism $\mathfrak{A}_B(S) \to \mathfrak{A}_B(S_1)$ is dense in $\mathfrak{A}_B(S_1)$.*

On the other hand, if S contains no open and closed subset which is disjoint with S_1, the restriction $\mathfrak{A}(S) \to \mathfrak{A}(S_1)$ is clearly injective. Hence we have

Proposition 10.2.2. *If $S_1 \subset S \subset M$, and if S contains no open and closed subset disjoint with S_1, then $\mathfrak{B}^*(S_1)$ is a dense submodule of $\mathfrak{B}^*(S)$, in other words, each perfect hyperfunction on S is approximated by a perfect hyperfunction on S_1 (with respect to the Mackey topology induced in $\mathfrak{B}^*(S)$* by *$\mathfrak{A}(S)$).*

This proposition presents a clear difference between the topology of $\mathfrak{B}^*(S)$ and that of $(\mathcal{E}_S)'$ of L. Schwartz. Roughly speaking, our topology of $\mathfrak{B}^*(S)$ is 'of non-localizable nature' contrarily to that of $(\mathcal{E}_S)'$. Further, this fact is linked with the fact that we cannot define a separative topology in $\mathfrak{B}(S)$ in a natural way.

10.3. A space S with analytic structure $\mathfrak{A}_S$ of dimension m will be called *of Stein type* if for any homomorphism $\sigma: \mathfrak{A}(S) \to \boldsymbol{C}$ as algebras over $\boldsymbol{C}$ there exists one and only one $p \in S$ such that σ is induced by setting $\sigma(\varphi)=\varphi(p)$ for every $\varphi \in \mathfrak{A}(S)$. (The meaning of the symbol $\varphi(p)$ will be obvious.) For instance, it is proved that every (paracompact) real analytic manifold is a space with analytic structure of Stein type.

Now, for any type $\boldsymbol{B}$, $\mathfrak{B}_B^*(S)$ will denote the inductive limit of $\{H_{\mathrm{rel}}^m(K, \mathfrak{A}_B)$; K=compact subset of $S\}$ by dilution homomorphisms (as defined in 5.1.). $\mathfrak{B}_B^*(S)$ clearly constitutes an $\mathfrak{A}(S)$-module.

Further, denoting with $\boldsymbol{B}'$ the analytic vector bundle complementary to $\boldsymbol{B}$, there is a canonically defined bilinear map $\mathfrak{A}_{B'}(K)\times H^m_{(S)}(K, \mathfrak{A}_B)\to H^m_{(S)}(K, \mathfrak{A}_{1^m})$, hence a bilinear map

$$\mathfrak{A}_{B'}(S)\times \mathfrak{B}^*_B(S) \longrightarrow \mathfrak{B}^*_{1^m}(S). \tag{10.3.1}$$

On the other hand, we have a canonical homomorphism $\mathfrak{B}_{1^m}(S)\to \boldsymbol{C}$ by (10.1.9), and hence, combining this homomorphism with the bilinear map (10.3.1), we can define a canonical inner product between $\mathfrak{A}_{B'}(S)$ and $\mathfrak{B}^*_B(S)$. The proposition 10.2.1 is now generalized as follows:

Proposition 10.3.1. *If S is of Stein type, $\mathfrak{A}_{B'}(S)$ and $\mathfrak{B}^*_B(S)$ constitute a pair of mutually dual vector spaces.* The Mackey topology induced in $\mathfrak{A}_{B'}(S)$ by $\mathfrak{B}^*_B(S)$ coincides with the ordinary topology in $\mathfrak{A}_{B'}(S)$.

In case S is a complex analytic manifold (i.e. a Stein manifold), this proposition reduces to a duality theorem of J.-P. Serre ([11]).

Consider a complex analytic manifold X of dimension m. Let Ω^0 $(=\mathfrak{A})$, Ω^1, $\cdots$, Ω^m and $d: \Omega^n\to\Omega^{n+1}$ $(n=0, 1, \cdots, m-1)$ be of the same meaning as in **10.1.** Let $\mathfrak{F}=\mathfrak{A}_B$ denote a locally free analytic sheaf over X.

Now, any complex analytic manifold of dimension m induces in a natural manner a structure of an oriented real analytic manifold of dimension $2m$. We shall denote with $\mathfrak{O}$ the sheaf of holomorphic functions on X regarded as a real analytic manifold of dimension $2m$. $\mathfrak{O}$ contains $\mathfrak{A}$ and the complex conjugate $\bar{\mathfrak{A}}$ of $\mathfrak{A}$ as subsheaves. $\bar{\mathfrak{A}}$ is the sheaf which determines the complex conjugate analytic structure of X, and will be called *the sheaf of anti-holomorphic functions on X*. Similarly, $\bar{\Omega}^0$ $(=\bar{\mathfrak{A}})$, $\bar{\Omega}^1$, $\cdots$, $\bar{\Omega}^m$ and $\bar{d}$ denote the sheaves of anti-holomorphic differential forms and the corresponding exterior derivation operator. Setting $\mathfrak{O}_B=\mathfrak{A}_B\otimes_{\mathfrak{A}}\mathfrak{O}$ and $\mathfrak{X}^n=\mathfrak{O}_B\otimes_{\bar{\mathfrak{A}}}\bar{\Omega}^n$, we have a well known exact sequence

$$0\longrightarrow\mathfrak{A}_B\longrightarrow\mathfrak{X}^0_B\xrightarrow{\bar{d}}\mathfrak{X}^1_B\xrightarrow{\bar{d}}\cdots\xrightarrow{\bar{d}}\mathfrak{X}^m_B\longrightarrow 0. \tag{10.3.2}$$

(Usually, the operators d, $\bar{d}$ above are denoted with d', d'' respectively, and the sequence (10.3.2) is called d''-resolution of $\mathfrak{A}_B$.)

Further, denote with $\mathfrak{B}$ the sheaf of hyperfunctions on X regarded as a real analytic manifold of dimension $2m$, and set $\mathfrak{B}_B=\mathfrak{A}_B\otimes_{\mathfrak{A}}\mathfrak{B}$, $\mathfrak{Y}^n_B=\mathfrak{B}_B\otimes_{\bar{\mathfrak{A}}}\bar{\Omega}^n$. Then, $\mathfrak{Y}^n_B$ is a hyperfine sheaf containing $\mathfrak{X}^n_B$ as a subsheaf. Moreover we have an exact sequence by $\mathfrak{A}$-homomorphisms

$$0\longrightarrow\mathfrak{A}_B\longrightarrow\mathfrak{Y}^0_B\xrightarrow{\bar{d}}\mathfrak{Y}^1_B\xrightarrow{\bar{d}}\cdots\xrightarrow{\bar{d}}\mathfrak{Y}^m_B\longrightarrow 0. \tag{10.3.3}$$

Clearly (10.3.3) yields a *hyperfine resolution* of $\mathfrak{A}_B$, and the injections $\mathfrak{X}^n_B\to\mathfrak{Y}^n_B$ induce a chain homomorphism from (10.3.2) to (10.3.3) over the identical mapping

$1: \mathfrak{A}_{B} \to \mathfrak{A}_{B}$.

10.4. *Integration, general case.* As we have announced in [8] and shall expound in detail in a subsequent paper, the notion of integration for hyperfunctions defined in **10.1.** will be generalized in the following case.

Let (X', Φ) denote an analytic fiber space of dimension r over a complex analytic manifold X of dimension m. Let F be a closed subset of X, F' a closed subset of $\Phi^{-1}(F)$ such that $\Phi^{-1}(K) \frown F'$ is compact whenever K is a compact subset of F. Further, let $\boldsymbol{B}$ be an analytic vector bundle over X, $\Phi^{-1}(\boldsymbol{B})$ the analytic vector bundle induced from $\boldsymbol{B}$ by Φ. On the other hand, for each $p \in X$ corresponds an r-dimensional complex analytic manifold $\Phi^{-1}(p)$. Denoting with $\boldsymbol{T}_s(\Phi^{-1}(p))$ $(s=0, 1, \cdots, r)$ the tangential tensor bundle of contravariant anti-symmetric tensors of order s, we can define an analytic vector bundle $\boldsymbol{T}_s' = \cup_{p \in X} \boldsymbol{T}_s(\Phi^{-1}(p))$ over X', a subbundle of the tangential tensor bundle over X' of contravariant anti-symmetric tensors of order s. Accordingly, an analytic vector bundle $\boldsymbol{B}'$ over X is defined by setting $\boldsymbol{B}' = \cup_{p \in X} B_p$, $B_p = \mathrm{Hom}_C(T'_{r,p}, \Phi(\boldsymbol{B})_p)$, where $T'_{r,p}$ and $\Phi(\boldsymbol{B})_p$ denote the fibers at p of $\boldsymbol{T}_r'$ and $\Phi(\boldsymbol{B})$, respectively.

Under these circumstances, we can derive the following homomorphism as a generalization of (10.1.1)

$$\Phi^*: \ H^{n+r}_{\mathrm{rel}}(F', \mathfrak{A}_{B'}) \longrightarrow H^n_{\mathrm{rel}}(F, \mathfrak{A}_B). \tag{10.4.1}$$

If, further, we have an analytic fiber space (X'', Φ') of dimension r' over X' and a closed subset F'' of $\Phi'^{-1}(F')$ such that $\Phi'^{-1}(K') \frown F''$ is compact whenever $K' \subset F'$ is compact, then, defining $\boldsymbol{B}''$ in an analogous way, we have homomorphisms $\Phi'^*: H^{n+r'}_{\mathrm{rel}}(F'', \mathfrak{A}_{B''}) \to H^n_{\mathrm{rel}}(F', \mathfrak{A}_{B'})$ and $(\Phi \circ \Phi')^*: H^{n+r+r'}_{\mathrm{rel}}(F'', \mathfrak{A}_{B''}) \to H^n_{\mathrm{rel}}(F, \mathfrak{A}_B)$, and obtain the relation

$$(\Phi \circ \Phi')^* = \Phi^* \circ \Phi'^*. \tag{10.4.2}$$

Consider now an analytic fiber space (M', Φ) of dimension r over a real analytic manifold M of dimension m, and an analytic vector bundle $\boldsymbol{B}$ over M. An analytic vector bundle $\boldsymbol{B}'$ over M' is then defined in an analogous way as above. Let S be a locally closed set in M, S' a closed subset of $\Phi^{-1}(S)$ such that $\Phi^{-1}(K) \frown S'$ is compact whenever $K \subset S$ is compact. We have then, setting $n=m$ in (10.4.1), a homomorphism

$$\Phi^*: \ \mathfrak{B}_{B'}(S') \longrightarrow \mathfrak{B}_B(S). \tag{10.4.3}$$

The image $\Phi^*(g') = g$ of a $g' \in \mathfrak{B}_{B'}(S')$ will be called *the integral of g' along fibers*, and denoted with

(10.4.4) $$g=\int_{\Phi^{-1}} g' \quad \text{or} \quad g(p)=\overbrace{\int\cdots\int}^{r\text{-fold}}_{p'\in\Phi^{-1}(p)} g'(p').$$

The formula (10.4.2) yields in this case the Fubini theorem for integration of hyperfunctions.

10.5. *Duality, another case.* In the first place, suppose that we have three sheaves $\mathfrak{F}_1$, $\mathfrak{F}_2$, $\mathfrak{F}$ over a topological space X, and a bilinear mapping f: $\mathfrak{F}_1\times\mathfrak{F}_2\to\mathfrak{F}$ (i. e. a homomorphism f: $\mathfrak{F}_1\otimes\mathfrak{F}_2\to\mathfrak{F}$ where $\mathfrak{F}_1\otimes\mathfrak{F}_2$ denotes the tensorial product of $\mathfrak{F}_1$ and $\mathfrak{F}_2$). As is well known, f induces a bilinear mapping f^*: $H^{n_1}(D_1, \mathfrak{F}_1)\times H^{n_2}(D_2, \mathfrak{F}_2)\to H^{n_1+n_2}(D_1\frown D_2, \mathfrak{F})$ for any $D_1, D_2\in\mathfrak{L}(X)$. The image of $(g_1, g_2)\in H^{n_1}(D_1, \mathfrak{F}_1)\times H^{n_2}(D_2, \mathfrak{F}_2)$ is usually denoted with $g_1\smile g_2$ and called the *cup product* of g_1 and g_2. Now, this notion of cup product is easily extended to the relative case; namely, we have a bilinear mapping

(10.5.1) $$\begin{aligned} f^*:\ & H^{n_1}(D_1 \bmod D_1', \mathfrak{F}_1)\times H^{n_2}(D_2 \bmod D_2', \mathfrak{F}_2)\\ &\longrightarrow H^{n_1+n_2}(D_1\frown D_2 \bmod (D_1'\frown D_2)\smile(D_1\frown D_2'), \mathfrak{F})\end{aligned}$$

for any D_j, $D_j'\in\mathfrak{L}(X)$, $D_j'\subset D_j$ $(j=1, 2)$. Further, the result (10.5.1) is immediately generalized to the case where D_j, D_j' are replaced by arbitrary subset E_j, E_j' of X satisfying $E_j'\subset E_j$.

Now consider a complex analytic manifold X of dimension m. Let $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, $\boldsymbol{B}$ be three analytic vector bundles over X, f a bilinear mapping from $\boldsymbol{B}_1\times\boldsymbol{B}_2$ to $\boldsymbol{B}$ (i. e. an analytic mapping from $\boldsymbol{B}_1\times\boldsymbol{B}_2$ to $\boldsymbol{B}$ such that f induces a bilinear mapping $B_{1,p}\times B_{2,p}\to B_p$ between fibers for any $p\in X$), or equivalently, a homomorphism from $\boldsymbol{B}_1\otimes\boldsymbol{B}_2$ to $\boldsymbol{B}$, where $\boldsymbol{B}_1\otimes\boldsymbol{B}_2$ denotes the tensorial product of $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ (whose fiber at $p\in X$ is $B_{1,p}\otimes B_{2,p}$). As we have $\mathfrak{A}_{B_1}\otimes_{\mathfrak{A}}\mathfrak{A}_{B_2}\simeq\mathfrak{A}_{B_1\otimes B_2}$ canonically, f induces a bilinear mapping $\mathfrak{A}_{B_1}\times\mathfrak{A}_{B_2}\to\mathfrak{A}_B$. Consequently, (10.5.1) yields a bilinear mapping for cohomology moduli

(10.5.2) $$f^*:\ H^{n_1}_{\mathrm{rel}}(S_1, \mathfrak{A}_{B_1})\times H^{n_2}_{\mathrm{rel}}(S_2, \mathfrak{A}_{B_2})\longrightarrow H^{n_1+n_2}_{\mathrm{rel}}(S_1\frown S_2, \mathfrak{A}_B)$$

for any locally closed S_1, S_2 $(\subset X)$, or more generally,

$$\begin{aligned} f^*:\ & H^{n_1}(E_1 \bmod E_1', \mathfrak{A}_{B_1})\times H^{n_2}(E_2 \bmod E_2', \mathfrak{A}_{B_2})\\ &\longrightarrow H^{n_1+n_2}(E_1\frown E_2 \bmod (E_1'\frown E_2)\smile(E_1\frown E_2'), \mathfrak{A}_B)\end{aligned}$$

for any $E_j'\subset E_j\subset X$ $(j=1, 2)$.

Now consider a real analytic manifold M of dimension m laid in a complex analytic manifold X of dimension m, and an analytic fiber space (X', M', Φ) over (X, M). Let $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, $\boldsymbol{B}$ be analytic vector bundles over X', f a bilinear mapping from $\boldsymbol{B}_1\times\boldsymbol{B}_2$ to $\boldsymbol{B}$. For any $D\in\mathfrak{L}(X)$, we have by (10.5.2) a canonical bilinear mapping

$$(10.5.3)\qquad f^*:\ H^m_{\mathrm{rel}}(\Phi^{-1}(M)\frown D, \mathfrak{A}_{B_1})\times H^r_{\mathrm{rel}}(D_1, \mathfrak{A}_{B_2})\longrightarrow H^{m'}_{\mathrm{rel}}(M'\frown D, \mathfrak{A}_B)$$

where we set $D_1=D\frown(\Phi^{-1}(X-M)\smile M')=D-(\Phi^{-1}(M)-M')$. The first and the last terms in (10.5.3) are $\mathfrak{B}_{B_1}(\Phi^{-1}(M)\frown D)$ and $\mathfrak{B}_{B_2}(M'\frown D)$, respectively. Taking $D\in\mathfrak{D}(\Phi^{-1}(M))$, and going to the inductive limit, we obtain

$$(10.5.4)\qquad f^*:\ \mathfrak{B}_{B_1}(\Phi^{-1}(M))\times H^r_{\mathrm{rel}}(M', \mathfrak{A}_{\Phi^{-1}(M), B_2})\longrightarrow\mathfrak{B}_B(M'),$$

where $\mathfrak{A}_{\Phi^{-1}(M), B_2}$ denotes the confinement of $\mathfrak{A}_{B_2}$ onto $\Phi^{-1}(M)$. Namely, we can define a *product of a hyperfunction over* $\Phi^{-1}(M)$ (i.e. a hyperfunction which is holomorphic on each fiber of $\Phi^{-1}(M)$) *and a hyperfunction over* M' *containg real holomorphic parameters on* M as a hyperfunction over M'. Specifically, if $\boldsymbol{B}$ and $\boldsymbol{B}'$ are analytic vector bundles of mutually complementry types, then we have a canonical bilinear mapping $\boldsymbol{B}\times\boldsymbol{B}'\to\boldsymbol{T}^{m'}$, and hence a *bilinear mapping*

$$\mathfrak{B}_{B'}(\Phi^{-1}(M))\times H^r_{\mathrm{rel}}(M', \mathfrak{A}_{\Phi^{-1}(M), B})\longrightarrow\mathfrak{B}_{T^m}(M').$$

Similarly we obtain, for any locally closed $S\subset M$ and any closed subset F of $\Phi^{-1}(S)$, a canonical bilinear mapping

$$\mathfrak{B}_{B'}(\Phi^{-1}(S))\times H^r_{\mathrm{rel}}(F, \mathfrak{A}_{\Phi^{-1}(M), B})\longrightarrow\mathfrak{B}_{T^m}(F).$$

Further, we can clearly replace the first term by $\mathfrak{B}_{B'}(\Phi^{-1}(S)\frown D)$ with any $D\in\mathfrak{D}(F)$. In case F is a compact set K, we can combine the integration $\mathfrak{B}_{T^m}(K)\to\boldsymbol{C}$ with the above bilinear mapping, and obtain an *inner multiplication*

$$\mathfrak{B}_{B'}(\Phi^{-1}(S)\frown D)\times H^r_{\mathrm{rel}}(K, \mathfrak{A}_{\Phi^{-1}(M), B})\longrightarrow\boldsymbol{C}.$$

In this case we have, as a generalization of proposition 10.1.1,

Proposition 10.5.1. *The inductive limit of* $\{\mathfrak{B}_{B'}(\Phi^{-1}(S)\frown D)\,;\ D\in\mathfrak{D}(K)\}$ *and* $H^r_{\mathrm{rel}}(K, \mathfrak{A}_{\Phi^{-1}(M), B})$ *constitute a pair of mutually dual vector spaces by the inner multilpication defined above.*

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Errata.

The author profits by this opportunity to correct some errors in his former paper [8] as follows: '$\mathrm{Dist}^n(S, \mathfrak{F})$' in the expression (9) on p. 606 should be corrected to '$\mathrm{Dist}^{n-p}(S, \mathfrak{F})$'; Proposition 2. (ii) on p. 606 should be supplemented by inserting 'If $\mathrm{Dist}^n(S, \mathfrak{F})=0$ for $n=0, 1, \cdots, m-1$, then' in the beginning; and the description on a generalization of integration on pp. 607-608 should be revised as in 10.4 of the present paper.