Scattering Matrices for Two-body Schrödinger Operators

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Introduction and summary

In this paper we study some properties of the scattering matrix for the Schrödinger operators $H_0=-\frac{1}{2}\Delta=-\frac{1}{2}\sum_{j=1}^{N}\partial^2/\partial x_j^2$ and $H=H_0+V$ in $L^2(R^N)$, $N\geqq 2$, where the potential $V(x)$ is a real-valued function. Let us begin with the short-range case, i.e., we assume $V(x)=0(|x|^{-1-\epsilon_0})$ as $|x|\to\infty$ for some $\epsilon_0>0$. In this case, it is well-known that the wave operators

$$W_{\pm}=s\text{-}\lim_{t\to\pm\infty} e^{itH}e^{-itH_0} \tag{0.1}$$

exist and are complete. The scattering operator S is defined by

$$S=W_+^*W_-, \tag{0.2}$$

which is unitary on $L^2(R^N)$. Because of the conservation law of energy, it is convenient to consider the Fourier transform $\hat{S}$ of S, i.e.,

$$\hat{S}=\mathcal{F}S\mathcal{F}^{-1}, \tag{0.3}$$

where $\mathcal{F}$ denotes the Fourier transformation,

$$\mathcal{F}f(\xi)=\hat{f}(\xi)=(2\pi)^{-N/2}\int e^{-ix\cdot\xi}f(x)dx.$$

$\hat{S}$ is known to be decomposable: for a. e. $\lambda>0$ there exists a unitary operator $\mathcal{S}(\lambda)$ on $L^2(S^{N-1})$ such that for a.e. $\lambda>0$ and $\omega\in S^{N-1}$,

$$(\hat{S}f)(\sqrt{2\lambda}\omega)=(\mathcal{S}(\lambda)f(\sqrt{2\lambda}\cdot))(\omega) \text{ for any } f\in L^2(R^N). \tag{0.4}$$

$\mathcal{S}(\lambda)$ is called the *scattering matrix* or *S-matrix*. The *transition matrix* $\mathcal{T}(\lambda)$ is defined through the relation

$$S(\lambda)=1-2\pi i(2\lambda)^{(N-2)/2}\mathcal{T}(\lambda), \tag{0.5}$$

which is known to be compact on $L^2(S^{N-1})$ (Agmon [2], Kuroda [14]). If $\mathcal{T}(\lambda)$ has an integral kernel $\mathcal{T}(\lambda, \omega, \omega')$, it is called the *scattering amplitude* and $|\mathcal{T}(\lambda, \omega, \omega')|^2$ is called the *differential cross section*, which is the most important quantity in scattering theory in the sense that it is the only one which can be observed through the physical experiment. Thus our first subject should be the existence of the scattering amplitude. Let us assume that

(S) $V(x)$ is a real-valued smooth function on R^N such that for all multi-indices α and some $\varepsilon_0>0$

$$|\partial_x^\alpha V(x)|\leqq C_\alpha\langle x\rangle^{-|\alpha|-1-\varepsilon_0},$$

where $\langle x\rangle=\sqrt{1+|x|^2}$.

THEOREM 0.1. *Under the assumption* (S), $\mathcal{T}(\lambda)$ *is an integral operator with a kernel* $\mathcal{T}(\lambda, \omega, \omega')$ *which is smooth for* $\lambda>0$, $\omega, \omega' \in S^{N-1}$ if $\omega\neq\omega'$.

We should remark here that this result has already been announced by Agmon [1] without proof. Amrein and Pearson [3] have also shown the existence of the scattering amplitude, but did not treat the regularity off the diagonal.

Our next concern is the behavior of the scattering amplitude at the diagonal. If $V(x)$ decays sufficiently rapidly as $|x|\to\infty$, $\mathcal{T}(\lambda, \omega, \omega')$ is known to be regular at the diagonal. More precisely, it is known that if $\varepsilon_0>N-1$, $\mathcal{T}(\lambda, \omega, \omega')$ is continuous on $S^{N-1}\times S^{N-1}$. Our second result asserts that if $V(x)$ decays more slowly, $\mathcal{T}(\lambda, \omega, \omega')$ does have singularities at the diagonal.

THEOREM 0.2. *Let* $0<\varepsilon_0<N-1$. *Then*:

i)
$$|\mathcal{T}(\lambda, \omega, \omega')|\leqq C|\omega-\omega'|^{-N+1+\varepsilon_0},$$

where the constant C *is independent of* λ *varying in a compact set of* $(0, \infty)$.

ii)
$$|\mathcal{T}(\lambda, \omega, \omega')-(2\pi)^{-N/2}\hat{V}(\sqrt{2\lambda}P(\omega-\omega'))|$$
$$\leqq C(|\omega-\omega'|)|\omega-\omega'|^{-N+1+\varepsilon_0},$$

where $\hat{V}$ *is the Fourier transform of* V, $P(x)=x-(x\cdot\omega)\omega$ *and* $C(\rho)\to 0$ *as* $\rho\to 0$.

Two remarks should be added here concerning Theorem 0.2-ii). First, under the assumption (S), $\hat{V}(\xi)$ is smooth for $\xi\neq 0$. Secondly, if $V(x)=|x|^{-1-\varepsilon_0}$ for $|x|>1$, $\hat{V}(\xi)$ behaves like $|\xi|^{-N+1+\varepsilon_0}$ as $|\xi|\to 0$. Since $|P(\omega-\omega')|^{-N+1+\varepsilon_0}$ has the singularity appearing on the right hand side (RHS) of Theorem 0.2-i), Theorem 0.2-ii) shows that the estimate i) is the best possible. From the above theorem, we can derive the following

COROLLARY. *Let* $V(x)$ *be a non trivial smooth homogeneous function of* *degree* $-1-\varepsilon_0$ for $|x|>R$ *for some constant* $R>0$. *Then the total cross section*

$$\iint_{S^{N-1}\times S^{N-1}} |\mathcal{T}(\lambda,\ \omega,\ \omega')|^2 d\omega d\omega'$$

is finite if and only if $\varepsilon_0>(N-1)/2$, *and is infinite if and only if* $0<\varepsilon_0\leqq(N-1)/2$.

Next we turn to the long-range potential. We assume that

(L) $V(x)$ is a real-valued C^∞ function on R^N such that for some $0<\varepsilon<1$ and all multi-indices α

$$|\partial_x^\alpha V(x)|\leqq C_\alpha\langle x\rangle^{-|\alpha|-\varepsilon}.$$

In this case, we have to modify the definition of wave operators. In [8] we have introduced the modified wave operators

$$W_J^\pm = s\text{-}\lim_{t\to\pm\infty} e^{itH} J e^{-itH_0}, \tag{0.6}$$

where J is a Fourier integral operator

$$Jf(x)=\iint e^{i(\varphi(x,\ \xi)-y\cdot\xi)} f(y)dy\bar{d}\xi \qquad (\bar{d}\xi=(2\pi)^{-N}d\xi) \tag{0.7}$$

with a phase function $\varphi(x,\ \xi)$ solving the eikonal equation

$$\frac{1}{2}|\nabla_x\varphi(x,\ \xi)|^2+V(x)=\frac{1}{2}|\xi|^2. \tag{0.8}$$

Our construction of the modified wave operators $W_J^\pm$ is slightly different from the usual one. The usual modified wave operators $W_\pm$ are defined through

$$W_\pm = s\text{-}\lim_{t\to\pm\infty} e^{itH} e^{-iW_t}, \tag{0.9}$$

where

$$e^{-iW_t}f(x)=\iint e^{i[(x-y)\cdot\xi-W(\xi,\ t)]} f(y)dy\bar{d}\xi \tag{0.10}$$

with $W(\xi,\ t)$ solving the Hamilton-Jacobi equation

$$\partial W/\partial t=\frac{1}{2}|\xi|^2+V(\partial W/\partial\xi). \tag{0.11}$$

It seems that, in general, $W_J^\pm$ do not coincide with $W_\pm$:

$$W_J^\pm\neq W_\pm. \tag{0.12}$$

(Our first remark in [8, § 4] seems to be wrong. We are indebted to Prof. K. Yajima for his indication of this point.)

Using these wave operators $W_J^{\pm}$, we can define the S-matrix $\mathcal{S}(\lambda)$ as in the short-range case. In the long-range case, however, it cannot be expected that $\mathcal{S}(\lambda)$ is equal to the identity plus a compact operator on $L^2(S^{N-1})$. Alternatively, regarding $\mathcal{S}(\lambda)$ as a distribution on $S^{N-1}\times S^{N-1}$, we shall show the following

THEOREM 0.3. *The singular support of $\mathcal{S}(\lambda)$ is contained in the diagonal of $S^{N-1}\times S^{N-1}$. Thus $\mathcal{S}(\lambda)$ has an integral kernel $\mathcal{S}(\lambda, \omega, \omega')$ if $\omega\neq\omega'$. This is smooth for $\lambda>0$, ω, $\omega' \in S^{N-1}$ if $\omega\neq\omega'$, and*

$$|\mathcal{S}(\lambda, \omega, \omega')|\leqq C|\omega-\omega'|^{-(N-1)/\varepsilon}, \tag{0.13}$$

where the constant C is independent of λ in a compact set in $(0, \infty)$.

Agmon also announced in [1] this theorem for the usual modified S-matrices $\mathcal{S}_D(\lambda)$ constructed from $W_{\pm}$ in (0.9). However, owing to (0.12), his announcement [1, Theorem 2] might not hold. It is an interesting problem to determine to what extent $W_J^{\pm}$ and $W_{\pm}$ are different. An elementary consideration shows that $\mathcal{S}(\lambda)=U_+(\lambda)^*\mathcal{S}_D(\lambda)U_-(\lambda)$ for some unitary operators $U_{\pm}(\lambda)$ in $L^2(S^{N-1})$. This is quite an interesting open problem.

It is also an interesting problem to calculate the singularities of $\mathcal{S}(\lambda, \omega, \omega')$ at the diagonal. For the Coulomb potential $V(x)=-c/|x|$, this has been done by Herbst [6] and Soffer [17]. For general long-range potentials, it is still an open problem.

Our final result is the reconstruction of the potential from the asymptotic behavior as $\lambda\to\infty$ of the scattering amplitude, which is a long-range version of the results obtained by Fadeev [5], Mochizuki [15] and Saitō [16]. This solves the inverse problem for the long-range potentials.

THEOREM 0.4. *Let $1>\varepsilon>1/2$ and $\xi\neq 0$ be fixed. Then*

$$\lim_{\substack{\lambda\to\infty \\ \sqrt{2\lambda}(\omega-\omega')=\xi}} -(2\pi)^{-(N-2)/2}i(2\lambda)^{(N-2)/2}\mathcal{S}(\lambda, \omega, \omega')=\hat{V}(\xi). \tag{0.14}$$

Since $\hat{V}(\xi)\in L^1(R^N)$ by our assumption (L) and the above convergence is locally uniform for $\xi\neq 0$, we can uniquely reconstruct the potential $V(x)$ from the scattering amplitude. This result has been announced in [11], and we shall give a detailed proof in this paper.

All of our results except for Theorem 0.4 can be extended to potentials with local singularities. Namely, if V is split into two parts: $V=V_1+V_2$, where V_1 satisfies (S) or (L) and V_2 is a compactly supported function belonging to the Stummel class, the above theorems 0.1~0.3 remain valid. For the sake of simplicity, we do not enter into such an argument here, however.

Our plan in this paper is as follows. We postpone the analysis of short-range S-matrices till the final section 6, where Theorem 0.2 will be proved. Theorem 0.1 is included in Theorem 0.3. The fundamental results concerning the eikonal and transport equations will be reviewed from [8] and [10] in section

2. In section 3, we derive a representation formula for the S-matrix $\mathcal{S}(\lambda)$. The proof of Theorems 0.3 and 0.4 will be given in sections 4 and 5, respectively. In these sections, the micro-local estimates for two-body resolvents proved in [9], [10] and [13] will play fundamental roles.

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§ 1. Micro-local estimates for resolvents.

Let $H=-\frac{1}{2}\Delta+V(x)$, where V satisfies the assumption (L), and let $R(z)=(H-z)^{-1}$. We denote by $L_s^2(s\in R^1)$ the weighted L^2 space:

$$L_s^2=L^2(R^N\,;\,\langle x\rangle^{2s}dx), \tag{1.1}$$

and denote its norm $\|\langle x\rangle^s f\|$ by $\|f\|_s$. Then by the limiting absorption principle (see e.g. Theorem 1.2 in [9]), $R(z)$ (Im $z\neq 0$) has a boundary value on $(0, \infty)$ belonging to $B(L_\gamma^2;\, L_{-\gamma}^2)$ for any $\gamma>\frac{1}{2}$:

$$R(\lambda\pm i0)=s\text{-}\lim_{\mu\downarrow 0} R(\lambda\pm i\mu), \qquad \lambda>0. \tag{1.2}$$

Let A_1, A_2 be the Fourier integral operators (F.I. Op.'s) (cf. [13]):

$$A_j f(x)=\iint e^{i(\varphi_j(x,\,\xi)-y\cdot\xi)}\, a_j(x,\,\xi)f(y)dy\bar{d}\xi \tag{1.3}$$

satisfying the following assumptions (1.4) and (1.5).

(1.4) $\varphi_j(x,\,\xi)$ is a real-valued $C^\infty(R^{2N})$-function such that for some constant $\varepsilon_0>0$, for $|\gamma|+|\delta|\leqq n_0(n_0\gg N)$ and for some $0<\tau\ll 1$,

$$\begin{cases} |\partial_x^\alpha\partial_\xi^\beta(\varphi_j(x,\,\xi)-x\cdot\xi)|\leqq C_{\alpha\beta}\langle x\rangle^{1-\varepsilon_0-|\alpha|}, \\ \sup_{x,\,\xi\in R^N}\left|\partial_x^\gamma\partial_\xi^\delta\left[\left(\frac{\partial^2}{\partial x_n\partial\xi_m}\varphi_j(x,\,\xi)\right)-I\right]\right|<\tau, \end{cases}$$

where I is the $N\times N$ identity matrix.

(1.5) $a_j(x,\,\xi)\in C^\infty(R^{2N})$, and there exists a constant $d>0$ such that

$$a_1(x,\,\xi)=a_2(x,\,\xi)=0 \qquad \text{if } |\xi|<d \text{ or } |x|<d.$$

Moreover, there exist constants $-1<\mu_1<\mu_2<1$, $0\leqq\delta<\rho\leqq 1$ and $m\geqq 0$ such that for any $L\geqq 0$

$$|\partial_x^\alpha \partial_\xi^\beta a_1(x, \xi)|$$

$$\leqq \begin{cases} C_{\alpha\beta L}\langle x\rangle^{-L} & \text{if} \quad \cos(x, \xi) \equiv \dfrac{x \cdot \xi}{|x| \cdot |\xi|} > \mu_1, \\ C_{\alpha\beta}\langle x\rangle^{m-\rho|\alpha|+\delta|\beta|} & \text{if} \quad \cos(x, \xi) \leqq \mu_1, \end{cases}$$

$$|\partial_x^\alpha \partial_\xi^\beta a_2(x, \xi)|$$

$$\leqq \begin{cases} C_{\alpha\beta L}\langle x\rangle^{-L} & \text{if} \quad \cos(x, \xi) < \mu_2, \\ C_{\alpha\beta}\langle x\rangle^{m-\rho|\alpha|+\delta|\beta|} & \text{if} \quad \cos(x, \xi) \geqq \mu_2. \end{cases}$$

The following estimate is our basic tool in the study of the scattering matrix.

THEOREM 1.1. *Let A_1, A_2 be as above. Let $\lambda_0>0$ be arbitrarily fixed. Then for any s, $k\geqq 0$, there exists a constant $C>0$ such that for any $\lambda>\lambda_0$*

$$\|A_1^*\left(\frac{d}{d\lambda}\right)^k R(\lambda+i0)A_2 f\|_s \leqq C\lambda^{-(k+1)/2}\|f\|_{-s}. \tag{1.6}$$

Proof. Since the case $k\geqq 1$ can be proved similarly to $k=0$ by the use of Theorem 1.8 in [7], we here prove the case $k=0$ only.

We define I_j by

$$I_j f(x) = \iint e^{i(\varphi_j(x,\xi)-y\cdot\xi)} f(y)\,dy\,\bar{d}\xi, \qquad j=1,2. \tag{1.7}$$

Then we can easily see (cf. [13], Theorem 3.2) that $B_1^*=I_1A_1^*$ and $B_2=A_2I_2^*$ are the pseudodifferential operators with symbols b_j $(j=1,2)$ which satisfies (1.5) with μ_1, μ_2 replaced by some constants $\tilde{\mu}_1$, $\tilde{\mu}_2$ such that $\mu_1<\tilde{\mu}_1<\tilde{\mu}_2<\mu_2$. Further, by virtue of (1.4), I_j and I_j^* have bounded inverses (cf. [13], Theorem 3.3).

Choose constants $\mu_\pm$ such that $\tilde{\mu}_1<\mu_-<\mu_+<\tilde{\mu}_2$ and split $B_1^*=(B_1^*)_++(B_1^*)_-$ and $B_2=(B_2)_++(B_2)_-$ so that

$$\begin{cases} \text{the symbol of } (B_1^*)_-=0 & \text{for } \cos(x, \xi)>\mu_-, \\ \text{the symbol of } (B_2)_+=0 & \text{for } \cos(x, \xi)<\mu_+. \end{cases}$$

Since the symbols of $(B_1^*)_+$ and $(B_2)_-$ are rapidly decreasing in x, we see that $(B_1^*)_+R(\lambda+i0)(B_2)_-$ verifies (1.6) with $k=0$ by [9], Theorem 1.2. We apply Theorems 3.5 and 3.12 in [9] to see that $(B_1^*)_-R(\lambda+i0)(B_2)_-$ and $(B_1^*)_+R(\lambda+i0)(B_2)_+$ have the same property. For $(B_1^*)_-R(\lambda+i0)(B_2)_+$, we apply Theorem 2 in [10]. Thus we have

$$\|B_1^*R(\lambda+i0)B_2 f\|_s \leqq C\lambda^{-1/2}\|f\|_{-s} \tag{1.8}$$

for any $\lambda>\lambda_0$ and $s\geqq 0$. Thus (1.6) follows from this and

$$A_1^* R(\lambda+i0)A_2 = I_1^{-1} B_1^* R(\lambda+i0) B_2 (I_2^*)^{-1}. \tag{1.9}$$ Q. E. D.

In the next section, we construct the above mentioned F.I.Op.'s by solving the eikonal and transport equations.

§ 2. Eikonal and transport equations.

In this section, we review some of the results of [8] and [10] concerning the classical orbits which we shall need later.

Let $\chi(x) \in C^\infty(R^N)$ satisfy $\chi(x)=1$ for $|x| \geqq 2$ and $=0$ for $|x| \leqq 1$. Let

$$V_\rho(t, x) = V(x)\chi(\rho x)\chi(\langle \log \langle t \rangle \rangle x / \langle t \rangle), \quad 0<\rho<1, \tag{2.1}$$

and let $(q, p)(t, s; x, \xi)$ be the classical orbits which satisfy the integrated form of the Hamilton's canonical equation

$$\begin{cases} q(t, s) = y + \int_s^t p(\tau, s)d\tau, \\ p(t, s) = \xi - \int_s^t \nabla_x V(\tau, q(\tau, s))d\tau. \end{cases} \tag{2.2}$$

Then there exist the inverse diffeomorphisms $x \to y(s, t; x, \xi)$ and $\xi \to \eta(t, s; x, \xi)$ of the mappings $y \to x = q(s, t; y, \xi)$ and $\eta \to \xi = p(t, s; x, \eta)$, respectively:

$$\begin{array}{ccc} \text{time } s & & \text{time } t \\ \begin{pmatrix} x \\ \eta(t, s; x, \xi) \end{pmatrix} & \xrightarrow{(q, p)(t, s; x, \eta)} & \begin{pmatrix} y(s, t; x, \xi) \\ \xi \end{pmatrix}. \end{array} \tag{2.3}$$

The estimates of the mappings q, p, y and η are summarized in Propositions 2.1 and 2.2 in [8]. Let

$$\phi(t; x, \xi) = u(t; x, \eta(t, 0; x, \xi)), \tag{2.4}$$

where

$$\begin{cases} u(t; x, \eta) \\ = x \cdot \eta + \int_0^t \{H_\rho - x \cdot \nabla_x H_\rho\}(\tau, 0; x, \eta), p(\tau, 0; x, \eta))d\tau, \\ H_\rho(t, x, \xi) = \frac{1}{2}|\xi|^2 + V_\rho(t, x). \end{cases} \tag{2.5}$$

Then $\partial_t \phi(t; x, \xi) = \frac{1}{2}|\xi|^2 + V_\rho(t, \nabla_\xi \phi(t; x, \xi))$, $\phi(0; x, \xi) = x \cdot \xi$, and

$$\begin{cases} \nabla_x \phi(t; x, \xi) = \eta(t, 0; x, \xi), \\ \nabla_\xi \phi(t; x, \xi) = y(0, t; x, \xi). \end{cases} \tag{2.6}$$

Using these, we have proved in [8] the existence of the limits

$$\phi_\pm(x,\ \xi)=\lim_{t\to\pm\infty}(\phi(t\,;\ x,\ \xi)-\phi(t\,;\ 0,\ \xi)) \tag{2.7}$$

for $(x,\ \xi)\in\Gamma_\pm(R,\ d,\ \sigma_0)=\{(x,\ \xi)\in R^{2N}|\ |x|\geqq R,\ |\xi|\geqq d,\ \pm\cos(x,\ \xi)\geqq-\sigma_0\}$ with d, $\sigma_0\in(0,1)$, and sufficiently large $R>1$. These $\phi_\pm$ satisfy the eikonal equation

$$\frac{1}{2}|\nabla_x\phi_\pm(x,\ \xi)|^2+V(x)=\frac{1}{2}|\xi|^2 \tag{2.8}$$

and the estimates

$$|\partial_x^\alpha\partial_\xi^\beta(\phi_\pm(x,\ \xi)-x\cdot\xi)|\leqq C_{\alpha\beta}|\xi|^{-1}\langle x\rangle^{1-|\alpha|-\epsilon} \tag{2.9}$$

for $(x,\ \xi)\in\Gamma_\pm(R,\ d,\ \sigma_0)$ (see Proposition 2.4 in [8]), if $R\gg1$ and $0<\rho\ll d$. Choose constants σ_0, σ_1 such that $-1<\sigma_0<\sigma_1<1$, and let $\psi_\pm(\sigma)\in C^\infty([-1,\ 1])$ satisfy

$$\begin{cases}0\leqq\psi_\pm(\sigma)\leqq1, \\ \psi_+(\sigma)=\begin{cases}1 & \text{for}\quad \sigma_1\leqq\sigma\leqq1, \\ 0 & \text{for}\quad -1\leqq\sigma\leqq(\sigma_0+\sigma_1)/2,\end{cases} \\ \psi_-(\sigma)=\begin{cases}0 & \text{for}\quad (\sigma_0+\sigma_1)/2\leqq\sigma\leqq1, \\ 1 & \text{for}\quad -1\leqq\sigma\leqq\sigma_0.\end{cases}\end{cases} \tag{2.10}$$

Set

$$\begin{aligned}\varphi(x,\ \xi)=\{&(\phi_+(x,\ \xi)-x\cdot\xi)\psi_+(\cos(x,\ \xi)) \\ &+(\phi_-(x,\ \xi)-x\cdot\xi)\psi_-(\cos(x,\ \xi))\}\chi(4\xi/d)\chi(4x/R)+x\cdot\xi.\end{aligned} \tag{2.11}$$

Then φ satisfies the eikonal equation (2.8) for $(x,\ \xi)\in\Gamma_+(R,\ d/2,\ -\sigma_1)\cup\Gamma_-(R,\ d/2,\ \sigma_0)$, and the estimate (2.9) for all $(x,\ \xi)\in R^{2N}$.

We next consider the transport equation

$$\nabla_x\varphi\cdot\nabla_x a+\frac{1}{2}\Delta_x\varphi\cdot a-\frac{1}{2}i\Delta_x a\equiv0 \qquad (\text{mod } B). \tag{2.12}$$

Here by $f\in B$ we mean that for any $L\geqq1$

$$\begin{aligned}&|\partial_x^\alpha\partial_\xi^\beta f(x,\ \xi)| \\ &\leqq\begin{cases}C_{\alpha\beta L}\langle x\rangle^{-L}\langle\xi\rangle^{-1} & \text{for } \cos(x,\ \xi)\in[-1,\ \sigma_0-\delta]\cup[\sigma_1+\delta,\ 1],\ |\xi|\geqq d,\ |x|\geqq R \\ C_{\alpha\beta}\langle x\rangle^{-1-|\alpha|}\langle\xi\rangle & \text{otherwise,}\end{cases}\end{aligned} \tag{2.13}$$

where $\delta>0$ is a sufficiently small constant. We construct a in the form of an asymptotic series: $a=\sum_{m=0}^\infty a^{(m)}$, $a^{(0)}=1$. Formally (2.12) is equivalent to

(2.14) $$\nabla_x\varphi\cdot\nabla_x a^{(m)}+\frac{1}{2}\Delta_x\varphi\cdot a^{(m-1)}-\frac{1}{2}i\Delta_x a^{(m-1)}\equiv 0 \qquad (\mathrm{mod}\ B)$$

for $m=1,2,\dots$. By virtue of (2.9) and the classical theory of first order partial differential equations, this equation can be solved for $(x, \xi)\in\Gamma\equiv\Gamma_+(R, d, -\sigma_1)\cup\Gamma_-(R, d, \sigma_0)$. The solution $a^{(m)}$ satisfies the estimate

(2.15) $$|\partial_x^\alpha\partial_\xi^\beta a^{(m)}(x, \xi)|\leqq C_{\alpha\beta}\langle\xi\rangle^{-1}\langle x\rangle^{-|\alpha|-\varepsilon m}, \quad m=1,2,\dots,$$

for $(x, \xi)\in\Gamma$. For a suitable choice of a sequence $\{\varepsilon_m\}_{m=0}^\infty$ tending to 0, $b(x, \xi)=\sum_{m=0}^\infty \chi(\varepsilon_m x)a^{(m)}(x, \xi)$ is absolutely convergent for $(x, \xi)\in\Gamma$. We then define $a(x, \xi)$ by

(2.16) $$a(x, \xi)=b(x, \xi)(\tilde{\psi}_+(\cos(x, \xi))+\tilde{\psi}_-(\cos(x, \xi)))\chi(2\xi/d)\chi(2x/R),$$

where $\tilde{\psi}_+(\sigma)=1$ for $\sigma\geqq\sigma_1+\delta$, $=0$ for $\sigma\leqq\sigma_1$ and $0\leqq\tilde{\psi}_+\leqq 1$, and $\tilde{\psi}_-(\sigma)$ is defined similarly. The properties of $\varphi(x, \xi)$ and $a(x, \xi)$ constructed above are summarized in the following

Theorem 2.1. *Let* $-1<\sigma_0<\sigma_1<1$, $d>0$ *and* $0<\delta\ll 1$. *Then:*

i) *For sufficiently large* $R>2$ *and small* $0<\rho\ll d$, φ *solves the eikonal equation*

(2.17) $$\frac{1}{2}|\nabla_x\varphi(x, \xi)|^2+V(x)=\frac{1}{2}|\xi|^2$$

for $|\xi|\geqq d/2$, $\cos(x, \xi)\in[-1, \sigma_0]\cup[\sigma_1, 1]$ *and* $|x|\geqq R/2$.

ii) *For any* $(x, \xi)\in R^{2N}$ *and* α, β, φ *satisfies the estimate*

(2.18) $$|\partial_x^\alpha\partial_\xi^\beta(\varphi(x, \xi)-x\cdot\xi)|\leqq C_{\alpha\beta}\langle x\rangle^{1-\varepsilon-|\alpha|}\langle\xi\rangle^{-1}.$$

Furthermore

(2.19) $$\varphi(x, \xi)=x\cdot\xi \qquad \textit{for} \quad |x|\leqq R/4 \quad \textit{or} \quad |\xi|\leqq d/4.$$

iii) *For any* $(x, \xi)\in R^{2N}$ *and* α, β, a *satisfies the estimate*

(2.20) $$\begin{cases} |\partial_x^\alpha\partial_\xi^\beta a(x, \xi)|\leqq C_{\alpha\beta}\langle x\rangle^{-|\alpha|}, \\ |\partial_x^\alpha\partial_\xi^\beta(a(x, \xi)-1)|\leqq C_{\alpha\beta}\langle x\rangle^{-|\alpha|-\varepsilon}\langle\xi\rangle^{-1} \\ \qquad \textit{for} \quad \cos(x, \xi)\in[-1, \sigma_0-\delta]\cup[\sigma_1+\delta, 1],\ |\xi|\geqq d,\ |x|\geqq R, \\ a(x, \xi)=0 \ \textit{for} \ \cos(x, \xi)\in[\sigma_0, \sigma_1] \ \textit{or} \ |x|\leqq R/2 \ \textit{or} \ |\xi|\leqq d/2. \end{cases}$$

iv) *Set*

(2.21) $$t(x, \xi)=e^{-i\varphi(x,\xi)}\left(-\frac{1}{2}\Delta+V(x)-\frac{1}{2}|\xi|^2\right)e^{i\varphi(x,\xi)}a(x, \xi).$$

Then for any $L\geqq 1$

$$|\partial_x^\alpha \partial_\xi^\beta t(x, \xi)| \leqq \begin{cases} C_{\alpha\beta L}\langle x\rangle^{-L}\langle\xi\rangle \\ \qquad \textit{for } \cos(x, \xi)\in[-1, \sigma_0-\delta]\cup[\sigma_1+\delta, 1],\ |\xi|\geqq d,\ |x|\geqq R, \\ C_{\alpha\beta}\langle x\rangle^{-1-|\alpha|}\langle\xi\rangle \quad \textit{otherwise.} \end{cases} \tag{2.22}$$

For the proof, see Theorem 2.5 in [8] and Theorem 2.3 in [10].

We next set

$$V_n(x)=V(x)\chi_0(x/n), \tag{2.23}$$

where $\chi_0\in C_0^\infty(R^N)$ with $\chi_0(x)=1$ for $|x|\leqq 1$ and $=0$ for $|x|\geqq 2$. Let φ^n, a^n and t^n be defined as in Theorem 2.1 with $V=V_n$. They satisfy the estimates in Theorem 2.1 uniformly in n. Moreover, we have

THEOREM 2.2. i) *There exist a real-valued function* $w^n(\xi)$ *and a constant* $A_n>n$ *such that*

$$\varphi^n(x, \xi)=x\cdot\xi-w^n(\xi) \tag{2.24}$$

for $|x|\geqq A_n$, $|\xi|\geqq d$ *and* $\cos(x, \xi)\in[-1, \sigma_0]\cup[\sigma_1, 1]$.

ii) *There exist constants* $C_{\alpha\beta}$ *such that for any* n *and* $L\geqq 1$

$$\begin{cases} |\partial_x^\alpha \partial_\xi^\beta(\varphi(x, \xi)-\varphi^n(x, \xi))|\leqq C_{\alpha\beta}n^{-\varepsilon_\alpha}\langle x\rangle^{1-\varepsilon_0-|\alpha|}\langle\xi\rangle^{-1}, \\ |\partial_x^\alpha \partial_\xi^\beta(a(x, \xi)-a^n(x, \xi))|\leqq C_{\alpha\beta}n^{-\varepsilon_\alpha}\langle x\rangle^{-\varepsilon_0-|\alpha|}\langle\xi\rangle^{-1}, \\ |\partial_x^\alpha \partial_\xi^\beta(t(x, \xi)-t^n(x, \xi))| \\ \leqq \begin{cases} C_{\alpha\beta L}n^{-\varepsilon_\alpha}\langle x\rangle^{-L}\langle\xi\rangle^{-1} \\ \qquad \text{for } \cos(x, \xi)\in[-1, \sigma_0-\delta]\cup[\sigma_1+\delta, 1],\ |\xi|\geqq d,\ |x|\geqq R, \\ C_{\alpha\beta}n^{-\varepsilon_\alpha}\langle x\rangle^{-1-\varepsilon_0-|\alpha|}\langle\xi\rangle^{-1} \quad \text{otherwise,} \end{cases} \end{cases} \tag{2.25}$$

where $0<\varepsilon-\varepsilon_0\ll 1$ and $0<\varepsilon_\alpha<\varepsilon_0$.

Proof. i) Omitting the subscript n, we have only to show

$$\nabla_x\varphi(x, \xi)=\xi \tag{2.26}$$

for $|x|\geqq A_n$, $|\xi|\geqq d$ and $\cos(x, \xi)\in[-1, \sigma_0]\cup[\sigma_1, 1]$. By the construction of φ, it suffices to show $\nabla_x\phi_+(x, \xi)=\xi$ for $|x|\geqq A_n$, $|\xi|\geqq d$ and $\cos(x, \xi)\in[\sigma_1, 1]$, since ϕ_- can be treated similarly. By (2.3) and (2.7), we have

$$\nabla_x\phi_+(x, \xi)=\lim_{t\to\infty}\eta(t, 0, x, \xi) \tag{2.27}$$

$$=\lim_{t\to\infty} p(0, t; y(0, t; x, \xi), \xi)$$

$$=\xi+\int_0^\infty (\nabla_x V_\rho)(\tau, q(\tau, 0; x, \eta(\infty, 0; x, \xi)))d\tau.$$

Since $|q(\tau, 0; x, \eta(\infty, 0; x, \xi))|\geqq C(|x|+\tau|\xi|)$ for $\cos(x, \xi)\in[\sigma_1, 1]$ and $|x|$ large enough (see e.g. (2.23) in [8]), (2.24) follows from the assumption that $V(x)=0$ for $|x|\geqq n$.

ii) We show the first estimate in (2.25). The others follow from this by careful re-examination of our construction of a and t. We denote the quantities corresponding to V_n by adding the sub- or superscript n. Then using the abbreviation $\bar{q}(\tau, \theta)=q(\tau, 0; \theta x, \eta(\infty, 0; \theta x, \xi))$ and $\bar{q}_n(\tau, \theta)=q_n(\tau, 0; \theta x, \eta_n(\infty, 0; \theta x, \xi))$, we have in view of (2.27) and (2.23).

$$\text{(2.28)}\qquad \{\phi_+(x, \xi)-\phi_+(0, \xi)\}-\{\phi_+^n(x, \xi)-\phi_+^n(0, \xi)\}$$

$$=x\cdot\int_0^1 d\theta\int_0^\infty d\tau\{(\nabla_x V_\rho)(\tau, \bar{q}(\tau, \theta))-(\nabla_x V_\rho)(\tau, \bar{q}_n(\tau, \theta))\}$$

$$=x\cdot\int_0^1 d\theta\int_0^\infty d\tau[\{1-\chi_0(\bar{q}(\tau, \theta)/n)\}(\nabla_x V_\rho)(\tau, \bar{q}(\tau, \theta))$$

$$+\chi_0(\bar{q}(\tau, \theta)/n)(\nabla_x V_\rho)(\tau, \bar{q}(\tau, \theta))$$

$$-\chi_0(\bar{q}_n(\tau, \theta)/n)(\nabla_x V_\rho)(\tau, \bar{q}_n(\tau, \theta))$$

$$-V_\rho(\tau, \bar{q}_n(\tau, \theta))n^{-1}(\nabla_x\chi_0)(\bar{q}_n(\tau, \theta)/n)].$$

The RHS is seen to satisfy the first estimate in (2.25) for $\cos(x, \xi)\in[\sigma_0, 1]$. But by (2.7), $\phi_+(0, \xi)=\phi_+^n(0, \xi)=0$, which implies (2.25) with $\varphi=\phi_+$ and $\varphi^n=\phi_+^n$. ϕ_- can be treated similarly, and the desired estimate for $\varphi-\varphi^n$ follows from (2.11) and interpolation. Q. E. D.

For our later purposes (section 6), we consider the short-range potentials satisfying the assumption (S) in the introduction. In this case, letting $\varphi(x, \xi)=x\cdot\xi$, we solve the transport equation

$$\text{(2.29)}\qquad \xi\cdot\nabla_x a+iV\cdot a-\frac{1}{2}i\Delta a\equiv 0 \qquad (\text{mod } B).$$

If we assume that a has the form $\sum_{m=0}^\infty a^{(m)}$, $a^{(0)}=1$, (2.29) is equivalent to

$$\text{(2.30)}\qquad \xi\cdot\nabla a^{(m)}+iV\cdot a^{(m-1)}-\frac{1}{2}i\Delta a^{(m-1)}\equiv 0 \qquad (\text{mod } B)$$

for $m=1, 2, \cdots$. We then have for $\cos(x, \xi)>-1+\delta$, $0<\delta<\frac{1}{4}$, $|\xi|\geqq d$,

$$\text{(2.31)}\qquad a^{(m)}(x, \xi)=-\frac{1}{i|\xi|}\int_0^\infty\{V(x+\hat{\xi}t)a^{(m-1)}(x+\hat{\xi}t, \xi)$$

$$-\frac{1}{2}\Delta a^{(m-1)}(x+\hat{\xi}t,\ \xi)\}dt, \qquad (\hat{\xi}=\xi/|\xi|),$$

and

$$|\partial_x^\alpha \partial_\xi^\beta a^{(m)}(x,\ \xi)| \leqq C_{\alpha\beta}|\xi|^{-1-|\beta|}\langle x\rangle^{-|\alpha|-m\varepsilon_0}. \tag{2.32}$$

In particular,

$$a^{(1)}(x,\ \xi)=-\frac{1}{i|\xi|}\int_0^\infty V(x+\hat{\xi}t)dt. \tag{2.23}$$

Taking a suitable sequence $\{\varepsilon_m\}$ tending to 0, we define

$$a(x,\ \xi)=\sum_{m=0}^\infty \chi(\varepsilon_m x)a^{(m)}(x,\ \xi)\chi(\xi/d)\chi(x/R) \tag{2.34}$$

for $\cos(x,\ \xi)>-1+\delta$, $|\xi|\geqq d$. Taking ρ_1, $\rho_2\in C^\infty([-1,\ 1])$ such that $\rho_2(\sigma)=1$ for $\sigma>2\delta$, $=0$ for $\sigma<\delta$, and $\rho_1(\sigma)=1$ for $\sigma>-1+2\delta$, $=0$ for $\sigma<-1+\delta$, we define

$$\begin{cases}\chi_1(x,\ \xi)=\rho_1\ (\cos(x,\ \xi)), \\ \chi_2^\pm(x,\ \xi)=\rho_2(\pm\cos(x,\ \xi)),\end{cases} \tag{2.35}$$

$$a_1(x,\ \xi)=a(x,\ \xi)\chi_1(x,\ \xi), \tag{2.36}$$

$$a_2(x,\ \xi)=a(x,\ \xi)\chi_2^+(x,\ \xi)+\overline{a(x,\ -\xi)}\chi_2^-(x,\ \xi), \tag{2.37}$$

$$t(x,\ \xi)=e^{-ix\cdot\xi}\left(-\frac{1}{2}\Delta+V(x)-\frac{1}{2}|\xi|^2\right)e^{ix\cdot\xi}a(x,\ \xi), \tag{2.38}$$

$$t_j(x,\ \xi)=e^{-ix\cdot\xi}\left(-\frac{1}{2}\Delta+V(x)-\frac{1}{2}|\xi|^2\right)e^{ix\cdot\xi}a_j(x,\ \xi), \qquad j=1,\ 2. \tag{2.39}$$

Then we easily have

Proposition 2.3. *$t_2(x,\ \xi)$ can be written as*

$$\begin{aligned} t_2(x,\ \xi) &= t_2^{(1)}(x,\ \xi)-t_2^{(2)}(x,\ \xi) \\ &= \{\chi_2^+(x,\ \xi)t(x,\ \xi)+\chi_2^-(x,\ \xi)\overline{t(x,\ -\xi)}\} \\ &\qquad -\{(i\xi\cdot\nabla\chi_2^+(x,\ \xi))a(x,\ \xi)+(i\xi\cdot\nabla\chi_2^-(x,\ \xi))\overline{a(x,\ -\xi)} \\ &\qquad +\nabla\chi_2^+(x,\ \xi)\cdot\nabla a(x,\ \xi)+\nabla\chi_2^-(x,\ \xi)\cdot\overline{\nabla a(x,\ -\xi)} \\ &\qquad +\frac{1}{2}(\Delta\chi_2^+(x,\ \xi))a(x,\ \xi)+\frac{1}{2}(\Delta\chi_2^-(x,\ \xi))\overline{a(x,\ -\xi)}\}. \end{aligned} \tag{2.40}$$

Further we have for any $L\geqq 1$

(2.41)
$$|\partial_x^\alpha \partial_\xi^\beta t_1(x,\xi)| \leqq \begin{cases} C_{\alpha\beta L}\langle x\rangle^{-L}\langle\xi\rangle^{1-|\beta|} & \text{for } \cos(x,\xi)\in[-1,-1+\delta]\cup[-1+2\delta,1], \\ C_{\alpha\beta}\langle x\rangle^{-1-|\alpha|}\langle\xi\rangle^{1-|\beta|} & \text{othewise,} \end{cases}$$

and

(2.42)
$$|\partial_x^\alpha \partial_\xi^\beta t_2(x,\xi)| \leqq \begin{cases} C_{\alpha\beta L}\langle x\rangle^{-L}\langle\xi\rangle^{1-|\beta|} & \text{for } \cos(x,\xi)\in[-1,-2\delta]\cup[2\delta,1], \\ C_{\alpha\beta}\langle x\rangle^{-1-|\alpha|}\langle\xi\rangle^{1-|\beta|} & \text{otherwise.} \end{cases}$$

§ 3. A representation formula of the scattering matrix.

In this section we shall prove a representation formula for the scattering matrix $\mathcal{S}(\lambda)$.

In Theorem 2.1, σ_0 and σ_1 are arbitrary as far as $-1<\sigma<\sigma_1<1$. We choose σ_0^j and σ_1^j $(j=1, 2)$ such that $-1<\sigma_0^1-\delta<\sigma_0^1<\sigma_1^1<\sigma_1^1+\delta<0<\sigma_0^2-\delta<\sigma_0^2<\sigma_1^2<\sigma_1^2+\delta<1$, and denote the corresponding φ, a and t by φ_j, a_j and t_j. We then define the Fourier integral operators J_1 and J_2 by

$$J_jf(x)=\iint e^{i(\varphi_j(x,\xi)-y\cdot\xi)}a_j(x,\xi)f(y)dyd\!\!{}^-\xi, \qquad j=1, 2, \tag{3.1}$$

for $f\in\mathcal{S}=\mathcal{S}(R^N)$, the space of rapidly decreasing functions. We further set

$$T_j=HJ_j-J_jH_0. \tag{3.2}$$

A simple calculation shows

$$T_jf(x)=\iint e^{i(\varphi_j(x,\xi)-y\cdot\xi)}t_j(x,\xi)f(y)dyd\!\!{}^-\xi. \tag{3.3}$$

We define $\mathcal{F}_0(\lambda)\in B(L_\gamma^2, L^2(S^{N-1}))$, $\gamma>\dfrac{1}{2}$, $\lambda>0$, by

$$(\mathcal{F}_0(\lambda)f)(\omega)=(2\lambda)^{(N-2)/4}(\mathcal{F}f)(\sqrt{2\lambda}\omega), \tag{3.4}$$

where $\mathcal{F}$ denotes the Fourier transformation. Then as is well-known (see e.g. [12]),

LEMMA 3.1. *Let $R_0(z)=(H_0-z)^{-1}$, $Im\ z\neq 0$. Then for any $f\in L_\gamma^2$, $\gamma>\dfrac{1}{2}$, $\lambda>0$*

$$(R_0(\lambda+i0)-R_0(\lambda-i0))f=2\pi i\mathcal{F}_0(\lambda)^*\mathcal{F}_0(\lambda)f. \tag{3.5}$$

We prepare one more lemma.

LEMMA 3.2. *For any $s>0$, $k\geqq 0$, there exists a constant C such that for $\lambda>d^2/2$*

$$\|\langle D\rangle^{-1}T_1^*\left(\frac{d}{d\lambda}\right)^k R(\lambda+i0)T_2\langle D\rangle^{-1}f\|_s\leqq C\lambda^{-(1+k)/2}\|f\|_{-s}. \tag{3.6}$$

Proof. Theorem 2.1 implies that $T_j\langle D\rangle^{-1}$ $(j=1, 2)$ are the F.I.Op.'s satisfying (1.4) and (1.5). Thus the lemma follows from Theorem 1.1. Q. E. D.

Our main theorem in this section is the following

THEOREM 3.3. *For a. e. $\lambda>d^2/2$, the scattering matrix $\mathcal{S}(\lambda)$ can be represented as*

$$\begin{aligned}\mathcal{S}(\lambda)-I=&-2\pi i\mathcal{F}_0(\lambda)J_1^*T_2\mathcal{F}_0(\lambda)^*\\&+2\pi i\mathcal{F}_0(\lambda)T_1^*R(\lambda+i0)T_2\mathcal{F}_0(\lambda)^*.\end{aligned} \tag{3.7}$$

Here d is the constant in Theorem 2.1.

Proof. We first note that the second term on the RHS of (3.7) is a well-defined bounded operator in $L^2(S^{N-1})$ by Lemma 3.2. Thus if (3.7) is proved in some weak topology of $L^2(S^{N-1})$, the first term on the RHS of (3.7) turns out to be a bounded operator in $L^2(S^{N-1})$, hence the expression (3.7) has a definite meaning.

Take $\Gamma\subset[d^2/2, \infty)$ and let

$$W_j^{\pm}(\Gamma)=s\text{-}\lim_{t\to\pm\infty} e^{itH}J_je^{-itH_0}E_{H_0}(\Gamma). \qquad j=1, 2, \tag{3.8}$$

where $E_{H_0}(\Gamma)$ is the resolution of the identity for H_0. By Theorem 1.1 in [8], (2.20), and the stationary phase method, $W_j^{\pm}(\Gamma)$'s exist; are partial isometries on $\mathcal{H}$; satisfy the intertwining property; and are identical. Thus we have, letting $S(\Gamma)=E_{H_0}(\Gamma)SE_{H_0}(\Gamma)$,

$$S(\Gamma)=W_2^+(\Gamma)^*W_2^-(\Gamma)=W_1^+(\Gamma)^*W_2^-(\Gamma). \tag{3.9}$$

Since $W_1^+(\Gamma)^*W_2^+(\Gamma)=E_{H_0}(\Gamma)$,

$$(S(\Gamma)-I)E_{H_0}(\Gamma)=W_1^+(\Gamma)^*(W_2^-(\Gamma)-W_2^+(\Gamma)). \tag{3.10}$$

Taking note of the relation

$$W_2^+(\Gamma)-W_2^-(\Gamma)=i\int_{-\infty}^{\infty} e^{itH}T_2e^{-itH_0}dtE_{H_0}(\Gamma) \tag{3.11}$$

and the intertwining property of $W_1^+(\Gamma)$, we have for f, $g\in\mathcal{D}\equiv\mathcal{S}(R^N)\cap E_{H_0}(\Gamma)L^2(R^N)$,

$$\begin{aligned}&((S(\Gamma)-I)E_{H_0}(\Gamma)f, g)\\&=((W_2^-(\Gamma)-W_2^+(\Gamma))f, W_1^+(\Gamma)g)\end{aligned} \tag{3.12}$$

$$=-i\int_{-\infty}^{\infty}(T_2e^{-itH_0}f,\ W_1^+(\Gamma)e^{-itH_0}g)dt.$$

Here we have used the fact that $\|T_je^{-itH_0}f\|\leqq C_{k,f}\langle t\rangle^{-k}$ for $f\in\mathcal{D}$, $t\in R^1$ and any $k\geqq 1$, which follows from Lemma 1.1 of [10] and (2.22). By a similar reasoning, we have

$$\begin{aligned}(3.13)\qquad &((S(\Gamma)-I)E_{H_0}(\Gamma)f,\ g)\\ &=i\int_0^{\infty}i\int_{-\infty}^{\infty}(T_2e^{-i\tau H_0}f,\ e^{i\sigma H}T_1e^{-i\sigma H_0}e^{-i\tau H_0}g)d\tau d\sigma\\ &\quad -i\int_{-\infty}^{\infty}(T_2e^{-i\tau H_0}f,\ J_1e^{-i\tau H_0}g)d\tau\\ &=\lim_{\mu,\mu'\downarrow 0}\Big[i\int_0^{\infty}e^{-\mu\sigma}i\int_{-\infty}^{\infty}e^{-\mu'|\tau|}(e^{i(\tau+\sigma)H_0}T_1^*e^{-i\sigma H}T_2e^{-i\tau H_0}f,\ g)d\tau d\sigma\\ &\quad -i\int_{-\infty}^{\infty}e^{-\mu'|\tau|}(e^{i\tau H_0}J_1^*T_2e^{-i\tau H_0}f,\ g)d\tau\Big].\end{aligned}$$

We now pass to the spectral representation for H_0. Let $\hat{\mathcal{H}}$ be the Hilbert space $L^2((0,\infty);L^2(S^{N-1}))$ and define $\mathcal{F}_0\in B(L^2(R^N);\hat{\mathcal{H}})$ by

$$(3.14)\qquad (\mathcal{F}_0f)(\lambda,\omega)=(\mathcal{F}_0(\lambda)f)(\omega)\ \text{ for } f\in C_0^{\infty}(R^N).$$

Then $\mathcal{F}_0$ is uniquely extended to a unitary operator and for $f,g\in C_0^{\infty}(R^N)$

$$(3.15)\qquad (f,\ g)=(\mathcal{F}_0f,\mathcal{F}_0g)_{\hat{\mathcal{H}}}=\int_0^{\infty}(\mathcal{F}_0(\lambda)f,\ \mathcal{F}_0(\lambda)g)_{L^2(S^{N-1})}d\lambda.$$

(See (3.5).) Therefore we have

$$\begin{aligned}(3.16)\qquad &(e^{i(\tau+\sigma)H_0}T_1^*e^{-i\sigma H}T_2e^{-i\tau H_0}f,\ g)\\ &=\int_0^{\infty}([\mathcal{F}_0T_1^*e^{-i\sigma(H-\lambda)}T_2e^{-i\tau(H_0-\lambda)}f](\lambda,\cdot),\ \mathcal{F}_0(\lambda)g(\cdot))_{L^2(S^{N-1})}d\lambda.\end{aligned}$$

Hence we have using Lemma 3.2 and $i\int_0^{\infty}e^{-\mu\sigma}e^{-i\sigma(H-\lambda)}d\sigma=R(\lambda+i\mu)$

$$\begin{aligned}(3.17)\qquad &\lim_{\mu\downarrow 0}i\int_0^{\infty}e^{-\mu\sigma}(e^{i(\tau+\sigma)H_0}T_1^*e^{-i\sigma H}T_2e^{-i\tau H_0}f,\ g)d\sigma\\ &=-\int_0^{\infty}([\mathcal{F}_0T_1^*R(\lambda+i0)T_2e^{-i\tau(H_0-\lambda)}f](\lambda,\cdot),\ \mathcal{F}_0(\lambda)g(\cdot))_{L^2(S^{N-1})}d\lambda.\end{aligned}$$

Again calculating $\lim_{\mu'\downarrow 0}\int_{-\infty}^{\infty}e^{-\mu'|\tau|}\cdots d\tau$, we see that the first term on the RHS of (3.13) is equal to

$$(3.18)\qquad 2\pi i\int_0^{\infty}([\mathcal{F}_0T_1^*R(\lambda+i0)T_2E_0'(\lambda)f](\lambda,\ \cdot),\ \mathcal{F}_0(\lambda)g(\cdot))_{L^2(S^{N-1})}d\lambda,$$

where $E_0'(\lambda)=(2\pi i)^{-1}(R_0(\lambda+i0)-R_0(\lambda-i0))=\mathcal{F}_0(\lambda)^*\mathcal{F}_0(\lambda)$. Lemma 3.2 implies that (3.18) is equal to

$$(3.19)\qquad 2\pi i\int_0^\infty (\mathcal{F}_0(\lambda)T_1^*R(\lambda+i0)T_2\mathcal{F}_0(\lambda)^*\mathcal{F}_0(\lambda)f,\ \mathcal{F}_0(\lambda)g)_{L^2(S^{N-1})}d\lambda.$$

Similarly, the second term on the RHS of (3.13) can be rewritten as

$$(3.20)\qquad -2\pi i\int_0^\infty (\mathcal{F}_0(\lambda)J_1^*T_2\mathcal{F}_0(\lambda)^*\mathcal{F}_0(\lambda)f,\ \mathcal{F}_0(\lambda)g)_{L^2(S^{N-1})}d\lambda.$$

We have thus completed the proof of the theorem. Q. E. D.

§ 4. Proof of Theorem 0.3.

We first note that the C^∞ function $\psi(x\,;\,\lambda,\,\omega)=\langle D_x\rangle e^{i\sqrt{2\lambda}\omega\cdot x}$ satisfies

$$(4.1)\qquad |\partial_\lambda^k\partial_\omega^\alpha\psi(x\,;\,\lambda,\,\omega)|\leqq C_{k\alpha}\lambda^{(1-2k+|\alpha|)/2}\langle x\rangle^{k+|\alpha|}.$$

Lemma 3.2 then implies that the function

$$(4.2)\qquad A(x\,;\,\lambda,\,\omega,\,\omega')$$
$$=\overline{\psi(x\,;\,\lambda,\,\omega)}[\langle D\rangle^{-1}T_1^*R(\lambda+i0)T_2\langle D\rangle^{-1}\psi(\cdot\,;\,\lambda,\,\omega')](x)$$

satisfies for any $L\geqq 1$

$$(4.3)\qquad |\partial_\lambda^k\partial_\omega^\alpha\partial_{\omega'}^\beta A(x\,;\,\lambda,\,\omega,\,\omega')|\leqq C\lambda^{(1-k+|\alpha|+|\beta|)/2}\langle x\rangle^{-L}.$$

Since for $\varphi\in L^2(S^{N-1})$

$$(4.4)\qquad \mathcal{F}_0(\lambda)T_1^*R(\lambda+i0)T_2\mathcal{F}_0(\lambda)^*\varphi(\omega)$$
$$=(2\pi)^{-N}(2\lambda)^{(N-2)/2}\iint A(x\,;\,\lambda,\,\omega,\,\omega')\varphi(\omega')d\omega'dx,$$

(4.3) implies that the second term on the RHS of (3.7) has a C^∞ integral kernel. The first term in (3.7) has the kernel $K(\lambda,\,\omega,\,\omega')$ such that

$$(4.5)\qquad i(2\pi)^{1-N}(2\lambda)^{(2-N)/2}K(\lambda,\,\omega,\,\omega')$$
$$=\int e^{-i(\varphi_1(x,\sqrt{2\lambda}\omega)-\varphi_1(x,\sqrt{2\lambda}\omega'))}q(x,\,\lambda,\,\omega,\,\omega')dx$$

where

$$(4.6)\qquad q(x,\,\lambda,\,\omega,\,\omega')=e^{i(\varphi_2(x,\sqrt{2\lambda}\omega')-\varphi_1(x,\sqrt{2\lambda}\omega'))}$$
$$\times\overline{a_1(x,\,\sqrt{2\lambda}\omega)}t_2(x,\,\sqrt{2\lambda}\omega').$$

By Theorem 2.1, q satisfies

(4.7) $$|\partial_x^\alpha q(x, \lambda, \omega, \omega')| \leqq C_\alpha \langle x \rangle^{-1-\varepsilon|\alpha|}\sqrt{\lambda}.$$

Set

(4.8) $$\Phi = \Phi(x, \lambda, \omega, \omega') = \varphi_1(x, \sqrt{2\lambda}\omega) - \varphi_1(x, \sqrt{2\lambda}\omega').$$

In Theorem 2.1, taking R large enough, we have

(4.9) $$|\nabla_x \nabla_\xi \varphi_1(\xi, x, \eta) - I| < \frac{1}{2},$$

where I is the $N \times N$ identity matrix and

(4.10) $$\nabla_\xi \varphi_1(\xi, x, \eta) = \int_0^1 (\nabla_\xi \varphi_1)(x, \eta + \theta(\xi - \eta)) d\theta.$$

Noting

(4.11) $$\Phi = \sqrt{2\lambda}(\omega - \omega') \cdot \nabla_\xi \varphi_1(\sqrt{2\lambda}\omega, x, \sqrt{2\lambda}\omega')$$

and using (4.9), we make a change of variable $y = \nabla_\xi \varphi_1(\sqrt{2\lambda}\omega, x, \sqrt{2\lambda}\omega')$ in (4.5). Then

(4.12) $$K(\lambda, \omega, \omega') = c \int e^{-i\sqrt{2\lambda}(\omega - \omega')\cdot y} \tilde{q}(y, \lambda, \omega, \omega') dy$$

for some function $\tilde{q}$ which satisfies (4.7) by Theorem 2.1. Thus by integration by parts it follows that $K(\lambda, \omega, \omega')$ is C^∞ when $\omega \neq \omega'$.

In order to estimate the behavior of $K(\lambda, \omega, \omega')$ near $\omega = \omega'$, it now suffices to prove the following

LEMMA 4.1. *Let $f(x, \xi) \in C^\infty(R^N \times R^N)$ satisfy*

(4.13) $$|\partial_x^\alpha f(x, \xi)| \leqq C_\alpha \langle x \rangle^{-\nu-\delta|\alpha|}$$

for some $0 < \nu < N$ and $0 < \delta \leqq 1$, and set

(4.14) $$\varphi(\xi) = \int e^{-ix\cdot\xi} f(x, \xi) dx.$$

Then

(4.15) $$|\varphi(\xi)| \leqq C|\xi|^{-(N-\nu)/\delta} \qquad \text{as} \quad |\xi| \to 0.$$

In fact Theorem 0.3 follows from (4.7) with $q = \tilde{q}$ and (4.12) by virtue of Lemma 4.1 with $f = \tilde{q}$, $\nu = 1$ and $\delta = \varepsilon$.

Proof of Lemma 4.1. Choose $\chi_0(t)$, $\chi_\infty(t) \in C^\infty(R^1)$ such that $\chi_0(t) = 1$ for $t \leqq 1$, $= 0$ for $t \geqq 2$ and

(4.16) $$\chi_0(t)+\chi_\infty(t)=1.$$

Set for $\sigma>0$

(4.17) $$\begin{cases}\varphi_0(\xi)=\int e^{-ix\cdot\xi}\chi_0(|x||\xi|^\sigma)f(x,\ \xi)dx,\\ \varphi_\infty(\xi)=\int e^{-ix\cdot\xi}\chi_\infty(|x||\xi|^\sigma)f(x,\ \xi)dx.\end{cases}$$

Then $\varphi(\xi)=\varphi_0(\xi)+\varphi_\infty(\xi)$. $\varphi_0(\xi)$ is estimated as

(4.18) $$\begin{aligned}|\varphi_0(\xi)|&\leqq C\int_{|x|\leqq 2|\xi|^{-\sigma}}(1+|x|)^{-\nu}dx\\ &\leqq C\int_0^{2|\xi|^{-\sigma}}(1+r)^{-\nu+N-1}dr\\ &\leqq C|\xi|^{-\sigma(N-\nu)}\end{aligned}$$

near $|\xi|=0$. For φ_∞, by integration by parts, we have

(4.19) $$|\varphi_\infty(\xi)|\leqq C|\xi|^{-|\alpha|}\int|\partial_x^\alpha(\chi_\infty(|x||\xi|^\sigma)f(x,\ \xi))|dx.$$

Since

(4.20) $$|\partial_x^\beta(\chi_\infty(|x||\xi|^\sigma))|\leqq C_\beta\langle x\rangle^{-|\beta|},$$

we have for any α

(4.21) $$|\partial_x^\alpha(\chi_\infty(|x||\xi|^\sigma)f(x,\ \xi))|\leqq C_\alpha\langle x\rangle^{-\nu-\delta|\alpha|}.$$

Inserting this into (4.19), we have

(4.22) $$\begin{aligned}|\varphi_\infty(\xi)|&\leqq C|\xi|^{-|\alpha|}\int_{|x|\geqq|\xi|^{-\sigma}}\langle x\rangle^{-\nu-\delta|\alpha|}dx\\ &\leqq C|\xi|^{-|\alpha|-\sigma(N-\nu-\delta|\alpha|)}\end{aligned}$$

for ξ near $|\xi|=0$, if $-\nu-\delta|\alpha|+N-1<-1$. Choosing $\sigma=\delta^{-1}$, we obtain (4.15).

Q. E. D.

§5. An approximation by short-range S-matrices.

In this section, we assume $1/2<\varepsilon_0<\varepsilon<1$. We set as in section 2, (2.23)

(5.1) $$V_n(x)=V(x)\chi_0(x/n)$$

and $H_n=H_0+V_n(x)$. We define J_j^n and $T_j^n(j=1,2,\ n=1,2,\cdots)$ by (3.1) and (3.3) with $\varphi_j=\varphi_j^n$, $a_j=a_j^n$ and $t_j=t_j^n$, where φ_j^n, a_j^n and t_j^n are constructed as in Theorem 2.1 with $V=V_n$. Then we can construct the scattering amplitude

$\mathcal{S}_n(\lambda, \omega, \omega')$ for H_n as in sections 3 and 4. Our purpose in this section is to show that $\mathcal{S}_n(\lambda, \omega, \omega')$ approximates $\mathcal{S}(\lambda, \omega, \omega')$ as $n\to\infty$ and to prove Theorem 0.4.

PROPOSITION 5.1. *Let the assumption* (L) *be satisfied. Then for any* $a>0$ *and* $\lambda_0>0$ *there are constants* $C_{a,\lambda_0}>0$ *and* $0<\varepsilon_1<\varepsilon_0$ *such that*

$$(5.2)\qquad (2\lambda)^{-(N-2)/2}|\mathcal{S}(\lambda, \omega, \omega')-\mathcal{S}_n(\lambda, \omega, \omega')|\leqq C_{a,\lambda_0}n^{-\varepsilon_1}$$

for $|\sqrt{2\lambda}(\omega-\omega')|\geqq a$ and $\lambda\geqq\lambda_0$.

Proof. By Theorem 3.3

$$(5.3)\qquad \mathcal{S}(\lambda)-\mathcal{S}_n(\lambda)=2\pi i\sum_{\ell=1}^{4}\mathcal{F}_0(\lambda)A_\ell^n(\lambda)\mathcal{F}_0(\lambda)^*,$$

where

$$(5.4)\qquad \begin{cases} A_1^n(\lambda)=-(J_1^*T_2-(J_1^n)^*T_2^n), \\ A_2^n(\lambda)=(T_1^*-(T_1^n)^*)R(\lambda+i0)T_2, \\ A_3^n(\lambda)=(T_1^n)^*(R(\lambda+i0)-R_n(\lambda+i0))T_2, \\ A_4^n(\lambda)=(T_1^n)^*R_n(\lambda+i0)(T_2-T_2^n). \end{cases}$$

By Theorem 3 of [10] and a simple variant of the proof of Theorem 1.1, we see that for any $s>0$

$$(5.5)\qquad ||\langle D\rangle^{-1}A_3^n(\lambda)\langle D\rangle^{-1}f||_s\leqq C\lambda^{-1}n^{-\varepsilon_1}||f||_{-s},\qquad \lambda\geqq d^2/2,$$

which implies similarly to the first part of section 4 that $\mathcal{F}_0(\lambda)A_3^n(\lambda)\mathcal{F}_0(\lambda)^*$ has a C^∞ integral kernel $B_3^n(\lambda, \omega, \omega')$ such that

$$(5.6)\qquad |B_3^n(\lambda, \omega, \omega')|\leqq Cn^{-\varepsilon_1}\lambda^{(N-2)/2}.$$

Next we consider $A_2^n(\lambda)$. By a simple manipulation we have

$$(5.7)\qquad \begin{aligned} &T_1f(x)-T_1^nf(x) \\ &=(2\pi)^{-N/2}\int e^{i\varphi_1(x,\xi)}\{b_n(x, \xi)+c_n(x, \xi)\}\hat{f}(\xi)d\xi, \end{aligned}$$

where

$$(5.8)\qquad \begin{cases} b_n(x, \xi)=t_1(x, \xi)-t_1^n(x, \xi), \\ c_n(x, \xi)=t_1^n(x, \xi)(1-e^{i(\varphi_1^n(x,\xi)-\varphi_1(x,\xi))}). \end{cases}$$

Since

$$(5.9)\qquad |\partial_x^\alpha\partial_\xi^\beta(1-e^{i(\varphi_1^n(x,\xi)-\varphi_1(x,\xi))})|\leqq C_{\alpha\beta}n^{-\varepsilon_\alpha}\langle\xi\rangle^{-1}\langle x\rangle^{-\varepsilon_0|\alpha|+(1-\varepsilon_0)(|\beta|+1)},$$

which follows from Theorem 2.2-ii), we see that the symbol $b_n(x, \xi)+c_n(x, \xi)$ satisfies (1.5). Thus we have in view of Theorem 1.1 and $1>\varepsilon_0>1/2$

$$\|A_2^n(\lambda)\langle D\rangle^{-1}f\|_s \leqq Cn^{-\varepsilon_1}\lambda^{-1/2}\|f\|_{-s} \tag{5.10}$$

for $s>0$ and $\lambda \geqq d^2/2$. This shows that $\mathcal{F}_0(\lambda)A_2^n(\lambda)\mathcal{F}_0(\lambda)^*$ has a C^∞ kernel $B_2^n(\lambda, \omega, \omega')$ such that

$$|B_2^n(\lambda, \omega, \omega')| \leqq Cn^{-\varepsilon_1}\lambda^{(N-2)/2}. \tag{5.11}$$

Similarly, $\mathcal{F}_0(\lambda)^*A_4^n(\lambda)\mathcal{F}_0(\lambda)^*$ has a C^∞ integral kernel $B_4^n(\lambda, \omega, \omega')$ which satisfies (5.11) with B_2^n replaced by B_4^n.

We finally consider $A_1^n(\lambda)$. Since

$$A_1^n(\lambda) = -(J_1^* - (J_1^n)^*)T_2 - (J_1^n)^*(T_2 - T_2^n), \tag{5.12}$$

and the symbols of $J_1^* - (J_1^n)^*$ and $T_2 - T_2^n$ are seen to satisfy (1.5) similarly to $A_2^n(\lambda)$, we have using Theorem 2.1-(2.18)

$$\begin{aligned} A_1^n(\lambda)f(x) & \\ &= \iint e^{i(x-y)\cdot\xi}a_1^n(x, \xi)f(y)dy\tilde{d}\xi, \end{aligned} \tag{5.13}$$

where the symbol $a_1^n(x, \xi)$ satisfies

$$|\partial_x^\alpha \partial_\xi^\beta a_1^n(x, \xi)| \leqq C_{\alpha\beta}n^{-\varepsilon_\alpha}\langle x\rangle^{(1-\varepsilon_0)|\beta|-\varepsilon_0|\alpha|+2}. \tag{5.14}$$

Thus $\mathcal{F}_0(\lambda)A_1^n(\lambda)\mathcal{F}_0(\lambda)^*$ has a kernel

$$\begin{aligned} A_1^n(\lambda, \omega, \omega') & \\ &= (2\pi)^{-N}(2\lambda)^{(N-2)/2}\int e^{i\sqrt{2\lambda}(\omega-\omega')\cdot x}a_1^n(x, \sqrt{2\lambda}\omega')dx. \end{aligned} \tag{5.15}$$

Taking notice of $|\sqrt{2\lambda}(\omega-\omega')| \geqq a$, integrating by parts with respect to x in (5.15) and using (5.14), we get

$$|A_1^n(\lambda, \omega, \omega')| \leqq C_a n^{-\varepsilon_1}(2\lambda)^{(N-2)/2}. \tag{5.16}$$

This completes the proof of the proposition. Q. E. D.

We denote by $\mathcal{S}_n^{(S)}(\lambda, \omega, \omega')$ the short-range type scattering amplitude for the compactly supported potential $V_n(x)$, i.e., the scattering amplitude defined as in sections 3 and 4 with $J_\pm^n = I$.

Lemma 5.2. *There is a real-valued function $P_n(\lambda, \omega, \omega')$ such that*

$$\mathcal{S}_n(\lambda, \omega, \omega') = e^{iP_n(\lambda,\omega,\omega')}\mathcal{S}_n^{(S)}(\lambda, \omega, \omega'), \tag{5.17}$$

$$|P_n(\lambda, \omega, \omega')| \leqq C_n/\sqrt{\lambda}. \tag{5.18}$$

Proof. Let $W_n^{\pm}=s\text{-}\lim_{t\to\pm\infty} e^{itH_n}e^{-itH_0}$. By Theorem 2.2-i) and the stationary phase method, we see that $J^n e^{-itH_0}f$ is asymptotically equal to $e^{-itH_0}e^{-iw_n(D_x)}f$ as $t\to\pm\infty$ for $f\in\mathcal{D}\equiv\mathcal{F}^{-1}(C_0^\infty(R^N-\{0\}))$. Thus, denoting the short-range type scattering operator for V_n by $S_n^{(S)}$, we have

$$S_n f=e^{iw_n(D)}S_n^{(S)}e^{-iw_n(D)}f. \tag{5.19}$$

Passing to the Fourier transform, from this we obtain (5.17) with $P_n(\lambda, \omega, \omega')=w_n(\sqrt{2\lambda}\omega)-w_n(\sqrt{2\lambda}\omega')$. (5.18) follows from (2.24) and (2.18). Q. E. D.

We prepare one more lemma.

LEMMA 5.3. *The following estimate holds uniformly in* $\lambda\geqq\lambda_0(>0)$ *and* ω, $\omega'\in S^{N-1}$:

$$\begin{aligned}&|(2\lambda)^{-(N-2)/2}(\mathcal{S}_n^{(S)}(\lambda, \omega, \omega')-\delta(\omega-\omega'))-c_N(\mathcal{F}V_n)(\sqrt{2\lambda}(\omega-\omega'))|\\&\leqq Cn^{N+3}/\sqrt{\lambda},\end{aligned} \tag{5.20}$$

where $c_N=(2\pi)^{(N+2)/2}i$.

Proof. In a way similar to section 3, we have

$$\begin{aligned}&\mathcal{S}_n^{(S)}(\lambda)-I\\&=-2\pi i\mathcal{F}_0(\lambda)V_n\mathcal{F}_0(\lambda)^*+2\pi i\mathcal{F}_0(\lambda)V_nR_n(\lambda+i0)V_n\mathcal{F}_0(\lambda)^*.\end{aligned} \tag{5.21}$$

Therefore we have using Theorem 1.2 in [9]

$$\begin{aligned}&|\mathcal{S}_n^{(S)}(\lambda, \omega, \omega')-\delta(\omega-\omega')\\&\quad -c_N(2\lambda)^{(N-2)/2}\int e^{-i\sqrt{2\lambda}(\omega-\omega')\cdot x}V_n(x)dx|\\&\leqq C\lambda^{(N-2)/2}\int|V_n(x)\{R_n(\lambda+i0)(V_n(y)e^{i\sqrt{2\lambda}\omega'\cdot y})\}(x)|dx\\&\leqq C\lambda^{(N-2)/2}\Big(\int_{|x|\leqq n}\langle x\rangle^{N+1}|R_n(\lambda+i0)(V_n(y)e^{i\sqrt{2\lambda}\omega'\cdot y})(x)|^2dx\Big)^{1/2}\\&\leqq C\lambda^{(N-2)/2}n^{(N+3)/2}\|R_n(\lambda+i0)(V_n(y)e^{i\sqrt{2\lambda}\omega'\cdot y})\|_{L^2_{-1}}\\&\leqq C\lambda^{(N-2)/2}n^{(N+3)/2}\|V_n\|_{L^2_1}/\sqrt{\lambda}\leqq C\lambda^{(N-2)/2}n^{N+3}/\sqrt{\lambda}.\end{aligned} \tag{5.22}$$

Q. E. D.

Now we can prove Theorem 0.4. Using Proposition 5.1 and Lemmas 5.2, 5.3, we have for $\xi=\sqrt{2\lambda}(\omega-\omega')$, $|\xi|\geqq a(>0)$, $\lambda\geqq\lambda_0(>0)$,

$$\begin{aligned}&|(2\lambda)^{-(N-2)/2}\mathcal{S}(\lambda, \omega, \omega')-c_N(\mathcal{F}V)(\xi)|\\&\leqq|(2\lambda)^{-(N-2)/2}(\mathcal{S}(\lambda, \omega, \omega')-\mathcal{S}_n(\lambda, \omega, \omega'))|\end{aligned} \tag{5.23}$$

$$+|e^{iP_n}((2\lambda)^{-(N-2)/2}\mathcal{S}_n^{(S)}(\lambda,\ \omega,\ \omega')-c_N(\mathcal{F}V_n)(\xi))|$$
$$+|(e^{iP_n}-1)c_N(\mathcal{F}V_n)(\xi)|+|c_N\mathcal{F}(V_n-V)(\xi)|$$
$$\leqq C_{a,\lambda_0}n^{-\varepsilon_1}+c_N n^{N+3}/\sqrt{\lambda}+C_{a}n/\sqrt{\lambda}+C_{a,\varepsilon}n^{-\varepsilon}.$$

This clearly implies (0.14). Using Lemma 4.1, we can easily see that $\mathcal{F}V\in L^1$.
Q. E. D.

Finally, as a simple corollary of Proposition 5.1 and Lemma 5.2, we record a theorem, which gives a calculation procedure of the differential cross section $|\mathcal{S}(\lambda,\ \omega,\ \omega')|^2$, $\omega\neq\omega'$.

THEOREM 5.4. *Let the assumption (L) with $1>\varepsilon>1/2$ be satisfied. Then for $\omega\neq\omega'$ and $\lambda>0$*

$$\lim_{n\to\infty}|\mathcal{S}_n^{(S)}(\lambda,\ \omega,\ \omega')|^2=|\mathcal{S}(\lambda,\ \omega,\ \omega')|^2. \tag{5.24}$$

§ 6. Proof of Theorem 0.2.

We use the phase function $\varphi(x,\ \xi)=x\cdot\xi$ and the amplitude functions $a_j(x,\ \xi)$, $j=1,2$, defined by (2.36)–(2.37). We define

$$J_jf(x)=\iint e^{i(x-y)\cdot\xi}a_j(x,\ \xi)f(y)dyd\hskip-0.1em\bar{}\,\xi \tag{6.1}$$

and

$$T_j=HJ_j-J_jH_0. \tag{6.2}$$

Then

$$T_jf(x)=\iint e^{i(x-y)\cdot\xi}t_j(x,\ \xi)f(y)dyd\hskip-0.1em\bar{}\,\xi, \tag{6.3}$$

where $t_j(x,\ \xi)$ are defined by (2.39). Arguing quite similarly to the proof of Theorem 3.3, we have

$$\mathcal{S}(\lambda)-I=-2\pi i\mathcal{F}_0(\lambda)J_1^*T_2\mathcal{F}_0(\lambda)^* \tag{6.4}$$
$$+2\pi i\mathcal{F}_0(\lambda)T_1^*R(\lambda+i0)T_2\mathcal{F}_0(\lambda)^*.$$

The second term on the RHS has a C^∞ integral kernel by Proposition 2.3 and Lemma 3.2. Furthermore by an argument similar to section 4, the first term on the RHS of (6.4) has a C^∞ integral kernel $K(\omega,\ \omega')$ when $\omega\neq\omega'$. $K(\omega,\ \omega')$ can be written down as follows:

$$(2\pi)^N K(\omega,\ \omega')=\int_{R^N}e^{-i\sqrt{2\lambda}(\omega-\omega')\cdot x}q(x,\ \omega,\ \omega')dx, \tag{6.5}$$

where

$$q(x, \omega, \omega')=\overline{a_1(x, \sqrt{2\lambda}\omega)}t_2(x, \sqrt{2\lambda}\omega'). \tag{6.6}$$

Since by (2.34), (2.35) and (2.38), $t_2^{(1)}(x, \xi)$ in (2.40) decays rapidly with respect to x, the singularity of $K(\omega, \omega')$ comes from $q_2(x, \omega, \omega')=-\overline{a_1(x, \sqrt{2\lambda}\omega)}t_2^{(2)}(x, \sqrt{2\lambda}\omega')$. In the following, we always assume $|\omega-\omega'|$ is sufficiently small but $\omega\neq\omega'$. By (2.32) and (2.35), on the support of $\partial_x^\alpha\chi_2^\pm(x, \omega')$, $\alpha\neq 0$, we have

$$a_1(x, \sqrt{2\lambda}\omega)=1+a^{(1)}(x, \sqrt{2\lambda}\omega)+r(x, \sqrt{2\lambda}\omega), \tag{6.7}$$

where

$$|\partial_x^\alpha\partial_\xi^\beta r(x, \xi)|\leqq C_{\alpha\beta}\langle x\rangle^{-|\alpha|-2\varepsilon_0}. \tag{6.8}$$

Using (2.40), we compute

$$\begin{aligned} q_2(x, \omega, \omega')&=-(1+\overline{a^{(1)}(x, \sqrt{2\lambda}\omega)}+\overline{r(x, \sqrt{2\lambda}\omega)})t_2^{(2)}(x, \sqrt{2\lambda}\omega') \\ &=-\left\{\left(i\sqrt{2\lambda}\omega'\cdot\nabla+\frac{1}{2}\Delta\right)(\chi_2^+(x, \omega')+\chi_2^-(x, \omega'))\right\} \\ &\quad-(i\sqrt{2\lambda}\omega'\cdot\nabla\chi_2^+(x, \omega'))(a^{(1)}(x, \sqrt{2\lambda}\omega')+\overline{a^{(1)}(x, \sqrt{2\lambda}\omega)}) \\ &\quad-(i\sqrt{2\lambda}\omega'\cdot\nabla\chi_2^-(x, \omega'))(\overline{a^{(1)}(x, -\sqrt{2\lambda}\omega')}+\overline{a^{(1)}(x, \sqrt{2\lambda}\omega)}) \\ &\quad-r_4(x, \omega, \omega') \\ &=-r_1-r_2-r_3-r_4, \end{aligned} \tag{6.9}$$

where r_4 satisfies

$$|\partial_x^\alpha r_4(x, \omega, \omega')|\leqq C_\alpha\langle x\rangle^{-1-2\varepsilon_0-|\alpha|}. \tag{6.10}$$

We write

$$K_j(\omega, \omega')=-\int e^{-i\sqrt{2\lambda}(\omega-\omega')\cdot x}r_j(x, \omega, \omega')dx, \qquad j=1,2,3,4. \tag{6.11}$$

By Lemma 4.1 and (6.10), we have

$$|K_4(\omega, \omega')|=o(|\omega-\omega'|^{-N+1+\varepsilon_0}) \tag{6.12}$$

as $|\omega-\omega'|\to 0$ but $\omega\neq\omega'$, where we have used $\varepsilon_0<N-1$. When $\omega\neq\omega'$, we can integrate by parts in the following integrals by the use of $L=|\omega-\omega'|^{-2}(\omega-\omega')\cdot\nabla_x$. and we have

$$K_1(\omega, \omega')=-\int e^{-i\sqrt{2\lambda}(\omega-\omega')\cdot x}r_1(x, \omega, \omega')dx \tag{6.13}$$

$$=-\frac{1}{2}i\sqrt{2\lambda}\int e^{-i\sqrt{2\lambda}(\omega-\omega')\cdot x}(\omega+\omega')\cdot\nabla(\chi_2^+(x,\ \omega')+\chi_2^-(x,\ \omega'))dx$$

$$=\lambda\int e^{-i\sqrt{2\lambda}(\omega-\omega')\cdot x}(\omega-\omega')\cdot(\omega+\omega')(\chi_2^+(x,\ \omega')+\chi_2^-(x,\ \omega'))dx$$

$$=0.$$

By (2.33), $a^{(1)}(x,\ \sqrt{2\lambda}\omega)$ is pure imaginary, hence $a^{(1)}(x,\ \sqrt{2\lambda}\omega)+\overline{a^{(1)}(x,\ \sqrt{2\lambda}\omega)}=0$. Thus we have

$$K_2(\omega,\ \omega')=o(|\omega-\omega'|^{-N+1+\epsilon_0}). \tag{6.14}$$

By a straightforward calculation

$$K_3(\omega,\ \omega')=-\int e^{-i\sqrt{2\lambda}(\omega-\omega')\cdot x}(\omega'\cdot\nabla\chi_2^-(x,\ \omega'))Q(x,\ \omega)dx, \tag{6.15}$$

$$Q(x,\ \omega)=\int_{-\infty}^{\infty}V(x+\omega t)dt. \tag{6.16}$$

Therefore we have for $|\omega-\omega'|$ near 0 and $\omega\neq\omega'$

$$|(2\pi)^N\mathcal{T}(\lambda,\ \omega,\ \omega')-K_3(\omega,\ \omega')|=o(|\omega-\omega'|^{-N+1+\epsilon_0}). \tag{6.17}$$

It remains to compute $K_3(\omega,\ \omega')$. First take notice that $Q(x,\ \omega)$ is translation invariant with respect to x:

$$Q(x+\omega s,\ \omega)=Q(x,\ \omega). \tag{6.18}$$

We set without loss of generality $\omega=(0,\cdots,0,1)$ and $x=(x',\ x_N)$. Then we have

$$Q(x,\ \omega)=\int_{-\infty}^{\infty}V(x',\ x_N)dx_N\equiv Q(x'), \tag{6.19}$$

and

$$K_3(\omega,\ \omega') \tag{6.20}$$

$$=-\int e^{-i\sqrt{2\lambda}(\omega-\omega')\cdot x}\frac{\partial}{\partial x_N}(\rho_2(-\hat{x}_N))Q(x')dx, \qquad \hat{x}_N=x_N/|x|.$$

We set

$$L(\omega,\ \omega') \tag{6.21}$$

$$=-\int e^{-i\sqrt{2\lambda}(\omega-\omega')\cdot(x',0)}\frac{\partial}{\partial x_N}(\rho_2(-\hat{x}_N))Q(x')dx,$$

and estimate $K_3(\omega,\ \omega')-L(\omega,\ \omega')$ for $|\omega-\omega'|$ near 0 but $\omega\neq\omega'$. We make a change of variable: $y=|\omega-\omega'|x$ and $\theta=(\omega-\omega')/|\omega-\omega'|$. Then

(6.22) $$K_3(\omega,\ \omega')=-\int|\omega-\omega'|^{-N}e^{-i\sqrt{2\lambda}\theta\cdot y}f(y,\ \omega-\omega')dy,$$

where

(6.23) $$f(y,\ \omega-\omega')=\frac{\partial}{\partial x_N}(\rho_2(-\hat{x}_N))Q(x')\Big|_{x=y/|\omega-\omega'|}.$$

Since $P=(\sqrt{2\lambda}|\theta|^2)^{-1}i\theta\cdot\nabla_y$ is well-defined for $\omega\neq\omega'$, by integration by parts we have for $\ell=0,1,2\cdots$

(6.24) $$K_3(\omega,\ \omega')=-\int|\omega-\omega'|^{-N}e^{-i\sqrt{2\lambda}\theta\cdot y}f_\ell(y,\ \omega-\omega')dy,$$

where $f_\ell(y,\ \omega-\omega')=({}^tP)^\ell f(y,\ \omega-\omega')$. By the definition of ρ_2 in section 2, $2\delta\geqq -x_N\geqq\delta$ on the support of $\frac{\partial}{\partial x_N}(\rho_2(-\hat{x}_N))$. Thus by (6.16) and the assumption (S),

(6.12) $$\left|\partial_x^\alpha\left(\frac{\partial}{\partial x_N}(\rho_2(-\hat{x}_N))Q(x')\right)\right|\leqq C_\alpha\langle x\rangle^{-1-\varepsilon_0-|\alpha|},$$

hence

(6.26) $$|\partial_y^\alpha f(y,\ \omega-\omega')|\leqq C_\alpha|\omega-\omega'|^{-|\alpha|}(1+|\omega-\omega'|^{-1}|y|)^{-1-\varepsilon_0-|\alpha|}.$$

Therefore

(6.27) $$|\text{the integrand of } K_3(\omega,\ \omega')|$$
$$\leqq\begin{cases}C_\ell|\omega-\omega'|^{-N+1+\varepsilon_0}\langle y\rangle^{-1-\varepsilon_0-\ell} & \text{for}\quad |y|\geqq 1,\\ C_0|\omega-\omega'|^{-N+1+\varepsilon_0}|y|^{-1-\varepsilon_0} & \text{for}\quad |y|\leqq 1.\end{cases}$$

The RHS is integrable on R_y^N since $N\geqq 2$ and $0<\varepsilon_0<N-1$. Similarly we have

(6.28) $$L(\omega,\ \omega')=-|\omega-\omega'|^{-N}\int e^{-i\sqrt{2\lambda}(\theta',0)\cdot y}f_\ell(y,\ (\omega-\omega')',\ 0)dy,$$

and (6.27) with K_3 replaced by L. Therefore

(6.29) $$|K_3(\omega,\ \omega')-L(\omega,\ \omega')|$$
$$\leqq\int|\omega-\omega'|^{-N}|e^{-i\sqrt{2\lambda}\theta\cdot y}f_\ell(y,\ \omega-\omega')$$
$$-e^{-i\sqrt{2\lambda}(\theta',0)\cdot y}f_\ell(y,\ (\omega-\omega')',\ 0)|dy.$$

Since the integrand satisfies (6.27) with K_3 replaced by K_3-L, we have by Lebesgue's dominated convergence theorem

(6.30) $$|K_3(\omega,\ \omega')-L(\omega,\ \omega')|=o(|\omega-\omega'|^{-N+1+\varepsilon_0}).$$

Noting

$$-\int_{-\infty}^{\infty}\frac{\partial}{\partial x_N}(\rho_2(-\hat{x}_N))dx_N=1, \tag{6.31}$$

we have

$$\begin{aligned} L(\omega,\ \omega') &= \int_{R^N} e^{-i\sqrt{2\lambda}(\omega-\omega')\cdot(x',0)}V(x)dx \\ &= (2\pi)^{-N/2}(\mathcal{F}V)(\sqrt{2\lambda}P(\omega-\omega')). \end{aligned} \tag{6.32}$$

This together with (6.17) and (6.30) proves Theorem 0.2-ii). Theorem 0.2-i) follows from Theorem 0.2-ii) and Lemma 4.1. Q. E. D.

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