学位論文

Fundamental inequalities in quantum many-body systems

(量子多体系における基礎不等式)

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Abstract

The analysis of quantum many-body systems can require enormous computational power because of *exponentially* increasing system parameters. However, we can often avoid such an exponential increase in practical calculations such as mean-field approximation and the density-matrix renormalization-group method. In these cases, the quantum states occupy only a tiny fraction of the total Hilbert space. Understanding such a 'small corner' of the total Hilbert space benefits us in terms of the efficiency of numerical algorithms, and hence many researchers have been studying them with great interest. More specifically, we want to know how to find an approximate description of quantum states in analyzing quantum many-body systems. A natural approach to this problem is to identify the constraints on these states, which can be derived in terms of a set of fundamental inequalities. Such constraints have been energetically investigated for short-range interacting systems, but there have been few studies for more general quantum systems, which are governed by the k-local Hamiltonians. The k-local Hamiltonians contain interactions of up to k-body couplings with finite k, that is, they include not only short-range interactions but also long-range interactions.

In the present thesis, we contribute to an essential development in the field by extending the systems from short-range interacting Hamiltonians to general k-local Hamiltonians. We exploit efficient theorems to analyze the k-local Hamiltonians and clarify what kind of constraints should be placed in the system. The results are summarized as follows:

Chapter 3 : Basic properties of the k-local Hamiltonians, especially in terms of the spectral aspect.

Chapter 4 : Entanglement structure of gapped ground states, which we characterize in terms of the reversibility.

Chapter 5 : Macroscopic superposition in low-lying energy states.

Chapter 6 : Fundamental constraints on the quantum dynamics due to the k-local Hamiltonians.

The first result provides us the fundamental tools to analyze systems with the k-local Hamiltonians, while the following three results characterize fundamental constraints on the quantum states. The essence of the k-locality lies in the fact that any local operators cannot cause global influences. This is trivial for classical systems, while it is not the case for quantum systems because of the non-local structure, namely quantum entanglement. Our results imply that the effect of the entanglement can exist only locally owing to the k-locality of systems. As further applications, we will be able to develop approximate descriptions of the quantum states with k-local Hamiltonians, which lead to a big breakthrough in the field of 'Hamiltonian complexity.'

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Contents

Chapter 1

Introduction and fundamental setup

1.1 Fundamental motivation

The lowest-energy states of quantum many-body systems, namely the ground states, allow us to analyze physical properties at zero or low temperatures, where quantum effects emerge most prominently. For this reason, it is one of the most fundamental problem to find a ground state of a quantum system. However, because of the quantum nature, we usually need $2^{\mathcal{O}(N)}$ parameters in order to describe an arbitrary quantum state of an N-partite system. This exponential increase of the number of parameter comes from the quantum entanglement [1–3]; a many-body quantum state is often not given by a simple product state but is given by an arbitrary superposition of $2^{\mathcal{O}(N)}$ product states. This means that for large N it is almost hopeless to find a ground state by directly diagonalizing the Hamiltonian [4–6].

In reality, however, we can often avoid the exponential increase in practical calculations of ground states. Ground states in several classes of the Hamiltonian only need $\mathrm{Poly}(N)$ pieces of parameters for their description [7–10]. More formally, there exists a set of quantum states G which contains every ground state in a certain class; for an arbitrary state $|\psi\rangle \in G$, there exists a classical expression $\{0,1\}^{\mathrm{Poly}(N)}$ which 'describes' $|\psi\rangle$ in the sense that we can efficiently compute any few-body observables.[*1] For example, the simplest class is the product state, where the number of parameters necessary for the description is $\mathcal{O}(N)$. We can describe the class of ground states of 1-local Hamiltonians in terms of the product state.

Practically, one of the most important and well-known candidates for such efficiently describable classes is the Hamiltonians which have non-degenerate gapped ground states [8, 10, 11], where the term "gapped" means that the energy difference between the ground state and the first excited state is bounded from below by a constant of $\mathcal{O}(1)$. Especially for one-dimensional systems, the properties of the gapped ground states have been completely understood; M. B. Hastings has first proved the entropic area law [12] which results in that the matrix product representation efficiently describes the gapped ground states. After that, as a remarkable achievement, Z. Landau, *et al.* have proved [13] that the computational class of the calculation of the ground energy is in the class P. In this way, the gapped ground states in quantum many-body systems can have good properties in terms of their description complexity. However, we are still far from the complete understanding of this problem beyond one-dimensional systems.

The possibility of the simple description comes from local natures of gapped quantum states [11], that is, *non-local properties of gapped quantum states should be highly suppressed*. This is deeply related to the fact that the Hamiltonians of the ground states in question consist of only local operators; in this thesis, we mean by the k-locality 'the locality of the Hamiltonian'.[*2]

[*1] Mathematically, few-body observables mean k-local operators with $k = \mathcal{O}(1)$. On k-local operators, see Subsection 1.2.2.

[*2] In fact, the locality is often defined in terms of the short-range interactions (in Section 2.2). We remark that the

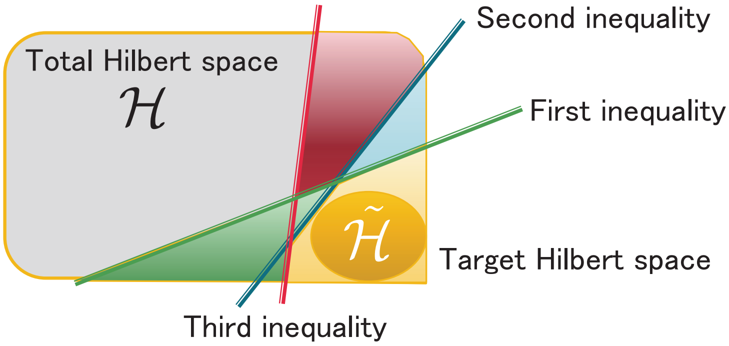

Figure 1.1: Schematic picture of the problem. The total Hilbert space, which we denote by $\mathcal{H}$, becomes exponentially large as the system size increases. In the realistic situation, however, it is often enough to consider only a small corner of the total Hibert space, which we denote by $\tilde{\mathcal{H}}$. The space $\tilde{\mathcal{H}}$ can be characterized by a set of fundamental inequalities; in the picture, three inequalities characterize $\tilde{\mathcal{H}}$. For example, when we consider one-dimensional short-range Hamiltonians with non-vanishing spectral gap just above their ground states, the entropic area law characterizes the Hilbert space $\tilde{\mathcal{H}}$ of the ground states, which leads to an efficient approximation of the states by the use of the matrix product states. In the thesis, we aim to establish a similar analysis for systems with general k-local Hamiltonians.

Here, in order to clarify whether there exists an efficient description for gapped ground states, we will have to take the following two steps:

1. We obtain the restrictions on the ground states due to the existence of the spectral gap in terms of fundamental inequalities (Fig. 1.1).

2. We find an efficient description of the ground states based on the restrictions.

In short-range interacting systems, for the first step, the entropic area law [12,14–19] and the exponential decay of bi-partite correlations [20–24] play essential roles, and for the second step, we are able to utilize the matrix product state (MPS) or the projected entangled pair state (PEPS) for an efficient description of the ground states [8,25–27], but neither of them has been rigorously proved yet in higher-dimensional systems.

In order to make a new breakthrough, in the present thesis, we mainly investigate ground states of general k-local Hamiltonians, which contain the interactions of up to k-body coupling with finite k. The reasons why we focus on this class of the Hamiltonians are the followings:

1. The preceding results are mainly on the short-range interacting systems, and hence we have little knowledge on long-range interacting systems; note that from the definition the class of the k-local Hamiltonians includes long-range interacting systems such as the infinite-range XY-model.

2. Although the k-local Hamiltonians may not be physically natural in themselves, they are quite common in discussing the Hamiltonian complexity. For example, the most important result on the QMA-completeness, which is a natural extension of the classical NP-complete, has been given for 5-local Hamiltonians by A. Kitaev [4].

3. We may be able to provide a new viewpoint on the locality analysis of the ground states through the investigation of the k-local Hamiltonians. Indeed, our new concept, the local reversibility, can serve as a new indicator of the locality for gapped ground states in short-range interacting systems.

k-locality is a more general concept in the sense that it contains the long-range interactions.

The final goal of the present study is to answer the following two questions: what properties completely characterize the gapped ground state; how do we construct a class of states which efficiently describes the gapped ground state? For the former question, we have obtained several candidates as shown in the following chapters, while on the latter question, it is still an open problem.[*3]

Because we are going to handle the qualitatively different Hamiltonian from the short-range Hamiltonians, we cannot apply the conventional techniques to our research. In analyzing ground states in the k-local Hamiltonians, we need to establish new frameworks on the following two points:

1. What can be the starting point of the analysis?
2. On what properties in the ground states should we investigate?

In short-range interacting systems, almost all the important results come from the Lieb-Robinson bound [23, 28–30], and we usually focus on bi-partite correlations and the entanglement entropy as the locality of the ground states. Mathematically, they are given in the form of fundamental inequalities (see Chapter 2). Thus, for systems with the k-local Hamiltonians, we also aim to obtain a set of inequalities which reflects the essence of the systems.

Based on the above motivations, we tackle the problems as follows:

1. (Chapter 2: review) We first put together previous studies on the short-range interacting systems [11]. We first give the Lieb-Robinson bound, which is the fundamental tool for the analysis. We then show several important applications of the Lieb-Robinson bound: the adiabatic continuation and the exponential decay of the bi-partite correlations. Third, we show the basic ideas of the proof for the one-dimensional area law by I. Arad, *et al.* [15]. A part of the techniques for the proof such as the Chebyshev polynomial and the effective Hamiltonian is also utilized in our main results.

2. (Chapter 3) As a fundamental research tool, we focus on the fact that no local operators can cause the global energy changes.[*4] It is mathematically expressed in the form that the energy excitation after an arbitrary local disturbance decays exponentially beyond a characteristic energy which is determined by the disturbance (Theorem 3.2). From this theorem, we also construct an effective Hamiltonian with approximately the same low-lying states as the original one, which possesses a convenient property for the analysis of the ground states (Theorem 3.4).

3. (Chapter 4) For the characterization of the ground states, we focus on the reversibility property of the ground states after an external disturbance. We find that if a state does not contain global quantum properties such as the topological order and the anomalous fluctuations, the state can be reversible only by the use of a local operator after a disturbance. We define such a property as the local reversibility (Definition 4.2.1) and prove it for arbitrary gapped ground states in the k-local Hamiltonians (Theorem 4.3), where we utilize Theorem 3.2 and basic techniques in the proof of the one-dimensional area law. We also show that the macroscopic superpositions cannot exist in the locally reversible states in terms of the quantum Fisher information (Theorem 4.6).[*5]

4. (Chapter 5) We refine the result on the macroscopic superpositions in Chapter 4. We generalize the result to low-lying states instead of the exact ground state (Theorems 5.3 and 5.4). In the proof of Theorem 5.4, Theorem 3.4 on the effective Hamiltonian plays the essential roles.

[*3] I conjecture that the tensor network states [27] may describe the gapped ground state efficiently, while it may be even possible that there exists no efficient description for the gapped ground states in the case of the general k-local Hamiltonians.

[*4] We notice that this fact itself has been already utilized for the proof the one-dimensional area law. However, there have been no previous studies to apply it to the analysis of the ground states in the k-local Hamiltonians.

[*5] The macroscopic superposition is here defined by the scaling of the Fisher information.

5. (Chapter 6) As a different expression of the k-locality from Theorem 3.2, we consider a Lieb-Robinson-like bound for the time-evolution due to the k-local Hamiltonians. Because the k-local Hamiltonian contains not only short-range interactions but also long-range interactions, the Lieb-Robinson itself cannot give a useful restriction to the time evolution. It comes from the fact that the Lieb-Robinson bound considers a restriction to the velocity of information transfer, whereas the long-range interaction can transfer information immediately. Therefore, instead of the information transfer, we take another standpoint which is based on information sharing. We mathematically formulate it in the form of an operator inequality (Theorem 6.2). Based on this result, we also point out the possibility that we can generalize the concept of the short-range entanglement (Theorem 6.4).

Based on these results, we have significantly deepened our knowledge on the systems with the k-local Hamiltonians, while we still have left many open problems. In Chapter 7, we summarize the results and show the future problems which have been left.

1.2 Setup and Notations

In this section, we give the basic definitions of the systems that we will study.

1.2.1 Definition of the system

We consider a spin system of finite volume with each spin having a D-dimensional Hilbert space and label each spin by $i = 1, 2, \dots N$. We assume that the dimension D does not depend on the system size N. We denote partial sets of sites by X, Y, Z and so on and the cardinality of X, that is, the number of sites contained in X, by $|X|$ (e.g. $X = \{i_1, i_2, \dots, i_{|X|}\}$). We also denote the complementary subsets of X, Y and Z by X^{c}, Y^{c} and Z^{c}, respectively; in other words, $X \oplus X^{\mathrm{c}}$ comprises the total system.

1.2.2 Definition of k-local operator

We here introduce the k-local operator, which is the central target of this thesis. We define the *k-local operator* $O^{(k)}$ as follows:

$$O^{(k)} = \sum_{|X| \le k} o_X, \tag{1.1}$$

where o_X is a local operator supported in the finite set X. The k-local operator contains the interactions of up to k-body coupling with finite k. More explicitly, this operator takes the form

$$O^{(k)} = \sum_{i_1 < i_2 < \cdots < i_k}^{N} \sum_{\mu_1, \dots, \mu_k} o_{i_1, \dots, i_k}^{\mu_1, \dots, \mu_k} s_{i_1}^{\mu_1} \otimes \cdots \otimes s_{i_k}^{\mu_k}, \tag{1.2}$$

where $\{s_i^{\mu}\}$ are operator bases at the site i; when we consider a 1/2-spin system for example, $\{s_i^{\mu}\}_{\mu} = \{\sigma_i^x, \sigma_i^y, \sigma_i^z\}$ with $\{\sigma_i^{\mu}\}_{\mu=x,y,z}$ the Pauli matrices.

For example, we can give a 3-local operator $O^{(3)}$ in a 7-spin system as

$$\begin{aligned} O^{(3)} =& 0.1 s_1^y + 0.2 s_3^z + 0.7 s_7^x \\ &+ 0.8 s_1^y s_7^x + 0.5 s_2^z s_3^y + s_4^y s_6^x + 0.7 s_3^z s_4^x + 0.8 s_1^y s_7^x + 0.7 s_2^x s_5^z \\ &+ 0.6 s_1^z s_2^y s_3^z + 1.6 s_4^x s_5^z s_7^y + 1.2 s_2^x s_4^y s_6^x + s_2^z s_3^x s_5^y + 1.4 s_3^y s_5^x s_6^z. \end{aligned} \tag{1.3}$$

Note that we do not put any restrictions to the connections among spins.

1.2.3 Definition of extensiveness

We next define the *extensiveness* as follows: the *k-local operator* $O^{(k)}$ satisfies the extensiveness if

$$\sum_{X:X\ni i} \|o_X\| \le g \quad \text{for} \quad i = 1, 2, \dots, N, \tag{1.4}$$

where $\|\cdots\|$ is the operator norm, that is, the maximum singular value of the operator, g is a constant of $\mathcal{O}(1)$ and $\sum_{X:X\ni i}$ denotes the summation with respect to the supports which contain the spin i. If a Hamiltonian satisfies the extensiveness, one spin's energy is bounded finitely.

We also note that if the operator $O^{(k)}$ satisfies the extensiveness it satisfies

$$\|O^{(k)}\| \le gN, \tag{1.5}$$

because of

$$\|O^{(k)}\| \le \sum_X \|o_X\| \le \sum_{i=1}^{N} \sum_{X\ni i} \|o_X\| \le gN, \tag{1.6}$$

where the final inequality comes from the extensiveness.

1.2.4 Definition of the Hamiltonian

In this thesis, we mainly consider Hamiltonians which are extensive k-local operators with $k = \mathcal{O}(1)$:

$$H = \sum_{|X|\le k} h_X \quad \text{with} \quad \sum_{X\ni i} \|h_X\| \le g \quad \text{for} \quad i = 1, 2, \dots, N. \tag{1.7}$$

It is worth mentioning that such a k-local Hamiltonian is the most general class that describes standard quantum systems; it includes not only short-range interacting systems but also long-range interacting systems such as the Lipkin-Meshcov-Glick (LMG) model [31, 32]:

$$H_{\mathrm{LMG}} = -\frac{J}{N} \sum_{i<j} (\sigma_i^x \sigma_j^x + \gamma \sigma_i^y \sigma_j^y) + \sum_{i=1}^{N} h\sigma_i^z \tag{1.8}$$

with J, γ and h being constants of $\mathcal{O}(1)$. We denote the eigenenergies of the Hamiltonian by $E_0 \le E_1 \le E_2 \le \cdots$ with the corresponding eigenstates $|E_0\rangle, |E_1\rangle, |E_2\rangle, \dots$, respectively.

For the fundamental parameters k and g, we often introduce a parameter λ as

$$\lambda \equiv \frac{1}{4gk} \tag{1.9}$$

for simplicity of the notation.

1.2.4 (a) Definition of the commuting Hamiltonian and the SC-Hamiltonian

We here define the commuting Hamiltonian H^{c} such that its summands in (1.7) satisfy the following properties:

$$[h_X, h_{X'}] = 0 \quad \forall X, X'. \tag{1.10}$$

This means that any components of the Hamiltonian commute with each other. At first glance, this class of Hamiltonian appears to show only trivial behavior. In fact, this class has many non-trivial Hamiltonians, such as Kitaev's toric code model [33] and the stabilizer operators for the graph state [36,37] (see Subsection 4.4.3 in Chapter 4).

We then introduce the summation of the commuting Hamiltonians:

$$H \equiv \sum_{m=1}^{n_{\rm sc}} \frac{H_m^{\rm c}}{n_{\rm sc}}, \tag{1.11}$$

where we assumed that each of the commuting Hamiltonians $\{H_m^{\rm c}\}_{m=1}^{n_{\rm sc}}$ is an extensive k-local operator as in Eq. (1.4). This class of Hamiltonian is much more general than that of the commuting Hamiltonians. As long as we know, the realistic Hamiltonian is always included in this class. Throughout this thesis, we refer to such Hamiltonians as SC Hamiltonian for brevity.

For example, the LMG Hamiltonian (1.8) can be decomposed as

$$H_{\rm LMG} = \frac{H_1^{\rm c} + H_2^{\rm c} + H_3^{\rm c}}{3} \tag{1.12}$$

with

$$H_1^{\rm c} \equiv -\frac{3J}{N}\sum_{i<j}\sigma_i^x\sigma_j^x, \quad H_2^{\rm c} \equiv -\frac{3J}{N}\sum_{i<j}\gamma\sigma_i^y\sigma_j^y, \quad H_3^{\rm c} \equiv 3h\sum_{i=1}^{N}\sigma_i^z, \tag{1.13}$$

where $k = 2$ and $g = \max(3J, 3\gamma J, 3h)$.

As another example, we consider a one-dimensional nearest-neighbor-coupling Hamiltonian with the periodic boundary condition[*6]:

$$H = \sum_{i=1}^{N} h_{\{i,i+1\}} \quad \text{with} \quad \|h_{\{i,i+1\}}\| \le \frac{g}{2}, \tag{1.14}$$

where $h_{\{i,i+1\}}$ is a coupling between the spins i and $i+1$. For simplicity, we assume that N is an even number. We can also decompose this Hamiltonian into two commuting Hamiltonians:

$$H = \frac{H_1 + H_2}{2} \tag{1.15}$$

with

$$H_1 = 2\sum_{i=1}^{N/2} h_{\{2i,2i+1\}}, \quad H_2 = 2\sum_{i=1}^{N/2} h_{\{2i-1,2i\}}. \tag{1.16}$$

Since we have $[h_{\{2i,2i+1\}}, h_{\{2i+2,2i+3\}}] = 0$, each of H_1 and H_2 is a commuting Hamiltonian.

[*6]We note that the same discussion is applicable to higher-dimensional cases.

Chapter 2

Reviews : Previous studies

2.1 Overview

In this chapter, we give an overview of previous studies in the field of the locality analysis [11] or the complexity problem [10], mainly on the short-range interacting systems. We review how the locality of the Hamiltonians reflects basic properties of the ground states. For a decade, the problem has been extensively investigated, especially under the assumption of a finite spectral gap between the ground state and the first excited state. As a fundamental theoretical tool, they have utilized the Lieb-Robinson bound [28], which characterizes how fast a piece of information propagates through a time-evolution of the Hamiltonian [29]. So far, the Lieb-Robinson bound is one of the most popular tools to analyze local properties in the ground states such as bi-partite correlations and the entanglement entropy.

While the Lieb-Robinson bound itself had been derived around forty years ago [28], the rapid developments started when Hastings resolved the long-standing problem of higher-dimensional "Lieb-Shulzt-Mattis (LSM)" theorem [20].[*1] Since then, various kinds of applications have been reported: the exponential decay of bi-partite correlations [22], the one-dimensional entropic area law [12], the adiabatic continuation [38–40], the stability of basic quantum properties [41–44] and so on. Such developments are quite important in understanding how hard it is to simulate quantum systems, namely the rapidly growing field of 'Hamiltonian complexity [10].'

In the field of the complexity problems, we have several important problems for short-range interacting Hamiltonians (see also Refs. [10, 11].):

1. Can we prove the entropic area law for non-degenerate gapped ground states in higher-dimensional systems [17, 18]? More generally, what are the sufficient conditions of the entropic area law? For example, the exponential decay of bi-partite correlations may imply the entropic area law; this has been proved only for the one-dimensional systems [16], whereas in the higher-dimensional cases the problem remains open.

2. What properties are necessary and sufficient to describe quantum states efficiently? We here mean by "efficiently" that we can find an approximate description of the states only by polynomially increasing parameters Poly(N). For example, we can utilize the projected entangled pair state (PEPS) [25, 26] as one of the candidates for the description.

[*1] The LSM theorem guarantees the upper bound of the spectral gap of order $(\log l_s)/l_s$ with l_s the system length under several assumptions on the Hamiltonians: finite-range interactions, the periodic boundary condition, specific symmetric conditions and so on.

3. As a related problem, what properties are necessary and sufficient conditions to characterize the non-degenerate gapped ground states? Sufficient conditions have been given in terms of the exponential decay of bi-partite correlations [22] and the entropic area law [19], while a necessary condition has been give in terms of the short-range entanglement (SRE) [45, 48]. We, however, do not have the complete knowledge on this problem (see Chapter 4 for more detail).

So far, except for one-dimensional systems [12, 16], we have not obtained enough knowledge for the above problems. They are also important not only for short-range Hamiltonians but also for long-range ones.

We, in the present chapter, particularly focus on the following problems: first, we review the Lieb-Robinson bound, which is one of the most important analytical tools. Second, we introduce the adiabatic continuation as a response of a ground state to variations of Hamiltonian's parameters. Third, we show a proof of the exponential decay of bi-partite correlations in gapped ground states. Finally, we give basic ideas on how to obtain the one-dimensional entropic area law.

Short-range Hamiltonian	k-local Hamiltonian
Lieb-Robinson bound (Section 2.3)	locality of the energy excitations (Chapter 3) Lieb-Robinson-like bound for information sharing (Chapter 6)
Exponential decay of correlations (Section 2.5)	Suppression of macroscopic superposition (Chapter 5)
Entropic area law (Section 2.6)	Local reversibility (Chapter 4)

Table 2.1: The comparisons of locality properties between short-range Hamiltonians and k-local Hamiltonians.

These problems are deeply related to our main results in the following chapters. In the present thesis, we mainly treat the k-local Hamiltonians instead of the short-range Hamiltonians. The k-local Hamiltonians contain not only short-range interactions but also long-range interactions (see the definition in Chapter 1), and hence many of the results in short-range interacting systems are no longer applicable to our systems. Instead, we prove other related statements as in Table 2.1. First, instead of the Lieb-Robinson bound, we find out in Chapter 3 that the excitation by the local disturbance reflects the k-locality. We utilize this statement as a fundamental theoretical tool for the analysis in the following chapters. In Chapter 6, we also derive a Lieb-Robinson-like bound for the time evolution due to the k-local Hamiltonians; in the derivation, we focus on the velocity of information sharing instead of the information transfer.[*2] We also argue in Chapter 5 that the exponential decay of bi-partite correlations should be replaced by the exponential suppression of the macroscopic superpositions, which can be characterized by fluctuations. In short-range interacting systems, the entropic area law gives the fundamental restrictions to the entanglement properties, whereas it may be no longer satisfied for long-range interacting systems. Therefore, as another expression of restrictions to the entanglement, we give a new approach which we refer to as the local reversibility. We note that the basic idea of the local reversibility is similar to that of the proof of the one-dimensional area law (see Section 4.3 in Chapter 4).

[*2]Note that the basic idea of the Lieb-Robinson bound comes from the restriction to the velocity of information transfer. However, for the time-evolution due to the k-local Hamiltonians, any two spins can couple with each other, and hence we can no longer obtain a strong bound for the information transfer.

2.2 Definition of the short-range interacting systems

In considering short-range interacting systems, we have to define the structure of the system explicitly (e.g. square lattice) [22]. We consider a finite-volume lattice system with spins (or sites) labelled as $i = 1, 2, \ldots, N$, where the Hilbert space of each spin is D-dimensional. We now define a set of the bonds Λ_b, i.e. pairs of spins $\{i_1, i_2\}$, $\{i_2, i_3\}$, $\{i_2, i_5\}$ and so on. The form of Λ_b decides the structure of the lattice. Based on this definition, we define the distance $\mathrm{dist}(X, Y)$ as the shortest-path length which one needs to connect X to Y, where X and Y are partial sets of sites as in Section 1.2 in Chapter 1.

In the following, we consider Hamiltonians which are extensive k-local operators with $k = \mathcal{O}(1)$:

$$H = \sum_{|X| \le k} h_X \quad \text{with} \quad \sum_{X \ni i} \|h_X\| \le g \quad \text{for} \quad i = 1, 2, \ldots, N. \tag{2.1}$$

In discussing short-range interacting Hamiltonians, it is often assumed that the interactions decay exponentially as

$$\sum_{X \ni \{i,i'\}} \|h_X\| \le \mathrm{const} \cdot e^{-\mathrm{const}\cdot \mathrm{dist}(i,i')}. \tag{2.2}$$

For simplicity, however, we here consider the finite-range interactions instead of the condition (2.2):

$$\sum_{X \ni \{i,i'\}} \|h_X\| = 0 \quad \text{for} \quad \mathrm{dist}(i, i') > d_H. \tag{2.3}$$

The condition (2.3) means that the two subsystems Y and Z do not directly interact with each other as long as $\mathrm{dist}(Y, Z) > d_H$. We note that there is no essential difference between (2.2) and (2.3).

Throughout this chapter, we set the ground energy to $E_0 = 0$.

2.3 Lieb-Robinson bound

In this section, we introduce the Lieb-Robinson bound and show its proof in the case of the finite-range interacting systems [22–24,28]. The Lieb-Robinson bound restricts the velocity of the information transfer through the time evolution of quantum many-body systems. In other words, if we want to send a piece of information from a subsystem X to a subsystem Y by the use of the time-evolution, we need a finite time which is proportional to the distance between X and Y. Mathematically, it can be expressed as follows.

Lieb-Robinson bound. Let A_X and B_Y be arbitrary operators on the subsystems X and Y, respectively. We then bound the norm of the commutator $[A_X(t), B_Y]$ from above by

$$\|[A_X(t), B_Y]\| \le \frac{2}{k} \|A_X\| \cdot \|B_Y\| \cdot |X| \frac{(2kg|t|)^{n_0}}{n_0!}, \tag{2.4}$$

with

$$n_0 = \left\lfloor \frac{\mathrm{dist}(X, Y)}{d_H} + 1 \right\rfloor, \tag{2.5}$$

where $A_X(t) \equiv e^{-iHt} A_X e^{iHt}$ and the Hamiltonian (2.1) is finite-range as in Eq. (2.3).

The above inequality implies that the norm of the commutator $[A_X(t), B_Y]$ is bounded from above

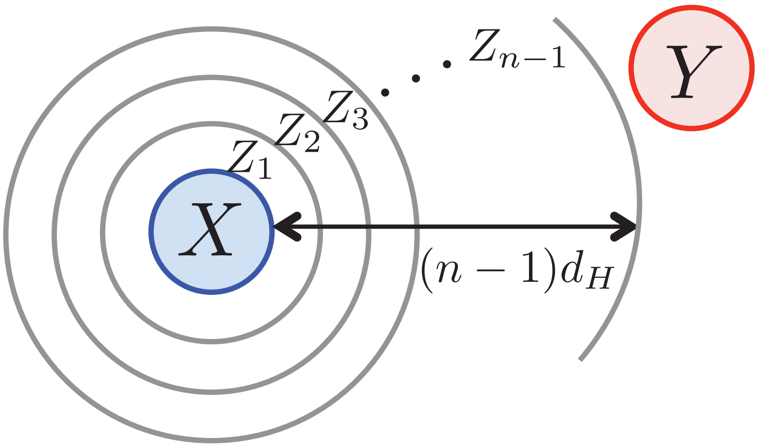

Figure 2.1: The regions $\{Z_m\}_{m=1}^n$ in the inequality (2.12) is defined as $\{Z_m : Z_m \cap Z_{m-1} \neq 0\}$. Because the interactions are short-ranged, each of the sites $\{i : i \in Z_1\}$ should satisfy $\mathrm{dist}(i, X) \leq d_H$; similarly, each of the sites $\{i : i \in Z_m\}$ satisfies $\mathrm{dist}(i, Z_m) \leq m d_H$.

by

$$\mathrm{const} \cdot \left(\frac{\mathcal{O}(|t|)}{\mathrm{dist}(X,Y)} \right)^{\mathrm{dist}(X,Y)}, \tag{2.6}$$

and hence as long as $t \lesssim \mathrm{dist}(X,Y)$, the norm is exponentially small with respect to the distance $\mathrm{dist}(X,Y)$. From the Lieb-Robinson bound, we can ensure that if an operator O is originally a short-range operator, the small time evolution also keeps the operator $O(t)$ short-range.

We often utilize the following simpler form instead of (2.4):

$$\|[A_X(t), B_Y]\| \leq \mathrm{const} \cdot \|A_X\| \cdot \|B_Y\| \cdot |X| \cdot |t| \exp\left(\frac{-\mathrm{dist}(X,Y) + v|t|}{\xi} \right) \tag{2.7}$$

with ξ and v constants of $\mathcal{O}(1)$. Although it is slightly weaker than the bound (2.4), it is mathematically easier to apply to other problems. In the case of the exponentially decaying interactions in (2.2), we obtain the bound (2.7) instead of (2.4).

We note that the Lieb-Robinson bound itself can be obtained for any Hamiltonians, not limited to the short-range ones. However, we cannot necessarily bound the norm of the commutator from above by an exponentially decaying term; for example, for systems with polynomially-decaying interactions, the Lieb-Robinson bound gives only a polynomially decaying bound with respect to the distance. The complete generalization of the Lieb-Robinson bound for arbitrary interacting Hamiltonians is given in Refs. [22–24].

Proof of the Lieb-Robinson bound. For simplicity, we consider the case $t > 0$ hereafter, but we can prove the case $t < 0$ in the same way. We start from the following inequality, which we prove below in Subsection 2.3.1:

$$\frac{d}{dt} \|[A_X(t), B_Y]\| \leq 2\|A_X\| \cdot \|[H_X(t), B_Y]\|, \tag{2.8}$$

where we define H_X as the partial Hamiltonian

$$H_X = \sum_{Z_1 : Z_1 \cap X \neq 0} h_{Z_1}. \tag{2.9}$$

By integrating the inequality (2.8) and utilizing the fact of $\|[A_X(t=0), B_Y]\| = \|[A_X, B_Y]\| = 0$, we obtain the following inequality:

$$\|[A_X(t), B_Y]\| \le 2\|A_X\| \sum_{Z_1 : Z_1 \cap X \neq 0} \int_0^t \|[h_{Z_1}(t_1), B_Y]\| dt_1. \tag{2.10}$$

Because we assume the finite-range interactions, an arbitrary site i in the set $\{Z_1 : Z_1 \cap X \neq 0\}$ satisfies $\mathrm{dist}(i_{\in Z_1}, X) \le d_H$ as shown in Fig. 2.1. If $\mathrm{dist}(X, Y) > d_H$, all the commutators $\{\|[h_{Z_1}, B_Y]\|\}$ vanish, and hence we apply the same process to $\|[h_{Z_1}(t), B_Y]\|$:

$$\|[A_X(t), B_Y]\| \le 4\|A_X\| \sum_{Z_1 : Z_1 \cap X \neq 0} \|h_{Z_1}\| \sum_{Z_2 : Z_2 \cap Z_1 \neq 0} \int_0^t \int_0^{t_1} \|[h_{Z_2}(t_2), B_Y]\| dt_2 dt_1. \tag{2.11}$$

Note that an arbitrary site i in the set Z_2 satisfies $\mathrm{dist}(i_{\in Z_2}, X) \le 2d_H$.

By iteratively applying this process, we have

$$\begin{aligned}
\|[A_X(t), B_Y]\| \le & 2^n \|A_X\| \sum_{Z_1 : Z_1 \cap X \neq 0} \|h_{Z_1}\| \sum_{Z_2 : Z_2 \cap Z_1 \neq 0} \|h_{Z_2}\| \cdots \sum_{Z_{n-1} : Z_{n-1} \cap Z_{n-2} \neq 0} \|h_{Z_{n-1}}\| \\
& \sum_{Z_n : Z_n \cap Z_{n-1} \neq 0} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \|[h_{Z_n}(t_n), B_Y]\| dt_n dt_{n-1} \cdots dt_1. \\
\le & 2^n \|A_X\| \sum_{Z_1 : Z_1 \cap X \neq 0} \|h_{Z_1}\| \sum_{Z_2 : Z_2 \cap Z_1 \neq 0} \|h_{Z_2}\| \cdots \sum_{Z_{n-1} : Z_{n-1} \cap Z_{n-2} \neq 0} \|h_{Z_{n-1}}\| \\
& \sum_{Z_n : Z_n \cap Z_{n-1} \neq 0} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} 2\|h_{Z_n}\| \cdot \|B_Y\| dt_n dt_{n-1} \cdots dt_1
\end{aligned} \tag{2.12}$$

under the assumption that X and Y satisfy the inequality $\mathrm{dist}(X, Y) > (n-1)d_H$, which reduces all the commutators $\{\|[h_{Z_{n-1}}, B_Y]\|\}$ to vanish. Then, the maximum value of such n is given by

$$n_0 = \left\lfloor \frac{\mathrm{dist}(X, Y)}{d_H} + 1 \right\rfloor. \tag{2.13}$$

We next calculate upper bounds of the summations in the inequality (2.12). Because of the extensiveness (2.1) of the Hamiltonian, we obtain

$$\sum_{Z_1 : Z_1 \cap X \neq 0} \|h_{Z_1}\| \le \sum_{i : i \in X} \sum_{Z_1 : Z_1 \ni i} \|h_{Z_1}\| \le \sum_{i : i \in X} g = g|X|. \tag{2.14}$$

Similarly, we have

$$\sum_{Z_2 : Z_1 \cap Z_2 \neq 0} \|h_{Z_2}\| \le \sum_{i : i \in Z_1} \sum_{Z_2 : Z_2 \ni i} \|h_{Z_2}\| \le \sum_{i : i \in Z_1} g = g|Z_1| \le kg, \tag{2.15}$$

where we utilize the k-locality of the Hamiltonian as in Eq. (2.1). By combining the inequalities (2.14) and (2.15) with (2.12), we obtain

$$\begin{aligned}
\|[A_X(t), B_Y]\| \le & 2^{n+1} \|A_X\| \cdot \|B_Y\| \cdot |X| k^{n-1} g^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} dt_n dt_{n-1} \cdots dt_1 \\
= & \frac{2}{k} \|A_X\| \cdot \|B_Y\| \cdot |X| \frac{(2kgt)^n}{n!}
\end{aligned} \tag{2.16}$$

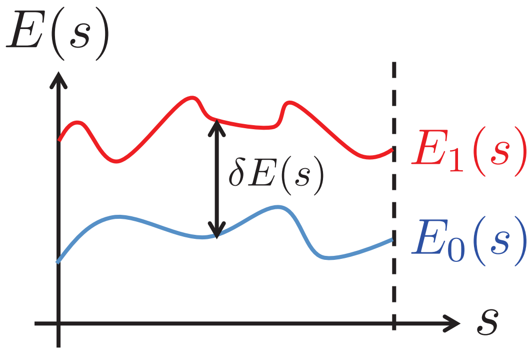

Figure 2.2: Schematic picture of the adiabatic continuation. We consider a Hamiltonian which depends on a parameter s, where we assume the spectral gap $\delta E(s)$ of $\mathcal{O}(1)$ just above the ground state. We want to know how to calculate the ground state $|E_0(s)\rangle$ from $|E_0(s=0)\rangle$.

for $\forall n \le n_0$. By choosing $n = n_0$ in the inequality (2.16), we can finally obtain the inequality (2.4).

2.3.1 Proof of the inequality (2.8)

In order to prove the inequality (2.8), we start from the following inequalities:

$$\begin{aligned}\|[A_X(t+\delta t), B_Y]\| &= \|[e^{-iH(t+\delta t)}A_X e^{iH(t+\delta t)}, B_Y]\| \\ &= \|[e^{-iHt}e^{-iH\delta t}A_X e^{iH\delta t}e^{iHt}, B_Y]\| \\ &= \|[(A_X - i\delta t[H, A_X] + \mathcal{O}(\delta t^2), B_Y(-t)]\| \\ &= \|[(A_X - i\delta t[H_X, A_X] + \mathcal{O}(\delta t^2), B_Y(-t)]\|, \end{aligned} \tag{2.17}$$

where H_X is defined in Eq. (2.9). We then obtain

$$\begin{aligned}&\|[e^{-iH_X\delta t}A_X e^{iH_X\delta t}, B_Y(-t)]\| + \mathcal{O}(\delta t^2) \\ =&\|[A_X, e^{iH_X\delta t}B_Y(-t)e^{-iH_X\delta t}]\| + \mathcal{O}(\delta t^2) \\ =&\|[A_X, B_Y(-t) + i\delta t[H_X, B_Y(-t)]\| + \mathcal{O}(\delta t^2) \\ \le&\|[A_X(t), B_Y]\| + \delta t\|[A_X, [H_X, B_Y(-t)]\| + \mathcal{O}(\delta t^2) \\ \le&\|[A_X(t), B_Y]\| + 2\delta t\|A_X\| \cdot \|[H_X(t), B_Y]\| + \mathcal{O}(\delta t^2). \end{aligned} \tag{2.18}$$

From (2.17) and (2.18), we obtain

$$\|[A_X(t+\delta t), B_Y]\| \le\|[A_X(t), B_Y]\| + 2\delta t\|A_X\| \cdot \|[H_X(t), B_Y]\| + \mathcal{O}(\delta t^2), \tag{2.19}$$

which gives

$$\frac{\|[A_X(t+\delta t), B_Y]\| - \|[A_X(t), B_Y]\|}{\delta t} \le 2\|A_X\| \cdot \|[H_X(t), B_Y]\| + \mathcal{O}(\delta t). \tag{2.20}$$

We thus prove the inequality (2.8).

2.4 Adiabatic continuation

As an important application of the Lieb-Robinson bound, we introduce the adiabatic continuation [11, 38, 39]. The adiabatic continuation considers the following problem; we first consider a Hamiltonian $H(s)$ which depends on a parameter s with the assumption that it has a non-degenerate ground state $|E_0(s)\rangle$ with a finite gap $\delta E(s) = \mathcal{O}(1)$. The problem is how to connect the ground state $|E_0(s = 0)\rangle$ to $|E_0(s)\rangle$ (see Fig. 2.2).

In the following, we assume that the Hamiltonian $H(s)$ smoothly changes with respect to the parameter s (e.g. $H(s) = H_0 + sV$). We formally describe the differential equation of the ground state $|E_0(s)\rangle$ as

$$\frac{d}{ds}|E_0(s)\rangle = iD(s)|E_0(s)\rangle, \tag{2.21}$$

where we refer to $D(s)$ as the adiabatic continuation operator. By the use of the operator $D(s)$, we can obtain the ground state $|E_0(s)\rangle$:

$$|E_0(s)\rangle = \mathcal{T}_s\big[e^{i\int_0^s D(s)ds}\big]|E_0(0)\rangle, \tag{2.22}$$

where $\mathcal{T}_s$ is the time-ordering operator. Now, the variation of the ground states can be described in the similar way to the time evolution. The goal of the problem is to obtain the specific form of $D(s)$ and prove that $D(s)$ should be a short-range operator as long as the Hamiltonian is short-ranged.

2.4.1 Derivation of the adiabatic continuation operator $D(s)$

We here derive the form of $D(s)$. We introduce the operator $V(s)$ which is defined as

$$V(s) \equiv \frac{dH(s)}{ds} \tag{2.23}$$

and assume that the operator $V(s)$ is short-ranged. We can now prove that the adiabatic continuation operator can be given as follows:

Adiabatic continuation. We can find a function $F(t)$ which gives the adiabatic continuation operator $D(s)$ as

$$iD(s) = \int_{-\infty}^{\infty} F\big(t\delta E(s)\big)e^{iH(s)t}V(s)e^{-iH(s)t}dt, \tag{2.24}$$

where the function $F(t)$ is an odd function, that is $F(t) = -F(-t)$, and satisfies the following two conditions:

$$F(t) \le e^{-\mathrm{const}\cdot t^{\alpha}} \quad \text{with} \quad 0 < \alpha < 1, \tag{2.25}$$

and

$$\tilde{F}(\omega) = \frac{1}{-\omega}, \quad \text{for} \quad |\omega| \ge 1, \tag{2.26}$$

where $\tilde{F}(\omega) \equiv \int_{-\infty}^{\infty} e^{i\omega t}F(t)dt$. Note that $\tilde{F}(0) = 0$ because $F(t)$ is an odd function.

The function $F(t)$ is a sub-exponentially decaying function with respect to t. Because we have not referred to the range $|\omega| < 1$ for $\tilde{F}(\omega)$, the choice of the function $F(t)$ is not unique. This degree of freedom reflects the uncertainty of the parameter α. The standard functional analysis [11] ensures the existence of $F(t)$ which satisfies (2.25) and (2.26).

Because of the form (2.24), we can prove that the adiabatic continuation operator should be short-ranged in the following sense:

$$\sum_{X \ni \{i,i'\}} \|D_X(s)\| \le \mathrm{const} \cdot e^{-\mathrm{const}\cdot \mathrm{dist}(i,i')^{\alpha}}, \tag{2.27}$$

where we denote $D(s)$ by $D(s) = \sum_X D_X(s)$. A qualitative explanation of the inequality (2.27) can be given as follows; first, because of the Lieb-Robinson bound, the operator $e^{iH(s)t}V(s)e^{-iH(s)t}$ in (2.24) is short-ranged as long as t is small. Second, although in the time range of $t \gg 1$ the operator $e^{iH(s)t}V(s)e^{-iH(s)t}$ may be no longer short-ranged, the function $F(t\delta E)$ is sub-exponentially small, and hence the contribution of such 'long-range' operators is sub-exponentially small. We thus reach the conclusion that the adiabatic continuation operator should be sub-exponentially short-ranged as in (2.27).

Proof of Eq. (2.24). We here prove Eq. (2.24) under the assumption of the existence of the function $F(t)$. First, because we have $E_n(s) - E_0(s) \ge \delta E$ for $n \ge 1$, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} e^{it(E_n(s)-E_0(s))} F\big(t\delta E(s)\big)dt &= \frac{1}{\delta E}\int_{-\infty}^{\infty} e^{it'\cdot\frac{E_n(s)-E_0(s)}{\delta E}} F(t')dt' \\ &= \frac{1}{E_n(s)-E_0(s)},\end{aligned} \tag{2.28}$$

where we utilized Eq. (2.26) with the inequality $\frac{E_n(s)-E_0(s)}{\delta E} \ge 1$ in the second equality.

We then expand $|E_0(s+ds)\rangle$ by the use of the standard perturbation theory:

$$|E_0(s+ds)\rangle = |E_0(s)\rangle + ds \sum_{n\neq 0} \frac{\langle E_n(s)|V(s)|E_0(s)\rangle}{E_n(s)-E_0(s)} |E_n(s)\rangle + \mathcal{O}(ds^2), \tag{2.29}$$

where we consider a finite volume system, and hence the second-order perturbation terms can be ignored in the limit of $ds \to 0$. We now substitute Eq. (2.28) for the denominator in Eq. (2.29):

$$\begin{aligned}|E_0(s+ds)\rangle &= |E_0(s)\rangle + ds \sum_{n\neq 0} \int_{-\infty}^{\infty} e^{it(E_n(s)-E_0(s))} F\big(t\delta E(s)\big)dt |E_n(s)\rangle\langle E_n(s)|V(s)|E_0(s)\rangle + \mathcal{O}(ds^2) \\ &= |E_0(s)\rangle + ds \sum_{n=0} \int_{-\infty}^{\infty} F\big(t\delta E(s)\big)\cdot |E_n(s)\rangle\langle E_n(s)| e^{iH(s)t}V(s)e^{-iH(s)t}dt|E_0(s)\rangle + \mathcal{O}(ds^2) \\ &= |E_0(s)\rangle + ds \int_{-\infty}^{\infty} F\big(t\delta E(s)\big)e^{iH(s)t}V(s)e^{-iH(s)t}dt|E_0(s)\rangle + \mathcal{O}(ds^2),\end{aligned} \tag{2.30}$$

where in the second equality we included the summand of $n = 0$ because it vanishes as

$$\begin{aligned}&\int_{-\infty}^{\infty} F\big(t\delta E(s)\big)|E_0(s)\rangle\langle E_0(s)|e^{iH(s)t}V(s)e^{-iH(s)t}dt|E_0(s)\rangle \\ =&\langle E_0(s)|V(s)|E_0(s)\rangle \int_{-\infty}^{\infty} F\big(t\delta E(s)\big)dt \cdot |E_0(s)\rangle = 0.\end{aligned} \tag{2.31}$$

Note that the property of the odd function gives $\tilde{F}(0) = \int_{-\infty}^{\infty} F\big(t\delta E(s)\big)dt = 0$. We thus prove Eq. (2.24) by combining Eq. (2.30) with the definition of $D(s)$ as in Eq. (2.21).

2.4.2 Lieb-Robinson bound for the adiabatic continuation operator $D(s)$

We here refer to the Lieb-Robinson bound for the adiabatic continuation. As in Subsection 2.3, we consider the commutator of two operators A_X and B_Y. We now define the parameter evolution $A_X(s)$

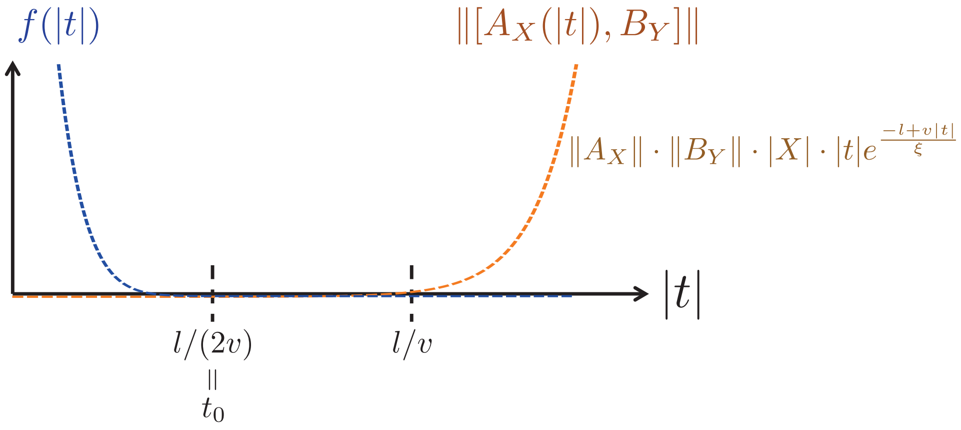

Figure 2.3: Schematic picture of the time dependence of the commutator $\|[A_X(t), B_Y]\|$ and the filter function $f(t)$.

as

$$A_X(s) \equiv U(s)^\dagger A_X U(s), \tag{2.32}$$

with

$$U(s) = \mathcal{T}_s\left[e^{i\int_0^s D(s)ds}\right]. \tag{2.33}$$

Because of (2.27), we can regard the unitary operator $U(s)$ as the time-evolution due to a sub-exponentially short-range Hamiltonian. For such a parameter evolution, we can also obtain the Lieb-Robinson bound as

$$\|[A_X(s), B_Y]\| \le \text{const} \cdot \|A_X\| \cdot \|B_Y\| \cdot |X| \cdot |s| \exp\left(\frac{-\text{dist}(X,Y)^\alpha + v|s|}{\xi}\right). \tag{2.34}$$

The essence of the proof is the same as that of (2.4).

2.5 Exponential decay of bi-partite correlation

We here consider an important application of the Lieb-Robinson bound to gapped ground state. We denote the spectral gap between the non-degenerate ground state and the first excited state as δE with the assumption that δE is smaller than a constant, namely $\delta E \lesssim 1$. In fact, this condition should be always satisfied for short-range interacting systems.

Exponential decay of bi-partite correlations. *[21–24] Let us consider two operators A_X and B_Y with the assumptions $\|A_X\| = \|B_Y\| = 1$ and $|X| = |Y| = \mathcal{O}(1)$. For any two operators A_X and B_Y, the bi-partite correlation can be exponentially bounded from above by*

$$\left|\langle E_0|A_X B_Y|E_0\rangle - \langle E_0|A_X|E_0\rangle\langle E_0|B_Y|E_0\rangle\right| \le e^{-\text{const}\cdot l\delta E}, \tag{2.35}$$

where $l = \text{dist}(X, Y)$.

Proof of the inequality (2.35). The sketch of the proof is given as follows. We start from the Lieb-

Robinson bound as

$$\|[A_X(t), B_Y]\| \le \text{const} \cdot |t| \exp\left(\frac{-l + v|t|}{\xi}\right). \tag{2.36}$$

We first expand the expectation value $\langle E_0|[A_X(t), B_Y]|E_0\rangle$ by the use of the spectral decomposition:

$$\begin{aligned}\langle E_0|[A_X(t), B_Y]|E_0\rangle &= \sum_{n\neq 0}\big(\langle E_0|A_X(t)P_nB_Y|E_0\rangle - \langle E_0|B_YP_nA_X(t)|E_0\rangle\big)\\ &= \sum_{n\neq 0}\big(e^{it(E_n-E_0)}\langle E_0|A_XP_nB_Y|E_0\rangle - e^{-it(E_n-E_0)}\langle E_0|B_YP_nA_X|E_0\rangle\big),\end{aligned} \tag{2.37}$$

where $P_n \equiv |E_n\rangle\langle E_n|$ for $n = 0, 1, 2, \ldots$. In order to prove the exponential decay of the correlation, we want to find a filter function $f(t)$ which approximately gives

$$\int_{-t_0}^{t_0} dt f(t)\langle E_0|[A_X(t), B_Y]|E_0\rangle \simeq 0, \tag{2.38}$$

and

$$\int_{-t_0}^{t_0} e^{-i\omega t} f(t)dt \simeq 0, \quad \int_{-t_0}^{t_0} e^{i\omega t} f(t)dt \simeq 1, \tag{2.39}$$

for $\omega > 0$, where $[-t_0, t_0]$ is an appropriate time range. If we find such a filter function, from (2.38) and (2.39), the integration of Eq. (2.37) with the filtering $f(t)$ reduces to

$$\sum_{n\neq 0}\langle E_0|A_XP_nB_Y|E_0\rangle \simeq 0, \tag{2.40}$$

and hence we obtain $\langle E_0|A_X(1 - |E_0\rangle\langle E_0|)B_Y|E_0\rangle \simeq 0$ because of $\sum_{n\neq 0} P_n = 1 - |E_0\rangle\langle E_0|$. This means that the correlation $\big|\langle E_0|A_XB_Y|E_0\rangle - \langle E_0|A_X|E_0\rangle\langle E_0|B_Y|E_0\rangle\big|$ is approximately equal to zero. Our task is to find such a filter function $f(t)$ and estimate how the approximations (2.38) and (2.39) depend on the spectral gap δE and the distance l.

We can prove that the choice of the filter function which we present in Subsection 2.5.1 gives

$$\begin{aligned}&\int_{-t_0}^{t_0} e^{-i\omega t} f(t)dt = e^{-\text{const}\cdot\omega l},\\ &\int_{-t_0}^{t_0} e^{i\omega t} f(t)dt = 1 + e^{-\text{const}\cdot\omega l},\end{aligned} \tag{2.41}$$

for $\omega \ge \delta E$, where $t_0 = l/(2v)$. We next multiply Eq. (2.37) by this filter function $f(t)$ and integrate it from $t = -t_0$ to $t = t_0$ (see Fig. 2.3); then, the left-hand side of Eq. (2.37) reduces to

$$\int_{-t_0}^{t_0} dt f(t)\langle E_0|[A_X(t), B_Y]|E_0\rangle \le \text{const} \cdot e^{-\text{const}\cdot l} \tag{2.42}$$

because of the Lieb-Robinson bound (2.36) and the definition of t_0. On the other hand, Eq. (2.41) with $E_n - E_0 \ge \delta E$ reduces the right-hand side of Eq. (2.37) to

$$\int_{-t_0}^{t_0} dt f(t)\sum_{n\neq 0} e^{-it(E_n-E_0)}\langle E_0|B_YP_nA_X|E_0\rangle = \text{const} \cdot e^{-\text{const}\cdot l\delta E}$$

$$\int_{-t_0}^{t_0} dt f(t) \sum_{n\neq 0} e^{it(E_n-E_0)}\langle E_0|A_X P_n B_Y|E_0\rangle = \text{const}\cdot e^{-\text{const}\cdot l\delta E} + \sum_{n\neq 0}\langle E_0|B_Y P_n A_X|E_0\rangle$$
$$= e^{-\text{const}\cdot l\delta E} + \langle E_0|B_Y(1-|E_0\rangle\langle E_0|)A_X|E_0\rangle. \qquad (2.43)$$

The inequalities (2.42) and (2.43), thereby yield

$$\langle E_0|A_X(1-|E_0\rangle\langle E_0|)B_Y|E_0\rangle \le e^{-\text{const}\cdot l\delta E} + e^{-\text{const}\cdot l}, \qquad (2.44)$$

which gives the inequality (2.35) because of $\delta E \lesssim 1$.

2.5.1 Choice of the filter function

In the above, we utilized the following filter function [22]:

$$f(t) = \frac{i}{2\pi}\lim_{\epsilon\to 0}\frac{e^{-\alpha t^2}}{t+i\epsilon} \qquad (2.45)$$

with $\alpha = \delta E/l$. We can prove that this function satisfies

$$\lim_{T\to\infty}\int_{-T}^{T} f(t)e^{i\omega t}dt = \begin{cases} 1+e^{-\text{const}\cdot\omega^2/\alpha} & \text{for} \quad \omega > 0, \\ e^{-\text{const}\cdot\omega^2/\alpha} & \text{for} \quad \omega < 0. \end{cases} \qquad (2.46)$$

We can ensure by straightforward calculation that the above definition of the filter function gives (2.41) and (2.42).

2.5.2 In the case of degenerate ground states

In the case where the ground states are degenerate, we can also obtain a similar bound to (2.35):

$$\mathrm{tr}\big[P_0(A_X B_Y - A_X P_0 B_Y)\big] \le e^{-\text{const}\cdot l\delta E}, \qquad (2.47)$$

where we denote the projection to the ground states' subspace by P_0. The projection operator P_0 is proportional to the uniformly mixed ground states, namely $\rho_0 \equiv P_0/m = m^{-1}\sum_{j=0}^{m-1}|E_j\rangle\langle E_j|$, where $\{|E_j\rangle\}_{j=0}^{m-1}$ are the degenerated ground states. The inequality (2.47) also gives the exponential decay of the correlation for ρ_0 in the following sense:

$$\big|\mathrm{tr}(\rho_0 B_Y A_X)\big| \le \frac{1}{m^2}e^{-\text{const}\cdot l\delta E} + \frac{1}{m^2}\left|\sum_{j,j'}^{m-1}(A_X)_{j,j'}(B_Y)_{j',j}\right|, \qquad (2.48)$$

where we assume $\mathrm{tr}(P_0 A_X) = \mathrm{tr}(P_0 B_Y) = 0$ and define $(A_X)_{j,j'} \equiv \langle E_j|A_X|E_{j'}\rangle$ for $j, j' = 0, 1, 2, \ldots, m-1$. We can see that the bipartite correlation can be bounded from above by the exponentially decaying term plus a term of matrix elements between the degenerated ground states.

2.6 Entropic area law for one-dimensional systems

We finally show the entropic area law in one-dimensional systems. When we assume the existence of the spectral gap of $\mathcal{O}(1)$, we empirically know that the ground state satisfies the entropic area law; that is, when we split the total system into two subsystems, the entanglement entropy with respect to this split is bounded from above by the boundary of the subregion. In one-dimensional cases, the boundary of the subregions has the cardinality of $\mathcal{O}(1)$, and hence the entropic area law means the inequality $S(|E_0\rangle) \lesssim 1$ with $S(|E_0\rangle)$ the entanglement entropy (see below on the definition of $S(|E_0\rangle)$).

The entropic area law for gapped ground states, however, has not been completely proved except for the one-dimensional case although it is also conjectured for higher-dimensional cases [19]. The first proof of the one-dimensional area law has been given by Hastings in 2007 [12]. However, the bound is exponentially large with respect to the spin dimension D as $S(|E_0\rangle) \le e^{\mathcal{O}(D)}$ and far from the optimal bound.[*3] Until now, the best upper bound of the entanglement entropy is given by $S(|E_0\rangle) \le \left(\log \mathcal{O}(D)\right)^3$ which has been proved by Arad, *et.al.* in 2013 [15]. We introduce the basic ideas of their proof in this section.

2.6.1 Several definitions

2.6.1 (a) 1D setting

We here consider a one-dimensional system and set $d_H = 1$, $k = 2$ and $g \le 1$ in the Hamiltonian (2.1) with (2.3), namely

$$H = \sum_{i=1}^{N-1} h_{\{i,i+1\}}, \quad \text{with} \quad \|h_{\{i-1,i\}}\| + \|h_{\{i,i+1\}}\| \le 1 \tag{2.49}$$

We assume a non-degenerate ground state $|E_0\rangle$ with a spectral gap δE. We spatially split the total space into two subsystems with their Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. We denote the reduced density matrix of the ground state in $\mathcal{H}_1$ by $\rho_0^{(1)}$:

$$\rho_0^{(1)} = \mathrm{tr}_{\mathcal{H}_2}(|E_0\rangle\langle E_0|), \tag{2.50}$$

where $\mathrm{tr}_{\mathcal{H}_2}(\cdots)$ denotes the partial trace operation with respect to the Hilbert space $\mathcal{H}_2$. We define the entanglement entropy of this split as

$$S(|E_0\rangle, \mathcal{H}_1) \equiv -\mathrm{tr}(\rho_0^{(1)} \log \rho_0^{(1)}). \tag{2.51}$$

In the following discussion, we omit the explicit notation of $\mathcal{H}_1$ and denote only $S(|E_0\rangle)$ for simplicity.

2.6.1 (b) Schmidt rank

We consider an operator O and define the Schmidt rank $\mathrm{SR}(O)$ as the minimal integer D_O such that

$$O = \sum_{m=1}^{D_O} O_1^m \otimes O_2^m. \tag{2.52}$$

For example, the Schmidt rank of the Hamiltonian is at most $\mathcal{O}(D^2)$ because we now assume the nearest-neighbor interactions and each of the site has a D-dimensional Hilbert space.

We also define the Schmidt rank of a state $|\psi\rangle$ as an integer D_ψ which appears in its Schmidt decomposition:

$$|\psi\rangle = \sum_{m=1}^{D_\psi} \mu_m |\psi_{1,m}\rangle \otimes |\psi_{2,m}\rangle. \tag{2.53}$$

[*3] Even then, if $D = \mathcal{O}(1)$, the entropy can be bounded from above by an $\mathcal{O}(1)$ constant.

2.6.1 (c) Approximate ground state projection (AGSP)

We here introduce the projection operator onto the ground state. It is usually difficult to construct the exact ground-state projection operator, and hence we consider an approximate one as

$$K|E_0\rangle \simeq |E_0\rangle \quad \text{and} \quad \|K\Pi^H_{[E_1,\infty)}\|^2 \simeq 0, \tag{2.54}$$

where $\Pi^H_{[E_1,\infty)}$ is the projection operator onto the eigenspace of the energies which are in $[E_1,\infty)$. In the following, we characterize the approximate ground state projection (AGSP) operators by several parameters $\{\delta_K, \Delta_K, D_K\}$. For a quantum state $|\tilde{E}_0\rangle$, the AGSP operator satisfies

$$K|\tilde{E}_0\rangle = |\tilde{E}_0\rangle, \tag{2.55}$$

with

$$\delta_K \equiv \||E_0\rangle - |\tilde{E}_0\rangle\|^2, \quad \Delta_K \equiv \|K(1 - |\tilde{E}_0\rangle\langle\tilde{E}_0|)\|^2 \quad \text{and} \quad D_K \equiv \mathrm{SR}(K). \tag{2.56}$$

Note that the state $|\tilde{E}_0\rangle$ is an approximate ground state if $\delta_K \simeq 0$. When $\delta_K = \Delta_K = 0$, the operator K is the exact ground state projection, namely $K = |E_0\rangle\langle E_0|$.

2.6.1 (d) The Young-Eckart theorem

We here introduce the Young-Eckart theorem [49] without the proof. Let us consider a normalized state $|\psi\rangle$ and give its Schmidt decomposition as

$$|\psi\rangle = \sum_{m=1} \mu_m |\psi_{1,m}\rangle \otimes |\psi_{2,m}\rangle, \tag{2.57}$$

where $\mu_1 \ge \mu_2 \ge \mu_3 \cdots$. We then consider another normalized state $|\phi\rangle$ with its Schmidt rank D_ϕ and define the overlap with the state $|\psi\rangle$ as

$$\||\phi\rangle - |\psi\rangle\|. \tag{2.58}$$

The Young-Eckart theorem gives the following inequality:

$$\sum_{m>D_\phi} \mu_m^2 \le \||\phi\rangle - |\psi\rangle\|^2. \tag{2.59}$$

2.6.2 Proof of one-dimensional area law (by Arad, Kitaev, Landau and Vazirani)

We here prove the following statement:

One-dimensional area law. For one-dimensional (1D) systems defined in Subsection 2.6.1 (a), we can obtain the upper bound of the entanglement entropy as

$$S(|E_0\rangle) \le \text{const} \cdot \frac{1}{\delta E}\left(\log \frac{D}{\delta E}\right)^3 \tag{2.60}$$

where D is the dimension of the Hilbert space of each spin.

Proof of 1D area law. We show the outline of the proof in Fig. 2.4. We first completely break the entanglement entropy of the ground state by performing a projection operator onto a product state with

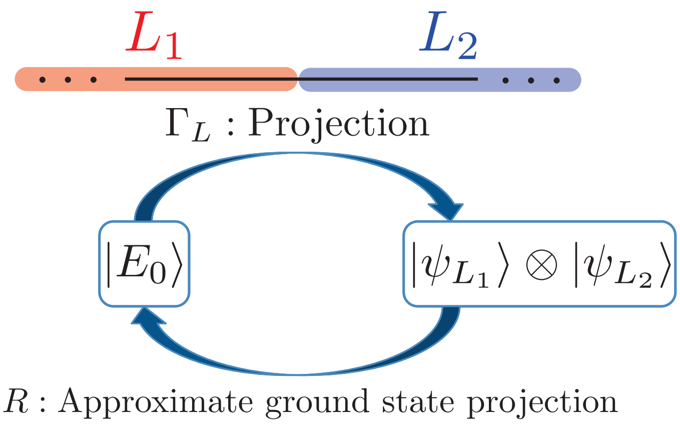

Figure 2.4: The schematic picture of the proof of the one-dimensional area law. We split the total system into the two subsystems L_1 and L_2. In the proof of the area law, we first perform a projection onto a product state which has the maximum overlap with $|E_0\rangle$. We then recover the original state from this product state by the use of the approximate ground state projector (AGSP). The entanglement entropy can be bounded from above by the entanglement generation of AGSP (the Schmidt rank of R) because the product state contains no entanglement.

respect to the partition. Note that the entanglement entropy is equal to zero for product states.

We second consider a reverse operator R from the product state to the ground state $|E_0\rangle$. The operator R is now taken as an approximate ground state projector (AGSP) K, namely $R = K \simeq |E_0\rangle\langle E_0|$. If the Schmidt rank of the AGSP operator K is sufficiently suppressed, we can prove that the ground state hardly contains the entanglement entropy.

The problem is now summarized as follows:

1. How do we mathematically relate the AGSP operator to the entanglement entropy? (Step 1 to Step 3 below)

2. How do we construct the optimal AGSP operator? (Step 4 to Step 6 below)

3. How do we estimate the Schmidt rank of the AGSP operator? (Step 7 below)

2.6.2 (a) Step 1: Upper bound of the entropy

We first relate the AGSP operator to the ground-state entropy $S(|E_0\rangle)$. For this purpose, we consider a sequence of AGSP $K_0, K_1, K_2, \ldots, K_s, \ldots$, whose errors Δ_{K_s} and δ_{K_s} decrease with respect to the index s, namely $\Delta_0 \geq \Delta_1 \geq \Delta_2 \geq \cdots$ and $\delta_0 \geq \delta_1 \geq \delta_2 \geq \cdots$; we choose K_s so that K_∞ may satisfy $\Delta_{K_\infty} = 0$, $\delta_{K_\infty} = 0$; in other words, K_∞ is the exact ground-state projector.

We define that AGSP operator K_s has a state $|\tilde{E}_{s,0}\rangle$ such that $K_s|\tilde{E}_{s,0}\rangle = |\tilde{E}_{s,0}\rangle$ with Eq. (2.56) (see Subsection 2.6.1 (c)). We then expand the state $|\tilde{E}_{s,0}\rangle$ by the use of the Schmidt decomposition:

$$|\tilde{E}_{s,0}\rangle = \sum_{m=1} \mu_{s,m}|\mathrm{Prod}_{s,m}\rangle, \tag{2.61}$$

where $\{|\mathrm{Prod}_{s,m}\rangle\}$ are product states with respect to the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, and we take $\{\mu_{s,m}\}$ in non-ascending order as $\mu_{s,1} \geq \mu_{s,2} \geq \mu_{s,3} \cdots$. In particular, we denote the exact ground state $|E_0\rangle = |\tilde{E}_{\infty,0}\rangle$ by

$$|E_0\rangle = \sum_{m=1} \mu_m|\mathrm{Prod}_m\rangle. \tag{2.62}$$

From the above definition (2.61), we can express the product state $|\mathrm{Prod}_{s,1}\rangle$ as

$$|\mathrm{Prod}_{s,1}\rangle = \mu_{s,1}|\tilde{E}_{s,0}\rangle + \sqrt{1-\mu_{s,1}^2}|\psi_{s,\perp}\rangle, \tag{2.63}$$

where $|\psi_{s,\perp}\rangle$ is a state orthogonal to $|\tilde{E}_{s,0}\rangle$. Note that the state $|\mathrm{Prod}_{s,1}\rangle$ has the maximum overlap with $|\tilde{E}_{s,0}\rangle$.

We now construct an approximate ground state by means of $K_s|\mathrm{Prod}_{s,1}\rangle$ and want to know how close it is to the exact ground state $|E_0\rangle$. We denote the norm of $K_s|\mathrm{Prod}_{s,1}\rangle$ by $\langle K_s\rangle_{P1}$, namely $\langle K_s\rangle_{P1}^2 = \mu_{s,1}^2 + (1-\mu_{s,1})^2\langle\psi_{s,\perp}|K_s^2|\psi_{s,\perp}\rangle$. We here define

$$\bar{\gamma}_s \equiv \left\| \frac{K_s}{\langle K_s\rangle_{P1}}|\mathrm{Prod}_{s,1}\rangle - |E_0\rangle \right\|, \tag{2.64}$$

which is the norm distance between the approximate ground state and the exact ground state. From the Young-Eckart theorem, we obtain

$$\sum_{m>D_{K_s}} \mu_m^2 \le \left\| \frac{K_s}{\langle K_s\rangle_{P1}}|\mathrm{Prod}_{s,1}\rangle - |E_0\rangle \right\|^2 = \bar{\gamma}_s^2, \tag{2.65}$$

where we utilized the fact that the Schmidt rank of $K_s|\mathrm{Prod}_{s,1}\rangle$ is equal to D_{K_s} because of the definition (2.56).

We next define

$$\gamma_s^2 \equiv \sum_{D_{K_s}<m\le D_{K_{s+1}}} \mu_m^2 \le \bar{\gamma}_s^2, \tag{2.66}$$

where the last inequality comes from (2.65). Therefore, if $\bar{\gamma}_s^2 \le 1/e$, we obtain the inequality $-\gamma_s^2\log\gamma_s^2 \le -\bar{\gamma}_s^2\log\bar{\gamma}_s^2$, and hence the upper bound of the entanglement entropy is given by

$$\begin{aligned} S(|E_0\rangle) &\le \log D_{K_0} - \sum_{s=0}^{\infty}\gamma_s^2 \log\frac{\gamma_s^2}{D_{K_{s+1}}} \\ &\le \log D_{K_0} - \sum_{s=0}^{\infty}\bar{\gamma}_s^2 \log\frac{\bar{\gamma}_s^2}{D_{K_{s+1}}}. \end{aligned} \tag{2.67}$$

From the above upper bound, for example, in the case where

$$D_{K_s} \le e^{c_0(s+s^*)\log\left[D(s+s^*)\right]} \quad \text{and} \quad \bar{\gamma}_s^2 \le e^{-c_\gamma(s+s^*)} \tag{2.68}$$

with $c_0 = \mathcal{O}(1)$ and $c_\gamma = \mathcal{O}(1)$, we obtain the upper bound of the entanglement entropy as

$$\begin{aligned} S(|E_0\rangle) &\le c_0 s^*\log(Ds^*) + \sum_{s=0}^{\infty} e^{-c_\gamma(s+s^*)}(s+s^*)\left[c_\gamma + c_0\log(Ds+Ds^*)\right] \\ &\le s^*\log(Ds^*)\left[c_0 + \mathcal{O}(e^{-c_\gamma s^*})\right], \end{aligned} \tag{2.69}$$

where we utilized the definition of the AGSP operator in Subsection 2.6.1 (c).

2.6.2 (b) Step 2: Upper bound of the norm $\bar{\gamma}_s$

As shown in the inequality (2.67), for the calculation of the entanglement entropy, we need to obtain the upper bound of the norm

$$\bar{\gamma}_s \equiv \left\| |E_0\rangle - \frac{K_s|\mathrm{Prod}_{s,1}\rangle}{\langle K_s\rangle_{P1}} \right\|. \tag{2.70}$$

For the calculation of $\bar{\gamma}_s$, we first obtain

$$\begin{aligned}
\left\| |\tilde{E}_{s,0}\rangle - \frac{K|\mathrm{Prod}_{s,1}\rangle}{\langle K_s\rangle_{P1}} \right\|^2 =& \left\| |\tilde{E}_{s,0}\rangle - \frac{\mu_{s,1}|\tilde{E}_{s,0}\rangle + \sqrt{1-\mu_{s,1}^2}K_s|\psi_{s,\perp}\rangle}{\langle K_s\rangle_{P1}} \right\|^2 \\
=& \frac{(\langle K_s\rangle_{P1} - \mu_{s,1})^2 + (1-\mu_{s,1}^2)\langle\psi_{s,\perp}|K_s^2|\psi_{s,\perp}\rangle}{\langle K_s\rangle_{P1}^2} \\
=& \frac{2\langle K_s\rangle_{P1}^2 - 2\mu_{s,1}\langle K_s\rangle_{P1}}{\langle K_s\rangle_{P1}^2} \\
=& 2 - 2\left(1 + \frac{1-\mu_{s,1}^2}{\mu_{s,1}^2}\langle\psi_{s,\perp}|K_s^2|\psi_{s,\perp}\rangle\right)^{-1/2} \\
\leq& \frac{1-\mu_{s,1}^2}{\mu_{s,1}^2}\langle\psi_{s,\perp}|K_s^2|\psi_{s,\perp}\rangle \leq \frac{1}{\mu_{s,1}^2}\Delta_{K_s},
\end{aligned} \tag{2.71}$$

where we utilized the inequality $\langle\psi_{s,\perp}|K_s^2|\psi_{s,\perp}\rangle \leq \Delta_{K_s}^2$, which comes from the definition (2.56). We thus obtain

$$\begin{aligned}
\bar{\gamma}_s = \left\| |E_0\rangle - \frac{K_s|\mathrm{Prod}_{s,1}\rangle}{\langle K_s\rangle_{P1}} \right\| &\leq \left\| |\tilde{E}_{s,0}\rangle - \frac{K_s|\mathrm{Prod}_{s,1}\rangle}{\langle K\rangle_{P1}} \right\| + \||\tilde{E}_{s,0}\rangle - |E_0\rangle\| \\
&\leq \frac{\sqrt{\Delta_{K_s}}}{\mu_{s,1}} + \sqrt{\delta_{K_s}},
\end{aligned} \tag{2.72}$$

where the last inequality comes from the inequality (2.71) and the definition $\delta_{K_s} \equiv \||\tilde{E}_{s,0}\rangle - |E_0\rangle\|^2$.

2.6.2 (c) Step 3: Upper bound of $\mu_{s,1}$ and the bootstrapping lemma

In Step 1, we have shown that the entropy bound can be obtained from the parameter $\bar{\gamma}_s$. In Step 2, we have obtained the upper bound of the parameter $\bar{\gamma}_s$ in terms of Δ_{K_s}, δ_{K_s} and $\mu_{s,1}$. We now want to derive the relationship between the coefficient $\mu_{s,1}$ and AGSP parameters $\{\delta_{K_s}, \Delta_{K_s}, D_{K_s}\}$, which allows us to obtain the upper bound of the entanglement entropy only by the AGSP parameters.

We here obtain the following statement called "the bootstrapping lemma"; if $\Delta_{K_s}D_{K_s} \leq 1/2$, $\mu_{s,1}$ is bounded from below by

$$\mu_{s,1} \geq \frac{1}{\sqrt{2D_{K_s}}}. \tag{2.73}$$

By combining the inequality (2.73) with (2.72), we have

$$\bar{\gamma}_s \leq \sqrt{2\Delta_{K_s}D_{K_s}} + \sqrt{\delta_{K_s}}. \tag{2.74}$$

Proof of the bootstrapping lemma

We first give the Schmidt decomposition of $K_s|\mathrm{Prod}_{s,1}\rangle$ by

$$K_s|\mathrm{Prod}_{s,1}\rangle = \sum_{m=1}^{D_{K_s}} \mu_m^{(K_s)} |\mathrm{Prod}_m^{(K_s)}\rangle. \tag{2.75}$$

Note that $K|\mathrm{Prod}_{s,1}\rangle$ is not normalized. We obtain

$$\begin{aligned}\langle \tilde{E}_{s,0}|K_s|\mathrm{Prod}_{s,1}\rangle &= \sum_{m=1}^{D_K} \mu_m^{(K_s)} \langle \tilde{E}_{s,0}|\mathrm{Prod}_m^{(K_s)}\rangle \\ &\le \sqrt{\sum_{m=1}^{D_{K_s}} (\mu_m^{(K_s)})^2} \sqrt{\sum_{m=1}^{D_{K_s}} |\langle \tilde{E}_{s,0}|\mathrm{Prod}_m^{(K_s)}\rangle|^2} \\ &= \| K_s|\mathrm{Prod}_{s,1}\rangle\| \sqrt{\sum_{m=1}^{D_{K_s}} |\langle \tilde{E}_{s,0}|\mathrm{Prod}_m^{(K_s)}\rangle|^2},\end{aligned} \tag{2.76}$$

where the first inequality comes from the Cauchy-Schwartz inequality. We now have

$$\begin{aligned}&\langle \tilde{E}_{s,0}|K|\mathrm{Prod}_{s,1}\rangle = \mu_{s,1}, \\ &\sum_{m=1}^{D_{K_s}} |\langle \tilde{E}_{s,0}|\mathrm{Prod}_m^{(K_s)}\rangle|^2 \le \sum_{m=1}^{D_{K_s}} \mu_{s,1}^2 = D_{K_s}\mu_{s,1}^2, \\ &\| K_s|\mathrm{Prod}_{s,1}\rangle\| = \langle K_s\rangle_{P1} \le \sqrt{\mu_{s,1}^2 + \Delta_{K_s}},\end{aligned} \tag{2.77}$$

where the first equality comes from the definition of $|\tilde{E}_{s,0}\rangle$ as in Eq. (2.61), the second inequality comes from the fact that for an arbitrary product state the overlap with $|\tilde{E}_{s,0}\rangle$ is smaller than $\langle \tilde{E}_{s,0}|\mathrm{Prod}_{s,1}\rangle = \mu_{s,1}$ and the third inequality comes from $\langle \psi_{s,\perp}|K_s^2|\psi_{s,\perp}\rangle \le \Delta_{K_s}^2$. Thus, the inequality (2.76) reduces to

$$\mu_{s,1} \le \sqrt{\mu_{s,1}^2 + \Delta_{K_s}} \sqrt{D_{K_s}\mu_{s,1}^2}, \tag{2.78}$$

which gives the inequality

$$\mu_{s,1}^2 \ge \frac{1}{D_{K_s}} - \Delta_{K_s} = \frac{1 - D_{K_s}\Delta_{K_s}}{D_{K_s}} \ge \frac{1}{2D_{K_s}}, \tag{2.79}$$

where we utilized $D_{K_s}\Delta_{K_s} \le 1/2$. This completes the proof.

2.6.2 (d) Step 4: The condition of AGSP

We prove that the entanglement entropy of the ground state follows the area law if we can construct the AGSP operator $K(r,l)$ which satisfies the following conditions:

$$D_{K(r,\tau)} \le (Dl)^{c_0 l}, \quad \Delta_{K(r,\tau)} \le e^{-c_0 r^2\sqrt{\delta E/l}}, \quad \delta_{K(r,l)} \le e^{-c_0 l}, \quad \forall l \ge r, \tag{2.80}$$

with c_0 an $\mathcal{O}(1)$ constant, where r and l are parameters of positive integers to characterize the AGSP operator. When we choose $l = r$, the above inequalities reduce to

$$D_{K(r,r)} \le (Dr)^{c_0 r}, \quad \Delta_{K(r,r)} \le e^{-c_0 r^{3/2}\sqrt{\delta E}}, \quad \delta_{K(r,r)} \le e^{-c_0 r}. \tag{2.81}$$

Using $K(r,\tau)$, we here choose the sequence of the AGSP operator $\{K_s\}$ as

$$K_s = K(s+s^*, s+s^*), \tag{2.82}$$

where s^* is an integer which we will appropriately set afterward. We now obtain

$$D_{K_s} \le e^{c_0(s+s^*)\log\left[D(s+s^*)\right]}, \quad \Delta_{K_s} \le e^{-c_0(s+s^*)^{3/2}\sqrt{\delta E}}, \quad \delta_{K_s} \le e^{-c_0(s+s^*)}, \tag{2.83}$$

which reduces the inequality (2.74) to

$$\begin{aligned}\bar{\gamma}_s &\le \sqrt{2D_{K_s}\Delta_{K_s}} + \sqrt{\delta_{K_s}} \\ &\le \sqrt{2}\exp\left\{-\frac{c_0}{2}(s+s^*)\left[(s+s^*)^{1/2}\sqrt{\delta E} - \log\left[D(s+s^*)\right]\right]\right\} + e^{-c_0(s+s^*)/2}.\end{aligned} \tag{2.84}$$

We then choose

$$s^* = \text{const} \times \frac{1}{\delta E}\left(\log\frac{D}{\delta E}\right)^2, \tag{2.85}$$

such that $(s+s^*)^{1/2}\sqrt{\delta E} - \log\left[D(s+s^*)\right] \ge \text{const} > 0$ for $s = 0, 1, 2, \ldots$, which reduces the inequality (2.84) to

$$\bar{\gamma}_s \le e^{-c_\gamma(s+s^*)} \tag{2.86}$$

with c_γ an $\mathcal{O}(1)$ constant. Because D_{K_s} and $\bar{\gamma}_s$ satisfy the inequality (2.68), we finally obtain the upper bound of the entropy as

$$S(|E_0\rangle) \le \text{const} \times s^* \log(Ds^*), \tag{2.87}$$

where we utilized the inequality (2.69). This gives the entropy bound as in the inequality (2.60) by combining the inequality (2.87) with the form of s^*.

2.6.2 (e) Step 5: The construction of AGSP

We, in the following step, show that we can indeed find the AGSP operator as in (2.80). In order to construct the AGSP, we utilize a polynomial of the Hamiltonian Poly(H). For example, one of the candidate for AGSP $\{K_s\}$ is given by

$$K_s = \sum_{m=0}^{\lceil\sqrt{s}\rceil} \frac{(-\sqrt{s}H)^m}{m!}. \tag{2.88}$$

In this case, K_s approaches $e^{-\sqrt{s}H}$, and we thereby obtain the exact ground-state projection in the limit of $s \to \infty$. In fact, we can show that this AGSP cannot give the area law according to the upper bound by the inequalities (2.67) and (2.74). For the proof of the area law, we have to construct an AGSP operator with a higher accuracy and a lower Schmidt rank.

For this purpose, we construct an mth-order polynomial $K(m,x)$ such that

$$K(m,0) = 1, \quad K(m,x) \le \epsilon_m, \tag{2.89}$$

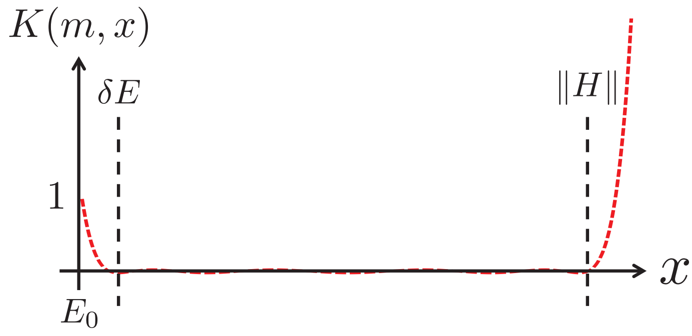

Figure 2.5: By the use of the Chebyshev polynomials, we can construct a function $K(m,x)$ which approximately satisfies $K(m,0)=1$ and $K(m,x)\simeq 0$ for $\delta E \le x \le \|H\|$. The error of the approximation is bounded from above as in (2.93).

for $\delta E \le x \le \|H\|$ with ϵ_m a positive number. From the definition $E_0=0$, this polynomial gives

$$\|K(m,H) - |E_0\rangle\langle E_0|\| \le \epsilon_m. \tag{2.90}$$

As the polynomial $K(m,x)$, we here utilize the Chebyshev polynomial (Fig. 2.5):

$$K(m,x) = \frac{T_m\left[\frac{2x-(\|H\|+\delta E)}{\|H\|-\delta E}\right]}{T_m\left[-\frac{\|H\|+\delta E}{\|H\|-\delta E}\right]}, \tag{2.91}$$

where $T_m(x)$ is the mth-order Chebyshev polynomial, which satisfies

$$|T_m(x)| \le 1 \quad \text{for} \quad -1 \le x \le 1,$$
$$|T_m(x)| \ge \frac{e^{2m\sqrt{(x+1)/(x-1)}}}{2} \quad \text{for} \quad x \le -1. \tag{2.92}$$

Note that $K(m,x)$ satisfies $K(m,0)=1$. The basic properties (2.92) give

$$\epsilon_m \le \frac{1}{T_m\left[-\frac{\|H\|+\delta E}{\|H\|-\delta E}\right]} \le 2e^{-2m\sqrt{\delta E/\|H\|}}. \tag{2.93}$$

We therefore conclude that the error of the AGSP $K(m,H)$ decreases as $e^{-2m\sqrt{\delta E/\mathcal{O}(N)}}$. However, in this case, we have to take m as large as $\mathcal{O}(\sqrt{N})$ for a good approximation, which may result in a high Schmidt rank of the AGSP operator. We thus have to achieve a good approximation with small m. We thereby consider an effective Hamiltonian instead of the original Hamiltonian.

2.6.2 (f) Step 6: Effective Hamiltonian (see also Section 3.4 in Chapter 3)

We here consider an effective Hamiltonian $\tilde{H}$ which has almost the same ground state as the original one. The points are the followings:

1. The effective Hamiltonian has the norm much smaller than that of the original Hamiltonian, namely $\|\tilde{H}\| \ll \|H\|$.

2. The Schmidt rank of polynomials of $\tilde{H}$ should be highly suppressed. This condition implies that the effective Hamiltonian $\tilde{H}$ should still contain the locality.

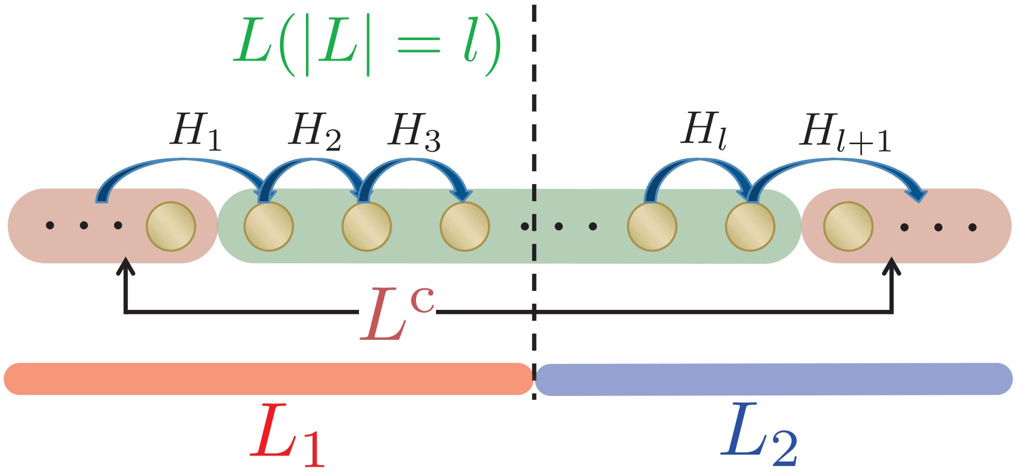

Figure 2.6: The total system is split into the two subsystems L and $L^{\rm c}$. We denote the number of spins in L by l. We decompose the total Hamiltonian by the use of the set of Hamiltonians $\{H_i\}_{i=1}^{l+1}$.

Note that because of the inequality (2.93) the norm of the Hamiltonian critically decides the error of the AGSP operators. By applying the AGSP construction in Step 5 to the effective Hamiltonian, we can prove below that the AGSP $K(m, \tilde{H})$ satisfies the second and the third inequalities in (2.80).

For the construction of such an effective Hamiltonian, we split the total system into the regions L and $L^{\rm c}$ as in Fig 2.6. We also define the number of spin in the region L as l, namely $l \equiv |L|$. We then decompose the total Hamiltonian H as

$$H = H_L + H_{L^{\rm c}} + H_{\partial L}. \tag{2.94}$$

By following the discussion in Section 3.4, we define the following effective Hamiltonian $\tilde{H}$:

$$\tilde{H} = H_L + \tilde{H}_{L^{\rm c}} + H_{\partial L}, \tag{2.95}$$

where

$$\tilde{H}_{L^{\rm c}} \equiv \sum_{E_n^{L^{\rm c}} \le \tau} E_n^{L^{\rm c}} \Pi_n^{H_{L^{\rm c}}} + \sum_{E_n^{L^{\rm c}} > \tau} \tau \Pi_n^{H_{L^{\rm c}}}. \tag{2.96}$$

Note that the norm of $\|\tilde{H}\|$ is given by

$$\|\tilde{H}\| \le \|H_L + H_{\partial L}\| + \|\tilde{H}_{L^{\rm c}}\| \le l + \tau. \tag{2.97}$$

We can prove that the ground state of $\tilde{H}$ is close to the original one $|E_0\rangle$ for large τ. For the ground state of $|\tilde{E}_0\rangle$ of $\tilde{H}$, Theorem 3.4 gives the following inequality:

$$\||E_0\rangle - |\tilde{E}_0\rangle\| \le e^{-{\rm const}\cdot(\tau - \tilde{c}\|H_{\partial L}\|)} \tag{2.98}$$

with $\tilde{c}$ a positive constant of $\mathcal{O}(1)$. Note that the norm of the boundary Hamiltonian is now of order 1.

We here utilize the operator $K(\tilde{H}, m)$ as the AGSP operator, where $K(m, x)$ is the same function as in Eq. (2.91). Note that $K(\tilde{H}, m)$ now depends on two parameters, namely m and $l = |L|$. Then, there exists a choice of $\tau \gtrsim l$ which gives the AGSP $K(\tilde{H}, m)$ with the errors

$$\Delta_{K(m,\tilde{H})} \le e^{-c_0 m\sqrt{\frac{\delta E}{l}}}, \quad \delta_{K(m,\tilde{H})} \le e^{-c_0 l} \tag{2.99}$$

with c_0 a positive constant of $\mathcal{O}(1)$, where in the first inequality we applied the inequality (2.93) and in the second inequality we utilized the inequality (2.98). We have thus proved that the AGSP $K(m, \tilde{H})$ satisfies the second and the third inequalities in (2.80) by choosing $m = r^2$. The last task is to prove the first inequality in (2.80) for the AGSP $K(m, \tilde{H})$.

2.6.2 (g) Step 7: Schmidt rank of $\tilde{H}^k$

The final step is the calculation of the Schmidt rank of $\tilde{H}^k$. We can prove the inequality

$$\log\big[SR(\tilde{H}^k)\big] \le \Big(\frac{c_1 k}{l} + c_2 l\Big)\log(Dl) \tag{2.100}$$

with c_1 and c_2 constants of $\mathcal{O}(1)$. By the use of this inequality, we have

$$SR\big[K(m,\tilde{H})\big] \le \text{const} \cdot (Dl)^{(c_1 m/l)+c_2 l}. \tag{2.101}$$

By choosing $m = r^2$ with $r \le l$, we can obtain the first inequality in (2.80). The exact proof of the inequality (2.100) is given in Ref. [14].

We here only show the qualitative reasoning of the inequality. We first decompose the total Hamiltonian into the following form:

$$\tilde{H} = H_1 + H_2 + H_3 + \cdots + H_{l+1}, \tag{2.102}$$

where H_1 contains all the spins in the left-hand side of L^{c} and H_l contains those in the right-hand side (see also Fig. 2.6). We can then write down each of the terms in $\tilde{H}^k$ as

$$H_{i_1} H_{i_2} H_{i_3} \cdots H_{i_k}, \tag{2.103}$$

where $\{i_m\}_{m=1}^k \in \{1, 2, 3, \dots l, l+1\}$. Now, we regard $\tilde{H}^k$ as an operator of information transfer from the region L_1 to L_2, where we roughly estimate the amount of the information by the logarithm of the Schmidt rank of $\tilde{H}^k$.

Between the left- and the right-hand sides of the boundary in the region L, the upper bound of information which can be sent to each other is given by $\mathcal{O}(l)\log D$, where the bound comes from the full dimensionality of the region L, namely $D^{\mathcal{O}(l)}$.

We then consider the information transfer from one side of L^{c} to the other side of L^{c}. Because the interaction of the Hamiltonian is the nearest neighboring, the information transfer should need at least $(l+1)$ steps; it means that the operator $\tilde{H}^k$ with $k \le l$ cannot send information between the two regions separated by the distance $l+1$, while the operator $\tilde{H}^{l+1}$ can send information of at most $\mathcal{O}(1)\log D$ because each of the Hamiltonians $\{H_i\}_{i=1}^{l+1}$ has the Schmidt rank of order $\mathcal{O}(D^2)$; note that $\{H_i\}_{i=1}^{l}$ describe the interactions between two D-dimensional spins. We therefore infer that $\tilde{H}^k = (\tilde{H}^{l+1})^{k/(l+1)}$ will send information of at most $\mathcal{O}(k/l)\log D$.

We have thus estimated the upper bound of the information transfer by $\tilde{H}^k$ between the regions L_1 and L_2 by $\big[\mathcal{O}(k/l) + \mathcal{O}(l)\big]\log D$.

Chapter 3

Fundamental tools for the locality analysis

In this chapter, we discuss the fundamental properties of the class of k-local Hamiltonians, which contains the interactions of up to k-body coupling. The main theorems in this chapter will be central tools to obtain the main results in the following chapters. After this chapter, we do not utilize the detail of the proofs; the readers only have to keep in mind the basic idea of the statements in order to understand the following chapters.

3.1 Introduction

When we analyze the real systems, the Hamiltonians do not take arbitrary forms but are strongly restricted; they may contain only short-range interactions, satisfy the translation invariance and so on. Such restrictions give us much useful information in analyzing properties of the systems. For example, when we assume the short-range interaction in a system, the Hamiltonian has to satisfy the Lieb-Robinson bound [28], which is a basic constraint on the time evolution of short-range interacting Hamiltonians, that is, on the velocity of the information transfer between two spins, and hence reflects the locality of short-range Hamiltonians. Based on this inequality, we can obtain many kinds of fundamental inequalities for the ground states such as the exponential decay of correlations or the entropic area law (see Chapter 2).

In the present thesis, we only assume the k-locality of the Hamiltonians, which will be the most general constraint in considering real systems. This class of the Hamiltonians includes not only the short-range interacting Hamiltonians but also long-range interacting Hamiltonians. In this case, the Lieb-Robinson bound can be no longer a strong tool to analyze the ground states because the long-range interaction can immediately transfer information between spins which are separated far.[*1] We therefore have to establish another restriction instead of the Lieb-Robinson bound in order to characterize the locality of the Hamiltonians. For this purpose, we focus on the spectrum of the Hamiltonians, which will reflect the k-locality.

To set the stage, let us consider the situation where we measure only one spin in the ground state. After the measurement, the state usually contains excited states because the measurement operator does not usually commute with the Hamiltonian. The terms in the Hamiltonian do not usually commute with each

[*1] Even for long-range Hamiltonians, we can obtain a useful constraint on the time evolution by focusing on the velocity of the information sharing instead of the information transfer (see Chapter 6).

other, and hence the spectrum becomes rather complicated. In the case of classical Hamiltonians, one-spin measurement affects the spectrum of only one spin, while for quantum Hamiltonians many other spins can be influenced simultaneously. In fact, however, the k-locality of the Hamiltonian imposes a strong restriction on the deformation of the spectral distribution by local disturbances (e.g. measurement). More mathematically, given a state whose energy distribution is supported on an interval such as $[-\infty, E)$, we investigate how the amplitudes of the excited states decay above E after the local disturbance. We prove that the decay is exponential, which indicates that the local disturbance cannot affect the system globally.

As an application, we also consider effective Hamiltonians which are constructed by truncating the high-energy spectrum of a partial Hamiltonian. If the truncation energy is high enough, we expect that the low-energy spectrum of the effective Hamiltonian is similar to that of the original Hamiltonian. We give a mathematical foundation to such an intuition by the use of the fundamental theorem on the energy distribution. This result has been utilized as a key tool to prove the improved one-dimensional area law [15] (see Chapter 2). It is also crucial in proving the main theorem on the macroscopic superposition in Chapter 5.

This chapter is organized as follows. In Section 3.2, we show the fundamental theorem on the general k-local Hamiltonians. This theorem is utilized as a basic tool throughout the thesis. In Section 3.3, under several special conditions, we derive stronger statements than the theorem in Section 3.2. In Section 3.4, we apply the main theorem to analyze the effective Hamiltonians where the high-energy spectrum in subregional Hamiltonian is truncated. We also show how we apply the main results in other chapters:

In Chapter 4, we utilize Theorem 3.2 for the proof of Theorem 4.3.

In Chapter 5, we utilize Theorem 3.4 for the proof of Theorem 5.4

In Chapter 6, we utilize Lemma 3.3.2 for the proof of Theorem 6.2.

3.2 Exponential decay of the excitation by the k-locality

In this section, we derive a fundamental property of the k-local Hamiltonians in terms of the energy excitation. We define an operator Γ_L supported in a region L. We here analyze the change in the energy distribution caused by the operator Γ_L as in Figure 3.1.

We consider Hamiltonians which are extensive k-local operators with $k = \mathcal{O}(1)$:

$$H = \sum_{|X| \le k} h_X \quad \text{with} \quad \sum_{X \ni i} \|h_X\| \le g \quad \text{for} \quad i = 1, 2, \dots, N. \tag{3.1}$$

In the following, we use the parameter λ of

$$\lambda \equiv \frac{1}{4gk} \tag{3.2}$$

for simplicity of the notation.

We now prove the following Theorem 3.2 on the energy excitation. This reflects the k-locality of the Hamiltonian. It is quite an important problem what properties are necessary and sufficient to characterize the k-locality of the Hamiltonians. We can only know that Theorem 3.2 gives one of the necessary properties.

Theorem 3.2. *Let us consider an arbitrary local operator Γ_L supported in a region L. The excitation due to the operator Γ_L is exponentially suppressed as follows:*

$$\|\Pi^H_{[E',\infty)} \Gamma_L \Pi^H_{(-\infty,E]}\| \le \|\Gamma_L\| e^{-\lambda(E'-E-3g|L|)}, \tag{3.3}$$

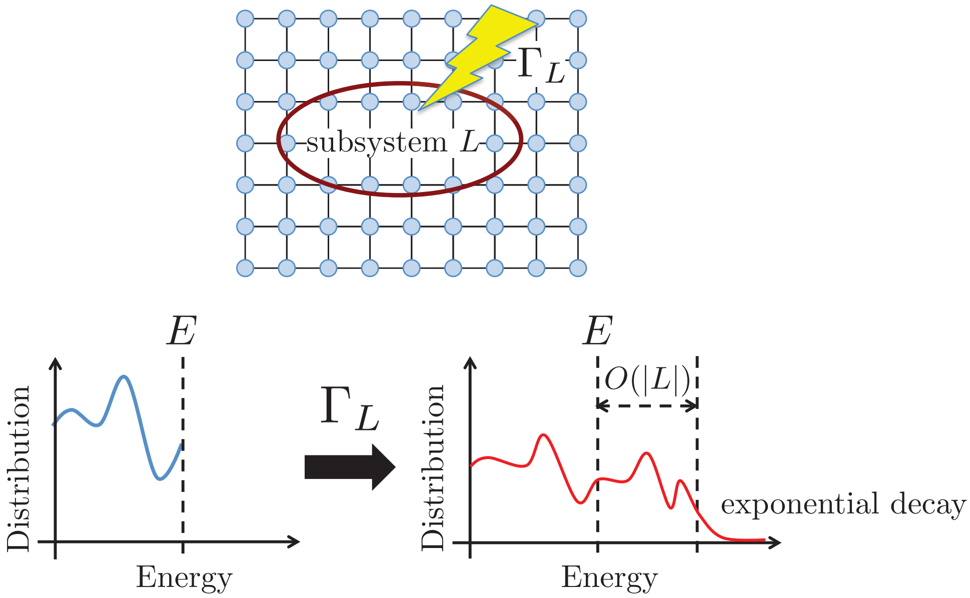

Figure 3.1: Let us consider a state which is in a superposition of energies below E (blue curve). After some local operation Γ_L in the region L, the energy distribution changes and there are non-zero probabilities of measuring energies larger than E (red curve). As in Theorem 3.2, however, the locality of the interactions implies that the energy excitation beyond $\mathcal{O}(|L|)$ exponentially decays. Note that we can take an arbitrarily large subregion L, as large as the system size, namely $|L| = \mathcal{O}(N)$.

where we define $\Pi^H_{[E',\infty)}$ and $\Pi^H_{(-\infty,E]}$ as projectors onto the subspaces of energies of H which are $[E',\infty)$ and $(-\infty, E]$, respectively.

Proof of Theorem 3.2. For the proof, we utilize the following inequality:

$$\|\Pi_{[E',\infty)}\Gamma_L\Pi_{(-\infty,E]}\| = \|\Pi_{[E',\infty)}e^{-\lambda H}e^{\lambda H}\Gamma_L e^{-\lambda H}e^{\lambda H}\Pi_{(-\infty,E]}\| \le e^{-\lambda(E'-E)}\|e^{\lambda H}\Gamma_L e^{-\lambda H}\|. \tag{3.4}$$

Our task is then to prove $\|e^{\lambda H}\Gamma_L e^{-\lambda H}\| \le \|\Gamma_L\|e^{3g\lambda|L|}$.

We first obtain

$$\begin{aligned}\frac{d}{dx}\|e^{xH}\Gamma_L e^{-xH}\| \le& \|e^{xH}[H,\Gamma_L]e^{-xH}\| \\ \le& \sum_{X:X\cap L\neq 0}\|e^{xH}[h_X,\Gamma_L]e^{-xH}\| \\ \le& 2\|e^{xH}\Gamma_L e^{-xH}\|\sum_{X:X\cap L\neq 0}\|e^{xH}h_X e^{-xH}\|.\end{aligned} \tag{3.5}$$

In order to calculate the right-hand side, we need to calculate the upper bound of $\|e^{xH}h_X e^{-xH}\|$. We can prove that in the special case of $|L| \le k$ the upper bound of $\|e^{xH}\Gamma_L e^{-xH}\|$ is given by

$$\|e^{xH}\Gamma_L e^{-xH}\| \le \frac{\|\Gamma_L\|}{1-2xgk} \quad \text{for} \quad x < \frac{1}{2gk}. \tag{3.6}$$

We give the proof of this inequality in Subsection 3.2.1. Note that this inequality can be applied to $\|e^{xH}h_X e^{-xH}\|$ because of $|X| \le k$.

By the use of the inequality (3.6), we obtain

$$\sum_{X:X\cap L\neq 0} \|e^{xH}h_X e^{-xH}\| \le \sum_{X:X\cap L\neq 0} \frac{\|h_X\|}{1-2xgk} \le \sum_{i:i\in L}\sum_{X:X\ni i} \frac{\|h_X\|}{1-2xgk} \le \frac{g|L|}{1-2xgk}, \tag{3.7}$$

where we utilized the extensiveness of the Hamiltonian (3.1) in the last inequality. Therefore, the inequality (3.5) reduces to

$$\frac{1}{\|e^{xH}\Gamma_L e^{-xH}\|}\frac{d}{dx}\|e^{xH}\Gamma_L e^{-xH}\| \le \frac{2g|L|}{1-2xgk} \tag{3.8}$$

for $x < 1/(2gk)$. We integrate the above inequality from $x = 0$ to $x = \lambda = 1/(4gk)$, and obtain

$$\log(\|e^{\lambda H}\Gamma_L e^{-\lambda H}\|) - \log(\|\Gamma_L\|) \le \frac{|L|\log 2}{k} = 4\log 2\cdot \lambda g|L| < 3\lambda g|L|. \tag{3.9}$$

This inequality reduces to $\|e^{xH}\Gamma_L e^{-xH}\| \le \|\Gamma_L\| e^{3\lambda g|L|}$. We thus prove the inequality (3.3).

3.2.1 Derivation of the inequality (3.6)

In order to calculate $\|e^{xH}\Gamma_L e^{-xH}\|$ as in the inequality (3.6), we expand $e^{xH}\Gamma_L e^{-xH}$ by the Hadamard lemma as

$$e^{xH}\Gamma_L e^{-xH} = \sum_{m=0}^{\infty}\frac{(-x)^m}{m!}O_m, \tag{3.10}$$

where $O_0 = \Gamma_L$ and $O_m = [\cdots[[\Gamma_L, \overbrace{H], H], \ldots], H}^{m}]$. The operator O_m is constructed from operators such as $h_{X_1}h_{X_2}\cdots h_{X_s}\Gamma_L h_{X_{s+1}}\cdots h_{X_m}$, where the index $s \in \{0, 1, \ldots, m\}$ denotes the position in the product where Γ_L appears. We then define $K^{(s)}_{X_1,X_2,\ldots,X_m} \equiv h_{X_1}h_{X_2}\cdots h_{X_s}\Gamma_L h_{X_{s+1}}\cdots h_{X_m}$ and denote O_m as

$$O_m = \sum_{s=0}^{m}\sum_{X_1,X_2,\ldots,X_m} n^{(s)}_{X_1,X_2,\ldots,X_m} K^{(s)}_{X_1,X_2,\ldots,X_m}, \tag{3.11}$$

where $n^{(s)}_{X_1,X_2,\ldots,X_m}$ is the number of times which $K^{(s)}_{X_1,X_2,\ldots,X_m}$ appears. We now bound $\|O_m\|$ by the use of $\overline{O}_m$, which is defined as

$$\overline{O}_m \equiv \sum_{s=0}^{m}\sum_{X_1,X_2,\ldots,X_m} n^{(s)}_{X_1,X_2,\ldots,X_m}\|K^{(s)}_{X_1,X_2,\ldots,X_m}\| \ge \|O_m\|. \tag{3.12}$$

We then derive the upper bound of $\overline{O}_{m+1}$ as a function of $\overline{O}_m$.

In order to obtain $\overline{O}_{m+1}$, we first calculate O_{m+1}:

$$O_{m+1} = \sum_{s=0}^{m}\sum_{X_1,X_2,\ldots,X_m} n^{(s)}_{X_1,X_2,\ldots,X_m}[K^{(s)}_{X_1,X_2,\ldots,X_m}, H]. \tag{3.13}$$

Each of the terms $[K^{(s)}_{X_1,X_2,\ldots,X_m}, H]$ is explicitly given by

$$\begin{aligned}[K^{(s)}_{X_1,X_2,\ldots,X_m}, H] =& [h_{X_1}, H]h_{X_2}\cdots h_{X_s}\Gamma_L h_{X_{s+1}}\cdots h_{X_m} \\ &+ h_{X_1}[h_{X_2}, H]\cdots h_{X_s}\Gamma_L h_{X_{s+1}}\cdots h_{X_m}\end{aligned}$$

$$+ h_{X_1}h_{X_2}\cdots h_{X_s}[\Gamma_L, H]h_{X_{s+1}}\cdots h_{X_m}$$
$$+\cdots + h_{X_1}h_{X_2}\cdots h_{X_s}\Gamma_L h_{X_{s+1}}\cdots [h_{X_m}, H]. \tag{3.14}$$

By expanding $[K^{(s)}_{X_1,X_2,\dots,X_m}, H]$, it is decomposed into terms such as

$$h_{X_1}h_{X_2}\cdots h_{X_s}\Gamma_L h_{X_{s+1}}\cdots h_{X_m}h_{X_{m+1}}. \tag{3.15}$$

Our task is now to take the summation of the norms of the above terms.

We here consider the first term in the right-hand side of Eq. (3.14), namely

$$[h_{X_1}, H]h_{X_2}\cdots h_{X_s}\Gamma_L h_{X_{s+1}}\cdots h_{X_m}. \tag{3.16}$$

Because we have

$$[h_{X_1}, H] = \sum_{X:X\bigcap X_1\neq 0}(h_{X_1}h_X - h_X h_{X_1}), \tag{3.17}$$

we obtain the summation of the norms which appear in $[h_{X_1}, H]h_{X_2}\cdots h_{X_s}\Gamma_L h_{X_{s+1}}\cdots h_{X_m}$:

$$\begin{aligned}
&\sum_{X:X\bigcap X_1\neq 0}\big(\|h_{X_1}h_X h_{X_2}\cdots h_{X_s}\Gamma_L h_{X_{s+1}}\cdots h_{X_m}\| + \|h_X h_{X_1}h_{X_2}\cdots h_{X_s}\Gamma_L h_{X_{s+1}}\cdots h_{X_m}\|\big)\\
\leq& 2\|h_{X_1}h_{X_2}\cdots h_{X_s}\Gamma_L h_{X_{s+1}}\cdots h_{X_m}\|\sum_{X:X\bigcap X_1\neq 0}\|h_X\|\\
=& 2\|K^{(s)}_{X_1,X_2\cdots X_m}\|\sum_{X:X\bigcap X_1\neq 0}\|h_X\|\\
\leq& 2g|X_1|\cdot\|K^{(s)}_{X_1,X_2\cdots X_m}\|,
\end{aligned} \tag{3.18}$$

where the last inequality comes from the following inequality:

$$\sum_{X:X\bigcap X_1\neq 0}\|h_X\| \leq \sum_{i\in X_1}\sum_{X\ni i}\|h_X\| \leq \sum_{i\in X_1} g \leq g|X_1|. \tag{3.19}$$

From similar calculations with respect to the other terms in Eq. (3.14), we can bound from above the summation of the norms of the terms in $[K^{(s)}_{X_1,X_2,\dots,X_m}, H]$ as

$$2g(|L| + \sum_{j=1}^{m}|X_j|)\cdot\|K^{(s)}_{X_1,X_2\cdots X_m}\| \leq 2g(|L| + mk)\cdot\|K^{(s)}_{X_1,X_2\cdots X_m}\|, \tag{3.20}$$

where we utilize the k-locality of the Hamiltonian, namely $|X_j| \leq k$ for $j = 1, 2, \dots, m$. We therefore obtain the upper bound of $\overline{O}_{m+1}$ as

$$\overline{O}_{m+1} \leq 2g(|L| + mk)\sum_{s=0}^{m}\sum_{X_1,X_2,\dots,X_m} n^{(s)}_{X_1,X_2,\dots,X_m}\|K^{(s)}_{X_1,X_2\cdots X_m}\| = 2gk\Big(\frac{|L|}{k} + m\Big)\overline{O}_m. \tag{3.21}$$

Using the notation $r = \frac{|L|}{k}$, we obtain

$$\overline{O}_m \leq \|\Gamma_L\|(2gk)^m r(r+1)\cdots(r+m-1), \tag{3.22}$$

where we utilized the equality of $\overline{O}_0 = \|\Gamma_L\|$. Because we are now considering the case of $|L| \leq k$, we have $r \leq 1$. The inequality (3.22) then reduces to

$$\overline{O}_m \leq \|\Gamma_L\|(2gk)^m m!. \tag{3.23}$$

We finally obtain

$$\|e^{xH}\Gamma_L e^{-xH}\| \leq \|\Gamma_L\| \sum_{m=0}^{\infty} \frac{x^m}{m!}\overline{O}_m \leq \|\Gamma_L\| \sum_{m=0}^{\infty} (2xgk)^m = \frac{\|\Gamma_L\|}{1-2xgk}, \tag{3.24}$$

for $x < 1/(2gk)$. This completes the proof of the inequality (3.6).

3.3 Theorem 3.2 under special conditions

In this section, we consider Theorem 3.2 under the following special conditions. First, we consider the commuting Hamiltonians, where $[h_X, h_{X'}] = 0$ for $\forall X, X'$ in Eq. (3.1). Under this condition, the energy excitation is finitely bounded.

Second, we consider the case of summation of the commuting Hamiltonians (SC Hamiltonian) in Eq. (1.11). In this case, we can obtain a slightly tighter upper bound than that of Theorem 3.2. This upper bound is crucial in proving a Lieb-Robinson-like bound for the time evolution of k-local Hamiltonians (see Chapter 6).

Third, we consider the case where the operator Γ_L commutes with the Hamiltonian in the region L; for example, Γ_L is a projection operator onto the eigenspace of H_L. We then know that the energy excitation depends only on the boundary Hamiltonian between the region L and the complementary region L^{c}. If we consider the short-range interacting systems, the norm of the boundary Hamiltonian only depends on the boundary of the region L.[*2]

3.3.1 In the case of the commuting Hamiltonians

We first consider the commuting Hamiltonians. We now prove the following lemma.

Lemma 3.3.1. *In the special case of the commuting Hamiltonians, the excitation by an arbitrary local operator Γ_L is finite, namely,*

$$\|\Pi^{H^{\mathrm{c}}}_{[E',\infty)}\Gamma_L \Pi^{H^{\mathrm{c}}}_{(-\infty,E]}\| = 0 \quad \text{for} \quad E' - E > 2g|L|. \tag{3.25}$$

Proof of Lemma 3.3.1. In order to prove Eq. (3.25), we utilize the same inequality as (3.4),

$$\|\Pi^{H^{\mathrm{c}}}_{[E',\infty)}\Gamma_L \Pi^{H^{\mathrm{c}}}_{(-\infty,E]}\| \leq e^{-x(E'-E)}\|e^{xH^{\mathrm{c}}}\Gamma_L e^{-xH^{\mathrm{c}}}\|, \tag{3.26}$$

for arbitrary real x and prove that in the limit of $x \to \infty$ the right-hand side approaches zero for $E' - E > 2g|L|$.

We expand $e^{xH^{\mathrm{c}}}\Gamma_L e^{-xH^{\mathrm{c}}}$ as in Eq. (3.10) and calculate the norm of O_m. We first obtain

$$O_1 = [\Gamma_L, H^{\mathrm{c}}] = \sum_{X: X \bigcap L \neq 0} [\Gamma_L, h_X]. \tag{3.27}$$

[*2] For general k-local Hamiltonians, however, the norm of the boundary Hamiltonian usually depends on the volume of the region.

We second obtain

$$O_2 = \sum_{X_1 \bigcap L \neq 0} \sum_{X_2 \bigcap L \neq 0} [[\Gamma_L, h_{X_1}], h_{X_2}], \tag{3.28}$$

where we utilize the commutativity of all the terms in the Hamiltonian. By repeating the same calculations, we have

$$O_m = \sum_{X_1 \bigcap L \neq 0} \cdots \sum_{X_m \bigcap L \neq 0} [\cdots[[\Gamma_L, h_{X_1}], \ldots], h_{X_m}], \tag{3.29}$$

and obtain

$$\|O_m\| \leq 2^m \|\Gamma_L\| \left(\sum_{X \bigcap L \neq 0} \|h_X\| \right)^m \leq (2g|L|)^m \|\Gamma_L\|, \tag{3.30}$$

where the last inequality comes from the extensiveness of the Hamiltonian.

From the inequality (3.30), we obtain $\|e^{xH^c} \Gamma_L e^{-xH^c}\| \leq \|\Gamma_L\| e^{2g|L|x}$, which reduces the inequality (3.26) to

$$\|\Pi^{H^c}_{[E',\infty)} \Gamma_L \Pi^{H^c}_{(-\infty,E]}\| \leq \|\Gamma_L\| e^{-x\left[(E'-E)-2g|L|\right]}. \tag{3.31}$$

Thus, if $E' - E > 2g|L|$, the right hand side approaches to zero in the limit of $x \to \infty$. This competes the proof.

3.3.2 In the case of the summation of the commuting Hamiltonians

We next consider the SC Hamiltonian. In this case, we also obtain a stronger restriction than that of Theorem 3.2.

<u>Lemma 3.3.2.</u> *Let us consider an arbitrary l_0-local operator $\Gamma^{(l_0)}$ and the SC Hamiltonians; note that we do not apply any restrictions on l_0. We then obtain*

$$\|\Pi^{H}_{[E',\infty)} \Gamma^{(l_0)} \Pi^{H}_{(-\infty,E]}\| \leq 2\|\Gamma^{(l_0)}\| e^{-\lambda'(E'-E-17gl_0)}, \tag{3.32}$$

where

$$\lambda' \equiv \frac{1}{24(k-1)g}. \tag{3.33}$$

In the case of $l_0 \leq k-1$, we have

$$\|\Pi^{H}_{[E',\infty)} \Gamma^{(l_0)} \Pi^{H}_{(-\infty,E]}\| \leq 2\|\Gamma^{(l_0)}\| e^{-2\lambda'(E'-E)}. \tag{3.34}$$

We notice that we cannot derive Lemma 3.3.2 directly from Theorem 3.2; to know this, let us expand the operator $\Gamma^{(l_0)}$ as

$$\Gamma^{(l_0)} = \sum_{|X| \leq l_0} \gamma_X \tag{3.35}$$

and apply Theorem 3.2 to each of the terms $\{\gamma_X\}$ as

$$\begin{aligned}\|\Pi^H_{[E',\infty)}\Gamma^{(l_0)}\Pi^H_{(-\infty,E]}\| &= \sum_{|X|\le l_0}\|\Pi^H_{[E',\infty)}\gamma_X\Pi^H_{(-\infty,E]}\| \\ &\le \sum_{|X|\le l_0}\|\gamma_X\|e^{-\lambda(E'-E-3g|X|)} \le e^{-\lambda(E'-E-3gl_0)}\sum_{|X|\le l_0}\|\gamma_X\|. \end{aligned} \tag{3.36}$$

If we could obtain the upper bound

$$\sum_{|X|\le l_0}\|\gamma_X\| \le e^{\mathcal{O}(|l_0|)}\|\Gamma^{(l_0)}\|, \tag{3.37}$$

the inequality (3.36) would reduce to

$$\|\Pi^H_{[E',\infty)}\Gamma^{(l_0)}\Pi^H_{(-\infty,E]}\| \le e^{-\lambda\left[E'-E-3gl_0-\mathcal{O}(|l_0|)\right]}, \tag{3.38}$$

and we could obtain a bound qualitatively equivalent to the inequality (3.32). However, the upper bound above is not generally satisfied as demonstrated by the following counterexample; let us consider the following 2-local operator

$$\Gamma^{(2)} = \frac{1}{N}\sum_{i<j}\gamma_{i,j}\sigma_i^z\otimes\sigma_j^z, \tag{3.39}$$

where $\{\gamma_{i,j}\}$ are uniform random numbers from -1 to 1. For this operator, we can obtain $\sum_{i<j}|\gamma_{i,j}|/N = \mathcal{O}(N)$, but $\|\Gamma^{(2)}\| = \mathcal{O}(\sqrt{N})$, and hence

$$\frac{1}{N}\sum_{i<j}\|\gamma_{i,j}\sigma_i^z\otimes\sigma_j^z\| = \mathcal{O}(\sqrt{N})\|\Gamma^{(2)}\|. \tag{3.40}$$

This does not satisfy the inequality (3.37).

It is an open problem whether we can derive the same upper bound as the inequality (3.32) without the SC condition; we conjecture that the general extensive k-local Hamiltonians satisfy it.

Proof of Lemma 3.3.2. The basic idea of the proof is almost the same as that of Theorem 3.2. We also consider the value of $e^{xH}\Gamma^{(l_0)}e^{-xH}$ and expand it as

$$e^{-xH}\Gamma^{(l_0)}e^{xH} = \sum_{m=0}^{\infty}\frac{x^m}{m!}O_m, \tag{3.41}$$

where $O_0=\Gamma^{(l_0)}$ and $O_m = [\cdots[[\Gamma^{(l_0)},\overbrace{H],H],\ldots],H]}^{m}$, namely $O_m=[O_{m-1},H]$. We then have to prove the inequality of $\|e^{-\lambda' H}\Gamma^{(l_0)}e^{\lambda' H}\| \le 2\|\Gamma^{(l_0)}\|e^{17\lambda' g l_0}$.

In order to calculate the upper bound of $\|e^{-xH}\Gamma^{(l_0)}e^{xH}\|$, we utilize the fact that we can decompose the Hamiltonian into the summation of commuting Hamiltonians, which gives us

$$\|[\Gamma^{(l_0)},H]\| \le \frac{1}{n^{\rm sc}}\sum_{m=1}^{n^{\rm sc}}\|[\Gamma^{(l_0)},H^{\rm c}_m]\|, \tag{3.42}$$

where $n_{\rm sc}$ is a positive integer and $\{H^{\rm c}_m\}$ is a set of commuting Hamiltonians. We can prove the following inequality for commutators $\{[\Gamma^{(l_0)}, H_m^{\rm c}]\}$:

$$\|[\Gamma^{(l_0)}, H_m^{\rm c}]\| \le 6gl_0\|\Gamma^{(l_0)}\|. \tag{3.43}$$

The proof of the inequality (3.43) using the SC condition of the Hamiltonian is given afterward. By the use of this inequality, we can also obtain

$$\|[\Gamma^{(l_0)}, H]\| \le 6gl_0\|\Gamma^{(l_0)}\|. \tag{3.44}$$

We can now bound the norm $\|O_{m+1}\|$ from above by $\|O_m\|$. We utilize the fact that the operator O_m can be described by a $[l_0 + (k-1)m]$-local operator from the definition. Therefore, from the inequality (3.44), we obtain

$$\|O_{m+1}\| = \|[O_m, H]\| \le 6g[l_0 + (k-1)m] \cdot \|O_m\|. \tag{3.45}$$

This inequality reduces to

$$\begin{aligned}\|O_m\| &\le (6g)^m\|\Gamma^{(l_0)}\| \cdot l_0(l_0+k-1)\big[l_0+2(k-1)\big]\cdots\big[l_0+(m-1)(k-1)\big] \\ &= \big[6g(k-1)\big]^m\|\Gamma^{(l_0)}\| \cdot r(r+1)(r+2)\cdots(r+m-1),\end{aligned} \tag{3.46}$$

where we utilize $\|O_0\| = \|\Gamma^{(l_0)}\|$ and r is defined as $r \equiv l_0/(k-1)$. The last term can be bounded from above by

$$\begin{aligned}r(r+1)(r+2)\cdots(r+m-1) \le &\lceil r\rceil(\lceil r\rceil+1)(\lceil r\rceil+2)\cdots(\lceil r\rceil+m-1) \\ =&m!\binom{\lceil r\rceil+m-1}{\lceil r\rceil} \le m!2^{\lceil r\rceil+m-1} \le m!2^{r+m},\end{aligned} \tag{3.47}$$

where $\lceil\cdots\rceil$ is the ceiling function and $\binom{\cdot}{\cdot}$ denotes the binomial coefficient.

We therefore obtain the norm of $e^{-xH}\Gamma^{(l_0)}e^{xH}$ by combining the inequalities (3.46) and (3.47):

$$\|e^{-xH}\Gamma^{(l_0)}e^{xH}\| \le 2^r\|\Gamma^{(l_0)}\|\sum_{m=0}^{\infty}\big[12xg(k-1)\big]^m. \tag{3.48}$$

By choosing $x = \lambda'$, we obtain

$$\begin{aligned}\|e^{-\lambda' H}\Gamma^{(l_0)}e^{\lambda' H}\| &\le 2^r\|\Gamma^{(l_0)}\|\sum_{m=0}^{\infty}2^{-m} \\ &= 2^{r+1}\|\Gamma^{(l_0)}\| = 2e^{24\log 2\cdot\lambda' gl_0} < 2e^{17\lambda' gl_0}.\end{aligned} \tag{3.49}$$

This completes the proof of the inequality (3.32).

In the case $l_0 \le k-1$, namely $r \le 1$, we can bound $r(r+1)(r+2)\cdots(r+m-1)$ from above by $m!$. We therefore obtain

$$O_m \le \big[6g(k-1)\big]^m m!, \tag{3.50}$$

which is followed by

$$\|e^{-2\lambda' H}\Gamma^{(l_0)}e^{2\lambda' H}\| \le \|\Gamma^{(l_0)}\| \sum_{m=0}^{\infty} 2^{-m} = 2\|\Gamma^{(l_0)}\|. \tag{3.51}$$

This completes the proof of the inequality (3.34).

3.3.2 (a) Proof of the inequality (3.43)

We here give the upper bound (3.43) for $\|[\Gamma^{(l_0)}, H^{\mathrm{c}}]\|$. We first decompose H^{c} as follows:

$$H^{\mathrm{c}} = H'^{\mathrm{c}} + \delta H^{\mathrm{c}}, \tag{3.52}$$

where

$$H'^{\mathrm{c}} \equiv \sum_{j=-\infty}^{\infty} \epsilon(j+1/2)\Pi^{H^{\mathrm{c}}}_{[\epsilon j, \epsilon j+\epsilon)}, \quad \delta H^{\mathrm{c}} \equiv H^{\mathrm{c}} - H'^{\mathrm{c}} \tag{3.53}$$

and $\Pi^{H^{\mathrm{c}}}_{[\epsilon j, \epsilon j+\epsilon)}$ is a projection operator onto the eigenspace of H^{c} with the eigenvalues $[\epsilon j, \epsilon j+\epsilon)$. We set the value of ϵ afterward. Note that the operator $\Pi^{H^{\mathrm{c}}}_{[\epsilon j, \epsilon j+\epsilon)}$ may be the null operator. From the definition, we have

$$\|\delta H^{\mathrm{c}}\| \le \frac{\epsilon}{2}. \tag{3.54}$$

We then obtain

$$\|[\Gamma^{(l_0)}, H^{\mathrm{c}}]\| = \|[\Gamma^{(l_0)}, H'^{\mathrm{c}} + \delta H^{\mathrm{c}}]\| \le \|[\Gamma^{(l_0)}, H'^{\mathrm{c}}]\| + \|[\Gamma^{(l_0)}, \delta H^{\mathrm{c}}]\|, \tag{3.55}$$

which necessitates that we calculate $\|[\Gamma^{(l_0)}, H'^{\mathrm{c}}]\|$ and $\|[\Gamma^{(l_0)}, \delta H^{\mathrm{c}}]\|$ separately.

We first obtain the norm of $[\Gamma^{(l_0)}, \delta H^{\mathrm{c}}]$ as follows:

$$\|[\Gamma^{(l_0)}, \delta H^{\mathrm{c}}]\| \le 2\|\Gamma^{(l_0)}\| \cdot \|\delta H^{\mathrm{c}}\| \le \epsilon \|\Gamma^{(l_0)}\|. \tag{3.56}$$

We second obtain

$$\begin{aligned}
[\Gamma^{(l_0)}, H'^{\mathrm{c}}] &= \sum_{j,j'} \Pi^{H^{\mathrm{c}}}_{[\epsilon j, \epsilon j+\epsilon)}(\Gamma^{(l_0)}H'^{\mathrm{c}} - H'^{\mathrm{c}}\Gamma^{(l_0)})\Pi^{H^{\mathrm{c}}}_{[\epsilon j', \epsilon j'+\epsilon)} \\
&= \sum_{j,j'} \epsilon(j'-j)\Pi^{H^{\mathrm{c}}}_{[\epsilon j, \epsilon j+\epsilon)}\Gamma^{(l_0)}\Pi^{H^{\mathrm{c}}}_{[\epsilon j', \epsilon j'+\epsilon)}.
\end{aligned} \tag{3.57}$$

Because we can obtain the norm of $[\Gamma^{(l_0)}, H'^{\mathrm{c}}]$ from the equality

$$\|[\Gamma^{(l_0)}, H'^{\mathrm{c}}]\| = \max_{|\psi\rangle} |\langle\psi|[\Gamma^{(l_0)}, H'^{\mathrm{c}}]|\psi\rangle|, \tag{3.58}$$

we, in the following, calculate the upper bound of $|\langle\psi|[\Gamma^{(l_0)}, H'^{\mathrm{c}}]|\psi\rangle|$ for arbitrary quantum states $|\psi\rangle$. From Eq. (3.57), we have

$$|\langle\psi|[\Gamma^{(l_0)}, H'^{\mathrm{c}}]|\psi\rangle| = \left|\sum_{j,j'} \epsilon(j'-j)\langle\psi|\Pi^{H^{\mathrm{c}}}_{[\epsilon j, \epsilon j+\epsilon)}\Gamma^{(l_0)}\Pi^{H^{\mathrm{c}}}_{[\epsilon j', \epsilon j'+\epsilon)}|\psi\rangle\right|$$

$$
\begin{aligned}
&\le \sum_{j,j'} \epsilon |j'-j| \cdot \|\langle\psi|\Pi^{H^c}_{[\epsilon j,\epsilon j+\epsilon)}\| \cdot \|\Pi^{H^c}_{[\epsilon j,\epsilon j+\epsilon)}\Gamma^{(l_0)}\Pi^{H^c}_{[\epsilon j',\epsilon j'+\epsilon)}\| \cdot \|\Pi^{H^c}_{[\epsilon j',\epsilon j'+\epsilon)}|\psi\rangle\| \\
&\equiv \epsilon \sum_{j,j'} |j'-j|\alpha_{j'}\alpha_j \Gamma^{(l_0)}_{j,j'},
\end{aligned} \tag{3.59}
$$

where $\alpha_j \equiv \|\Pi^{H^c}_{[\epsilon j,\epsilon j+\epsilon)}|\psi\rangle\|$ and $\Gamma^{(l_0)}_{j,j'} \equiv \|\Pi^{H^c}_{[\epsilon j,\epsilon j+\epsilon)}\Gamma^{(l_0)}\Pi^{H^c}_{[\epsilon j',\epsilon j'+\epsilon)}\|$; note that $\sum_j \alpha_j^2 = 1$.

Because we are now considering the commuting Hamiltonian, Lemma 3.3.1 gives us

$$
\begin{aligned}
&\|\Pi^{H^c}_{[E',\infty)}\Gamma^{(l_0)}\Pi^{H^c}_{(-\infty,E]}\| \le \|\Gamma^{(l_0)}\| \quad \text{for} \quad |E'-E| \le 2gl_0, \\
&\|\Pi^{H^c}_{[E',\infty)}\Gamma^{(l_0)}\Pi^{H^c}_{(-\infty,E]}\| = 0 \quad \text{for} \quad |E'-E| > 2gl_0,
\end{aligned} \tag{3.60}
$$

where the first inequality comes from the trivial bound of $\|\Pi^{H^c}_{[E',\infty)}\Gamma^{(l_0)}\Pi^{H^c}_{(-\infty,E]}\| \le \|\Pi^{H^c}_{[E',\infty)}\| \cdot \|\Gamma^{(l_0)}\| \cdot \|\Pi^{H^c}_{(-\infty,E]}\| = \|\Gamma^{(l_0)}\|$. This gives the following inequality:

$$
\begin{aligned}
&\Gamma^{(l_0)}_{j,j'} \le \|\Gamma^{(l_0)}\| \quad \text{for} \quad |j'-j| \le 1 + \frac{2gl_0}{\epsilon}, \\
&\Gamma^{(l_0)}_{j,j'} = 0 \quad \text{for} \quad |j'-j| > 1 + \frac{2gl_0}{\epsilon}.
\end{aligned} \tag{3.61}
$$

Because of the inequality (3.61), we have

$$
\begin{aligned}
|\langle\psi|[\Gamma^{(l_0)}, H^c]|\psi\rangle| = \epsilon \sum_{|j'-j|\le 1+2gl_0/\epsilon} |j'-j|\alpha_{j'}\alpha_j\Gamma^{(l_0)}_{j,j'} &\le \epsilon\|\Gamma^{(l_0)}\| \sum_{|j'-j|\le 1+2gl_0/\epsilon} |j'-j| \frac{\alpha_{j'}^2+\alpha_j^2}{2} \\
&\le \epsilon\|\Gamma^{(l_0)}\| \sum_{|j'-j|\le 1+2gl_0/\epsilon} |j'-j|\alpha_j^2 \\
&\le \epsilon\|\Gamma^{(l_0)}\| \sum_{j'=-\lfloor 1+\frac{2gl_0}{\epsilon}\rfloor}^{\lfloor 1+\frac{2gl_0}{\epsilon}\rfloor} |j'| \sum_j \alpha_j^2 \\
&\le \epsilon\|\Gamma^{(l_0)}\| \left\lfloor 1+\frac{2gl_0}{\epsilon}\right\rfloor \left(\left\lfloor 1+\frac{2gl_0}{\epsilon}\right\rfloor + 1\right),
\end{aligned} \tag{3.62}
$$

where $\lfloor\cdots\rfloor$ denotes the floor function.

The inequalities (3.56) and (3.62) reduce the inequality (3.55) to

$$
\|[\Gamma^{(l_0)}, H^c]\| \le \epsilon\|\Gamma^{(l_0)}\| + \epsilon\|\Gamma^{(l_0)}\| \left\lfloor 1+\frac{2gl_0}{\epsilon}\right\rfloor \left(\left\lfloor 1+\frac{2gl_0}{\epsilon}\right\rfloor + 1\right). \tag{3.63}
$$

We here choose $\epsilon = 2gl_0 + \delta\epsilon$ ($\delta\epsilon > 0$) and obtain

$$
\|[\Gamma^{(l_0)}, H^c]\| \le (6gl_0 + 3\delta\epsilon)\|\Gamma^{(l_0)}\|. \tag{3.64}
$$

By taking the limit of $\delta\epsilon \to +0$, we finally obtain $\|[\Gamma^{(l_0)}, H^c]\| \le 6gl_0\|\Gamma^{(l_0)}\|$.

3.3.3 In the case where Γ_L commutes with the Hamiltonian in the region L

We finally consider the special case where the operator Γ_L commutes with the Hamiltonian in the region L. We here decompose the total Hamiltonian as

$$
H = H_L + H_{L^c} + H_{\partial L} \tag{3.65}
$$

with the definitions of

$$H_L \equiv \sum_{X:X\in L} h_X, \quad H_{L^c} \equiv \sum_{X:X\in L^c} h_X,$$
$$H_{\partial L} \equiv H - H_L - H_{L^c} = \sum_{X:X\cap L\neq 0 \bigwedge X\cap L^c \neq 0} \|h_X\|. \tag{3.66}$$

Note that $H_{\partial L}$ is the boundary Hamiltonian between the regions L and L^c.

Lemma 3.3.3. *If the operator Γ_L commutes with H_L, namely $[\Gamma_L, H_L] = 0$, we obtain*

$$\|\Pi^H_{[E',\infty)}\Gamma_L \Pi^H_{(-\infty,E]}\| \leq \|\Gamma_L\| e^{-\lambda(E'-E-3\overline{H}_{\partial L})}, \tag{3.67}$$

where $\bar{H}_{\partial L}$ is defined as

$$\overline{H}_{\partial L} \equiv \sum_{X:X\cap X_0\neq 0 \bigwedge X\cap X_0^c \neq 0} \|h_X\|. \tag{3.68}$$

In the case where the Hamiltonian is short-ranged, $\overline{H}_{\partial L}$ is proportional to the surface of L.

Proof of Lemma 3.3.3. The proof can be given in the same way as Theorem 3.2. However, the commutativity of Γ_L changes the form of the inequality (3.5) to

$$\frac{d}{dx}\|e^{xH}\Gamma_L e^{-xH}\| \leq 2\|e^{xH}\Gamma_L e^{-xH}\| \sum_{X:X\cap L\neq 0 \bigwedge X\cap L^c \neq 0} \|e^{xH} h_X e^{-xH}\|, \tag{3.69}$$

which comes from the following equality

$$[H, \Gamma_L] = [H_L + H_{L^c} + H_{\partial L}, \Gamma_L] = [H_{\partial L}, \Gamma_L] = \sum_{X:X\cap L\neq 0 \bigwedge X\cap L^c \neq 0} [h_X, \Gamma_L], \tag{3.70}$$

where we utilized $[H_L, \Gamma_L] = [H_{L^c}, \Gamma_L] = 0$. After the same calculation as in the proof of Theorem 3.2, we can obtain the inequality (3.67).

3.4 Effective Hamiltonian

We here show an application of Theorem 3.2. When we analyze a quantum spin system, we often approximate the original Hamiltonian H by another Hamiltonian $\tilde{H}$. This Hamiltonian $\tilde{H}$ is identical to H in a local region L, while it can be very different in the other region L^c. Our main motivation to consider such an effective Hamiltonian is to bound the norm of the Hamiltonian; the norm of the original Hamiltonian scales as $\mathcal{O}(N)$, but we can control the norm of the effective one $\tilde{H}$. The norm of the Hamiltonian is one of the crucial parameters in approximating the ground space projector by the use of a low-degree polynomial of H. This is utilized to prove the one-dimensional area law [15] and the exponential suppression of the macroscopic superposition (see Chapter 5).

In constructing the effective Hamiltonian $\tilde{H}$, we want it to satisfy the following properties:

1. The construction of $\tilde{H}$ should be as simple as possible.
2. The effective Hamiltonian $\tilde{H}$ is approximately identical to the original Hamiltonian at least in an interested energy range.

For the construction of such a Hamiltonian $\tilde{H}$, we truncate the energy level of the Hamiltonian of the region outside the particular region (see Fig. 3.2). We here split the total system into the two subsystems

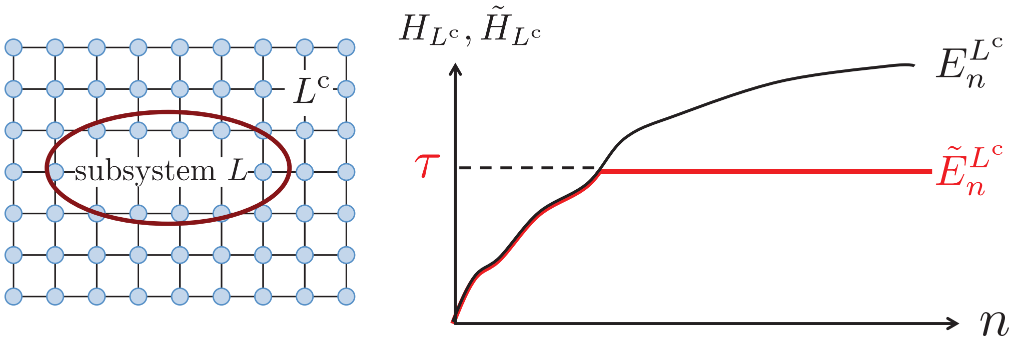

Figure 3.2: Schematic picture of our effective Hamiltonian. In the effective Hamiltonian $\tilde{H}$, we truncate the energy spectrum of over τ in the Hamiltonian $H^{L^{\mathrm{c}}}$, whereas the other part of the Hamiltonian is the same as the original Hamiltonian, which is given by $H_L + H_{\partial L}$. As long as we focus on the low-energy spectrum, the effective Hamiltonian looks almost the same as the original Hamiltonian. We show in Theorem 3.4 how accurately we can approximate the low-energy behavior by the use of the effective Hamiltonian.

L and L^{c} and decompose the total Hamiltonian as in Eq. (3.65).

By following the above discussion, we consider the following effective Hamiltonian $\tilde{H}$:

$$\tilde{H} = H_L + \tilde{H}_{L^{\mathrm{c}}} + H_{\partial L}, \tag{3.71}$$

where

$$\tilde{H}_{L^{\mathrm{c}}} \equiv H_{L^{\mathrm{c}}} \Pi^{H_{L^{\mathrm{c}}}}_{(-\infty,\tau)} + \tau \cdot \Pi^{H_{L^{\mathrm{c}}}}_{[\tau,\infty)}; \tag{3.72}$$

in other words, we uniformly truncate the energy levels above τ. Note that the norm $\|\tilde{H}\|$ is given by

$$\|\tilde{H}\| \le \|H_L + H_{\partial L}\| + \|\tilde{H}_{L^{\mathrm{c}}}\| \le g|L| + \tau, \tag{3.73}$$

where we bounded the first term $\|H_L + H_{\partial L}\|$ from above by the use of the extensiveness of the Hamiltonian.

If the truncation energy τ becomes large, the effective Hamiltonian is close to the original Hamiltonian, and we expect that the low-energy behavior of both Hamiltonians are approximately identical. We now want to know the τ-dependence of the accuracy of the low-energy spectrum of $\tilde{H}$ compared to that of the original Hamiltonian. We intuitively expect that in the case of $\tau \gg \|H_{\partial L}\|$ we can obtain the good approximation. We indeed prove the following theorem, which ensures the exponentially accurate approximation with respect to the value of τ.

Theorem 3.4. We define $\overline{H}_{\partial L}$ as in Eq. (3.68). The effective Hamiltonian satisfies the following inequalities:

$$\begin{aligned} \|(H - \tilde{H})\Pi^{H}_{(-\infty,E]}\| &\le \frac{7}{\lambda} e^{-\lambda(\delta\tau - \delta E - 5\overline{H}_{\partial L})}, \\ \|(H - \tilde{H})\Pi^{\tilde{H}}_{(-\infty,E]}\| &\le \frac{7}{\lambda} e^{-\lambda(\delta\tau - \delta\tilde{E} - 15\overline{H}_{\partial L})}, \end{aligned} \tag{3.74}$$

where $\delta\tau \equiv \tau - E_0^{L\mathrm{c}}$, $\delta E \equiv E - E_0$ and $\delta\tilde{E} \equiv E - \tilde{E}_0$. We denote the eigenvalues of H and $\tilde{H}$ by

$E_0 \le E_1 \le E_2 \le \cdots$ and $\tilde{E}_0 \le \tilde{E}_1 \le \tilde{E}_2 \le \cdots$, respectively. We then obtain

$$E_j - \frac{7}{\lambda} e^{-\lambda(\delta\tau - \delta\tilde{E}_j - 15\overline{H}_{\partial L})} \le \tilde{E}_j \le E_j, \tag{3.75}$$

where we defined $\delta\tilde{E}_j \equiv \tilde{E}_j - \tilde{E}_0$.

3.4.1 Proof of the inequalities (3.74)

We here prove the first part of Theorem 3.4. From the definition of the effective Hamiltonian, we calculate $\|(H - \tilde{H})\Pi^H_{(-\infty,E]}\|$ as

$$\begin{aligned}
\|(H - \tilde{H})\Pi^H_{(-\infty,E]}\| &= \left\|(H^{L^c} - \tau)\Pi^{H_{L^c}}_{[\tau,\infty)}\Pi^H_{(-\infty,E]}\right\| \\
&= \left\|\sum_{j=0}^{\infty}(H^{L^c} - \tau)\Pi^{H^{L^c}}_{[\tau+j\epsilon,\tau+(j+1)\epsilon)}\Pi^H_{(-\infty,E]}\right\| \\
&\le \sum_{j=0}^{\infty}\left\|(H^{L^c} - \tau)\Pi^{H^{L^c}}_{[\tau+j\epsilon,\tau+(j+1)\epsilon)}\right\| \cdot \left\|\Pi^{H^{L^c}}_{[\tau+j\epsilon,\tau+(j+1)\epsilon)}\Pi^H_{(-\infty,E]}\right\| \\
&\le \sum_{j=0}^{\infty}(j+1)\epsilon \cdot \left\|\Pi^{H^{L^c}}_{[\tau+j\epsilon,\infty)}\Pi^H_{(-\infty,E]}\right\|.
\end{aligned} \tag{3.76}$$

We now estimate the upper bound of $\left\|\Pi^{H^{L^c}}_{[\tau+j\epsilon,\infty)}\Pi^H_{(-\infty,E]}\right\|$, using the inequality (3.84) in Lemma 3.4.3:

$$\left\|\Pi^{H^{L^c}}_{[\tau+j\epsilon,\infty)}\Pi^H_{(-\infty,E]}\right\| \le \frac{e^{3/2}}{e-1} e^{-\lambda(\tau + j\epsilon - E_0^{L^c} - \delta E - 5\overline{H}_{\partial L})} = \frac{e^{3/2}}{e-1} e^{-\lambda(\delta\tau - \delta E - 5\overline{H}_{\partial L})} e^{-\lambda\epsilon j}, \tag{3.77}$$

which gives

$$\begin{aligned}
\|(H - \tilde{H})\Pi^H_{(-\infty,E]}\| &\le \frac{e^{3/2}}{e-1} e^{-\lambda(\delta\tau - \delta E - 5\overline{H}_{\partial L})} \sum_{j=0}^{\infty} \epsilon(j+1)e^{-\lambda\epsilon j} \\
&= \frac{e^{3/2}}{e-1} e^{-\lambda(\delta\tau - \delta E - 5\overline{H}_{\partial L})} \frac{\epsilon \cdot e^{2\lambda\epsilon}}{(e^{\lambda\epsilon} - 1)^2}.
\end{aligned} \tag{3.78}$$

By choosing $\epsilon = 1/\lambda$, we obtain

$$\|(H - \tilde{H})\Pi^H_{(-\infty,E]}\| \le \frac{e^{3/2}}{e-1} \cdot \frac{e^2}{\lambda(e-1)^2} e^{-\lambda(\delta\tau - \delta E - 5\overline{H}_{\partial L})} < \frac{7}{\lambda} e^{-\lambda(\delta\tau - \delta E - 5\overline{H}_{\partial L})}, \tag{3.79}$$

which is the first inequality in (3.74). The second inequality can be proved in the same way; we utilize the inequality (3.85) instead of (3.84) below.

3.4.2 Proof of the inequality (3.75)

We prove the second part of Theorem 3.4. Because of $H \ge \tilde{H}$, we can apply the Weyl inequality (see Appendix A.1) and obtain

$$E_j \ge \tilde{E}_j \tag{3.80}$$

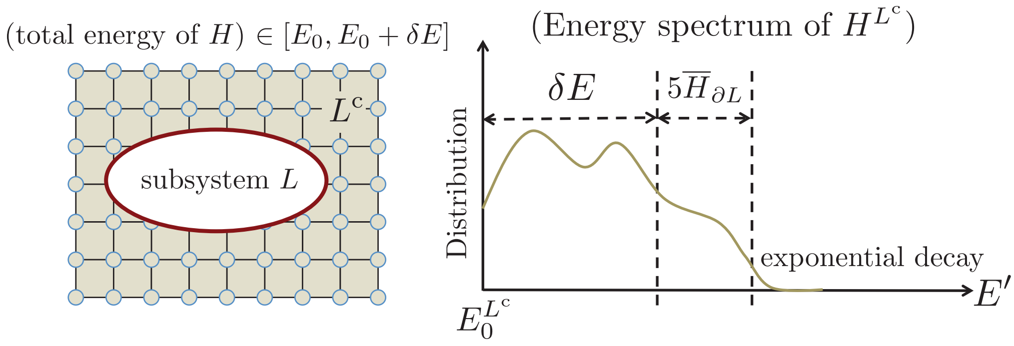

Figure 3.3: Schematic picture of what Lemma 3.4.3 means. The inequality (3.84) implies a restriction on the energy distribution in the local region L^{c} when the total system is in a superposition of energies less than $E = E_0 + \delta E$. When we observe the energy spectrum of $H^{L^{\mathrm{c}}}$, the eigenenergies larger than $E_0^{L^{\mathrm{c}}} + \delta E + 5\overline{H}_{\partial L}$ decay exponentially.

for $\forall j$, where $H \geq \tilde{H}$ comes from

$$H - \tilde{H} = \sum_{E_n^{L^{\mathrm{c}}} \geq \tau} (E_n^{L^{\mathrm{c}}} - \tau)\Pi^{H^{L^{\mathrm{c}}}}_{E_n^{L^{\mathrm{c}}}} \geq 0. \tag{3.81}$$

We then prove the inequality of $E_j - \frac{7}{\lambda} e^{-\lambda(\delta\tau - \delta\tilde{E}_j - 15\overline{H}_{\partial L})} \leq \tilde{E}_j$. We define $\tilde{P}_j$ as the projection operator onto the eigenspace of $\tilde{H}$ in the range of $[\tilde{E}_0, \tilde{E}_j]$ and start from the following inequality:

$$E_j \leq \|\tilde{P}_j H \tilde{P}_j\|. \tag{3.82}$$

This is the same as the inequality (A.4) and is proved in Appendix A.1. From the inequality (3.74), we have

$$\begin{aligned} E_j \leq \|\tilde{P}_j(H - \tilde{H} + \tilde{H})\tilde{P}_j\| &\leq \|\tilde{P}_j(H - \tilde{H})\tilde{P}_j\| + \|\tilde{P}_j \tilde{H} \tilde{P}_j\| \\ &\leq \tilde{E}_j + \frac{7}{\lambda} e^{-\lambda(\delta\tau - \delta\tilde{E}_j - 15\overline{H}_{\partial L})}. \end{aligned} \tag{3.83}$$

Thus, we prove the inequality (3.75).

3.4.3 The energy distribution of the subsystem L^{c} with respect to H

In the proof of Theorem 3.4, we had to prove the inequality (3.77). It is related to the energy distribution of the subsystem L^{c} under the condition that the total energy is involved in an interval $(-\infty, E]$. We here prove the exponential decay of the distribution of $H^{L^{\mathrm{c}}}$ beyond the interval of the total energy (see Fig. 3.3).

Lemma 3.4.3. *We define $\overline{H}_{\partial L}$ as in Eq. (3.68). Let $\Pi^{H_{L^{\mathrm{c}}}}_{[E',\infty)}$ be the projection operator onto the subspace of energies $[E', \infty)$ of $H_{L^{\mathrm{c}}}$. We then obtain*

$$\|\Pi^{H_{L^{\mathrm{c}}}}_{[E',\infty)} \Pi^{H}_{(-\infty,E]}\| \leq \frac{e^{3/2}}{e-1} e^{-\lambda(\delta E'_{L^{\mathrm{c}}} - \delta E - 5\overline{H}_{\partial L})}, \tag{3.84}$$

where $\delta E'_{L^{\mathrm{c}}} \equiv E' - E_0^{L^{\mathrm{c}}}$, $\delta E \equiv E - E_0$ and $E_0^{H_{L^{\mathrm{c}}}}$ is the ground energy of the Hamiltonian $H^{L^{\mathrm{c}}}$. The *same inequality is satisfied for* $\Pi^{\tilde{H}}_{(-\infty,E]}$*:*

$$\|\Pi^{H_{L^c}}_{[E',\infty)}\Pi^{\tilde{H}}_{(-\infty,E]}\| \le \frac{e^{3/2}}{e-1}e^{-\lambda(\delta E'_{L^c}-\delta\tilde{E}-15\overline{H}_{\partial L})}, \tag{3.85}$$

where $\Pi^{\tilde{H}}_{(-\infty,E]}$ *is the projection operator onto the subspace of energies* $(-\infty,E]$ *of* $\tilde{H}$ *and* $\delta\tilde{E}\equiv E-\tilde{E}_0$*.*

Proof of Lemma 3.4.3. We first consider a normalized quantum state $|\psi\rangle$ and construct the following quantum state $|\phi\rangle$:

$$|\phi\rangle \equiv \Pi^{H_{L^c}}_{[E',\infty)}\Pi^{H}_{(-\infty,E]}|\psi\rangle. \tag{3.86}$$

Note that this state $|\phi\rangle$ may not be normalized. The norm $\|\Pi^{H_{L^c}}_{[E',\infty)}\Pi^{H}_{(-\infty,E]}\|$ is now given by

$$\|\Pi^{H_{L^c}}_{[E',\infty)}\Pi^{H}_{(-\infty,E]}\| = \max_{|\psi\rangle}\|\phi\|, \tag{3.87}$$

where $\|\phi\|$ denotes the norm of the state $|\phi\rangle$. We then utilize the following inequality which is proved in Subsection 3.4.3 (a):

$$\|\phi\| \le \frac{e^{3/2}}{e-1}e^{-\lambda(\langle H\rangle_\phi - E - 3\overline{H}_{\partial L})}, \tag{3.88}$$

where $\langle H\rangle_\phi$ is defined as

$$\langle H\rangle_\phi \equiv \frac{\langle\phi|H|\phi\rangle}{\|\phi\|^2}. \tag{3.89}$$

In order to obtain an upper bound of $\|\phi\|^2$, we have to calculate a lower bound of $\langle H\rangle_\phi$:

$$\langle H\rangle_\phi = \frac{1}{\|\phi\|^2}\big(\langle\phi|H_L|\phi\rangle + \langle\phi|H_{L^c}|\phi\rangle + \langle\phi|H_{\partial L}|\phi\rangle\big). \tag{3.90}$$

From the definition of $|\phi\rangle$, we obtain

$$\begin{aligned}&\frac{1}{\|\phi\|^2}\langle\phi|H_L|\phi\rangle \ge E_0^L, \quad \frac{1}{\|\phi\|^2}\langle\phi|H_{L^c}|\phi\rangle \ge E',\\ &\frac{1}{\|\phi\|^2}\langle\phi|H_{\partial L}|\phi\rangle \ge -\|H_{\partial L}\| \ge -\overline{H}_{\partial L}.\end{aligned} \tag{3.91}$$

These inequalities give us

$$\langle H\rangle_\phi \ge E_0^L + E' - \overline{H}_{\partial L} \ge E_0 + E' - E_0^{L^c} - 2\overline{H}_{\partial L}, \tag{3.92}$$

where the last inequality comes from

$$E_0 \le \big(\langle E_0^L|\otimes\langle E_0^{L^c}|\big)H\big(|E_0^L\rangle\otimes|E_0^{L^c}\rangle\big) \le E_0^L + E_0^{L^c} + \overline{H}_{\partial L}. \tag{3.93}$$

By substituting $\langle H\rangle_\phi$ in the inequality (3.88) by (3.92), we can obtain the inequality (3.84).

We can prove the inequality (3.85) for $\tilde{H}$ in the same way; the only difference is that the inequality

(3.88) changes to

$$\|\tilde{\phi}\| \le \frac{e^{3/2}}{e-1} e^{-\lambda(\langle \tilde{H}\rangle_\phi - E - 13\overline{H}_{\partial L})}, \tag{3.94}$$

where $|\tilde{\phi}\rangle \equiv \Pi^{H_{L^c}}_{[E',\infty)} \Pi^{\tilde{H}}_{(-\infty,E]} |\psi\rangle$.

3.4.3 (a) Proof of the inequalities (3.88) and (3.94)

We start from the following equality:

$$\langle\phi|H|\phi\rangle = \langle\phi|\Pi^H_{(-\infty,\Delta]} H \Pi^H_{(-\infty,\Delta]}|\phi\rangle + \sum_{j=1}^{\infty} \langle\phi|\Pi^H_{[\Delta+(j-1)\epsilon,\Delta+j\epsilon)} H \Pi^H_{[\Delta+(j-1)\epsilon,\Delta+j\epsilon)}|\phi\rangle, \tag{3.95}$$

where Δ and ϵ are parameters which we set afterward. We here obtain

$$\begin{aligned}
\langle\phi|H|\phi\rangle &\le \Delta \|\Pi^H_{(-\infty,\Delta]}|\phi\rangle\|^2 + \sum_{j=1}^{\infty} (\Delta + j\epsilon) \|\Pi^H_{[\Delta+(j-1)\epsilon,\Delta+j\epsilon)}|\phi\rangle\|^2 \\
&= \Delta\left(\|\Pi^H_{(-\infty,\Delta]}|\phi\rangle\|^2 + \sum_{j=1}^{\infty} \|\Pi^H_{[\Delta+(j-1)\epsilon,\Delta+j\epsilon)}|\phi\rangle\|^2 \right) + \epsilon \sum_{j=1}^{\infty} j \|\Pi^H_{[\Delta+(j-1)\epsilon,\Delta+j\epsilon)}|\phi\rangle\|^2 \\
&= \Delta \|\phi\|^2 + \epsilon \sum_{j=1}^{\infty} j \|\Pi^H_{[\Delta+(j-1)\epsilon,\Delta+j\epsilon)}|\phi\rangle\|^2.
\end{aligned} \tag{3.96}$$

Utilizing the commutation relation $[\Pi^{H_{L^c}}_{[E',\infty)}, H_{L^c}] = 0$ and the inequality (3.67) in Lemma 3.3.3, we obtain

$$\begin{aligned}
\|\Pi^H_{[\Delta+(j-1)\epsilon,\Delta+j\epsilon)}|\phi\rangle\|^2 &= \|\Pi^H_{[\Delta+(j-1)\epsilon,\Delta+j\epsilon)} \Pi^{H_{L^c}}_{[E',\infty)} \Pi^H_{(-\infty,E]}|\psi\rangle\|^2 \\
&\le \|\Pi^H_{[\Delta+(j-1)\epsilon,\infty)} \Pi^{H_{L^c}}_{[E',\infty)} \Pi^H_{(-\infty,E]}\|^2 \\
&\le e^{-2\lambda\left[\Delta - E + (j-1)\epsilon - 3\overline{H}_{\partial L}\right]},
\end{aligned} \tag{3.97}$$

where the operator $\Pi^{H_{L^c}}_{[E',\infty)}$ corresponds to the disturbance operator Γ_L in the inequality (3.67). By the use of this inequality, we have

$$\begin{aligned}
\sum_{j=1}^{\infty} j \|\Pi^H_{[\Delta+(j-1)\epsilon,\Delta+j\epsilon)}|\phi\rangle\|^2 &\le e^{-2\lambda(\Delta - E - 3\overline{H}_{\partial L})} \sum_{j=1}^{\infty} j e^{-2\lambda(j-1)\epsilon} \\
&= e^{-2\lambda(\Delta - E - 3\overline{H}_{\partial L})} \frac{e^{4\epsilon\lambda}}{(e^{2\epsilon\lambda} - 1)^2}.
\end{aligned} \tag{3.98}$$

This inequality reduces the inequality (3.96) to

$$\langle\phi|H|\phi\rangle \le \Delta \|\phi\|^2 + \epsilon \cdot e^{-2\lambda(\Delta - E - 3\overline{H}_{\partial L})} \frac{e^{4\epsilon\lambda}}{(e^{2\epsilon\lambda} - 1)^2}. \tag{3.99}$$

By choosing $\epsilon = 1/(2\lambda)$, we obtain

$$\langle\phi|H|\phi\rangle \le \Delta \|\phi\|^2 + \frac{e^2}{2\lambda(e-1)^2} e^{-2\lambda(\Delta - E - 3\overline{H}_{\partial L})}. \tag{3.100}$$

From the definition (3.89), we have $\langle\phi|H|\phi\rangle = \|\phi\|^2 \cdot \langle H\rangle_\phi$, which reduces the above inequality to

$$\|\phi\|^2 \le \frac{e^2}{2\lambda(e-1)^2(\langle H\rangle_\phi - \Delta)} e^{2\lambda(\langle H\rangle_\phi - \Delta)} e^{-2\lambda(\langle H\rangle_\phi - E - 3\overline{H}_{\partial L})}. \tag{3.101}$$

By choosing Δ so that $\langle H\rangle_\phi - \Delta = 1/(2\lambda)$, we finally obtain

$$\|\phi\| \le \frac{e^{3/2}}{e-1} e^{-\lambda(\langle H\rangle_\phi - E - 3\overline{H}_{\partial L})}. \tag{3.102}$$

This completes the proof.

We can prove the inequality (3.94) in the same way. Here, in the inequality (3.97), we utilize the inequality (3.103) in Lemma 3.4.4 instead of the inequality (3.67) in Lemma 3.3.3.

3.4.4 Locality of the excitation in the effective Hamiltonian $\tilde{H}$

We here derive the same inequality as the one in Lemma 3.3.3 for the effective Hamiltonian in Eq. (3.71). In this case, the effective Hamiltonian no longer satisfies the k-locality in the region L^{c}. Even so, we can prove the following lemma.

Lemma 3.4.4. Let an operator $\Gamma_{L^{\mathrm{c}}}$ satisfy $[\Gamma_{L^{\mathrm{c}}}, H_{L^{\mathrm{c}}}] = 0$. It satisfies

$$\|\Pi^{\tilde{H}}_{[E',\infty)} \Gamma_{L^{\mathrm{c}}} \Pi^{\tilde{H}}_{(-\infty,E]}\| \le \|\Gamma_{L^{\mathrm{c}}}\| e^{-\lambda(E'-E-13\overline{H}_{\partial L})}. \tag{3.103}$$

We define $\overline{H}_{\partial L}$ as in Eq. (3.68).

Proof of Lemma 3.4.4. We utilize the same inequality as (3.4) and calculate the norm of $e^{\lambda\tilde{H}}\Gamma_{L^{\mathrm{c}}}e^{-\lambda\tilde{H}}$. We have to prove the upper bound of $\|e^{\lambda\tilde{H}}\Gamma_{L^{\mathrm{c}}}e^{-\lambda\tilde{H}}\| \le e^{13\lambda\overline{H}_{\partial L}}$. For this purpose, we utilize the Dyson expansion (see Subsection A.2); for any two operators O_1 and O_2, we obtain

$$e^{x(O_1+O_2)} = \sum_{j=0}^{\infty} G_j(x) e^{xO_1}, \quad e^{-x(O_1+O_2)} = e^{-xO_1} \sum_{j=0}^{\infty} G'_j(x), \tag{3.104}$$

where $G_0(x) \equiv 1$, $G'_0(x) \equiv 1$ and

$$\begin{aligned}
&G_j(x) \equiv \int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{j-1}} dx_j O_2(x_j) O_2(x_{j-1}) \cdots O_2(x_1) \quad \text{for} \quad j \ge 1, \\
&G'_j(x) \equiv (-1)^j \int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{j-1}} dx_j O_2(x_1) O_2(x_2) \cdots O_2(x_j) \quad \text{for} \quad j \ge 1, \\
&O_2(x) \equiv e^{xO_1} O_2 e^{-xO_1}.
\end{aligned} \tag{3.105}$$

We thereby obtain

$$\begin{aligned}
e^{\lambda\tilde{H}}\Gamma_{L^{\mathrm{c}}}e^{-\lambda\tilde{H}} &= \sum_{j=0}^{\infty}\sum_{j'=0}^{\infty} G_j(\lambda) e^{\lambda(\tilde{H}_{L^{\mathrm{c}}}+H_L)} \Gamma_{L^{\mathrm{c}}} e^{-\lambda(\tilde{H}_{L^{\mathrm{c}}}+H_L)} G'_{j'}(\lambda) \\
&= \sum_{j=0}^{\infty}\sum_{j'=0}^{\infty} G_j(\lambda) \Gamma_{L^{\mathrm{c}}} G'_{j'}(\lambda),
\end{aligned} \tag{3.106}$$

where we let $O_1 = \tilde{H}_{L^{\mathrm{c}}} + H_L$ and $O_2 = H_{\partial L}$ and utilize $[\Gamma_{L^{\mathrm{c}}}, \tilde{H}_{L^{\mathrm{c}}} + H_L] = 0$. We therefore obtain

$$\|e^{\lambda\tilde{H}}\Gamma_{L^{\mathrm{c}}}e^{-\lambda\tilde{H}}\| \le \|\Gamma_{L^{\mathrm{c}}}\| \sum_{j=0}^{\infty}\sum_{j'=0}^{\infty} \|G_j(\lambda)\| \cdot \|G'_{j'}(\lambda)\|. \tag{3.107}$$

From the definition of $G_j(\lambda)$ and $G'_j(\lambda)$, we obtain

$$
\begin{aligned}
\|G_j(\lambda)\| &\le \int_0^\lambda dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{j-1}} dx_j \|H_{\partial L}(x_j)\| \cdot \|H_{\partial L}(x_{j-1})\| \cdots \|H_{\partial L}(x_1)\| \\
&= \frac{1}{j!}\left(\int_0^\lambda dx \|H_{\partial L}(x)\|\right)^j, \\
\|G'_j(\lambda)\| &\le \int_0^\lambda dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{j-1}} dx_j \|H_{\partial L}(x_1)\| \cdot \|H_{\partial L}(x_2)\| \cdots \|H_{\partial L}(x_j)\| \\
&= \frac{1}{j!}\left(\int_0^\lambda dx \|H_{\partial L}(x)\|\right)^j, \\
H_{\partial L}(x) &\equiv e^{x(\tilde{H}_{L^c}+H_L)} H_{\partial L} e^{-x(\tilde{H}_{L^c}+H_L)}.
\end{aligned} \tag{3.108}
$$

We now have to obtain the upper bound of $\int_0^\lambda dx \|H_{\partial L}(x)\|$. From the calculation in Subsection 3.4.4 (a), we obtain

$$
\int_0^\lambda dx \|H_{\partial L}(x)\| \le (2+4\sqrt{3}) \log 2 \cdot \lambda \overline{H}_{\partial L}. \tag{3.109}
$$

By combining the inequalities (3.108) and (3.109), we obtain

$$
\|G_j(\lambda)\| \le \frac{\left[(2+4\sqrt{3}) \log 2 \cdot \lambda \overline{H}_{\partial L}\right]^j}{j!}, \tag{3.110}
$$

which reduces the inequality (3.107) to

$$
\|e^{\lambda \tilde{H}} \Gamma_{L^c} e^{-\lambda \tilde{H}}\| \le \|\Gamma_{L^c}\| e^{2(2+4\sqrt{3}) \log 2 \cdot \lambda \overline{H}_{\partial L}} < \|\Gamma_{L^c}\| e^{13 \lambda \overline{H}_{\partial L}}. \tag{3.111}
$$

This completes the proof of Lemma 3.4.4.

3.4.4 (a) Upper bound of $\int_0^\lambda \|H_{\partial L}(x)\| dx$

In this subsection, we prove the inequality (3.109), which gives the upper bound of $\int_0^\lambda \|H_{\partial L}(x)\| dx$. We first decompose the integrand as

$$
\begin{aligned}
\|H_{\partial L}(x)\| &= \|e^{x(\tilde{H}_{L^c}+H_L)} H_{\partial L} e^{-x(\tilde{H}_{L^c}+H_L)}\| \\
&\le \sum_{X: X\cap L \neq 0 \bigwedge X \cap L^c \neq 0} \|e^{x(\tilde{H}_{L^c}+H_L)} h_X e^{-x(\tilde{H}_{L^c}+H_L)}\|.
\end{aligned} \tag{3.112}
$$

In order to calculate $\|e^{x(\tilde{H}_{L^c}+H_L)} h_X e^{-x(\tilde{H}_{L^c}+H_L)}\|$, we obtain the upper bound of

$$
|\langle \psi | e^{x(\tilde{H}_{L^c}+H_L)} h_X e^{-x(\tilde{H}_{L^c}+H_L)} | \psi \rangle|, \tag{3.113}
$$

which also gives the upper bound of

$$
\|e^{x(\tilde{H}_{L^c}+H_L)} h_X e^{-x(\tilde{H}_{L^c}+H_L)}\| = \max_{|\psi\rangle} |\langle \psi | e^{x(\tilde{H}_{L^c}+H_L)} h_X e^{-x(\tilde{H}_{L^c}+H_L)} | \psi \rangle|. \tag{3.114}
$$

From the definition of the effective Hamiltonian, we have

$$
e^{x \tilde{H}_{L^c}} = \Pi^{H_{L^c}}_{(-\infty,\tau)} e^{x H_{L^c}} + \Pi^{H_{L^c}}_{[\tau,\infty)} e^{x\tau}, \tag{3.115}
$$

where we utilized the equalities $\tilde{H}_{L^c}\Pi^{H_{L^c}}_{(-\infty,\tau)} = H_{L^c}\Pi^{H_{L^c}}_{(-\infty,\tau)}$ and $\tilde{H}_{L^c}\Pi^{H_{L^c}}_{[\tau,\infty)} = \tau\Pi^{H_{L^c}}_{[\tau,\infty)}$. We then have

$$
\begin{aligned}
&e^{x(\tilde{H}_{L^c}+H_L)}h_X e^{-x(\tilde{H}_{L^c}+H_L)} \\
=&\Big(\Pi^{H_{L^c}}_{(-\infty,\tau)}e^{xH_{L^c}} + \Pi^{H_{L^c}}_{[\tau,\infty)}e^{x\tau}\Big)e^{xH_L}h_X e^{-xH_L}\Big(\Pi^{H_{L^c}}_{(-\infty,\tau)}e^{-xH_{L^c}} + \Pi^{H_{L^c}}_{[\tau,\infty)}e^{-x\tau}\Big),
\end{aligned}
\tag{3.116}
$$

which is decomposed into the following three terms:

$$
\begin{aligned}
&\Pi^{H_{L^c}}_{(-\infty,\tau)}e^{x(H_L+H_{L^c})}h_X e^{-x(H_L+H_{L^c})}\Pi^{H_{L^c}}_{(-\infty,\tau)}, \\
&\Pi^{H_{L^c}}_{[\tau,\infty)}e^{xH_L}h_X e^{-xH_L}\Pi^{H_{L^c}}_{[\tau,\infty)}, \\
&\big(\Pi^{H_{L^c}}_{[\tau,\infty)}e^{x\tau}\big)e^{xH_L}h_X e^{-xH_L}\big(\Pi^{H_{L^c}}_{(-\infty,\tau)}e^{-xH_{L^c}}\big) + \text{h.c.}
\end{aligned}
\tag{3.117}
$$

This gives the inequalities

$$
\begin{aligned}
&|\langle\psi|e^{x(\tilde{H}_{L^c}+H_L)}h_X e^{-x(\tilde{H}_{L^c}+H_L)}|\psi\rangle| \\
\le&\big|\langle\psi|\Pi^{H_{L^c}}_{(-\infty,\tau)}e^{x(H_L+H_{L^c})}h_X e^{-x(H_L+H_{L^c})}\Pi^{H_{L^c}}_{(-\infty,\tau)}|\psi\rangle\big| + \big|\langle\psi|\Pi^{H_{L^c}}_{[\tau,\infty)}e^{xH_L}h_X e^{-xH_L}\Pi^{H_{L^c}}_{[\tau,\infty)}|\psi\rangle\big| \\
&+ 2\big|\langle\psi|\big(\Pi^{H_{L^c}}_{[\tau,\infty)}e^{x\tau}\big)e^{xH_L}h_X e^{-xH_L}\big(\Pi^{H_{L^c}}_{(-\infty,\tau)}e^{-xH_{L^c}}\big)|\psi\rangle\big| \\
\le&\|\langle\psi|\Pi^{H_{L^c}}_{(-\infty,\tau)}\|^2\cdot\|e^{x(H_L+H_{L^c})}h_X e^{-x(H_L+H_{L^c})}\| + \|\langle\psi|\Pi^{H_{L^c}}_{[\tau,\infty)}\|^2\cdot\|e^{xH_L}h_X e^{-xH_L}\| \\
&+ 2e^{x\tau}\big|\langle\psi|\Pi^{H_{L^c}}_{[\tau,\infty)}e^{xH_L}h_X e^{-xH_L}\big(\Pi^{H_{L^c}}_{(-\infty,\tau)}e^{-xH_{L^c}}\big)|\psi\rangle\big| \\
\le&\frac{\|h_X\|}{1-2gkx} + 2e^{x\tau}|\langle\psi|\Pi^{H_{L^c}}_{[\tau,\infty)}e^{xH_L}h_X e^{-xH_L}\big(\Pi^{H_{L^c}}_{(-\infty,\tau)}e^{-xH_{L^c}}\big)|\psi\rangle|,
\end{aligned}
\tag{3.118}
$$

where we utilized the equality $\|\langle\psi|\Pi^{H_{L^c}}_{(-\infty,\tau)}\|^2 + \|\langle\psi|\Pi^{H_{L^c}}_{[\tau,\infty)}\|^2 = 1$ and applied the inequality (3.6) as

$$
\|e^{x(H_L+H_{L^c})}h_X e^{-x(H_L+H_{L^c})}\| \le \frac{\|h_X\|}{1-2gkx}, \quad \|e^{xH_L}h_X e^{-xH_L}\| \le \frac{\|h_X\|}{1-2gkx}.
\tag{3.119}
$$

Note that x takes a value from 0 to λ, where $\lambda < 1/(2gk)$.

We now calculate the upper bound of $2e^{x\tau}|\langle\psi|\Pi^{H_{L^c}}_{[\tau,\infty)}e^{xH_L}h_X e^{-xH_L}\big(\Pi^{H_{L^c}}_{(-\infty,\tau)}e^{-xH_{L^c}}\big)|\psi\rangle|$ in the inequality (3.118). We first obtain

$$
\begin{aligned}
&2e^{x\tau}|\langle\psi|\Pi^{H_{L^c}}_{[\tau,\infty)}e^{xH_L}h_X e^{-xH_L}\big(\Pi^{H_{L^c}}_{(-\infty,\tau)}e^{-xH_{L^c}}\big)|\psi\rangle| \\
=&2e^{x\tau}\sum_{j=0}^{\infty}|\langle\psi|\Pi^{H_{L^c}}_{[\tau,\infty)}e^{xH_L}h_X e^{-xH_L}\big(\Pi^{H_{L^c}}_{[\tau-\epsilon(j+1),\tau-\epsilon j)}e^{-xH_{L^c}}\big)|\psi\rangle| \\
\le&2\sum_{j=0}^{\infty}e^{x\epsilon(j+1)}\|\langle\psi|\Pi^{H_{L^c}}_{[\tau,\infty)}\|\cdot\|\Pi^{H_{L^c}}_{[\tau,\infty)}e^{xH_L}h_X e^{-xH_L}\Pi^{H_{L^c}}_{[\tau-\epsilon(j+1),\tau-\epsilon j)}\|\cdot\|\Pi^{H_{L^c}}_{[\tau-\epsilon(j+1),\tau-\epsilon j)}|\psi\rangle\| \\
\le&\frac{2e^{x\epsilon}\cdot\|h_X\|\cdot\|\langle\psi|\Pi^{H_{L^c}}_{[\tau,\infty)}\|}{1-2gksx}\sum_{j=0}^{\infty}e^{-(s-1)x\epsilon j}\|\Pi^{H_{L^c}}_{[\tau-\epsilon(j+1),\tau-\epsilon j)}|\psi\rangle\|,
\end{aligned}
\tag{3.120}
$$

where s is a constant in the range $1 < s < 1/(2gkx) = 2\lambda/x$, and therefore in the range $1 < s < 2$ because of $0 \le x \le \lambda$, and we utilized the inequality

$$
\begin{aligned}
&\big\|\Pi^{H_{L^c}}_{[\tau,\infty)}\big[e^{xH_L}h_X e^{-xH_L}\big]\Pi^{H_{L^c}}_{[\tau-\epsilon(j+1),\tau-\epsilon j)}\big\| \\
=&\big\|\Pi^{H_{L^c}}_{[\tau,\infty)}e^{-sxH_{L^c}}\big[e^{x(H_L+sH_{L^c})}h_X e^{-x(H_L+sH_{L^c})}\big]e^{sxH_{L^c}}\Pi^{H_{L^c}}_{[\tau-\epsilon(j+1),\tau-\epsilon j)}\big\| \\
\le&\|e^{x(H_L+sH_{L^c})}h_X e^{-x(H_L+sH_{L^c})}\|\cdot e^{-sx\epsilon j} \\
\le&\frac{\|h_X\|\cdot e^{-sx\epsilon j}}{1-2gksx},
\end{aligned} \tag{3.121}
$$

using the inequality (3.6) in the last inequality. In the inequality (3.120), the summation with respect to j reduces to

$$
\begin{aligned}
\sum_{j=0}^{\infty} e^{-(s-1)x\epsilon j}\|\Pi^{H_{L^c}}_{[\tau-\epsilon(j+1),\tau-\epsilon j)}|\psi\rangle\| &\le \left(\sum_{j=0}^{\infty}\|\Pi^{H_{L^c}}_{[\tau-\epsilon(j+1),\tau-\epsilon j)}|\psi\rangle\|^2\right)^{1/2}\cdot\left(\sum_{j=0}^{\infty}e^{-2(s-1)x\epsilon j}\right)^{1/2} \\
&= \frac{\|\Pi^{H_{L^c}}_{(-\infty,\tau)}|\psi\rangle\|}{(1-e^{-2(s-1)x\epsilon})^{1/2}},
\end{aligned} \tag{3.122}
$$

where we utilized the Cauchy-Schwarz inequality in the first inequality.

From the inequalities (3.120) and (3.122), we obtain

$$
\begin{aligned}
&2e^{x\tau}|\langle\psi|\Pi^{H_{L^c}}_{[\tau,\infty)}e^{xH_L}h_X e^{-xH_L}\big(\Pi^{H_{L^c}}_{(-\infty,\tau)}e^{xH_{L^c}}\big)|\psi\rangle| \\
\le&\frac{2\|h_X\|\cdot\|\langle\psi|\Pi^{H_{L^c}}_{[\tau,\infty)}\|\cdot\|\Pi^{H_{L^c}}_{(-\infty,\tau)}|\psi\rangle\|}{1-2gksx}\cdot\frac{e^{x\epsilon}}{(1-e^{-2(s-1)x\epsilon})^{1/2}} \\
=&\frac{\|h_X\|}{1-2gksx}\cdot\frac{e^{x\epsilon}}{(1-e^{-2(s-1)x\epsilon})^{1/2}},
\end{aligned} \tag{3.123}
$$

where we utilized the inequality

$$
2\|\langle\psi|\Pi^{H_{L^c}}_{[\tau,\infty)}\|\cdot\|\Pi^{H_{L^c}}_{(-\infty,\tau)}|\psi\rangle\| \le \|\langle\psi|\Pi^{H_{L^c}}_{[\tau,\infty)}\|^2+\|\Pi^{H_{L^c}}_{(-\infty,\tau)}|\psi\rangle\|^2 = 1. \tag{3.124}
$$

By choosing the parameter ϵ as

$$
\epsilon = \frac{\log s}{2(s-1)x}, \tag{3.125}
$$

we obtain

$$
\frac{e^{x\epsilon}}{(1-e^{-2(s-1)x\epsilon})^{1/2}} = \frac{s^{\frac{1}{2(s-1)}}}{\big(1-\frac{1}{s}\big)^{1/2}}, \tag{3.126}
$$

which, with $\lambda \equiv 1/(4gk)$, reduces the inequality (3.123) to

$$
2e^{x\tau}|\langle\psi|\Pi^{H_{L^c}}_{[\tau,\infty)}e^{xH_L}h_X e^{-xH_L}\big(\Pi^{H_{L^c}}_{(-\infty,\tau)}e^{xH_{L^c}}\big)|\psi\rangle| \le \frac{\|h_X\|}{1-\frac{sx}{2\lambda}}\cdot\frac{s^{\frac{1}{2(s-1)}}}{\big(1-\frac{1}{s}\big)^{1/2}}. \tag{3.127}
$$

By combining the inequalities (3.118) and (3.127), we finally obtain

$$|\langle\psi|e^{x(\tilde{H}_{L^c}+H_L)}h_X e^{-x(\tilde{H}_{L^c}+H_L)}|\psi\rangle| \leq \frac{\|h_X\|}{1-\frac{x}{2\lambda}} + \frac{\|h_X\|}{1-\frac{sx}{2\lambda}} \cdot \frac{s^{\frac{1}{2(s-1)}}}{\left(1-\frac{1}{s}\right)^{1/2}}, \tag{3.128}$$

which is followed by

$$\|H_{\partial L}(x)\| \leq \frac{\overline{H}_{\partial L}}{1-\frac{x}{2\lambda}} + \frac{\overline{H}_{\partial L}}{1-\frac{sx}{2\lambda}} \cdot \frac{s^{\frac{1}{2(s-1)}}}{\left(1-\frac{1}{s}\right)^{1/2}}. \tag{3.129}$$

By integrating it from $x=0$ to $x=\lambda$, we have

$$\int_0^\lambda dx\|H_{\partial L}(x)\| \leq \lambda\overline{H}_{\partial L}\left(2\log 2 - \frac{2}{s}\log(1-s/2)\cdot\frac{s^{\frac{1}{2(s-1)}}}{\left(1-\frac{1}{s}\right)^{1/2}}\right). \tag{3.130}$$

By choosing the parameter s as $3/2$, we finally arrive at

$$\int_0^\lambda dx\|H_{\partial L}(x)\| \leq (2+4\sqrt{3})\log 2\cdot\lambda\overline{H}_{\partial L}. \tag{3.131}$$

This completes the proof of (3.109).

Chapter 4

Many-body entanglement and reversibility after external disturbance

In this chapter, we explore how the k-locality of the Hamiltonian influences the ground states' structures. We analyze it in terms of the reversibility property.

4.1 Introduction

Ground states define quantum phases of matter at zero temperature. They are very often studied in many-body physics under a large variety of different approaches. As a result from interactions between condensed matter and quantum information, it has been understood that the features of the quantum phases can be characterized through the quantum entanglement [3]. One of the important achievements [45, 48] in this context has been the establishment of roles of short- and long-range entanglement as essential tools for characterizing different quantum orders occurring in systems. The entanglement is also a crucial parameter to determine whether we can develop an efficient description of the ground states, for example, by the use of the matrix product state [8, 9].

For this reason, there is a strong motivation to find essential parameters that govern the entanglement in quantum many-body systems. Preceding studies have indicated that the spectral gap just above the ground state places a strong restriction on the entanglement pattern, e.g. whether it is short- or long-ranged [45]. Outstanding issues in this context include the relation between the existence of the finite gap and the range of correlations in the system [20–23] and the celebrated 'area law' fulfilled for the entanglement entropy [12, 14, 15, 19] (see also Chapter 2).

We, however, have not obtained the complete knowledge between the spectral gap and the ground state properties:

1. We have little knowledge on the ground states in long-range interacting systems; we do not even know an efficient tool to characterize them.
2. For short-range interacting systems, the present approaches cannot give a necessary and sufficient condition which the gapped ground states satisfy.

Until our new proposal described below, there had been mainly three ways to solve the second problem; the exponential decay of correlation, the entropic area law, and the short-range entanglement (SRE). The first

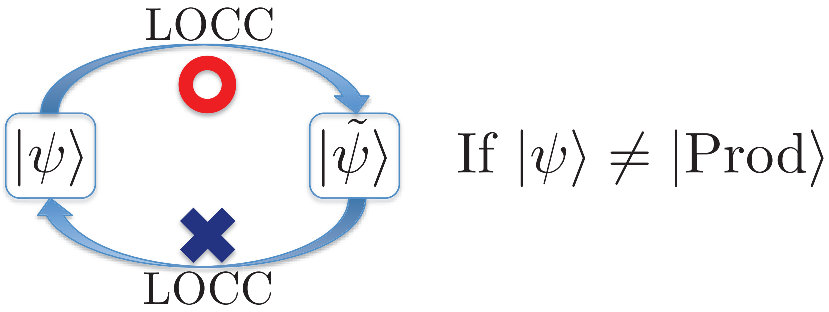

Figure 4.1: If an LOCC transformation changes a state $|\psi\rangle$ to another state $|\tilde{\psi}\rangle$, the inverse transformation by LOCC is usually prohibited. The reversibility is completely determined by the entanglement. If an inverse LOCC transformation is possible for any kind of LOCC, the state $|\psi\rangle$ must be a product state, which has no entanglement. In our new setting, we consider k-local operations instead of LOCC and expect the entanglement to play important roles in this case, too.

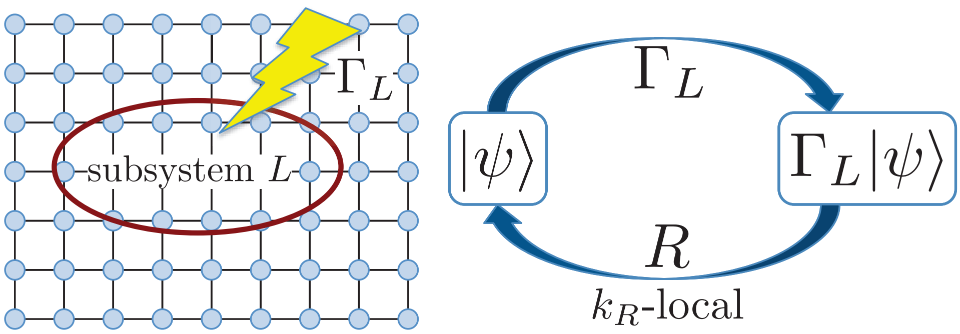

Figure 4.2: Schematic picture of the local reversibility. We disturb a quantum state $|\psi\rangle$ by an operator Γ_L supported in a subsystem L. We then try to recover the state $|\psi\rangle$ from $\Gamma_L|\psi\rangle$ by the use of a k_R-local operator R. If the state $|\psi\rangle$ is locally reversible, we can recover the original state by an operator R with $k_R = \mathcal{O}(\sqrt{|L|})$. The property "local reversibility" is expected to restrict the entanglement properties of the system. This comes from the fact that the entanglement cannot be recovered by local operations once it is broken.

two properties are the necessary conditions for the gapped ground state to satisfy[*1] ; as the contraposition, we have a class of states which can never be non-degenerate gapped ground states (see Subsection 4.6.2). The third property, the short-range entanglement (SRE), is quite an efficient tool to differentiate the quantum phases between trivially and non-trivially structured phases [45–47]; for example, the Landau symmetry-breaking phases (e.g. the ferromagnetic phase) are trivial and the topologically ordered phases are non-trivial. However, SRE is too strict to characterize the whole gapped phases; there indeed exists a class of gapped ground states which are not classified into the SRE state. For example, Kitaev's model [33] on a sphere is a commuting local Hamiltonian and has a non-degenerate ground state with an $\mathcal{O}(1)$ gap, but is not classified into SRE [50].

We, in the present chapter, focus on reversibility of ground states against external disturbance. It is well-known that the entanglement is essentially an irreversible entity; the irreversibility has been intensively investigated, especially in terms of the LOCC [51–53] (Fig. 4.1). We thus expect that if the entanglement pattern (e.g. SRE and LRE) reflects the essence of ground states, reversibility sheds a new light on the understanding of ground states. For this purpose, we take a new approach to defining the reversibility of quantum states, which we refer to as *local reversibility*.

[*1] In fact, the entropic area law can be rigorously proved only for one-dimensional systems (see Section 2.6 in Chapter 2), but the higher-dimensional area law is also conjectured [19].

4.2 Our new approach

We here try to probe a ground state through its response to an external disturbance. In order to characterize reversibility of general quantum states, we consider a generic disturbance (see below for more precise definitions) induced by the action of a non-trivial operator on a part of the system, say L (see Fig.4.2), and see how it changes a state $|\psi\rangle$. We then want to recover the original state $|\psi\rangle$ by the use of a k_R-local operator R. Because of the quantum entanglement, generic states are locally unrecoverable after the disturbance. We address the fundamental question: how large k_R should we take in order for the k_R-local operator R to bring back the original state $|\psi\rangle$? If the necessary value of k_R is of $\mathcal{O}(\sqrt{|L|})$, we call the state $|\psi\rangle$ *locally reversible*; that is, we only have to access quite a small number of spins compared with the subsystem's size $|L|$. We shall see that the spectral gap gives the size of k_R enough to recover the ground state after the disturbance.

4.2.1 Mathematical definition of the local reversibility

We here make the above discussion explicit:

Definition 4.2.1. *Let us consider an arbitrary quantum state $|\psi\rangle$, an operator Γ_L supported in a subsystem L, and a k_R-local operator R. We define that the state $|\psi\rangle$ is locally reversible iff for $\forall \Gamma_L$ and $\forall k_R$, there exists a k_R-local operator R which satisfies*

$$\|R\Gamma_L|\psi\rangle - |\psi\rangle\| \le \frac{\|\Gamma_L\|}{|\langle\psi|\Gamma_L|\psi\rangle|} f\left(\frac{k_R}{\sqrt{|L|}}\right), \tag{4.1}$$

where $f(x)$ decays faster than any power-law decays.

The local reversibility implies that the restoration error of the state $|\psi\rangle$ drops super-polynomially as k_R increases beyond $\sqrt{|L|}$. Note that we can take the size of the subsystem arbitrarily large, as large as the system size ($|L| = N$). In Appendix B, we prove that the choice of $\sqrt{|L|}$ is optimal in the sense that $k_R = \sqrt{|L|}$ is the necessary and sufficient size for the product states to be reversed.

4.2.1 (a) Is the term $1/|\langle\psi|\Gamma_L|\psi\rangle|$ necessary?

The term $\langle\psi|\Gamma_L|\psi\rangle|$ in the denominator of Eq. (4.1) means that we need large k_R as $|\langle\psi|\Gamma_L|\psi\rangle|$ becomes small; if the converted state $\Gamma_L|\psi\rangle$ is almost orthogonal to the original state $|\psi\rangle$, the reverse conversion becomes quite difficult. This is rational because after a state is completely broken in the region L we expect that the reverse operator with at least $k_R = \mathcal{O}(|L|)$ is necessary.

For example, let us consider the product state $|\psi\rangle = |00\cdots 0\rangle$ and the conversion operator $\Gamma_L = |++\cdots+\rangle\langle++\cdots+|$ with $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. Now, we have

$$\Gamma_L|\psi\rangle = |++\cdots+\rangle \quad \text{with} \quad |\langle\psi|\Gamma_L|\psi\rangle| = O(2^{-N/2}), \tag{4.2}$$

and there does not exist a reverse operator R with $k_R = \mathcal{O}(\sqrt{N})$ which converts $|++\cdots+\rangle$ back into $|\psi\rangle = |00\cdots 0\rangle$. In this case, however, the situation is consistent because of the term $|\langle\psi|\Gamma_L|\psi\rangle|^{-1}$; the inequality (4.1) now reduces to

$$\|R\Gamma_L|\psi\rangle - |\psi\rangle\| \le O(2^{N/2}) f(k_R/\sqrt{N}), \tag{4.3}$$

and the reverse operator with $k_R = \mathcal{O}(\sqrt{N})$ cannot make the right-hand side approach to zero.

Locally reversible	Non locally reversible
Product state	GHZ state
Graph state with finite vertices	macroscopically-entangled state
Short-range-entangled state	topologically-ordered ground state
W state	

Table 4.1: Several examples; see also Sections 4.5 and 4.6. We mean by 'with finite vertices' a graph each of whose nodes has at most $\mathcal{O}(1)$ neighboring nodes. The short-range entanglement is characterized by local-unitary operations; the local-unitary operation is defined by unitary quantum circuits whose depth is $\mathcal{O}(1)$. If a state can be connected to a product state by the use of a local-unitary evolution, the state is short-range-entangled. The macroscopic entanglement, which is related to macroscopic superposition, can be characterized by the Fisher information $\mathcal{F}(\rho, A)$ [56–58]; it is defined by a quantum state ρ and an additive operator A (e.g. $A = \sum_{i=1}^{N} \sigma_i^z$); if a state ρ is macroscopically entangled, there exists A which gives a scaling $\mathcal{O}(N^p)$ of $\mathcal{F}(\rho, A)$ with $p > 1$.

4.2.2 Examples of the local reversibility (LR): relation to the macroscopic quantumness

In Table 4.1, we show several examples of LR states and non-LR states; we will give the details in Sections 4.5 and 4.6. We consider several typical states with multipartite entanglement and classify them into LR or non-LR. Within the examples which we considered, non-LR quantum states may define the class of states with macroscopic quantum phenomena. Indeed, the macroscopic entanglement, which can be characterized by the scaling of the Fisher information [55–58], is prohibited for locally-reversible states; the macroscopic entanglement typically appears at the quantum critical point with spontaneous symmetry breaking. Even for locally undetectable phenomena (e.g. the topological order), we can still see their non-LR by taking the subsystem L large enough (see Subsection 4.6.2).

4.3 Reversibility in ground states

We here present the main theorem on which our approach is based. We first give more precise definitions on our Hamiltonian. We consider quantum systems described by the class of Hamiltonians given by an extensive k-local operator as

$$H = \sum_{|X| \le k} h_X \quad \text{with} \sum_{X:X \ni i} \|h_X\| \le g \quad \forall i, \tag{4.4}$$

where $\| \cdots \|$ is the operator norm, g is a constant of $\mathcal{O}(1)$, and $\sum_{X:X\ni i}$ denotes the summation with respect to the supports which contain the spin i. We fix the ground state energy $E_0 = 0$, assume its *non-degeneracy*, and denote the other eigenenergies by $0 = E_0 < E_1 \le E_2 \le \cdots$ with the corresponding eigenstates $|E_0\rangle, |E_1\rangle, |E_2\rangle, \ldots$, respectively. We then denote the spectral gap just above the ground state by $\delta E \equiv E_1 - E_0$. We now have all the ingredients to show the main theorem.

Theorem 4.3. For $\forall \Gamma_L$ and $\forall k_R$, there exists a k_R-local operator R which satisfies

$$\|R\Gamma_L|E_0\rangle - |E_0\rangle\| \le \frac{6\|\Gamma_L\|}{|\langle E_0|\Gamma_L|E_0\rangle|} e^{-2n_R/\xi}, \tag{4.5}$$

where $n_R \equiv \lfloor k_R/k \rfloor$ and

$$\xi \equiv \sqrt{1 + \frac{2E_c}{\delta E}}, \quad E_c = \frac{3g|L|}{2} + 8gkn_R. \tag{4.6}$$

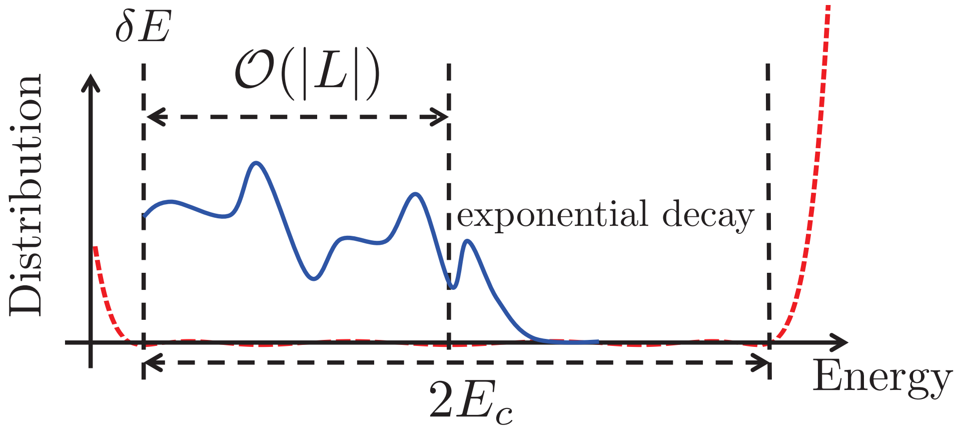

Figure 4.3: Schematic picture of the proof. After applying the operator Γ_L to the ground state $|E_0\rangle$, the energies of order $\mathcal{O}(|L|)$ at most are excited (blue curve) according to Theorem 3.2. We then filter out the excited states by the polynomial $F_R(x)$ (red curve) which approximately behaves as a boxcar function in the range $[\delta E, \delta E + E_c]$. Although the function $F_R(x)$ rapidly increases for $x \geq 2E_c + \delta E$, this can be cancelled by the exponential decay of the energy excitation.

All the technical details of the proof are reported in the next section. We can find that from Theorem 4.3 the spectral gap of $\mathcal{O}(1)$ promises the local reversibility of the ground states. This theorem represents a strong necessary condition which ground states should satisfy and allows us to treat important classes of systems such as the quantum critical points and long-range interacting systems. For short-range interacting systems, the local reversibility also complements the insufficiency of the pre-existing conditions; necessary conditions for ground states with finite gaps have been given by the exponential decay of bi-partite correlations or the related entropic area law. These properties, however, do not give a sufficient condition for the ground states; locally-hidden macroscopicity such as the topological order cannot be prohibited by them. Therefore, in order to characterize the Hilbert space of unique gapped ground states, we need at least the three properties: the exponential decay of correlations, the entropic area law, and the local reversibility.

Let us mention the relationship to the area law of the entanglement entropy (on the proof of the area law, see Section 2.6 in Chapter 2). In the proof of the one-dimensional area law [14, 15], we consider a disturbance Γ_L to the ground state $|E_0\rangle$ in the form of the projection to a product state with the maximum overlap with $|E_0\rangle$. We then recover the original state $|E_0\rangle$ from the product state by the use of an approximate ground-state projection (AGSP) operator K, namely

$$|E_0\rangle \xrightarrow[\text{Max. overlap with } |E_0\rangle]{\Gamma_L} |\mathrm{Prod}\rangle \xrightarrow[\text{AGSP}]{K} |E_0\rangle. \tag{4.7}$$

We can therefore make a correspondence between the AGSP operator K and our reverse operator R. The area law is derived from a restriction on the Schmidt rank of K. Thus, its fundamental philosophy is parallel to ours in the sense that we investigate the restriction on the recovery operation after the destruction of the entanglement.

4.4 Proof of Theorem 4.3

In this section, we give the full proof of the main Theorem 4.3. We first show the outline of the proof and then explain the details.

4.4.1 Outline of the proof

For the proof, we construct a k_R-local operator R by the use of a polynomial of H. For this purpose, we construct a polynomial $F_R(x)$ which approximately satisfies

$$F_R(H)|E_0\rangle = |E_0\rangle, \quad F_R(H)\Pi^H_{[E_1,\infty)}\Gamma_L|E_0\rangle = 0, \tag{4.8}$$

where $\Pi^H_{[E_1,\infty)}$ is the projection operator onto the subspace of energies which satisfy $E \geq E_1 = \delta E$. In Fig. 4.3, we sketch our reasoning in the proof. We first focus on the fact that the operator Γ_L causes the energy excitation of order of $\mathcal{O}(|L|)$ at most as shown in Theorem 3.2 (we will show it again). We next filter out the excited states by the use of a polynomial of the Hamiltonian $F_R(H)$ which approximately gives a boxcar filtering. Our task is to prove that the degree of such a polynomial $F_R(x)$ can be taken as $\mathcal{O}(\sqrt{|L|})$; then, the operator $F_R(H)$ is also $\mathcal{O}(\sqrt{|L|})$-local.

Theorem 3.2. The excitation by an arbitrary local operator Γ_L is suppressed as follows:

$$\|\Pi^H_{[E',\infty)}\Gamma_L\Pi^H_{(-\infty,E]}\| \leq \|\Gamma_L\|e^{-\lambda(E'-E-3g|L|)}, \tag{4.9}$$

where λ is given by $1/(4gk)$. We emphasize that this lemma gives the essence of the locality of k-local Hamiltonians instead of the well-known Lieb-Robinson bound [28].

In constructing the filter function $F_R(x)$, we utilize the technique of constructing the approximate ground state projector, which was one of the crucial techniques for the proof of the entropic area law [14, 15]; we utilize the following polynomial:

$$F_R(x) = \frac{T_{n_R}(\frac{x-\delta E}{E_c} - 1)}{T_{n_R}(\frac{-\delta E}{E_c} - 1)}, \tag{4.10}$$

where $T_{n_R}(x)$ is the Chebyshev polynomial [54] of degree $n_R \equiv \lfloor k_R/k \rfloor$. The Chebyshev polynomial has the values of $|T_n(x)| \leq 1$ for $-1 \leq x \leq 1$, while it increases rapidly for $|x| \geq 1$ as $e^{2n\sqrt{(x+1)/(x-1)}} \leq 2T_n(x) \leq (2x)^n$. The function $F_R(H)$ is k_R-local at most because H^{n_R} is (kn_R)-local; note that $kn_R = k\lfloor k_R/k \rfloor \leq k_R$. The polynomial $F_R(x)$ satisfies $F_R(0) = 1$ and therefore $F_R(H)|E_0\rangle = F_R(0)|E_0\rangle = |E_0\rangle$ because of $E_0 = 0$. We are now able to prove the following inequality

$$\|F_R(H)\Pi^H_{[\delta E,\infty)}\Gamma_L|E_0\rangle\| \leq 6\|\Gamma_L\|e^{-2n_R/\xi} \tag{4.11}$$

after algebra by the use of the inequality (4.9), where ξ is defined in Eq. (4.6). From the above inequality (4.11), we choose $R = F_R(H)/|\langle E_0|\Gamma_L|E_0\rangle|$ and obtain the inequality (4.5). This completes the proof of Theorem 4.3. $\square$

In the special case where the state $|E_0\rangle$ is a gapped ground state of commuting Hamiltonian (e.g. the graph state), we obtain a slightly tighter upper bound. We mean by "commuting Hamiltonian" that $[h_X, h_{X'}] = 0$ for $\forall X, X'$ in the Hamiltonian of $H = \sum_{|X|\leq k} h_X$.

Corollary 4.4. Let us define $n_R \equiv \lfloor k_R/k \rfloor$ and assume the commuting Hamiltonian. Then, a non-degenerate gapped ground state satisfies the local reversibility as

$$\|R\Gamma_L|E_0\rangle - |E_0\rangle\| \leq \frac{2\|\Gamma_L\|}{|\langle E_0|\Gamma_L|E_0\rangle|}e^{-2n_R/\xi'}, \tag{4.12}$$

where $\xi' \equiv \sqrt{2g|L|/\delta E}$.

Proof of Corollary 4.4. The proof is essentially the same as Theorem 4.3. We only show the key point and give its details in Subsection 4.4.3. The only difference comes from the following point: if the

Hamiltonian is commuting, the excitation by any local operators Γ_L is finite, namely,

$$\|\Pi^H_{[E',\infty)}\Gamma_L\Pi^H_{(-\infty,E]}\| = 0 \quad \text{for} \quad E - E' > 2g|L|, \tag{4.13}$$

which is given in Lemma 3.3.1 (see Chapter 3).

Because the energy excitation is finite, we only have to filter out the excited states in the range $[\delta E, \delta E + 2g|L|]$ (see also Fig. 4.3). We then take E_c as $2E_c + \delta E = 2g|L|$ in the polynomial (4.10) instead of $E_c = 3g|L|/2 + 8gkn_R$. After the same calculations as those in the proof of Theorem 4.3, we can obtain the inequality (4.12).

4.4.2 Proof of the inequality (4.11)

We here prove the inequality (4.11). First, from the definition of $F_R(x)$ and the basic properties of the Chebyshev polynomials [14, 15], we have the following inequalities (see Subsection 4.4.2 (a)):

$$|F_R(x)| \le 2e^{-2n_R/\xi} \quad \text{for} \quad \delta E \le x \le 2E_c + \delta E \tag{4.14}$$

and

$$|F_R(x)| \le \left(\frac{2x - 2\delta E}{E_c} - 2\right)^{n_R} e^{-2n_R/\xi} \quad \text{for} \quad x \ge 2E_c + \delta E. \tag{4.15}$$

In order to calculate the upper bound of $\|F_R(H)\Pi^H_{[\delta E,\infty)}\Gamma_L|E_0\rangle\|$, we decompose it into

$$\|F_R(H)\Pi^H_{[\delta E,2E_c+\delta E)}\Gamma_L|E_0\rangle\| + \|F_R(H)\Pi^H_{[2E_c+\delta E,\infty)}\Gamma_L|E_0\rangle\| \tag{4.16}$$

and calculate each term separately. The inequality (4.14) gives the upper bound of the first term as

$$\|F_R(H)\Pi^H_{[\delta E,2E_c+\delta E)}\Gamma_L|E_0\rangle\| \le 2\|\Gamma_L\|e^{-2n_R/\xi}. \tag{4.17}$$

We next calculate the upper bound of the second term $\|F_R(H)\Pi^H_{[2E_c+\delta E,\infty)}\Gamma_L|E_0\rangle\|$. For this purpose, we decompose it as

$$\|F_R(H)\Pi^H_{[2E_c+\delta E,\infty)}\Gamma_L|E_0\rangle\| \le \sum_{j=1}^{\infty} \|F_R(H)\Pi^H_{[2E_c+\delta E+(j-1)\epsilon,2E_c+\delta E+j\epsilon)}\Gamma_L|E_0\rangle\|, \tag{4.18}$$

where ϵ is a positive constant which will be set afterward. We then utilize Theorem 3.2; by applying the inequality (4.9) to the norm $\|\Pi^H_{[2E_c+\delta E+(j-1)\epsilon,2E_c+\delta E+j\epsilon)}\Gamma_L|E_0\rangle\|$, we obtain

$$\begin{aligned}
&\|F_R(H)\Pi^H_{[2E_c+\delta E+(j-1)\epsilon,2E_c+\delta E+j\epsilon)}\Gamma_L|E_0\rangle\| \\
\le&\|F_R(H)\Pi^H_{[2E_c+\delta E+(j-1)\epsilon,2E_c+\delta E+j\epsilon)}\| \cdot \|\Pi^H_{[2E_c+\delta E+(j-1)\epsilon,2E_c+\delta E+j\epsilon)}\Gamma_L|E_0\rangle\| \\
\le&\|\Gamma_L\|F_R(2E_c + \delta E + j\epsilon)e^{-\lambda(2E_c+\delta E+(j-1)\epsilon-3g|L|)} \\
\le&\|\Gamma_L\|e^{-2n_R/\xi}e^{\lambda\epsilon}e^{-\lambda(2E_c+\delta E+j\epsilon-3g|L|)/2},
\end{aligned} \tag{4.19}$$

where, from the second line to the third line, we utilized the fact that the function $F_R(x)$ monotonically increases for $x \geq 2E_c + \delta E$, and from the third line to the fourth line, we utilized the following inequality:

$$F_R(x)e^{-\lambda(x-3g|L|)} \leq e^{-2n_R/\xi} \cdot e^{-\lambda(x-3g|L|)/2} \tag{4.20}$$

for $x \geq 2E_c + \delta E$ (see Subsection 4.4.2 (b)). By the use of the inequality (4.19) and the definition of E_c in (4.6), we obtain

$$\begin{aligned} \|F_R(H)\Pi^H_{[2E_c+\delta E,\infty)}\Gamma_L|E_0\rangle\| \leq& \|\Gamma_L\|e^{-2n_R/\xi}e^{\lambda(\epsilon-\delta E/2)-2n_R}\sum_{j=1}^{\infty}e^{-\lambda\epsilon j/2} \\ \leq& \|\Gamma_L\|e^{-2n_R/\xi}\frac{e^{\lambda\epsilon}}{e^{\lambda\epsilon/2}-1} = 4\|\Gamma_L\|e^{-2n_R/\xi}, \end{aligned} \tag{4.21}$$

where we chose $\epsilon = 2\log 2/\lambda$ and have $e^{\lambda\epsilon}/(e^{\lambda\epsilon/2}-1) = 4$. By applying the inequalities (4.17) and (4.21) to (4.16), we finally obtain the inequality (4.11). This completes the proof. □

4.4.2 (a) Derivation of the inequalities (4.14) and (4.15)

The Chebyshev polynomial $T_n(x)$ satisfies

$$|T_n(x)| \leq 1 \quad \text{for} \quad -1 \leq x \leq 1, \tag{4.22}$$

$$|T_n(x)| \geq \frac{1}{2}\exp\left(2n\sqrt{\frac{x+1}{x-1}}\right) \quad \text{for} \quad x \leq -1. \tag{4.23}$$

Because of the above inequalities, we obtain

$$T_n\left(\frac{-\delta E}{E_c} - 1\right) \geq \frac{1}{2}\exp\left(2n\sqrt{\frac{\delta E}{2E_c+\delta E}}\right) \equiv \frac{1}{2}e^{2n/\xi} \tag{4.24}$$

$$T_n\left(\frac{x-\delta E}{E_c} - 1\right) \leq 1 \tag{4.25}$$

for $\delta E \leq x \leq \delta E + 2E_c$. We thus prove the inequality (4.14) from the definition of $F_R(x)$ in Eq. (4.10).

We second prove the inequality (4.15). For this purpose, we obtain the upper bound of $|T_n(x)|$ for $x \geq 1$ as

$$T_n(x) \leq \frac{(2x)^n}{2}. \tag{4.26}$$

We can prove it from the following inequality:

$$(2x-y)^n + y^n \leq (2x)^n, \tag{4.27}$$

for $x \geq 1$ and $0 \leq y \leq 1$. Note that the inequality (4.26) is given by choosing $y = x - \sqrt{x^2-1}$ in (4.27) because $T_n(x) = \left[(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\right]/2$. The proof is given as follows; because the function $(2x)^n - ((2x-y)^n + y^n)$ monotonically increases for $x \geq 1$ and $0 \leq y \leq 1$, its minimum value is obtained for $x = 1$ and $y = 0$, which gives $(2x)^n - ((2x-y)^n + y^n) = 0$. We thus prove the inequality (4.27).

From the inequality (4.26), we obtain

$$T_n\left(\frac{x-\delta E}{E_c} - 1\right) \leq \frac{1}{2}\left(\frac{2x-2\delta E}{E_c} - 2\right)^n \tag{4.28}$$

for $x \geq 2E_c + \delta E$. By combining the inequalities (4.24) and (4.28), we obtain the inequality (4.15).

4.4.2 (b) Derivation of the inequality (4.20)

From the inequality (4.15), we have

$$F_R(x)e^{-\lambda(x-3g|L|)} \le e^{-2n_R/\xi}\left(\frac{2x-2\delta E}{E_c}-2\right)^{n_R} e^{-\lambda(x-3g|L|)} \tag{4.29}$$

for $x \ge 2E_c + \delta E$. We then consider the logarithm of $\left[(2x-2\delta E)/E_c - 2\right]^{n_R} e^{-\lambda(x-3g|L|)/2}$, namely,

$$g(x) \equiv -\frac{\lambda}{2}(x-3g|L|) + n_R \log\left(\frac{2x-2\delta E}{E_c}-2\right) \tag{4.30}$$

and prove that $g(x)$ is negative for $x \ge 2E_c + \delta E$. After we prove this statement, we have

$$F_R(x)e^{-\lambda(x-3g|L|)} \le e^{-2n_R/\xi} e^{g(x)-\lambda(x-3g|L|)/2} \le e^{-2n_R/\xi} e^{-\lambda(x-3g|L|)/2}, \tag{4.31}$$

which completes the proof of the inequality (4.20).

We now calculate the function $g(x)$. We first have

$$\frac{dg(x)}{dx} = -\frac{\lambda}{2} + \frac{n_R}{x-E_c-\delta E} \le -\frac{\lambda}{2} + \frac{n_R}{E_c} = -\frac{\lambda}{2} + \frac{\lambda}{2+\frac{3g\lambda|L|}{2n_R}} < 0 \tag{4.32}$$

for $x \ge 2E_c + \delta E$; note that $E_c = 3g|L|/2 + 8gkn_R = 3g|L|/2 + 2n_R/\lambda$. We also have

$$g(2E_c+\delta E) = -2n_R - \frac{\lambda \delta E}{2} + n_R \log 2 < 0. \tag{4.33}$$

We thus prove $g(x) \le 0$ for $x \ge 2E_c + \delta E$.

4.4.3 Details of the proof of Corollary 4.4

We here show the details of the proof of Corollary 4.4. As in the proof of Theorem 4.3, we have to find a polynomial $F'_R(x)$ which approximately satisfies

$$F'_R(H)|E_0\rangle = |E_0\rangle, \quad F'_R(H)\Pi^H_{[\delta E,\infty)}\Gamma_L|E_0\rangle = 0. \tag{4.34}$$

We construct the operator R by the use of the following polynomial:

$$F'_R(x) = \frac{T_{n_R}(\frac{x-\delta E}{E'_c}-1)}{T_{n_R}(\frac{-\delta E}{E'_c}-1)}, \tag{4.35}$$

where we choose $2E'_c + \delta E = 2g|L|$.

From Eq. (4.13), we have $\Pi^H_{(2g|L|,\infty)}\Gamma_L|E_0\rangle = 0$ and

$$\|F'_R(H)\Pi^H_{[\delta E,\infty)}\Gamma_L|E_0\rangle\| = \|F'_R(H)\Pi^H_{[\delta E,2g|L|]}\Gamma_L|E_0\rangle\| = \|F'_R(H)\Pi^H_{[\delta E,2E'_c+\delta E]}\Gamma_L|E_0\rangle\|. \tag{4.36}$$

We can prove an inequality similar to (4.14) as

$$|F'_R(x)| \le 2e^{-2n_R/\xi'} \tag{4.37}$$

for $\delta E \le x \le 2E'_c + \delta E$, where $\xi' = \sqrt{(2E'_c+\delta E)/\delta E} = \sqrt{2g|L|/\delta E}$. This gives the upper bound of

(4.36) as

$$\|F'_R(H)\Pi^H_{[\delta E,\infty)}\Gamma_L|E_0\rangle\| \le 2\|\Gamma_L\|e^{-2n_R/\xi'}. \tag{4.38}$$

By choosing $R = F'_R(H)/|\langle E_0|\Gamma_L|E_0\rangle|$, we can prove the inequality (4.12). This completes the proof of Corollary 4.4.

4.5 Examples of locally reversible states

First, we will explore the local reversibility of exemplarily non-degenerate gapped many-body ground states, which are mentioned in Table 4.1.

4.5.1 Product states

The first example is the product state. For an arbitrary product state, we can always construct a commuting Hamiltonian which has the product state as its non-degenerate gapped ground state. From Theorem 4.3 (or Corollary 4.4), we prove the local reversibility of the product states.

4.5.2 Graph states

The second example is the graph state with a finite number of edges [34–37], which means that each node has at most $\mathcal{O}(1)$ neighboring nodes. The graph state is a non-degenerate gapped ground states of the summation of the following commuting stabilizers $\{g_i\}_{i=1}^N$ [36, 37]:

$$H = \sum_{i=1}^{N} g_i, \quad g_i = \sigma_i^x \otimes (\sigma_{j_1}^z \sigma_{j_2}^z \cdots \sigma_{j_{k_i}}^z), \tag{4.39}$$

where $[g_i, g_{i'}] = 0$ for $\forall i, i'$, $\{\sigma^x, \sigma^y, \sigma^z\}$ are the Pauli matrices, and $\{j_1, j_2, \ldots, j_{k_i}\}$ are nodes which connect to the node i. From the assumption of $k_i = \mathcal{O}(1)$, the summation of the stabilizers is also $\mathcal{O}(1)$-local. We therefore conclude that the graph states with finite edges satisfy the local reversibility.

4.5.3 Short-range entanglement (SRE)

The third example is a state with the short-range entanglement (SRE) [45]. It can be transformed to a product state $|\mathrm{Prod}\rangle$ by a local-unitary evolution U_{loc}, which is a set of unitary quantum circuits of depth of $\mathcal{O}(1)$, where each circuit consists of local non-overlapping unitaries $\{U_i\}$.

We can prove that the state $|\psi\rangle = U_{\mathrm{loc}}|\mathrm{Prod}\rangle$ with SRE is also locally reversible. The proof is given as follows; as mentioned above, we can always find the Hamiltonian H_{Prod} which gives $|\mathrm{Prod}\rangle$ as its non-degenerate gapped ground state. Note that the Hamiltonian H_{Prod} can be expressed by a sum of one-spin operators. The state $|\psi\rangle = U_{\mathrm{loc}}|\mathrm{Prod}\rangle$ then becomes the ground state of $U_{\mathrm{loc}} H_{\mathrm{Prod}} U_{\mathrm{loc}}^\dagger$. Because the local unitary maintains the locality of the Hamiltonian H_{Prod}, the Hamiltonian $U_{\mathrm{loc}} H_{\mathrm{Prod}} U_{\mathrm{loc}}^\dagger$ is also $\mathcal{O}(1)$-local. Therefore, Theorem 4.3 gives the local reversibility of $|\psi\rangle$.

The SRE is often defined with respect to time evolution of a short-range Hamiltonian instead of the finite-depth unitary circuit; we denote it as $\tilde{U}_{\mathrm{loc}} \equiv \mathcal{T}[e^{-i\int_0^t H(t)dt}]$ with $t = \mathcal{O}(1)$ and $\mathcal{T}$ the time ordering operator. We can also prove the local reversibility of $\tilde{U}_{\mathrm{loc}}|\mathrm{Prod}\rangle$ by the use of the Lieb-Robinson technique [28, 29].

We note that SRE does not give a necessary condition for the local reversibility; that is, the long-range entanglement (LRE) does not necessarily imply the breaking of the local reversibility. For example, if we

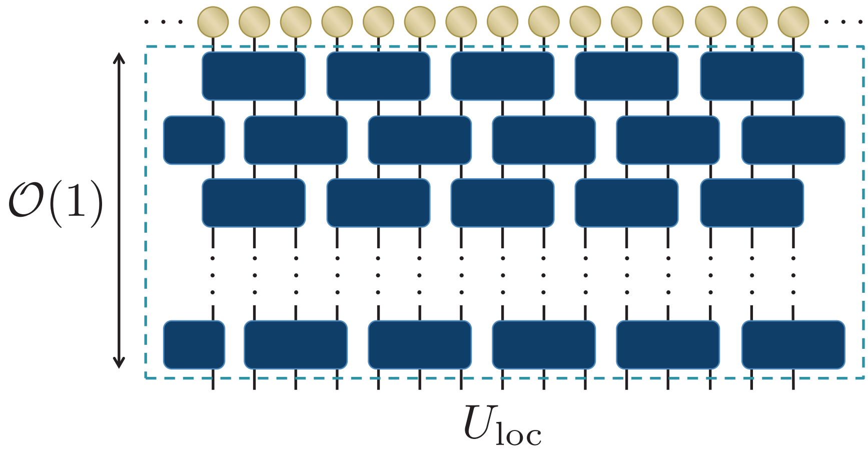

Figure 4.4: Schematic picture of the local unitary operation. The local unitary operation is defined by a constant depth unitary circuit. One layer consists of a set of unitary operations; each unitary operator (indigo box) is applied to a small number of spins and does not overlap with each other. The depth of the quantum circuit is bounded from above by an $\mathcal{O}(1)$ constant.

consider gapped ground states of infinite-range interacting Hamiltonians, they are usually not classified into SRE but satisfies the local reversibility according to Theorem 4.3. In Subsection 4.5.4, we also prove that the W state is locally reversible.

4.5.4 The local reversibility of the W state

We here consider the local reversibility of the W state, namely $|W\rangle \equiv S^+|0_N\rangle$, where $S^\pm \equiv \frac{1}{\sqrt{N}}\sum_i^N \sigma_i^\pm$, $\sigma_i^\pm \equiv (\sigma_i^x \pm i\sigma_i^y)/\sqrt{2}$ and $|0_N\rangle \equiv |\overbrace{00\cdots0}^{N}\rangle$, where $|0\rangle$ is the eigenstate of σ^z, namely $\sigma^z|0\rangle = |0\rangle$. We can prove that the W state is locally reversible but in a slightly weaker sense; there exists a k_R-local operator R which satisfies

$$\|R\Gamma_L|W\rangle - |W\rangle\| \le \frac{c_1(1+|L|/\sqrt{N})^3 \cdot \|\Gamma_L\|}{|\langle W|\Gamma_L|W\rangle|} e^{-c_2(k_R-4)/\sqrt{|L|}} \tag{4.40}$$

for $\forall\Gamma_L$ and $\forall k_R$, where c_1 and c_2 are constants of $\mathcal{O}(1)$. We thus need an $\mathcal{O}(\sqrt{|L|}\log|L|)$-local operator in order to recover the W state in the case of $|L| \gg \sqrt{N}$.

We here prove the inequality (4.40). The outline of the proof is given as follows; we first consider the reverse operator $\tilde{R}$ from $S^-\Gamma_L S^+|0_N\rangle$ to $|0_N\rangle$. Note that $S^-\Gamma_L S^+$ is supported in a region with the cardinality $|L|+2$ because $S^\pm$ is 1-local. We then choose $S^+\tilde{R}S^-$ as the reverse operator R from $\Gamma_L|W\rangle$ to $|W\rangle$. The problem is how to construct the operator $\tilde{R}$ properly.

We first introduce the decomposition of $S^\pm$ as

$$S^\pm \equiv S_L^\pm + S_{L^c}^\pm, \tag{4.41}$$

where $S_L^\pm \equiv \sum_{i\in L}\sigma_i^+/\sqrt{N}$ and $S_{L^c}^\pm \equiv \sum_{i\in L^c}\sigma_i^+/\sqrt{N}$ with L^c the complementary subsystem of L. In order to construct the operator $\tilde{R}$, we consider two operators $\tilde{R}_1$ and $\tilde{R}_2$ and let $\tilde{R} = \tilde{R}_2\tilde{R}_1$. We can first prove that the 2-local operator $\tilde{R}_1 \equiv 1 - S_{L^c}^+ S_{L^c}^-/c_L$ gives

$$\tilde{R}_1 S^-\Gamma_L S^+|0_N\rangle = \tilde{\Gamma}_L|0_L\rangle \otimes |0_{L^c}\rangle \tag{4.42}$$

with

$$\langle 0_L|\tilde{\Gamma}_L|0_L\rangle = \langle W|\Gamma_L|W\rangle, \tag{4.43}$$

where $\tilde{\Gamma}_L \equiv S_L^-\Gamma_L S_L^+ + c_L\Gamma_L$, $c_L \equiv \frac{N-|L|}{N}$ and $|0_N\rangle \equiv |0_L\rangle \otimes |0_{L^c}\rangle$. We give the proof afterward.

From Eq. (4.42), we obtain

$$\tilde{R}S^-\Gamma_L S^+|0_N\rangle = \tilde{R}_2\tilde{R}_1 S^-\Gamma_L S^+|0_N\rangle = |0_N\rangle + \big(\tilde{R}_2\tilde{\Gamma}_L|0_L\rangle - |0_L\rangle\big) \otimes |0_{L^c}\rangle. \tag{4.44}$$

Next, Corollary 4.4 promises the existence of a $k_{\tilde{R}}$-local operator $\tilde{R}_2$ which satisfies

$$\|\tilde{R}_2\tilde{\Gamma}_L|0_L\rangle - |0_L\rangle\| \leq \frac{c_1\|\tilde{\Gamma}_L\|}{\langle W|\Gamma_L|W\rangle} e^{-c_2 k_{\tilde{R}}/\sqrt{|L|}}, \tag{4.45}$$

where c_1 and c_2 are constants of $\mathcal{O}(1)$. We have used the fact that the product state $|0_L\rangle$ is a gapped ground state of a commuting Hamiltonian. Since $\tilde{R}_1$ is 2-local and $\tilde{R}_2$ is $k_{\tilde{R}}$-local, the operator $\tilde{R} = \tilde{R}_2\tilde{R}_1$ is now $(k_{\tilde{R}} + 2)$-local.

We, from Eq. (4.44), have

$$\begin{aligned} S^+\tilde{R}S^-\Gamma_L S^+|0_N\rangle =&|W\rangle + S^+(\tilde{R}_2\tilde{\Gamma}_L|0_L\rangle - |0_L\rangle) \otimes |0_{L^c}\rangle \\ =&|W\rangle + S_L^+(\tilde{R}_2\tilde{\Gamma}_L|0_L\rangle - |0_L\rangle) \otimes |0_{L^c}\rangle + (\tilde{R}_2\tilde{\Gamma}_L|0_L\rangle - |0_L\rangle) \otimes S_{L^c}^+|0_{L^c}\rangle, \end{aligned} \tag{4.46}$$

which, with the inequality (4.45), gives

$$\begin{aligned} \|S^+\tilde{R}S^-\Gamma_L|W\rangle - |W\rangle\| \leq& \|S_L^+\| \cdot \|\tilde{R}_2\tilde{\Gamma}_L|0_L\rangle - |0_L\rangle\| + \|\tilde{R}_2\tilde{\Gamma}_L|0_L\rangle - |0_L\rangle\| \cdot \|S_{L^c}^+|0_{L^c}\rangle\|. \\ \leq& \frac{c_1\|\tilde{\Gamma}_L\|(\|S_L^+\| + \sqrt{c_L})}{\langle W|\Gamma_L|W\rangle} e^{-c_2 k_{\tilde{R}}/\sqrt{|L|}}. \end{aligned} \tag{4.47}$$

By utilizing the inequalities

$$\begin{aligned} &\|S_L^+\| \leq |L|/\sqrt{N}, \\ &\|\tilde{\Gamma}_L\| = \|S_L^-\Gamma_L S_L^+ + c_L\Gamma_L\| \leq \|\Gamma_L\|(c_L + |L|^2/N) \end{aligned} \tag{4.48}$$

and $c_L \leq 1$, we obtain

$$\begin{aligned} \|S^+\tilde{R}S^-\Gamma_L|W\rangle - |W\rangle\| \leq& \frac{c_1\|\Gamma_L\|(\sqrt{c_L} + |L|/\sqrt{N})(c_L + |L|^2/N)}{\langle W|\Gamma_L|W\rangle} e^{-c_2 k_{\tilde{R}}/\sqrt{|L|}} \\ \leq& \frac{c_1\|\Gamma_L\|(1 + |L|/\sqrt{N})^3}{\langle W|\Gamma_L|W\rangle} e^{-c_2 k_{\tilde{R}}/\sqrt{|L|}}. \end{aligned} \tag{4.49}$$

Therefore, when we choose a k_R-local reverse operator as $R = S^+\tilde{R}S^-$, we can take $k_{\tilde{R}} = k_R - 4$ in the inequality (4.49) because the operator $S^+\tilde{R}S^-$ is $(k_{\tilde{R}} + 4)$-local. This completes the proof of the inequality (4.40).

4.5.4 (a) Proofs of Eqs. (4.42) and (4.43)

We first obtain

$$S^-\Gamma_L S^+|0_N\rangle =(S_L^- + S_{L^c}^-)\Gamma_L(S_L^+ + S_{L^c}^+)|0_N\rangle$$

$$
\begin{aligned}
&= S_L^- \Gamma_L S_L^+ |0_N\rangle + \Gamma_L S_{L^c}^- S_{L^c}^+ |0_N\rangle + S_L^- \Gamma_L S_{L^c}^+ |0_N\rangle + \Gamma_L S_L^+ S_{L^c}^- |0_N\rangle \\
&= (S_L^- \Gamma_L S_L^+ + c_L \Gamma_L)|0_N\rangle + S_L^- \Gamma_L S_{L^c}^+ |0_N\rangle,
\end{aligned} \tag{4.50}
$$

where we utilize the equalities $S_L^+ S_{L^c}^- |0_N\rangle = 0$ and $\Gamma_L S_{L^c}^- S_{L^c}^+ |0_N\rangle = \frac{N-|L|}{N} \Gamma_L |0_N\rangle \equiv c_L \Gamma_L |0_N\rangle$.

When we consider the operator $\tilde{R}_1 = 1 - S_{L^c}^+ S_{L^c}^- / c_L$, we obtain

$$
\tilde{R}_1 S_L^- \Gamma_L S_L^+ |0_N\rangle = S_L^- \Gamma_L S_L^+ |0_N\rangle, \quad \tilde{R}_1 c_L \Gamma_L |0_N\rangle = c_L \Gamma_L |0_N\rangle, \quad \text{and} \quad \tilde{R}_1 S_L^- \Gamma_L S_{L^c}^+ |0_N\rangle = 0. \tag{4.51}
$$

These equalities give

$$
\tilde{R}_1 S^- \Gamma_L S^+ |0_N\rangle = (S_L^- \Gamma_L S_L^+ + c_L \Gamma_L)|0_N\rangle \equiv \tilde{\Gamma}_L |0_L\rangle \otimes |0_{L^c}\rangle. \tag{4.52}
$$

This competes the proof of Eq. (4.42). Because of $\langle 0_N | S_L^- \Gamma_L S_{L^c}^+ |0_N\rangle = 0$, from Eq. (4.50), we have

$$
\langle W | \Gamma_L | W \rangle = \langle 0_N | S^- \Gamma_L S^+ | 0_N \rangle = \langle 0_N | (S_L^- \Gamma_L S_L^+ + c_L \Gamma_L) | 0_N \rangle = \langle 0_L | \tilde{\Gamma}_L | 0_L \rangle. \tag{4.53}
$$

We thus obtain Eq. (4.43).

4.6 Examples of states that are non-locally reversible

As has been shown in Table 4.1, there are mainly two classes which do not satisfy the local reversibility: anomalously fluctuating states and topologically ordered states. Both of them show the macroscopic quantumness; the former ones can be characterized by the use of local operators, while the latter ones cannot, where we mean by "locally characterized" the fact that we can detect the macroscopicity by the use of a few k-local observables.

4.6.1 Anomalously fluctuated states

We here discuss the relationship between the local reversibility and the macroscopic entanglement which can be characterized by the Fisher information [55–58]. The Fisher information $\mathcal{F}(\rho, A)$ is defined for a quantum state ρ and an additive operator A (e.g. magnetization $\sum_{i=1}^N \sigma_i^z$). In the case where the state ρ is a pure state, namely $\rho = |\psi\rangle\langle\psi|$, the Fisher information is known to reduce to the variance $4(\Delta A)^2$ with respect to $|\psi\rangle$. As a measure of the macroscopic entanglement, we usually take the maximum value of $(\Delta A)^2$, namely $\max_A (\Delta A)^2$; if the macroscopic entanglement exists, the observable A can largely fluctuate as $\max_A (\Delta A)^2 = \mathcal{O}(N^p)$ with $p > 1$. We also comment that such an anomalous fluctuation usually appears at a quantum critical point; for example, the quantum critical point of the transverse Ising model has $p = 7/4$. We can prove that if there exists an anomalous fluctuation with $p > 1$, the state $|\psi\rangle$ does not satisfy local reversibility. The contraposition implies the following statement:

Theorem 4.6. Let $|\psi\rangle$ be a quantum state which satisfies the local reversibility. Then, for an arbitrary operator A which is a sum of one-spin operators $\{a_i\}_{i=1}^N$ with $\|a_i\| = 1$, the variance of A with respect to the state $|\psi\rangle$ is suppressed as $\mathcal{O}(N^p)$ with $p \le 1$.

We show the proof in Subsection 4.6.1 (a). This theorem characterizes a class of non-LR states; for example, the GHZ state has the value of $p = 2$ [56] and turns out to be non-LR.

4.6.1 (a) Proof of Theorem 4.6

We first prove that for an arbitrary LR state $|\psi\rangle$ we have

$$\|\Pi^A_{(-\infty,x]} P_\psi \Pi^A_{[x+\Delta x,\infty)}\| \le f(\Delta x/\sqrt{|L|}), \tag{4.54}$$

where $P_\psi \equiv |\psi\rangle\langle\psi|$ and $f(x)$ decays faster than any power laws. The inequality (4.54) implies that the spectrum of the operator A is superpolynomially localized with the localization width smaller than $\mathcal{O}(\sqrt{N})$. We can thus prove Theorem 4.6, namely $(\Delta A)^2 = \mathcal{O}(|L|^p)$ with $p \le 1$.

For the proof of the inequality (4.54), we utilize the inequality

$$\max(\langle\psi|\Pi^A_{[x+\Delta x/2,\infty)}|\psi\rangle, \langle\psi|\Pi^A_{(-\infty,x+\Delta x/2]}|\psi\rangle) \ge 1/2. \tag{4.55}$$

We first consider the case $\langle\psi|\Pi^A_{[x+\Delta x/2,\infty)}|\psi\rangle \ge 1/2$ and start from the local reversibility in the form

$$\|R\Pi^A_{[x+\Delta x/2,\infty)}|\psi\rangle - |\psi\rangle\|) \le \frac{\tilde{f}(k_R/\sqrt{|L|})}{\langle\psi|\Pi^A_{[x+\Delta x/2,\infty)}|\psi\rangle} \le 2\tilde{f}(k_R/\sqrt{|L|}), \tag{4.56}$$

where R is a k_R-local operator and the function $\tilde{f}(x)$ decays super-polynomially. Because the operator A corresponds to a 1-local Hamiltonian with $g = 1$*2, we can apply Lemma 3.3.1 to it, which gives

$$\|\Pi^A_{(-\infty,x]} R \Pi^A_{[x+\Delta x/2,\infty)}\| = 0 \quad \text{for} \quad k_R < \Delta x/4. \tag{4.57}$$

We now choose k_R as $k_R = \lceil \Delta x/4 \rceil - 1$ with $\lceil \cdots \rceil$ denoting the ceiling function. We then have

$$\begin{aligned} &\|\Pi^A_{(-\infty,x]} P_\psi \Pi^A_{[x+\Delta x,\infty)}\| \\ =&\left\|\Pi^A_{(-\infty,x]}(R\Pi^A_{[x+\Delta x/2,\infty)}P_\psi - P_\psi)\Pi^A_{[x+\Delta x,\infty)}\right\| \le 2\tilde{f}\left(\frac{\lceil \Delta x/4 \rceil - 1}{\sqrt{|L|}}\right). \end{aligned} \tag{4.58}$$

In the same way, in the case $\langle\psi|\Pi^A_{(-\infty,x+\Delta x/2]}|\psi\rangle \ge 1/2$, we can also obtain the same inequality as above. This completes the proof of (4.54).

We can also prove Corollary 4.7 in the following section 4.7 by applying the same process to the case where $f(x)$ is given by Theorem 4.3.

4.6.2 Topologically ordered states

Although bi-partite correlations or the Fisher information can characterize various kinds of macroscopic quantum properties, they cannot detect the locally-hidden macroscopicity such as the topological order. As an example, we consider the ground states of Kitaev's model [33] on a circular ring, where any bi-partite correlations vanish in a finite distance. In this model, there are two degenerate ground states $\{|E_0\rangle, T|E_0\rangle\}$ which can be characterized by a topologically non-trivial loop operator T along the ring. We here show that a ground state $|E_0\rangle$ is not locally reversible. It is known that the state $|E_0\rangle$ is robust to any disturbance Γ_L as long as $|L| \ll l_s$, where l_s is the system length. However, if we take a subregion of size $|L|$ as large as $\mathcal{O}(l_s)$, the state is no longer robust as will be shown below.

We here define $|E_0^\pm\rangle \equiv \frac{1}{\sqrt{2}}(|E_0\rangle \pm T|E_0\rangle)$. The topological order has the properties [29,41] $\langle E_0|o_X|E_0\rangle = \langle E_0|To_XT|E_0\rangle$ and $\langle E_0|o_XT|E_0\rangle = 0$ for any local operators o_X as long as $|X| \le cl_s$ with c a constant of $\mathcal{O}(1)$. These equalities give $\langle E_0^+|R|E_0^-\rangle = 0$ for any k_R-local operators R with $k_R \le cl_s$. We now choose

*2Note that a 1-local Hamiltonian is always the commuting Hamiltonian.

the disturbance operator $\Gamma_L = (T+1)/\sqrt{2}$ and obtain $\Gamma_L|E_0\rangle = |E_0^+\rangle$, where $|L| = \mathcal{O}(l_s)$. We then have $\|R|E_0^+\rangle - |E_0\rangle\| \ge 1/\sqrt{2}$ for $\forall R$ with $k_R \le cl_s$. Therefore, the state $|E_0\rangle$ is reversible only by the use of $\mathcal{O}(l_s)$-local operator and does not satisfy the local reversibility.

4.7 Practical applications: fluctuation, critical exponents and mean-field approximation

We here show several applications which can be immediately derived from Theorem 4.3. We first show the exponential suppression of the interference term between two distinct eigen-subspaces of an additive operator A_L; $A_L = \sum_{i\in L} a_i$ with $\|a_i\| = 1$. By applying this result to a quantum critical point, we also obtain a fundamental inequality for quantum critical exponents.

Corollary 4.7. *Let $\Pi^A_{[x,x')}$ be the projection operator onto the subspace of the eigenvalues of A_L which are in $[x,x')$. Interference term due to the overlap between $\Pi^A_{(-\infty,x]}$ and $\Pi^A_{[x+\Delta x,\infty)}$ is exponentially suppressed for $\forall L$ as*

$$\|\Pi^A_{(-\infty,x]} P_0 \Pi^A_{[x+\Delta x,\infty)}\| \le 2e^{-\Delta x/\xi'_L}, \tag{4.59}$$

where $\xi'_L \equiv \sqrt{\frac{c_1|L|}{\delta E}}$, $P_0 = |E_0\rangle\langle E_0|$, $\Delta x \ge 0$ and $c_1 = \mathcal{O}(1)$.

The proof is given in the same way as in Theorem 4.6 (see Subsection 4.6.1 (a)). This gives a different kind of exponential restriction from the well-known exponential decay of correlation [22, 23] in gapped ground states of short-range interacting systems. Such an exponential bound on the superposition will play essential roles in characterizing ground states in long-range interacting systems as well as systems on the expander graph [59], where bi-partite correlations may be no longer meaningful.

The inequality (4.59) reduces to a trade-off relationship between the spectral gap and the fluctuation ΔA_L with respect to $|E_0\rangle$:

$$\delta E \cdot (\Delta A_L)^2 \le c_2|L|, \tag{4.60}$$

where $c_2 = \mathcal{O}(1)$. We then consider quantum critical points and define $A_L = \sum_{i=1}^N a_i$ with L a total system and $\{a_i\}_{i=1}^N$ order parameters (e.g. magnetization). We then introduce the critical exponents z, η, γ and ν as in Refs. [60–64]; z is the dynamical critical exponent, η is the anomalous critical exponent, γ is the susceptibility critical exponent, and ν is the correlation length exponent. By applying the finite-size scaling ansatz to the inequality (4.60), we can obtain

$$z \ge 1 - \frac{\eta}{2} = \frac{\gamma}{2\nu}, \tag{4.61}$$

where the second equality comes from the Fisher equality $2 - \eta = \gamma/\nu$ [64]. The inequality (4.60) is also applicable to a long-range interacting system. Let us consider the Lipkin-Meshcov-Glick model [31], namely

$$H_{\mathrm{LMG}} = -\frac{J}{N}\sum_{i<j}(\sigma_i^x\sigma_j^x + \gamma\sigma_i^y\sigma_j^y) + \sum_{i=1}^N h\sigma_i^z \tag{4.62}$$

with $|\gamma| \le 1$. At the critical point $J = |h|$, we have the scalings of [32]

$$\delta E \propto N^{-1/3} \quad \text{and} \quad (\Delta M_x)^2 \propto N^{4/3}, \tag{4.63}$$

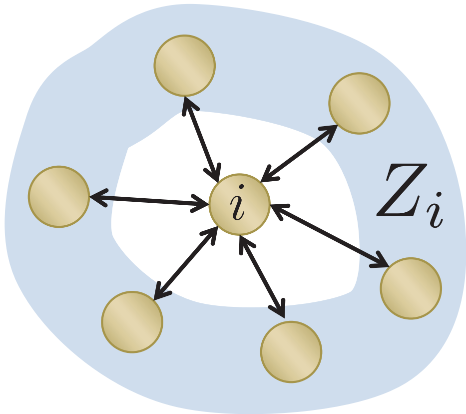

Figure 4.5: Mean-field approximation and the local reversibility. We consider a lattice system with a 2-local Hamiltonian. We then consider the error of the mean-field approximation between the spin i and the neighboring spins $j \in Z_i$. Because of the inequality (4.65) which comes from the local reversibility, we have the upper bound of the error as $\mathcal{O}(1/\sqrt{|Z_i|\delta E})$.

where M_x is the magnetization in the x direction, $M_x = \sum_{i=1}^{N} \sigma_i^x$. Thus, the spectral gap and the fluctuation can give the sharp upper bounds to each other.

We finally discuss the product-state approximation of the reduced density matrix $\rho_{i,j}$ of the ground state [65]:

$$\|\rho_{i,j} - \rho_i \otimes \rho_j\| \tag{4.64}$$

for $i, j = 1, 2, 3, \ldots, N$, where ρ_i and ρ_j are reduced density matrices with respect to the spins i and j, respectively. Because we are now considering the ground state of k-local Hamiltonians, we cannot obtain a tight upper bound for $\|\rho_{i,j} - \rho_i \otimes \rho_j\|$ for particular i and j. However, if we take the average, we can obtain the meaningful upper bound as we will show in the following.

If a state $|\psi\rangle$ satisfies the local reversibility or the inequality (4.60), we can prove the upper bound of the average value of $\|\rho_{i,j} - \rho_i \otimes \rho_j\|$ with respect to the spins j in a region L:

$$\frac{1}{|L|} \sum_{j \in L} \|\rho_{i,j} - \rho_i \otimes \rho_j\| \le \text{const} \times \frac{1}{\sqrt{|L|\delta E}} \quad \text{for} \quad \forall i. \tag{4.65}$$

We can ensure that this upper bound is qualitatively optimal by considering the following LR state[*3]:

$$\frac{1}{\sqrt{2}}|0\rangle|00\cdots 0\rangle + \frac{1}{\sqrt{2}}|1\rangle|\mathrm{W}\rangle. \tag{4.66}$$

If we take the average over all spin pairs in the region L, we conjecture the following inequality instead of (4.65):

$$\frac{1}{|L|^2} \sum_{i,j \in L} \|\rho_{i,j} - \rho_i \otimes \rho_j\| \le \text{const} \times \frac{1}{|L|\delta E}. \tag{4.67}$$

This approximation is important in discussing the validity of the mean-field approximation. For simplicity, let us assume that our system is defined by a 2-local Hamiltonian with nearest-neighbor interactions on a lattice. We define the region Z_i as the set of spins which are neighbors of the spin i

[*3] Note that the state also gives the upper limit of the monogamy inequality of the entanglement.

(Fig. 4.5). We then obtain a bound on the error of the mean-field approximation as follows:

$$\begin{aligned}\Big|\frac{1}{|Z_i|}\sum_{j\in Z_i}\langle h_{ij}\rangle_{\rm exact}-\frac{1}{|Z_i|}\sum_{j\in Z_i}\langle h_{ij}\rangle_{\rm MF}\Big| &\le \frac{1}{|Z_i|}\sum_{j\in Z_i}\|h_{ij}\|\cdot\|\rho_{i,j}-\rho_i\otimes\rho_j\| \\ &\le \frac{1}{|Z_i|}\sum_{j\in Z_i}\|h_{ij}\|\cdot\sum_{j\in Z_i}\|\rho_{i,j}-\rho_i\otimes\rho_j\| \\ &\le \text{const}\times\frac{1}{\sqrt{|Z_i|\delta E}},\end{aligned} \tag{4.68}$$

where we utilize the inequality (4.65) by choosing $L=Z_i$ and utilize the extensiveness of the Hamiltonian as $\sum_{j\in Z_i}\|h_{ij}\|\le g$. This result is consistent with the well-known fact that the mean-field approximation becomes exact in the infinite-dimensional lattice ($|Z_i|\to\infty$).

4.7.1 Derivation of inequality (4.61)

We here derive the inequality (4.61) in Section 4.7 under the scaling ansatz (4.69) [**?**, 60, 61, 63]. In the derivation of this inequality, we start from the inequality (4.60).

We first define the variance $(\Delta A_t)^2$ which depends on the time as $(\Delta A_t)^2\equiv\frac{1}{2}\big\langle\{A(t)-\langle A\rangle,\langle A-\langle A\rangle\}\big\rangle$, where $A(t)=e^{-iHt}Ae^{iHt}$ and $\{\cdots,\cdots\}$ is the anticommutator. The variance $(\Delta A_t)^2$ reduces to the summation of the correlation functions:

$$(\Delta A_t)^2=\sum_{i,j=1}^{N}\left\langle\frac{1}{2}\{a_i(t),a_j\}-\langle a_i\rangle\langle a_j\rangle\right\rangle\equiv\sum_{i,j=1}^{N}C_{i,j}(t),$$

where $a_i(t)\equiv e^{-iHt}a_ie^{iHt}$ for $i=1,2,\dots N$. Note that $(\Delta A_{t=0})^2$ is equal to $(\Delta A)^2=\langle A^2\rangle-\langle A\rangle^2$. In the following, we denote $C_{i,j}(t)=C(\boldsymbol{r},t)$ under the assumption of the translation symmetry.

Now, we adopt the following scaling ansatz [63]:

$$S(\boldsymbol{q},\omega)=\xi^{2-\eta}D(\boldsymbol{q}\xi,\omega\xi^z), \tag{4.69}$$

where ξ is the correlation length and $S(\boldsymbol{q},\omega)$ is the spatial-temporal Fourier component of $C(\boldsymbol{r},t)$, namely

$$S(\boldsymbol{q},\omega)=\int_{\boldsymbol{r}}\int_t C(\boldsymbol{r},t)e^{-i(\boldsymbol{q}\cdot\boldsymbol{r}+\omega t)}d\boldsymbol{r}dt. \tag{4.70}$$

We also define $S(\boldsymbol{q})$ as

$$S(\boldsymbol{q})=\frac{1}{2\pi}\int_{-\infty}^{\infty}S(\boldsymbol{q},\omega)d\omega. \tag{4.71}$$

We can see that the static fluctuation $(\Delta A_{t=0})^2$ is equal to $NS(\boldsymbol{q}=0)$ by expanding $S(\boldsymbol{q}=0)$.

We then obtain the scaling of $S(\boldsymbol{q}=0)$ as $\xi^{2-\eta-z}$ by taking the scaling (4.69) for $S(\boldsymbol{q},\omega)$, and hence we have $(\Delta A_{t=0})^2/N\propto\xi^{2-\eta-z}$. We also have the scaling of the energy gap as $\delta E_0\propto\xi^{-z}$ [61] by the use of the dynamical critical exponent z. At a critical point, where the correlation length is as large as the system length, the inequality (4.60) reduces to

$$-z\le-(2-\eta-z) \tag{4.72}$$

in the infinite volume limit ($N\to\infty$). This gives the inequality (4.61).

4.7.2 Proof of the inequality (4.65)

In the following, we set $i = 1$ for simplicity. We then calculate the upper bound of

$$\sum_{j \in L} \|\rho_{1,j} - \rho_1 \otimes \rho_j\|. \tag{4.73}$$

First, we can find a set of projectors $\{P_1^{(m)}\}$ onto the spin $i = 1$ which satisfies

$$\|\rho_{1,j} - \rho_1 \otimes \rho_j\| \leq \sum_{m=1}^{D^2} \|P_1^{(m)}(\rho_{1,j} - \rho_1 \otimes \rho_j)P_1^{(m)}\|, \tag{4.74}$$

where D is the dimension of the spin dimension. In the case of the spin-1/2 systems ($D = 2$), we can take $P_1^{(1)} = |0_1\rangle\langle 0_1|$, $P_1^{(2)} = |1_1\rangle\langle 1_1|$, $P_1^{(3)} = |+_1\rangle\langle +_1|$, $P_1^{(4)} = |-_1\rangle\langle -_1|$. [*4]

The inequality (4.74) gives

$$\sum_{j \in L} \|\rho_{1,j} - \rho_1 \otimes \rho_j\| \leq \sum_{m=1}^{D^2} \sum_{j \in L} \|P_1^{(m)}(\rho_{1,j} - \rho_1 \otimes \rho_j)P_1^{(m)}\|, \tag{4.76}$$

and we have to obtain the upper bound of $\sum_{j \in L} \|P_1^{(m)}(\rho_{1,j} - \rho_1 \otimes \rho_j)P_1^{(m)}\|$ for arbitrary m. Note that we have

$$P_1^{(m)}(\rho_{1,j} - \rho_1 \otimes \rho_j)P_1^{(m)} = P_1^{(m)} \otimes \big(\rho_j^{(m)} - \langle\psi|P_1^{(m)}|\psi\rangle \cdot \rho_j\big), \tag{4.77}$$

where $\rho_j^{(m)} \equiv \mathrm{tr}_1(P_1^{(m)}\rho_{1,j}P_1^{(m)})$ with $\mathrm{tr}_i(\cdots)$ a partial trace operation only for the spin i. We now define the projector $P_j^{(m)}$ as the one which gives the maximum absolute eigenvalue of $\rho_j^{(m)} - \langle\psi|P_1^{(m)}|\psi\rangle \cdot \rho_j$. We then have

$$\begin{aligned}
\|P_1^{(m)}(\rho_{1,j} - \rho_1 \otimes \rho_j)P_1^{(m)}\| &= \|P_1^{(m)} \otimes \big(\rho_j^{(m)} - \langle\psi|P_1^{(m)}|\psi\rangle \cdot \rho_j\big)\| \\
&= s_j^{(m)} \cdot \mathrm{tr}\Big[P_j^{(m)}\big(\rho_j^{(m)} - \langle\psi|P_1^{(m)}|\psi\rangle \cdot \rho_j\big)\Big] \\
&= s_j^{(m)} \cdot \big[\langle\psi|P_1^{(m)}P_j^{(m)}|\psi\rangle - \langle\psi|P_1^{(m)}|\psi\rangle \cdot \langle\psi|P_j^{(m)}|\psi\rangle\big],
\end{aligned} \tag{4.78}$$

where $s_j^{(m)} \equiv \mathrm{sign}\Big\{\mathrm{tr}\big[P_j^{(m)}(\rho_j^{(m)} - \langle\psi|P_1^{(m)}|\psi\rangle \cdot \rho_j)\big]\Big\}$.

We define the additive operator A as

$$A \equiv \sum_{j \in L} s_j^{(m)} \cdot P_j^{(m)}, \tag{4.79}$$

where $[A, P_1^{(m)}] = 0$. We then obtain

$$\begin{aligned}
&\sum_{j \in L} \|P_1^{(m)}(\rho_{1,j} - \rho_1 \otimes \rho_j)P_1^{(m)}\| \\
=&\langle\psi|P_1^{(m)}A|\psi\rangle - \langle\psi|P_1^{(m)}|\psi\rangle \cdot \langle\psi|A|\psi\rangle
\end{aligned}$$

[*4] We define $\delta\rho_{1,j} \equiv \rho_{1,j} - \rho_1 \otimes \rho_j$ and utilize $\langle 0_1|\delta\rho_{1,j}|1_1\rangle = \langle 1_1|\delta\rho_{1,j}|0_1\rangle$:

$$\begin{aligned}
\|\delta\rho_{1,j}\| &\leq \|\langle 0_1|\delta\rho_{1,j}|0_1\rangle\| + \|\langle 1_1|\delta\rho_{1,j}|1_1\rangle\| + \|\langle 0_1|\delta\rho_{1,j}|1_1\rangle + \langle 1_1|\delta\rho_{1,j}|0_1\rangle\| \\
&= \|\langle 0_1|\delta\rho_{1,j}|0_1\rangle\| + \|\langle 1_1|\delta\rho_{1,j}|1_1\rangle\| + \|\langle +_1|\delta\rho_{1,j}|+_1\rangle - \langle -_1|\delta\rho_{1,j}|-_1\rangle\| \\
&\leq \|\langle 0_1|\delta\rho_{1,j}|0_1\rangle\| + \|\langle 1_1|\delta\rho_{1,j}|1_1\rangle\| + \|\langle +_1|\delta\rho_{1,j}|+_1\rangle\| + \|\langle -_1|\delta\rho_{1,j}|-_1\rangle\|
\end{aligned} \tag{4.75}$$

$$
\begin{aligned}
&=(1-\langle\psi|P_1^{(m)}|\psi\rangle)\cdot\langle\psi|P_1^{(m)}AP_1^{(m)}|\psi\rangle-\langle\psi|P_1^{(m)}|\psi\rangle\cdot\langle\psi|Q_1^{(m)}AQ_1^{(m)}|\psi\rangle \\
&=p(1-p)\cdot\langle\psi_P|A|\psi_P\rangle-p(1-p)\langle\psi_Q|A|\psi_Q\rangle \\
&=p(1-p)\big(\langle\psi_P|A|\psi_P\rangle-\langle\psi_Q|A|\psi_Q\rangle\big),
\end{aligned}
\tag{4.80}
$$

where we utilize the equality of $\langle\psi|P_1^{(m)}A|\psi\rangle = \langle\psi|P_1^{(m)}AP_1^{(m)}|\psi\rangle$ and define $Q_1^{(m)} \equiv 1-P_1^{(m)}$, $p \equiv \langle\psi|P_1^{(m)}|\psi\rangle$, $\sqrt{p}|\psi_P\rangle \equiv P_1^{(m)}|\psi\rangle$, and $\sqrt{1-p}|\psi_Q\rangle \equiv (1-P_1^{(m)})|\psi\rangle$.

We finally obtain

$$
\begin{aligned}
(\Delta A)^2 &= p\langle\psi_P|A^2|\psi_P\rangle+(1-p)\langle\psi_Q|A^2|\psi_Q\rangle-\big[p\langle\psi_P|A|\psi_P\rangle+(1-p)\langle\psi_Q|A|\psi_Q\rangle\big]^2 \\
&= p^2(\langle\psi_P|A^2|\psi_P\rangle-\langle\psi_P|A|\psi_P\rangle^2)+(1-p)^2(\langle\psi_Q|A^2|\psi_Q\rangle-\langle\psi_Q|A|\psi_Q\rangle^2) \\
&\quad + p(1-p)(\langle\psi_P|A^2|\psi_P\rangle-2\langle\psi_P|A|\psi_P\rangle\langle\psi_Q|A|\psi_Q\rangle+\langle\psi_Q|A^2|\psi_Q\rangle) \\
&\geq p(1-p)\big(\langle\psi_P|A|\psi_P\rangle-\langle\psi_Q|A|\psi_Q\rangle\big)^2.
\end{aligned}
\tag{4.81}
$$

By combining the inequalities (4.80) and (4.81), we arrive at

$$
\sum_{j\in L}\|P_1^{(m)}(\rho_{1,j}-\rho_1\otimes\rho_j)P_1^{(m)}\| \leq \sqrt{p(1-p)}\cdot\Delta A \leq \text{const}\times\sqrt{\frac{|L|}{\delta E}}, \tag{4.82}
$$

where the last inequality comes from (4.60). Thus, the inequality (4.74) reduces to

$$
\sum_{j\in L}\|\rho_{1,j}-\rho_1\otimes\rho_j\| \leq \sum_{m=1}^{D^2}\sum_{j\in L}\|P_1^{(m)}(\rho_{1,j}-\rho_1\otimes\rho_j)P_1^{(m)}\| \leq \text{const}\times\sqrt{\frac{|L|}{\delta E}}. \tag{4.83}
$$

This completes the proof.

Chapter 5

Macroscopic superposition in low-energy states

In this chapter, we show fundamental bounds on the macroscopic superposition in the ground states. This result is a generalization of the previous chapter.

5.1 Introduction

In the previous chapter, we have shown the inequality (4.59) in Corollary 4.7, which is on the exponential suppression of the interference term between two distinct subspaces $\Pi^A_{(-\infty,x]}$ and $\Pi^A_{[x+\Delta x,\infty)}$:

$$\|\Pi^A_{(-\infty,x]}|E_0\rangle\langle E_0|\Pi^A_{[x+\Delta x,\infty)}\| \le \exp\left(-\mathrm{const}\cdot\frac{\Delta x}{\sqrt{|L|/\delta E}}\right) \quad \text{for} \quad \forall L, \tag{5.1}$$

where we let $A_L = \sum_{i\in L} a_i$ with $\|a_i\| = 1$ and denote by $\Pi^A_{[x,x')}$ the projection operator onto the subspace of the eigenvalues of A_L which are in $[x, x')$. We derived this inequality based on the local reversibility and Theorem 4.3 (see also Subsection 4.6.1 (a)), assuming the non-degeneracy of the ground state; we therefore cannot apply it to the degenerate cases ($\delta E = E_1 - E_0 = 0$). The primary reason of the assumption is that the local reversibility can be only applied to pure states. We therefore have no information for the case where the ground state is degenerate or almost degenerate.

The degenerate ground states, however, should also satisfy strong inequalities when there exists a finite gap just above them. For example, in the case of the short-range interacting Hamiltonians, the subspace of the ground states should satisfy the exponential clustering bound (see Subsection 2.5.2 in Chapter 2); correlation functions in the ground-state subspace can be given by a term of the exponential decay with a term of matrix elements between the low-lying states [21]:

$$\mathrm{Cor}(A_X, B_Y) = \frac{1}{m^2}e^{-\mathrm{const}\cdot\mathrm{dist}(X,Y)\cdot\delta E} + \frac{1}{m^2}\left|\sum_{j,j'}^{m-1}(A_X)_{j,j'}(B_Y)_{j',j}\right|, \tag{5.2}$$

where $(A_X)_{j,j'} = \langle E_j|A_X|E_{j'}\rangle$ for $j, j' = 0, 1, 2, \ldots, m-1$.[*1] We assumed the m-fold degeneracy of the ground states and the correlation is taken with respect to the uniformly mixed ground states, namely $m^{-1}\sum_{j=0}^{m-1}|E_j\rangle\langle E_j|$. For the degenerate ground states, we should consider the subspace of low-lying

[*1] Now, δE is given by $E_m - E_0$, which is equal to the spectral gap between the ground states and the first excited state.

states instead of a single ground state. From this preliminary results on the short-range interacting Hamiltonians, we expect a similar structure in the case of the general k-local Hamiltonians.

The purpose of this chapter is a complete generalization of Corollary 4.7. We here again consider the extensive k-local Hamiltonians:

$$H = \sum_{|X| \le k_H} h_X \quad \text{with} \quad \sum_{X \ni i} \|h_X\| \le g \quad \text{for} \quad i = 1, 2, \ldots, N, \tag{5.3}$$

but we do not assume the non-degeneracy of the ground state: $E_0 \le E_1 \le E_2 \le \cdots$. Moreover, in order to make the discussion more general than that in Chapter 4, we generalize A_L to an extensive k-local operator with $k = \mathcal{O}(1)$, namely

$$A_L \equiv \sum_{X: X \in L, |X| \le k} a_X \quad \text{with} \quad \sum_{X: X \in i} \|a_X\| \le g \quad \forall i \in L. \tag{5.4}$$

We here consider the subspace of low-lying states:

$$P_{m-1} \equiv \sum_{j=0}^{m-1} |E_j\rangle\langle E_j| \tag{5.5}$$

instead of $|E_0\rangle\langle E_0|$, where the eigenenergy E_{m-1} is supposed to be close to the ground energy.[*2] We now want to prove a similar statement to (5.1) in this extended setting:

$$\|\Pi^A_{(-\infty,x]} P_{m-1} \Pi^A_{[x+\Delta x,\infty)}\| \le \exp\left(-\text{const} \cdot \frac{\Delta x}{\sqrt{|L|/\delta E}}\right) \quad \text{for} \quad \forall L. \tag{5.6}$$

We note that this statement does not necessarily imply no quantum macroscopicity. Indeed, if the ground states have the degeneracy, the ground-state eigenspace may contain a macroscopic quantum structure; a remarkable example is the topological order, which is usually associated with the degeneracy with a finite gap of order 1

We show the organization of this chapter as follows. We first show a trade-off relationship between the spectral gap and the Fisher information. The Fisher information is one of the most famous measures of the quantum macroscopicity. This is also deeply related to the norm of the interference term $\Pi^A_{(-\infty,x]} P_{m-1} \Pi^A_{[x+\Delta x,\infty)}$ as is shown in subsection 5.2.5. Our main Theorem 5.3 gives a trade-off inequality with respect to the spectral gap and the Fisher information, which implies the polynomial decay of $\|\Pi^A_{(-\infty,x]} P_{m-1} \Pi^A_{[x+\Delta x,\infty)}\|$ with respect to Δx. Although it is weaker than the expected bound as in (5.6), we nevertheless want to emphasize it in the following points:

1. We can prove Theorem 5.3 in quite a simple way, only by applying the variational principle and the extensiveness of the Hamiltonian.

2. We can prove Theorem 5.3 in more general settings than the k-locality of the Hamiltonian.

Finally in Section 5.4, we improve the polynomial bound in Section 5.3 to the exponential bound in (5.6). The bound (5.61) in Theorem 5.4 gives the complete generalization of Corollary 4.7.

[*2] In fact, we can take the energy E_{m-1} as large as we want, but in that case, we could not obtain a meaningful bound for the energy space P_{m-1}.

5.2 Basic properties of the Fisher information: measure of the macroscopic superposition

In this section, we review the basic properties of the quantum Fisher information [58], which is often used as a measure of the macroscopic superposition. As our original result, in Subsection 5.2.5, we also discuss a polynomial decay of the off-diagonal elements of density matrix in terms of the Fisher information. Moreover, in the subsequent section, we prove that the ground states satisfy a trade-off relationship between the Fisher information and the spectral gap just above the ground states.

5.2.1 Definition of the Fisher information

The Fisher information is defined for a quantum state ρ and an operator O [58], which usually takes as an extensive k-local operator. We denote the spectral decomposition of ρ by

$$\rho = \sum_{j=1}^{D^N} \pi_j \Pi_j^\rho, \tag{5.7}$$

where D^N is the dimension of the total Hilbert space and $\{\pi_j, \Pi_j^\rho\}$ are the eigenvalues and the eigenstates of ρ, respectively.[*3] The Fisher information $\mathcal{F}(\rho, O)$ is defined as

$$\mathcal{F}(\rho, O) = 2 \sum_{j,j'=1}^{D^N} \frac{(\pi_j - \pi_{j'})^2}{\pi_j + \pi_{j'}} \mathrm{tr}(\Pi_j^\rho O \Pi_{j'}^\rho O). \tag{5.8}$$

The Fisher information characterizes the macroscopic entanglement in a subsystem L when we choose O as an extensive operator A_L in Eq. (5.4). In this case, the Fisher information $\mathcal{F}(\rho, A_L)$ takes a value of $\mathcal{O}(|L|^p)$ with $0 \le p \le 2$. If there exists an operator A_L which gives p larger than 1, the state ρ has a macroscopic superposition.

5.2.2 The Fisher information for pure states and uniformly mixed states

We here calculate the Fisher information $\mathcal{F}(\rho, O)$ for two important classes of states, namely the pure states and the uniformly mixed states. For a pure state, namely

$$\rho = \sum_{j=1}^{D^N} \pi_j \Pi_j^\rho, \tag{5.9}$$

with $\pi_1 = 1$ and $\pi_j = 0$ for $j = 2, 3, \ldots, D^N$, we have

$$F(\rho, O) = 4\mathrm{tr}(\Pi_1^\rho O \sum_{j \ge 2} \Pi_j^\rho O) = 4\mathrm{tr}(\Pi_1^\rho O (1 - \Pi_1^\rho) O) = 4(\Delta O)^2 \tag{5.10}$$

after a straightforward algebra. In this case, the Fisher information is proportional to the variance $(\Delta O)^2$ with respect to ρ.

For a uniformly mixed state as

$$\rho = \sum_{j=1}^{m} \frac{1}{m} \Pi_j^\rho, \tag{5.11}$$

[*3]Note that the state ρ is an arbitrary quantum state.

we only have to pick up the terms of $\{1 \le j \le m, j' \ge m+1\}$ and $\{1 \le j' \le m, j \ge m+1\}$ in Eq. (5.8), which gives

$$F(\rho, O) = \frac{4}{m} \sum_{j=1}^{m} \sum_{j' \ge m+1} \mathrm{tr}(\Pi_j^\rho O \Pi_{j'}^\rho O) = 4\mathrm{tr}\big[\rho O(1 - m\rho)O\big], \tag{5.12}$$

where we utilized the equality $\sum_{j=1}^{m} \Pi_j^\rho = m\rho$.

5.2.3 The physical meaning of the Fisher information

We here show that the quantum Fisher Information is given by the convex roof of the variance:

$$F(\rho, O) = 4 \min_{\{p_j, |\psi_j\rangle\}} \sum_j p_j (\Delta O)_j^2, \tag{5.13}$$

where we take the minimum average of $(\Delta O)^2$ for any decompositions of $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$. Note that $\{|\psi_j\rangle\}$ may not be orthogonal to each other. Indeed, in the case where ρ is a pure state, the Fisher information reduces to $4(\Delta O)^2$ as in Eq. (5.10).

The equality (5.13) has been conjectured in Ref. [66] and completely proved in Ref. [67]. In this section, we prove this relationship only in the case of the uniformly mixed states. For the proof, we first show that for an arbitrary decomposition of ρ we have

$$F(\rho, O) \le \sum_j p_j (\Delta O)_j^2. \tag{5.14}$$

We then prove that we can achieve the equality by choosing an appropriate decomposition of ρ.

We first prove the inequality (5.14). We, in the following, consider a uniformly mixed state of the form

$$\rho = \sum_{j=1}^{m} \frac{1}{m} \Pi_j^\rho. \tag{5.15}$$

From Eq. (5.12), we obtain

$$\begin{aligned} F(\rho, O) &= 4\mathrm{tr}\big[\rho(O^2 - mO\rho O)\big] \\ &= 4\sum_{j=1}^{m} \frac{\langle O^2\rangle_j}{m} - \sum_{j,j'=1}^{m} \frac{\mathrm{tr}(\Pi_j O \Pi_{j'} O)}{m} \\ &= 4\sum_{j=1}^{m} \frac{\langle O^2\rangle_j - \langle O\rangle_j^2}{m} - \sum_{j \ne j'} \frac{|\langle j|O|j'\rangle|^2}{m} \\ &\le 4\sum_{j=1}^{m} \frac{1}{m}(\Delta O)_j^2, \end{aligned} \tag{5.16}$$

where we define the states $\{|j\rangle\}$ by $\Pi_j^\rho \equiv |j\rangle\langle j|$ and $\langle O\rangle_j = \langle j|O|j\rangle$. We thus prove the inequality (5.14).

We second consider the condition that we achieve the equality of (5.14). From the third line in the inequality (5.16), we can give the condition in the form

$$\sum_{j \ne j'} \frac{|\langle j|O|j'\rangle|^2}{m} = 0. \tag{5.17}$$

We then prove that there exists a decomposition of ρ which achieves this equality. First, because the set of states $\{|j\rangle\}_{j=1}^m$ are the eigenstates of ρ, the states

$$|\psi_j\rangle = \sum_{j'=1} U_{j,j'}|j'\rangle \tag{5.18}$$

are also the eigenstates of ρ, where $\{U_{j,j'}\}_{j,j'}$ is an arbitrary unitary matrix. Second, we take the unitary U which diagonalizes the matrix $\{\langle j|O|j'\rangle\}_{j,j'}$. We then obtain

$$\langle\psi_j|O|\psi_{j'}\rangle = \langle\psi_j|O|\psi_j\rangle \cdot \delta_{j,j'}. \tag{5.19}$$

By the use of the base $\{\psi_j\}$, we achieve Eq. (5.17). This completes the proof of Eq. (5.13).

In the case of the general quantum states, as has been proved in Ref. [66,67], we can also construct the state decomposition $\{p_j, |\psi_j\rangle\}$ so that the average of the variance may be equal to the Fisher information.

5.2.4 Several examples

For example, let us consider the GHZ state, namely $|\mathrm{GHZ}_\pm\rangle = (|00\cdots0\rangle \pm |11\cdots1\rangle)/\sqrt{2}$. We now consider a pure state, and hence the Fisher information reduces to the variance $(\Delta A)^2$ because of Eq. (5.10). We then choose the operator A as

$$A = \sum_{i=1}^N \sigma_z^i, \tag{5.20}$$

where the region L is the total system. For the above choice of A, we can obtain the scaling of $(\Delta A)^2$ as $\mathcal{O}(N^2)$, namely $p = 2$. This means that the GHZ state contains a macroscopic superposition, which is consistent with the fact that the two states $|00\cdots0\rangle$ and $|11\cdots1\rangle$ are macroscopically distinct.

On the other hand, a mixed state of the GHZ states of the form

$$\rho = \frac{1}{2}(|\mathrm{GHZ}_+\rangle\langle\mathrm{GHZ}_+| + |\mathrm{GHZ}_-\rangle\langle\mathrm{GHZ}_-|) \tag{5.21}$$

has the scaling of $(\Delta A)^2$ as $\mathcal{O}(N^2)$, but has the scaling of the Fisher information $\mathcal{F}(\rho, A)$ as $\mathcal{O}(N^p)$ with $p \leq 1$. We can prove this as follows. The state ρ in Eq. (5.21) is also decomposed into the mixed state of the form $\frac{1}{2}(|00\cdots0\rangle\langle00\cdots0| + |11\cdots1\rangle\langle11\cdots1|)$. Because of the inequality (5.13) in the previous subsection, we can obtain

$$\mathcal{F}(\rho, A) \leq \frac{(\Delta A)^2_{00\cdots0} + (\Delta A)^2_{11\cdots1}}{2}, \tag{5.22}$$

where $(\Delta A)^2_{00\cdots0}$ and $(\Delta A)^2_{11\cdots1}$ are the variances with respect to the states $|00\cdots0\rangle$ and $|11\cdots1\rangle$, respectively. We can prove that the product states always have the scaling of $(\Delta A_L)^2 = \mathcal{O}(|L|^p)$ with $p \leq 1$ for $\forall A$, and hence the Fisher information $\mathcal{F}(\rho, A_L)$ also scales as $\mathcal{O}(|L|^p)$ with $p \leq 1$.

Because we consider general k-local operators in Eq. (5.4), we can detect the macroscopic entanglement of more general states than in Ref. [55, 56], where they mainly considered the cases where A is a sum of one-spin operators $\{a_i\}_{i=1}^N$ with $\|a_i\| = 1$. For example, let us consider a resonating valence bond (RVB) state of the form

$$|B_{1,2}\rangle|B_{3,4}\rangle\cdots|B_{2N-1,2N}\rangle + |B_{2,3}\rangle|B_{4,5}\rangle\cdots|B_{2N,1}\rangle, \tag{5.23}$$

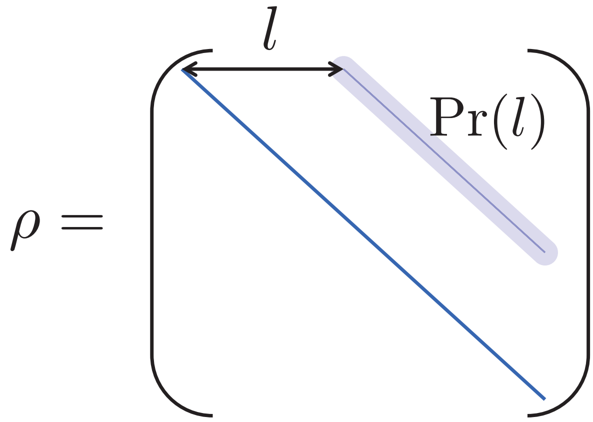

Figure 5.1: Off-diagonal elements of the density matrix. We take the base of the matrix by the use of eigenstates of A. We define $\mathrm{Pr}(l)$ as the weight (5.27) of the off-diagonal elements with distance l such as $|\langle x|\rho|x+l\rangle|^2$. Now, the sum of $\mathrm{Pr}(l)$ with respect to l is normalized to unity. The Fisher information gives the variance of the distance l as $\mathcal{F}(\rho, A) = \sum_l l^2 \cdot \mathrm{Pr}(l)$. If the Fisher information has the value of $\mathcal{O}(|L|)$, the off-diagonal element decays at least faster than $\mathcal{O}(|L|/l^2)$ because of the Chebyshev inequality.

where $|B_{i,i+1}\rangle$ is the Bell state $(|00\rangle + |11\rangle)/\sqrt{2}$ with respect to the spins i and $i+1$. For this state, any one-spin operators have the fluctuation of $\mathcal{O}(N^p)$ with $p \leq 1$. However, if we take

$$A = \sum_{i=1}^{N}(\sigma^x_{2i-1}\sigma^x_{2i} - \sigma^y_{2i-1}\sigma^y_{2i}) - \sum_{i=1}^{N}(\sigma^x_{2i}\sigma^x_{2i+1} - \sigma^y_{2i}\sigma^y_{2i+1}), \tag{5.24}$$

with the periodic boundary condition $\sigma_{N+1} = \sigma_1$, we can detect the macroscopicity of the RVB state by means of its variance of $\mathcal{O}(N^2)$.

5.2.5 Relationship between the interference term and the Fisher information

We here relate the Fisher information to the interference term in the density matrix. We consider a uniformly mixed state ρ of m states. From Eq. (5.10), the quantum Fisher information $\mathcal{F}(\rho, A)$ is given by

$$\begin{aligned}\mathcal{F}(\rho, A) &= 4\mathrm{tr}(\rho A^2 - m\rho A\rho A) \\ &= 4m \cdot \mathrm{tr}(\rho^2 A^2 - \rho A\rho A) = 2m \cdot \mathrm{tr}\big[\rho(\rho A^2 - 2A\rho A + A^2\rho)\big],\end{aligned} \tag{5.25}$$

where we utilized the equality of $\rho^2 = \rho/m$. We next expand it by the eigenstates of A, which we denote as $\{|x\rangle\}$ with the corresponding eigenvalues $\{x\}$:

$$\mathcal{F}(\rho, A) = 2m\sum_{x,y}(x-y)^2|\langle x|\rho|y\rangle|^2 = 2m\sum_l l^2 \sum_x |\langle x|\rho|x+l\rangle|^2 \equiv 2\sum_l l^2 \mathrm{Pr}(l), \tag{5.26}$$

where

$$\mathrm{Pr}(l) \equiv m\sum_x |\langle x|\rho|x+l\rangle|^2 \tag{5.27}$$

and $-\|A\| \leq l \leq \|A\|$. Note that $\{\Pr(l)\}$ is normalized in the following sense:

$$\sum_l \Pr(l) = m \sum_l \sum_x |\langle x|\rho|x+l\rangle|^2 = m \cdot \mathrm{tr}(\rho^2) = 1. \tag{5.28}$$

We can regard $\Pr(l)$ as a probabilistic distribution of the off-diagonal distance l (see Fig. 5.1).

We therefore conclude that the Fisher information characterizes the spread of the off-diagonal elements in the density matrix. Because of $\sum_l l \cdot \Pr(l) = 0$, the Fisher information is equal to the variance of l, namely

$$\mathcal{F}(\rho, A) = \sum_l l^2 \cdot \Pr(l). \tag{5.29}$$

We can apply the Chebyshev inequality[*4] and obtain the restriction to the distribution $\Pr(l)$ as

$$\sum_{|l|\geq l_0} \Pr(l) \leq \frac{\mathcal{F}(\rho, A)}{l_0^2}. \tag{5.31}$$

Note that this also implies the inequality

$$m\|\Pi^A_{(-\infty,x]}\rho\Pi^A_{[x+\Delta x,\infty)}\| \leq \frac{\mathcal{F}(\rho, A)}{\Delta x^2} \tag{5.32}$$

because of $\sum_{|l|\geq \Delta x} \Pr(l) \geq m\|\Pi^A_{(-\infty,x]}\rho\Pi^A_{[x+\Delta x,\infty)}\|_{\mathrm{F}}$ and Eq. (5.33), where $\|\cdots\|_{\mathrm{F}}$ denotes the Frobenius norm of the operator, that is, $\|O\|_{\mathrm{F}} = \mathrm{tr}(O^\dagger O)$.

We thus conclude that the Fisher information decides how far the off-diagonal elements spread in the basis of the extensive operator A. From the inequality (5.31), the Fisher information tells us the polynomial decay of the off-diagonal elements for $\Delta x \gtrsim \sqrt{\mathcal{F}(\rho, A)}$.

5.2.5 (a) The Frobenius norm and the operator norm

We here prove

$$\begin{aligned}\|\Pi^A_{(-\infty,x]}\rho\Pi^A_{[x+\Delta x,\infty)}\| &= \|\Pi^A_{(-\infty,x]}\rho\Pi^A_{[x+\Delta x,\infty)}\|_{\mathrm{F}} \\ &= \mathrm{tr}(\rho\Pi^A_{(-\infty,x]}\rho\Pi^A_{[x+\Delta x,\infty)}),\end{aligned} \tag{5.33}$$

in the case where ρ is a uniformly mixed state as in Eq. (5.11).

We first have the inequality of $\|O\| \leq \|O\|_{\mathrm{F}}$ for an arbitrary operator O, which is a basic relationship of the matrix norm. We then prove the inequality of

$$\|\Pi^A_{(-\infty,x]}\rho\Pi^A_{[x+\Delta x,\infty)}\| \geq \|\Pi^A_{(-\infty,x]}\rho\Pi^A_{[x+\Delta x,\infty)}\|_{\mathrm{F}} \tag{5.34}$$

in order to prove the equality (5.33). The proof comes from the following inequalities:

$$\|\Pi^A_{(-\infty,x]}\rho\Pi^A_{[x+\Delta x,\infty)}\| \geq \langle j|\Pi^A_{(-\infty,x]}\rho\Pi^A_{[x+\Delta x,\infty)}|j\rangle \tag{5.35}$$

[*4] The Chebyshev inequality relates the probability distribution to the variance. When we define a stochastic variable x with the average μ and the variance σ^2, we can bound the probability of $\Pr(|x-\mu| \geq k\sigma)$ from above by

$$\Pr(|x-\mu| \geq k\sigma) \leq \frac{1}{k^2}, \tag{5.30}$$

where k is an arbitrary positive constant.

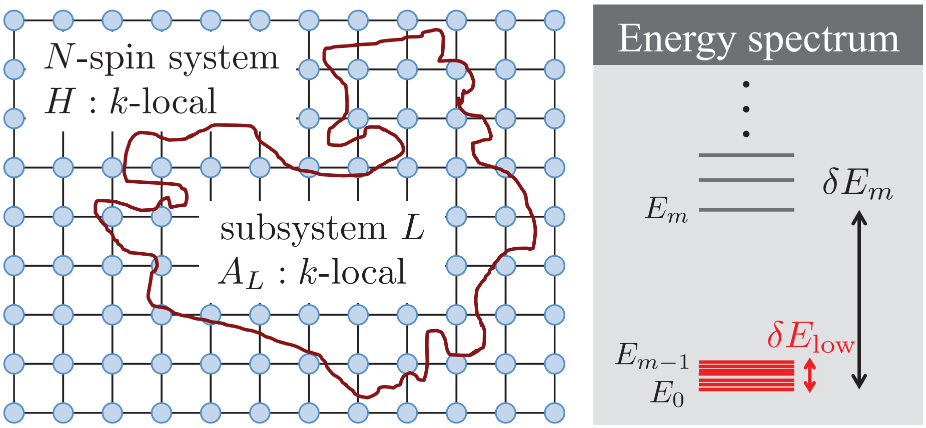

Figure 5.2: Schematic picture of Theorem 5.3. We here consider the low-lying state (5.37) below the eigenenergy E_{m-1} and investigate a relationship between the spectral gap δE_m and the macroscopic superposition in terms of the Fisher information (5.8). We consider a subsystem L and a k-local extensive operator A_L as in Eq. (5.4) and give a trade-off relationship between the spectral gap δE_m and the Fisher information $\mathcal{F}(\rho_{\rm low}, A_L)$ as in the inequality (5.38).

and

$$\|\Pi^A_{(-\infty,x]}\rho\Pi^A_{[x+\Delta x,\infty)}\| \ge \frac{1}{m}\sum_{j=1}^m \mathrm{tr}\big(\Pi_j\Pi^A_{(-\infty,x]}\rho\Pi^A_{[x+\Delta x,\infty)}\big) = \mathrm{tr}(\rho\Pi^A_{(-\infty,x]}\rho\Pi^A_{[x+\Delta x,\infty)})$$
$$= \|\Pi^A_{(-\infty,x]}\rho\Pi^A_{[x+\Delta x,\infty)}\|_{\rm F}, \tag{5.36}$$

where we define $\{|j\rangle\}$ as the eigenstates of ρ. We thereby proved the equality (5.33).

5.3 Trade-off relationship between the Fisher information and the spectral gap

We here define the spectral gap $\delta E_m \equiv E_m - E_0$ and derive a trade-off relationship between the Fisher information and the spectral gap. In the following, we denote $E_{m-1} - E_0$ by $\delta E_{\rm low}$ and the uniformly mixed state of $\{E_j\}_{j=0}^{m-1}$ by $\rho_{\rm low}$ (Fig. 5.2):

$$\rho_{\rm low} \equiv \frac{1}{m}\sum_{j=0}^{m-1} |E_j\rangle\langle E_j| = \frac{P_{m-1}}{m}. \tag{5.37}$$

Theorem 5.3. The spectral gap δE_m and the Fisher information $\mathcal{F}(\rho_{\rm low}, A_L)$ satisfy the following inequality:

$$\delta E_m \cdot \mathcal{F}(\rho_{\rm low}, A_L) \le \Lambda|L| + 8g^2|L|^2\delta E_{\rm low}, \tag{5.38}$$

where $\Lambda \equiv 16g^3k^2$.

When the eigenenergies $\{E_j\}_{j=0}^{m-1}$ satisfy $E_0 = E_1 = E_2 = \cdots = E_{m-1}$, that is, they are exactly degenerate, the inequality (5.38) reduces to

$$\delta E_m \cdot \mathcal{F}(\rho_{\rm low}, A_L) \le \Lambda|L| \tag{5.39}$$

because the energy $\delta E_{\rm low}$ is equal to zero. On the other hand, if the ground state is not degenerate, the Fisher information reduces to the variance $4(\Delta A_L)^2$ with respect to the ground state and the inequality (5.39) reduces to

$$\delta E_1 \cdot (\Delta A_L)^2 \le \frac{\Lambda}{4}|L|, \tag{5.40}$$

where $(\Delta A_L)^2 = \langle E_0|A_L^2|E_0\rangle - (\langle E_0|A_L|E_0\rangle)^2$. When the spectral gap is $\mathcal{O}(1)$, the Fisher information of the ground state always scales as $\mathcal{O}(|L|^p)$ with $p \le 1$ for arbitrary operators A_L. This means that in gapped ground states the macroscopic superposition cannot exist for arbitrary subsystems L.

5.3.1 Outline of the proof

We prove Theorem 5.3 from the following two statements, which we will prove later:

Statement 1. Let $\langle \cdots \rangle_{\rm low}$ be the expectation value with respect to $\rho_{\rm low}$. For any Hermitian operators O, the spectral gap δE_m is bounded from above by the Fisher information $\mathcal{F}(\rho_{\rm low}, O)$:

$$\delta E_m \le \frac{2\langle H_O\rangle_{\rm low} + 8\langle O^2\rangle_{\rm low}\delta E_{\rm low}}{F(\rho_{\rm low}, O)}, \tag{5.41}$$

where H_O is defined by $H_O \equiv -[[H, O], O]$. This is just a kind of variational principle; we do not need the k-locality of the Hamiltonian to prove this statement.

Statement 2. When we take the operator O as A_L in Eq. (5.4), we can obtain the upper bound of $\|H_{A_L}\|$ as

$$\|H_{A_L}\| \le 8g^3k^2|L| = \frac{\Lambda}{2}|L|. \tag{5.42}$$

The inequality (5.42) is more important in the sense that the k-locality of the Hamiltonian plays an essential role in the proof of the statement 2.

From these two statements, we can derive Theorem 5.3 by substituting A_L for O in Eq. (5.41) and utilizing the inequalities (5.42) as well as $\langle A_L^2\rangle_{\rm low} \le \|A_L\|^2 \le (g|L|)^2$, where $\|A_L\| \le g|L|$ comes from the extensiveness of the operator A_L as

$$\|A_L\| \le \sum_{X\in L}\|a_X\| \le \sum_{i\in L}\sum_{X\ni i}\|a_X\| \le g|L|. \tag{5.43}$$

This completes the proof of Theorem 5.3.

5.3.1 (a) Proof of the statement 1.

We first construct a quantum state ρ by the use of O:

$$\rho \equiv \frac{O\rho_{\rm low}O}{\langle O^2\rangle_{\rm low}}, \tag{5.44}$$

where the term $\langle O^2\rangle_{\rm low}$ is the normalization factor. We can prove the positivity of ρ as follows;

$$\langle\psi|\rho|\psi\rangle = \frac{\langle\psi|O\rho_{\rm low}O|\psi\rangle}{\langle O^2\rangle_{\rm low}} = \frac{1}{m\langle O^2\rangle_{\rm low}}\sum_{j=0}^{m-1}|\langle E_j|O|\psi\rangle|^2 \ge 0, \tag{5.45}$$

where $|\psi\rangle$ is an arbitrary quantum state.

We next prove

$$\delta E_m \leq \frac{\mathrm{tr}\big[H(\rho-\rho_{\mathrm{low}})\big]+\delta E_{\mathrm{low}}}{1-m\cdot\mathrm{tr}(\rho\rho_{\mathrm{low}})}. \tag{5.46}$$

We first denote $\langle E_j|\rho|E_j\rangle = p_j \geq 0$ and obtain

$$\begin{aligned}\mathrm{tr}(H\rho) &\geq E_0 \sum_{0\leq j\leq m-1} p_j + E_m \sum_{j\geq m} p_j \\ &= mE_0\cdot\mathrm{tr}(\rho_{\mathrm{low}}\rho) + E_m\big[1-m\cdot\mathrm{tr}(\rho_{\mathrm{low}}\rho)\big],\end{aligned} \tag{5.47}$$

where we utilized the equality of $m\cdot\mathrm{tr}(\rho_{\mathrm{low}}\rho) = \sum_{0\leq j\leq m-1} p_j = 1-\sum_{j\geq m} p_j$. The inequalities (5.47) and $\mathrm{tr}(H\rho_{\mathrm{low}}) = \sum_{0\leq j\leq m-1}\frac{1}{m}E_j \leq E_{m-1}$ give

$$\begin{aligned}\mathrm{tr}\big[H(\rho-\rho_{\mathrm{low}})\big] \geq& (E_m-E_0)\big[1-m\cdot\mathrm{tr}(\rho_{\mathrm{low}}\rho)\big]-(E_{m-1}-E_0) \\ =&\delta E_m\big[1-m\cdot\mathrm{tr}(\rho_{\mathrm{low}}\rho)\big]-\delta E_{\mathrm{low}}.\end{aligned} \tag{5.48}$$

We have thus proved the inequality (5.46).

We next obtain an upper bound of $\mathrm{tr}\big[H(\rho-\rho_{\mathrm{low}})\big]$ in (5.46). From the definitions of ρ and H_O, we have

$$\begin{aligned}\mathrm{tr}\big[H(\rho-\rho_{\mathrm{low}})\big] \leq& \frac{1}{\langle O^2\rangle_{\mathrm{low}}}\sum_{j=0}^{m-1}\frac{1}{m}\langle E_j|OHO|E_j\rangle - E_0 \\ =&\frac{1}{\langle O^2\rangle_{\mathrm{low}}}\sum_{j=0}^{m-1}\frac{1}{2m}\langle E_j|(HO^2+O^2H+H_O)|E_j\rangle - E_0 \\ \leq&\frac{1}{\langle O^2\rangle_{\mathrm{low}}}\sum_{j=0}^{m-1}\frac{1}{2m}\langle E_j|(2E_{m-1}O^2+H_O)|E_j\rangle - E_0 \\ =&\delta E_{\mathrm{low}} + \frac{\langle H_O\rangle_{\mathrm{low}}}{2\langle O^2\rangle_{\mathrm{low}}},\end{aligned} \tag{5.49}$$

where we utilized the inequality $\mathrm{tr}(H\rho_{\mathrm{low}}) \geq E_0$ in the first line and the equality $H_O = -HO^2 + 2OHO - O^2H$ in the second line. We then apply the upper bound (5.49) and the equality $\mathrm{tr}(\rho\rho_{\mathrm{low}}) = \frac{1}{\langle O^2\rangle_{\mathrm{low}}}\mathrm{tr}(O\rho_{\mathrm{low}}O\rho_{\mathrm{low}})$ to the inequality (5.46):

$$\delta E_m \leq \frac{\langle H_O\rangle_{\mathrm{low}} + 4\langle O^2\rangle_{\mathrm{low}}\delta E_{\mathrm{low}}}{2\langle O^2\rangle_{\mathrm{low}} - 2\mathrm{tr}(O\rho_{\mathrm{low}}O\rho_{\mathrm{low}})}. \tag{5.50}$$

We finally utilize the fact that in the case of the uniformly mixed states, the Fisher information reduces to the form of (5.12). We then obtain

$$2\mathrm{tr}(\rho_{\mathrm{low}}O^2 - n\rho_{\mathrm{low}}O\rho_{\mathrm{low}}O) = \frac{\mathcal{F}(\rho_{\mathrm{low}},O)}{2}. \tag{5.51}$$

By combining the inequality (5.50) and Eq. (5.51), we prove the statement 1.

5.3.1 (b) Proof of the statement 2.

Because H_{A_L} is given by $-[[H, A_L], A_L]$, we first consider the following equality:

$$[H, A_L] = \sum_{X_1 \in L} \sum_{X_2 \cap X_1 \neq 0} [h_{X_2}, a_{X_1}] = \sum_{X_1 \in L} \sum_{X_2 \cap X_1 \neq 0} (h_{X_2} a_{X_1} - a_{X_1} h_{X_2}). \tag{5.52}$$

We thus obtain

$$[[H, A_L], A_L] = \sum_{X_1 \in L} \sum_{X_2 \cap X_1 \neq 0} \sum_{X_3 \cap (X_1 \cup X_2) \neq 0} ([h_{X_2} a_{X_1}, a_{X_3}] - [a_{X_1} h_{X_2}, a_{X_3}]). \tag{5.53}$$

This gives the inequality of

$$\|H_{A_L}\| \leq 4 \sum_{X_1 \in L} \sum_{X_2 \cap X_1 \neq 0} \sum_{X_3 \cap (X_1 \cup X_2) \neq 0} \|a_{X_1}\| \cdot \|h_{X_2}\| \cdot \|a_{X_3}\|. \tag{5.54}$$

From the extensiveness of A_L, we calculate the sum over X_3 as

$$\sum_{X_3 \cap (X_1 \cup X_2) \neq 0} \|a_{X_3}\| \leq \sum_{i \in (X_1 \cup X_2)} \sum_{X_3 \ni i} \|a_{X_3}\| \leq \sum_{i \in (X_1 \cup X_2)} g \leq g(|X_1| + |X_2|) \leq 2gk. \tag{5.55}$$

This reduces the inequality (5.54) to

$$\|H_{A_L}\| \leq 8gk \sum_{X_1 \in L} \sum_{X_2 \cap X_1 \neq 0} \|a_{X_1}\| \cdot \|h_{X_2}\|. \tag{5.56}$$

We similarly calculate the sums over X_2 and X_1 as

$$\sum_{X_2 \cap X_1 \neq 0} \|h_{X_2}\| \leq \sum_{i \in X_1} \sum_{X_2 \ni i} \|h_{X_2}\| \leq g|X_1| \leq gk, \tag{5.57}$$

and

$$\sum_{X_1 \in L} \|a_{X_1}\| \leq \sum_{i \in L} \sum_{X_1 \ni i} \|a_{X_1}\| \leq g|L|. \tag{5.58}$$

From the inequalities (5.57) and (5.58), we reduce the inequality (5.56) to (5.42). This completes the proof. □

5.4 Macroscopic superposition in low-energy states

In Theorem 5.3, we have obtained the upper bound of the Fisher information in terms of the spectral gap. By combining the inequality (5.38) with the inequality (5.31) in Subsection 5.2.5, we can prove that the off-diagonal terms of the density matrix at least polynomially decay. We here prove a stronger bound showing that the norm of the off-diagonal block in fact decays exponentially instead of polynomially (Fig. 5.3). We also consider the low-energy state of $\rho_{\rm low} \equiv \frac{1}{m} \sum_{j=0}^{m-1} |E_j\rangle\langle E_j|$ as in Fig. 5.2.

Theorem 5.4. Let $\Pi^A_{[x,x')}$ be the projection operator onto the subspace of the eigenvalues of A_L which are in $[x, x')$. The off-diagonal block between the subspaces $\Pi^A_{(-\infty,x]}$ and $\Pi^A_{[x+\Delta x,\infty)}$ is exponentially

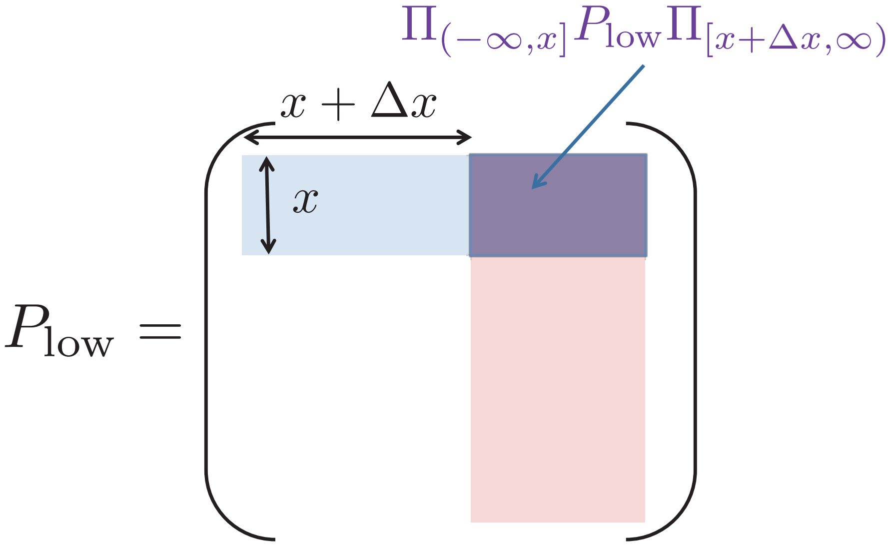

Figure 5.3: Schematic picture of Theorem 5.4. In this theorem, we show the upper bound of an off-diagonal block. The off-diagonal element $\|\Pi^A_{(-\infty,x]}P_{m-1}\Pi^A_{[x+\Delta x,\infty)}\|$, which is featured in the purple region, decays exponentially as $e^{-\mathcal{O}(\Delta x)/\sqrt{L/\delta E_m}}$. This is exponentially stronger than the bound (5.32) in Section 5.3, which gives the polynomial decay of the weight of the off-diagonal elements.

suppressed for $\forall L$*:*

$$m\cdot\|\Pi^A_{(-\infty,x]}\rho_{\rm low}\Pi^A_{[x+\Delta x,\infty)}\| \le 13\exp\left(-\frac{c_L\Delta x}{\xi_L}\right)+\frac{(6+2\lfloor c_L\Delta x\rfloor)(\delta E_{\rm low}+e^{-|L|})}{\delta E_m}, \tag{5.59}$$

where $\Delta x\ge 0$*,*

$$\xi_L\equiv\sqrt{\frac{1}{2}+\frac{(c_0+g)|L|}{2\delta E_m}} \tag{5.60}$$

and $c_L^{-1}\equiv 4gk\log\bigl(8+\frac{8g}{c_0}+\frac{8E_m}{c_0|L|}\bigr)$ *with* c_0 *a positive constant of* $\mathcal{O}(1)$ *depending only on* k *and* g*.*

In the special case where the ground states are exactly degenerate (e.g. $E_0=E_1=\cdots=E_{m-1}=0$), we can bound the off-diagonal block $m\|\Pi^A_{(-\infty,x]}\rho_{\rm low}\Pi^A_{[x+\Delta x,\infty)}\|$ by $e^{-c\Delta x/\sqrt{|L|/\delta E_m}}$ with c a positive constant of $\mathcal{O}(1)$. In the case where the ground states are almost degenerate (e.g. $\delta E_{\rm low}=e^{-\mathcal{O}(N)}$), we can also obtain the same upper bound.

In the following, we set $E_0=0$ for simplicity and prove the inequality

$$\|\Pi^A_{(-\infty,x]}P_{m-1}\Pi^A_{[x+\Delta x,\infty)}\| \le 13\exp\left(-\frac{c_L\Delta x}{\xi_L}\right)+\frac{(6+2\lfloor c_L\Delta x\rfloor)(E_{m-1}+e^{-|L|})}{E_m}, \tag{5.61}$$

where $P_{m-1}=m\cdot\rho_{\rm low}=\sum_{j=0}^{m-1}|E_j\rangle\langle E_j|$. In the proof, we only have to consider the case $E_m\ge 2e^{-|L|}$ because the right-hand side of the inequality (5.61) is larger than 1 for $E_m\le 2e^{-|L|}$ and we have the trivial bound of $\|\Pi^A_{(-\infty,x]}P_{m-1}\Pi^A_{[x+\Delta x,\infty)}\|\le 1$.

5.4.1 Effective Hamiltonian

Before we show the proof, we first introduce the following effective Hamiltonian $\tilde{H}$:

$$\tilde{H}\equiv H_L+\tilde{H}_{L^{\rm c}}\quad\text{with}\quad \tilde{H}_{L^{\rm c}}\equiv H_{L^{\rm c}}\Pi^{H_{L^{\rm c}}}_{[0,t)}+t\cdot\Pi^{H_{L^{\rm c}}}_{[t,\infty)}, \tag{5.62}$$

where

$$H_L \equiv \sum_{X:X\cap L\neq 0} h_X \quad \text{and} \quad H_{L^c} \equiv \sum_{X:X\in L^c} h_X. \tag{5.63}$$

In this Hamiltonian, we truncate the energy eigenspace of H_{L^c} which is higher than t. If the parameter t is sufficiently large, we expect that the Hamiltonians H and $\tilde{H}$ give the same low-energy subspace; indeed, from Theorem 3.4 in Chapter 3, we can obtain

$$\begin{aligned}
&\max(\|(H-\tilde{H})P_j\|, \|(H-\tilde{H})\tilde{P}_j\|) \le e^{-(t-E_j-c_2|L|)/c_1}, \\
&E_s - e^{-(t-E_j-c_2|L|)/c_1} \le \tilde{E}_s \le E_s \quad \text{for} \quad s \le j,
\end{aligned} \tag{5.64}$$

where c_1 and c_2 are positive constants of order 1 and depend only on k and g. We denote by $\{\tilde{E}_j, |\tilde{E}_j\rangle\}$ the eigenvalues and the eigenstates of the effective Hamiltonian, respectively, and $\tilde{P}_j \equiv \sum_{s=0}^{j} |\tilde{E}_s\rangle\langle\tilde{E}_s|$. In the following, we utilize the notations

$$c_1 + c_2 \equiv c_0 \quad \text{and} \quad \epsilon_L \equiv e^{-|L|}. \tag{5.65}$$

We here let $t = c_0|L| + E_m$, and thereby the inequalities in (5.64) give us

$$\begin{aligned}
&\max(\|(H-\tilde{H})P_{m-1}\|, \|(H-\tilde{H})\tilde{P}_{m-1}\|) \le \epsilon_L, \\
&E_s - \epsilon_L \le \tilde{E}_s \le E_s \quad \text{for} \quad s \le m.
\end{aligned} \tag{5.66}$$

Because we consider the case $E_m \ge 2\epsilon_L$, we have

$$\frac{E_m}{2} \le \tilde{E}_m \le E_m. \tag{5.67}$$

5.4.2 Outline of the proof

For the proof, we first approximate the low-energy space P_{m-1} by the use of a polynomial of the Hamiltonian. We then calculate the upper bound of the norm of the off-diagonal block based on the polynomial. In constructing such a polynomial, we utilize a polynomial of $\tilde{H}$ instead of the original Hamiltonian H; this is because the approximation error of the low-energy space depends on the norm of the Hamiltonian. The norm of $\tilde{H}$ is of $\mathcal{O}(|L|)$, while the norm of the original Hamiltonian H is of $\mathcal{O}(N)$. Thus, the polynomial of $\tilde{H}$ gives a better approximation than that of H.

We then decompose the norm $\|\Pi^A_{(-\infty,x]} P_{m-1} \Pi^A_{[x+\Delta x,\infty)}\|$ into

$$\begin{aligned}
&\|\Pi^A_{(-\infty,x]} P_{m-1} \Pi^A_{[x+\Delta x,\infty)}\| \\
\le&\|\Pi^A_{(-\infty,x]} \tilde{P}_{m-1} \Pi^A_{[x+\Delta x,\infty)}\| + \|P_{m-1} - \tilde{P}_{m-1}\| \\
\le&\|\Pi^A_{(-\infty,x]} K_n(\tilde{H}) \Pi^A_{[x+\Delta x,\infty)}\| + \|\tilde{P}_{m-1} - K_n(\tilde{H})\| + \|P_{m-1} - \tilde{P}_{m-1}\|,
\end{aligned} \tag{5.68}$$

where $K_n(\tilde{H})$ is an nth-order polynomial which approximates the low-energy space $\tilde{P}_{m-1}$; we set the value of n afterward. We here adopt the following polynomial $K_n(x)$:

$$K_n(x) = \frac{T_n(\frac{x-\tilde{E}_m}{E_c} - 1)}{T_n(\frac{\tilde{E}_0-\tilde{E}_m}{E_c} - 1)} \tag{5.69}$$

with

$$2E_c + \tilde{E}_m = \|\tilde{H}\|. \tag{5.70}$$

This polynomial is the same as the one in Eq. (4.20) except the parameter E_c, which is used for the proof of Theorem 4.3. The value of $K_n(x)$ is approximately equal to 1 at $x \simeq \tilde{E}_0$ and close to zero in the range $\tilde{E}_m \le x \le \|\tilde{H}\|$. That is, the operator $K_n(H)$ works as an approximate projector onto the low-energy subspace. The error of the approximation $\|\tilde{P}_{m-1} - K_n(\tilde{H})\|$ and the norm $\|\Pi^A_{(-\infty,x]} P_{m-1} \Pi^A_{[x+\Delta x,\infty)}\|$ depend on the order n of the polynomial; if n is large, the approximation is good but the term $\|\Pi^A_{(-\infty,x]} K_n(\tilde{H}) \Pi^A_{[x+\Delta x,\infty)}\|$ becomes large, while if n is small, the term $\|\Pi^A_{(-\infty,x]} K_n(\tilde{H}) \Pi^A_{[x+\Delta x,\infty)}\|$ becomes small but the approximation is not good. We thus have to choose the order n so that this trade-off relationship is optimized.

After algebra, we can prove the following inequalities:

$$\|P_{m-1} - \tilde{P}_{m-1}\| \le \frac{6(E_{m-1} + \epsilon_L)}{E_m}, \tag{5.71}$$

$$\|\tilde{P}_{m-1} - K_n(\tilde{H})\| \le \frac{2n(E_{m-1} + \epsilon_L)}{E_m} + 2e^{-n/\xi_L}, \tag{5.72}$$

$$\|\Pi^A_{(-\infty,x]} K_n(\tilde{H}) \Pi^A_{[x+\Delta x,\infty)}\| \le e^{-\Delta x/(4gk) + c'n - n/\xi_L}, \tag{5.73}$$

where $c' \equiv \log\left(8 + \frac{8g}{c_0} + \frac{8E_m}{c_0|L|}\right)$. These inequalities reduce the inequality (5.68) to

$$\|\Pi^A_{(-\infty,x]} P_{m-1} \Pi^A_{[x+\Delta x,\infty)}\| \le (e^{-\Delta x/(4gk) + c'n} + 2)e^{-n/\xi_L} + \frac{(6+2n)(E_{m-1} + \epsilon_L)}{E_m}. \tag{5.74}$$

By choosing $n = \lfloor \Delta x/(4c'gk) \rfloor \equiv \lfloor c_L \Delta x \rfloor$, we can obtain the main inequality (5.61), where the first term in (5.74) is bounded from above by $3e^{1/\xi_L} \cdot e^{-c_L \Delta x/\xi_L} \le 13 e^{-c_L \Delta x/\xi_L}$ because $\xi_L \ge 1/\sqrt{2}$. This completes the proof of Theorem 5.4. □

5.4.3 Proof of the inequality (5.71)

We start from the following inequality:

$$\begin{aligned}
\|P_{m-1} - \tilde{P}_{m-1}\| &= \|(P_{m-1} + Q_{m-1})(P_{m-1} - \tilde{P}_{m-1})\| \\
&= \|P_{m-1} - P_{m-1}\tilde{P}_{m-1} + Q_{m-1}\tilde{P}_{m-1}\| \\
&= \|P_{m-1}\tilde{Q}_{m-1} + Q_{m-1}\tilde{P}_{m-1}\| \le \|P_{m-1}\tilde{Q}_{m-1}\| + \|Q_{m-1}\tilde{P}_{m-1}\|,
\end{aligned} \tag{5.75}$$

where $Q_{m-1} \equiv 1 - P_{m-1}$ and $\tilde{Q}_{m-1} \equiv 1 - \tilde{P}_{m-1}$. We thus have to find upper bounds of $\|P_{m-1}\tilde{Q}_{m-1}\|$ and $\|Q_{m-1}\tilde{P}_{m-1}\|$ separately.

We first treat $\|Q_{m-1}\tilde{P}_{m-1}\|$ as follows; because of the inequality (5.66), we have

$$\begin{aligned}
\epsilon_L \ge \|(H - \tilde{H})\tilde{P}_{m-1}\| &\ge \|H\tilde{P}_{m-1}\| - \|\tilde{H}\tilde{P}_{m-1}\| \\
&\ge \|H\tilde{P}_{m-1}\| - |\tilde{E}_{m-1}| \\
&\ge \|H(P_{m-1} + Q_{m-1})\tilde{P}_{m-1}\| - |\tilde{E}_{m-1}| \\
&\ge \|HQ_{m-1}\tilde{P}_{m-1}\| - \|HP_{m-1}\tilde{P}_{m-1}\| - |\tilde{E}_{m-1}| \\
&\ge E_m\|Q_{m-1}\tilde{P}_{m-1}\| - E_{m-1} - |\tilde{E}_{m-1}|.
\end{aligned} \tag{5.76}$$

This gives

$$\|Q_{m-1}\tilde{P}_{m-1}\| \leq \frac{E_{m-1}+|\tilde{E}_{m-1}|+\epsilon_L}{E_m} \leq \frac{2(E_{m-1}+\epsilon_L)}{E_m}, \tag{5.77}$$

where we utilized the inequality (5.66) of $|\tilde{E}_{m-1}| \leq E_{m-1}+\epsilon_L$. In the same way, we can obtain

$$\|P_{m-1}\tilde{Q}_{m-1}\| \leq \frac{E_{m-1}+|\tilde{E}_{m-1}|+\epsilon_L}{\tilde{E}_m} \leq \frac{2(E_{m-1}+\epsilon_L)}{E_m/2}, \tag{5.78}$$

where we utilized the inequality (5.67) of $\tilde{E}_m \leq E_m/2$. We thereby prove the inequality (5.71) by combining the inequalities (5.75), (5.77) and (5.78).

5.4.4 Proof of the inequality (5.72)

We here decompose the term $\|\tilde{P}_{m-1}-K_n(\tilde{H})\|$ into

$$\begin{aligned}&\|\big[\tilde{P}_{m-1}-K_n(\tilde{H})\big](\tilde{P}_{m-1}+\tilde{Q}_{m-1})\|\\ \leq&\|\big[\tilde{P}_{m-1}-K_n(\tilde{H})\big]\tilde{P}_{m-1}\|+\|\big[\tilde{P}_{m-1}-K_n(\tilde{H})\big]\tilde{Q}_{m-1}\|\end{aligned} \tag{5.79}$$

and calculate the upper bounds of $\|\big[\tilde{P}_{m-1}-K_n(\tilde{H})\big]\tilde{P}_{m-1}\|$ and $\|\big[\tilde{P}_{m-1}-K_n(\tilde{H})\big]\tilde{Q}_{m-1}\|$ separately by the use of the basic properties of the Chebyshev polynomial in Subsection 4.4.2 (a).

We calculate the first term in (5.79) as follows:

$$\|\big[\tilde{P}_{m-1}-K_n(\tilde{H})\big]\tilde{P}_{m-1}\| = \left\|\sum_{j=0}^{m-1}\big[1-K_n(\tilde{E}_j)\big]|\tilde{E}_j\rangle\langle\tilde{E}_j|\right\| = 1-K_n(\tilde{E}_{m-1}), \tag{5.80}$$

where we utilized the fact that for $\tilde{E}_0 \leq x \leq \tilde{E}_m$ the function $K_n(x)$ monotonically decreases with $K_n(\tilde{E}_0)=1$. We can bound the function $K_n(\tilde{E}_{m-1})$ from below by

$$K_n(\tilde{E}_{m-1}) = \frac{T_n(\frac{\tilde{E}_{m-1}-\tilde{E}_m}{E_c}-1)}{T_n(\frac{\tilde{E}_0-\tilde{E}_m}{E_c}-1)} \geq 1-n\frac{(\tilde{E}_{m-1}-\tilde{E}_0)/E_c}{(\tilde{E}_m-\tilde{E}_0)/E_c} = 1-n\frac{\tilde{E}_{m-1}-\tilde{E}_0}{\tilde{E}_m-\tilde{E}_0}, \tag{5.81}$$

where we utilized the inequality:

$$\frac{T_n(-1-x+y)}{T_n(-1-x)} = \frac{T_n(1+x-y)}{T_n(1+x)} \geq 1-\frac{n}{x}y \tag{5.82}$$

for $x \geq 0$ and $0 \leq y \leq x$. Note that $T_n(1+x)$ is a concave function for $x>0$, which yields

$$\begin{aligned}T_n(1+x-y) &\geq T_n(1+x)-y\frac{d}{dx}T_n(1+x)\\ &= T_n(1+x)-y\frac{n}{\sqrt{x(x+2)}}\cdot\frac{\big(x+1+\sqrt{x(x+2)}\big)^n-\big(x+1-\sqrt{x(x+2)}\big)^n}{2}\\ &\geq T_n(1+x)-y\frac{n}{x}T_n(1+x)\end{aligned} \tag{5.83}$$

for $x \geq 0$ and $0 \leq y \leq x$. By the use of the inequalities (5.80) and (5.81), we can obtain

$$\|\big[\tilde{P}_{m-1}-K_n(\tilde{H})\big]\tilde{P}_{m-1}\| \leq n\frac{\tilde{E}_{m-1}-\tilde{E}_0}{\tilde{E}_m-\tilde{E}_0} \leq n\frac{E_{m-1}+\epsilon_L}{\tilde{E}_m} \leq \frac{2n(E_{m-1}+\epsilon_L)}{E_m}, \tag{5.84}$$

where we utilized the inequalities $\tilde{E}_{m-1} \leq E_{m-1}$, $-\epsilon_L \leq \tilde{E}_0 \leq E_0 = 0$ and $\tilde{E}_m \leq E_m/2$, which are derived from the inequalities (5.66) and (5.67).

We then obtain an upper bound of the second term in (5.79) as

$$\| [\tilde{P}_{m-1} - K_n(\tilde{H})] \tilde{Q}_{m-1} \| = \| K_n(\tilde{H}) \tilde{Q}_{m-1} \| \leq \frac{1}{T_n(\frac{\tilde{E}_0 - \tilde{E}_m}{E_c} - 1)}, \tag{5.85}$$

which is derived from the basic property of the Chebyshev polynomial, namely $|T_n(x)| \leq 1$ for $-1 \leq x \leq 1$. We also calculate $T_n(\frac{\tilde{E}_0 - \tilde{E}_m}{E_c} - 1)$ by the use of the inequality (4.23) as

$$\begin{aligned} \frac{1}{T_n(\frac{\tilde{E}_0 - \tilde{E}_m}{E_c} - 1)} &\leq 2 \exp\left(-2n \sqrt{\frac{\tilde{E}_m - \tilde{E}_0}{\tilde{E}_m - \tilde{E}_0 + 2E_c}} \right) \\ &= 2 \exp\left(-2n \sqrt{\frac{\tilde{E}_m - \tilde{E}_0}{\|\tilde{H}\| - \tilde{E}_0}} \right) \\ &\leq 2 \exp\left(-2n \sqrt{\frac{\tilde{E}_m}{\|\tilde{H}\|}} \right) \leq 2 \exp\left(-n \sqrt{\frac{2E_m}{\|\tilde{H}\|}} \right), \end{aligned} \tag{5.86}$$

where the definition of E_c is given in Eq. (5.70), we can bound $(\tilde{E}_m - \tilde{E}_0)/(\|\tilde{H}\| - \tilde{E}_0) \geq \tilde{E}_m/\|\tilde{H}\|$ because of $0 \leq \tilde{E}_m \leq \|\tilde{H}\|$ and $\tilde{E}_0 \leq 0$, and the last inequality comes from $\tilde{E}_m \geq E_m/2$. From the definition of $\tilde{H}$, we can bound the norm of the effective Hamiltonian from above by

$$\|\tilde{H}\| \leq \|H_L + H_{\partial L}\| + \|\tilde{H}_{L^c}\| \leq g|L| + t = (g + c_0)|L| + E_m. \tag{5.87}$$

This upper bound for $\|\tilde{H}\|$ reduces the inequalities (5.85) and (5.86) to

$$\begin{aligned} \| [\tilde{P}_{m-1} - K_n(\tilde{H})] \tilde{Q}_{m-1} \| &\leq \frac{1}{T_n(\frac{\tilde{E}_0 - \tilde{E}_m}{E_c} - 1)} \\ &\leq 2 \exp\left(-n \sqrt{\frac{2E_m}{(g + c_0)|L| + E_m}} \right) \equiv 2e^{-n/\xi_L}. \end{aligned} \tag{5.88}$$

By combining the inequalities (5.84) and (5.88), we finally prove the inequality (5.72).

5.4.5 Proof of the inequality (5.73)

We first follow the same calculation as in (3.4):

$$\begin{aligned} \|\Pi^A_{(-\infty,x]} K_n(\tilde{H}) \Pi^A_{[x+\Delta x,\infty)}\| &\leq \|\Pi^A_{(-\infty,x]} e^{\lambda A_L} e^{-\lambda A_L} K_n(\tilde{H}) e^{\lambda A_L} e^{-\lambda A_L} \Pi^A_{[x+\Delta x,\infty)}\| \\ &\leq e^{-\lambda \Delta x} \|\Pi^A_{(-\infty,x]} K_n(e^{-\lambda A_L} \tilde{H} e^{\lambda A_L}) \Pi^A_{[x+\Delta x,\infty)}\| \\ &\leq e^{-\lambda \Delta x} \|K_n(e^{-\lambda A_L} \tilde{H} e^{\lambda A_L})\|, \end{aligned} \tag{5.89}$$

where we used the definition $\lambda \equiv 1/(4gk)$. We, in the following, find an upper bound of $\|K_n(e^{-\lambda A_L} \tilde{H} e^{\lambda A_L})\|$. For this purpose, we now focus on the fact that nth order Chebyshev polynomial has the form

$$T_n(x) = 2^{n-1} \Pi_{j=1}^n (x - x_j) \quad \text{with} \quad |x_j| \leq 1. \tag{5.90}$$

We then bound $\|K_n(e^{-\lambda A_L}\tilde{H}e^{\lambda A_L})\|$ from above by

$$\begin{aligned}
\|K_n(e^{-\lambda A_L}\tilde{H}e^{\lambda A_L})\| &\le \frac{2^{n-1}}{T_n(\frac{\tilde{E}_0-\tilde{E}_m}{E_c}-1)}\left(\frac{\|e^{-\lambda A_L}\tilde{H}e^{\lambda A_L}\|+\tilde{E}_m}{E_c}+2\right)^n \\
&\le 2^n e^{-n/\xi_L}\left(\frac{\|e^{-\lambda A_L}\tilde{H}e^{\lambda A_L}\|+\tilde{E}_m+2E_c}{E_c}\right)^n \\
&\le e^{-n/\xi_L}\left(\frac{4\|e^{-\lambda A_L}\tilde{H}e^{\lambda A_L}\|+4\|\tilde{H}\|}{\|\tilde{H}\|-\tilde{E}_m}\right)^n,
\end{aligned} \tag{5.91}$$

where we utilized the inequality in (5.88) and the form of E_c as in Eq. (5.70).

We then obtain an upper bound of $\|e^{-\lambda A_L}\tilde{H}e^{\lambda A_L}\|$. From the definitions of H_L and H_{L^c} in Eq. (5.63), we have the commutation relation $[\tilde{H}_{L^c}, A_L]=0$ and obtain

$$e^{-\lambda A_L}\tilde{H}e^{\lambda A_L} = e^{-\lambda A_L}H_L e^{\lambda A_L} + \tilde{H}_{L^c}. \tag{5.92}$$

The inequality (3.6) gives

$$\begin{aligned}
\|e^{-\lambda A_L}H_L e^{\lambda A_L}\| &\le \sum_{X:X\cap L\neq 0}\|e^{-\lambda A_L}h_X e^{-\lambda A_L}\| \\
&\le \sum_{X:X\cap L\neq 0}\frac{\|h_X\|}{1-2\lambda gk} \le 2g|L| \quad \text{for} \quad \lambda=\frac{1}{4gk}.
\end{aligned} \tag{5.93}$$

The inequalities (5.92) and (5.93) give the upper bound of $\|e^{-\lambda A_L}\tilde{H}e^{\lambda A_L}\|$ as

$$\|e^{-\lambda A_L}\tilde{H}e^{\lambda A_L}\| \le \|e^{-\lambda A_L}H_L e^{\lambda A_L}\| + \|\tilde{H}_{L^c}\| \le 2g|L|+t. \tag{5.94}$$

From the inequality (5.94), we have

$$\begin{aligned}
\frac{\|e^{-\lambda A_L}\tilde{H}e^{\lambda A_L}\|+\|\tilde{H}\|}{\|\tilde{H}\|-\tilde{E}_m} &\le \frac{2g|L|+2\|\tilde{H}\|}{\|\tilde{H}\|-\tilde{E}_m} \\
&\le \frac{2g|L|+2\tilde{E}_m}{c_0|L|+E_m-\tilde{E}_m}+2 \\
&\le \frac{2g+2E_m/|L|}{c_0}+2,
\end{aligned} \tag{5.95}$$

where in the first and the second inequalities we utilized $t=c_0|L|+E_m\le\|\tilde{H}\|$ and the last inequality comes from $E_m-\tilde{E}_m\ge 0$. Thus, the inequality (5.95) reduces the inequality (5.91) to

$$\|K_n(e^{-\lambda A_L}\tilde{H}e^{\lambda A_L})\| \le e^{-n/\xi_L}\cdot e^{c'n} \tag{5.96}$$

for $c' \equiv \log\big(8+\frac{8g+8E_m/|L|}{c_0}\big)$. We therefore prove the inequality (5.73) from the inequalities (5.89) and (5.96).

Chapter 6

Lieb-Robinson like bound for time evolution in terms of information sharing

6.1 The motivation

In this chapter, we consider time evolutions due to k-local Hamiltonians. The main purpose here is to understand how the k-locality can place a restriction to the time evolutions. The most well-known example is the Lieb-Robinson bound for short-range interacting Hamiltonians [28–30]. It gives an upper bound of the commutator between spatially separated operators A_i and $B_{i'}$ on spins i and i', respectively:

$$\|[A_i(t), B_{i'}]\| \leq \text{const} \cdot \exp\left(\frac{-\text{dist}(i,i') + vt}{\xi}\right), \tag{6.1}$$

where v and ξ are positive constants of $\mathcal{O}(1)$, $\text{dist}(i,i')$ is the spatial distance between the spins i and i', and we assume $\|A_i\| = \|B_{i'}\| = 1$. The physical meaning of the Lieb-Robinson bound is that information cannot be transferred instantly from one spin to another spin. In other words, the velocity of the information transfer should be finitely bounded. This can be mathematically given in terms of the information theory. The amount of the information which can be sent by the time evolution is characterized by the Holevo capacity [1,2]. We can indeed prove that the Holevo capacity is also bounded from above [29] in a similar fashion to the Lieb-Robinson bound (6.1).

We have various motivations to investigate fundamental limits to the time evolutions. The Lieb-Robinson bound is not only physically interesting but also contains practically important applications:

1. We can figure out whether a time evolution (e.g. $e^{iHt}|\text{Prod}\rangle$) can be classically simulated or not.[*1] For one-dimensional short-range interacting systems for example, It has been proved [68] that the classical simulation is possible as long as $t \lesssim \log N$, for which the evolution of the state can be well approximated by that of a matrix product state.

2. We empirically know that basic properties of the non-degenerate ground state with a finite gap can be characterized through finite-time evolution over the time $t = \mathcal{O}(1/\delta E)$ [11]. In this context, many fundamental properties have been proved for gapped ground states in short-range interacting systems: the exponential decay of bi-partite correlations, the entropic area law and so on.

[*1] We mean by 'classically simulated' that we can simulate the dynamics by the use of the classical computer [10].

3. We can also analyze the adiabatic continuation [38–40] as follows (see also Section 2.4 in Chapter 2). Let us consider the evolution due to the change of an internal parameter instead of the time evolution; consider, for example, a Hamiltonian $H(s) = H + sV$ and its ground state $|E_0(s)\rangle$, where H and V are short-range interacting Hamiltonians. As long as the ground state $|E_0(s)\rangle$ is non-degenerate and gapped as we change the parameter s, the evolution of the state can be described similarly to the time evolution:

$$\frac{d}{ds}|E_0(s)\rangle = D_s|E_0(s)\rangle, \tag{6.2}$$

   where D_s is a short-range interacting operator. We therefore can conclude that the parameter evolution should also satisfy the Lieb-Robinson bound. The adiabatic continuation by Eq. (6.2) is crucial to investigate the stability of various properties in ground states: the quantum topological order [41, 42], the entropic area law [43] and so on.

4. Finally, by the use of the time-evolution, we can define a class of quantum states which have common simple structures. For example, after the time evolution of a product state over the time of $\mathcal{O}(1)$, namely $e^{iHt}|\mathrm{Prod}\rangle$ with $t = \mathcal{O}(1)$, the resulting state gives a class of the short range entanglement (SRE). This class of states is useful in characterizing quantum phases in many-body physics [45, 48].

The above statements have been mainly found for short-range interacting systems and the Lieb-Robinson bound has played the crucial roles in analyzing them. Our goal is to extend the results to long-range interacting systems.

For systems with general k-local Hamiltonians which include long-range interactions, however, we cannot expect the same achievements by the use of the Lieb-Robinson bound. The primary reasons is that the Lieb-Robinson bound focuses on the velocity of the information transfer, whereas long-range interacting systems can transport information immediately. Because of this, the Lieb-Robinson bound can no longer give the exponentially strong restriction as in (6.1) for long-range interacting systems [22, 23].

We here consider a qualitatively new bound for the time evolution in terms of the velocity of the information sharing instead of the information transfer. This bound gives exponentially strong restrictions for long-range interacting systems as well, which will enable us to obtain novel strong statements on various kinds of fundamental properties in quantum many-body systems with long-range interactions.

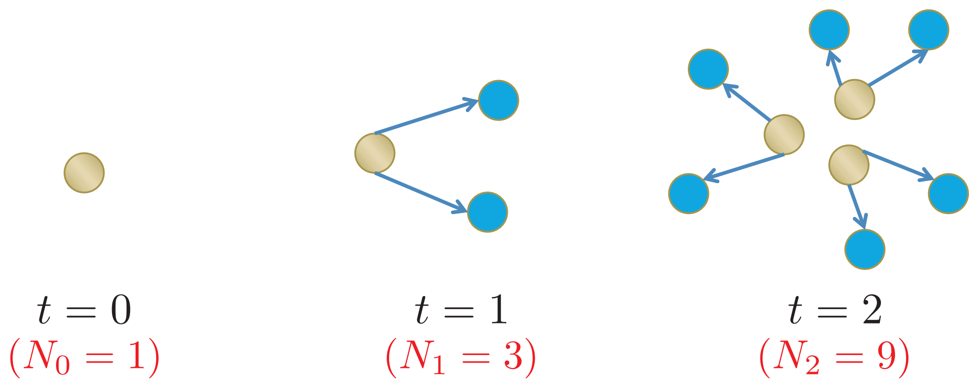

Figure 6.1: Classical information sharing. We start from one information source and consider the process that each element which has received the information can send the information to other κ_c elements per unit of time; we above consider the case $\kappa_c = 2$. The number of receivers N_n increases as 3^n because of Eq. (6.4), namely $N_0 = 1$, $N_1 = 3$, $N_2 = 9$ and so on.

In order to make our concept clear, we first consider a classical process in which a source transfers information to receivers. The total system consists of N elements, each of which sends the information to other $\kappa_{\rm c}$ elements per unit of time. Let N_n denote the number of elements which share the information at time n (Fig. 6.1). The time-dependence of N_n is given by

$$N_{n+1} = \kappa_{\rm c} N_n + N_n = (\kappa_{\rm c} + 1) N_n, \tag{6.3}$$

which gives

$$N_n = (\kappa_{\rm c} + 1)^n \tag{6.4}$$

because of $N_0 = 1$. This means that the number of elements which share the information can be classically bounded from above by $e^{\mathcal{O}(t)}$. We expect that quantum systems will also have a similar bound.

In order to mathematically apply the above discussion to quantum cases, we consider the time evolution of the locality of operators. We first consider an l_0-local operator Γ^{l_0}, where Γ^{l_0} contains the operators up to l_0-body couplings. After a time evolution, the operator $\Gamma^{l_0}(t_0)$ will be no longer an l_0-local operator but may be approximated by an l-local operator, where l is greater than l_0. We now regard l_0 and l as the numbers of particles which share the information at $t = 0$ and $t = t_0$, respectively. We then expect that the approximation is exponentially improved beyond $l \gtrsim l_0 e^{\mathcal{O}(t_0)}$. Our main purpose is to give a mathematical foundation to such a conceptual reasoning.

This chapter is organized as follows: In Section 6.2, we first give the main result on how accurate we can construct an l-local operator in order to approximate the time evolution of an l_0-local operator. In Section 6.3, we discuss the stability of the topological order after time evolutions over $t = \mathcal{O}(1)$. Finally in Section 6.4, we consider the time evolution of a product state, namely $e^{iHt}|{\rm Prod}\rangle$, and prove the exponential spectral concentration for k-local operators with $k = \mathcal{O}(1)$.

6.2 Upper bound on the velocity of Information sharing

We here consider systems which are governed by a sum of commuting Hamiltonians (an SC Hamiltonian):

$$H \equiv \sum_{m=1}^{n_{\rm sc}} \frac{H_m^{\rm c}}{n_{\rm sc}}, \tag{6.5}$$

where we assume that each of the commuting Hamiltonians $\{H_m^{\rm c}\}_{m=1}^{n_{\rm sc}}$ is an extensive k-local Hamiltonian as

$$H^{\rm c} = \sum_{|X| \le k} h_X \quad \text{with} \sum_{X: X \ni i} \|h_X\| \le g \quad \forall i. \tag{6.6}$$

For the details of the SC Hamiltonians, see Subsection 1.2.4 in Chapter 1.

We then consider a time-evolution of an l_0-local operator $\Gamma^{(l_0)}$:

$$\Gamma^{(l_0)}(t) = e^{-iHt} \Gamma^{(l_0)} e^{iHt}. \tag{6.7}$$

We approximate it to an l-local operator $\Gamma_t^{(l)}$ and estimate the error $\|\Gamma^{(l_0)}(t) - \Gamma_t^{(l)}\|$. The discussion in Section 6.1 suggests that we need $l = l_0 e^{\mathcal{O}(t)}$ in order to approximate $\Gamma^{(l_0)}(t)$ well. Indeed, we prove the following theorem:

Theorem 6.2. Let $\mathcal{U}(l)$ be the set of l-local operators and consider an arbitrary l_0-local operator $\Gamma^{(l_0)} \in \mathcal{U}(l_0)$. There exists an l-local operator $\Gamma_t^{(l)}$ which approximates the operator $\Gamma^{(l_0)}(t)$ with the following

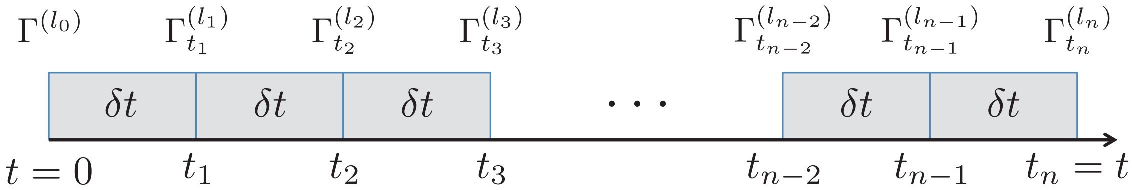

Figure 6.2: Schematic picture of the proof of Theorem 6.2. For small-time evolutions, we can obtain the approximation of $\Gamma^{(l_0)}(t)$ directly by the use of the time expansion as in Eq. (6.10). For long-time evolutions, we split the total time range $[0,t]$ into n intervals with a length δt. In each of the intervals, we can apply the result for small-time evolutions, and hence we can connect the approximations as follows; we first approximate $\Gamma^{(l_0)}(\delta t)$ by $\Gamma_{t_1}^{(l_1)}$, second approximate $\Gamma_{t_1}^{(l_1)}(\delta t)$ by $\Gamma_{t_2}^{(l_2)}$ and so on. Then, we can finally prove that the operator $\Gamma_{t_n}^{(l_n)}$ satisfies the inequality (6.8) by choosing the set $\{l_m, \Gamma_{t_m}^{(l_m)}\}_{m=1}^n$ appropriately.

error:

$$\min_{\Gamma_t^{(l)} \in \mathcal{U}(l)} \left(\|\Gamma^{(l_0)}(t) - \Gamma_t^{(l)}\| \right) \le 8\|\Gamma^{(l_0)}\| \lceil \kappa t \rceil \exp\left[-\frac{1}{\xi}\left(\frac{l}{l(t)} - l_0 \right) \right] \tag{6.8}$$

with $\kappa = 24g(k-1)$, $\xi = (k-1)/\log 2$ *and* $l(t) = 2^{\lceil \kappa t \rceil} - 1$.

Because the function $l(t)$ scales as $e^{\mathcal{O}(t)}$, the time-evolution of $\Gamma^{(l_0)}$ can be well approximated by an $(l_0 e^{\mathcal{O}(t)})$-local operator. Note that we do not assume any conditions for $\Gamma^{(l_0)}$.

Proof of Theorem 6.2. For the proof, we first obtain the upper bound for a small-time evolution:

$$\min_{\Gamma_t^{(l)} \in \mathcal{U}(l)} \left(\|\Gamma^{(l_0)}(t) - \Gamma_t^{(l)}\| \right) \le 2^{\frac{l_0}{k-1}} \cdot \frac{(\kappa t/2)^{\frac{l-l_0}{k-1}}}{1-\kappa t/2} \|\Gamma^{(l_0)}\| \tag{6.9}$$

for $t < 2/\kappa$. We will show the proof in Subsection 6.2.1. In the proof of (6.9), we utilize the direct expansion of $e^{-iHt}\Gamma^{(l_0)}e^{iHt}$ according to the Hadamard lemma of the form

$$e^{-iHt}\Gamma^{(l_0)}e^{iHt} = \sum_{m=0}^{\infty} \frac{(it)^m}{m!} O_m, \tag{6.10}$$

where $O_0 = \Gamma^{(l_0)}$ and $O_m = [\cdots[[\Gamma^{(l_0)}, \overbrace{H], H], \ldots], H]}^{m}$. We terminate the expansion (6.10) at $m = m_0$ so that the expanded operator may be l-local. We then estimate the error due to the termination to prove the bound (6.9). We notice that the SC-condition of the Hamiltonian is necessary for the derivation of (6.9).

We, however, cannot utilize the expansion (6.10) in order to obtain a meaningful bound for $t > 2/\kappa$. In obtaining the inequality (6.8), we will have to utilize the property that e^{-iHt} is a unitary operator.[*2] For this purpose, we split the time range $[0,t]$ into n intervals (Fig. 6.2) such that

$$\frac{t}{n} \le \frac{1}{\kappa}. \tag{6.11}$$

We here choose

$$n = \lceil \kappa t \rceil \tag{6.12}$$

[*2]Note that the bound (6.9) can be also applied to the imaginary time evolution $e^{Ht}\Gamma^{(l_0)}e^{-Ht}$ without the unitarity condition. The approximation by the finite expansion of $e^{Ht}\Gamma^{(l_0)}e^{-Ht}$ will becomes less accurate beyond a certain time t_c, which comes from the fact that the norm of $e^{Ht}\Gamma^{(l_0)}e^{-Ht}$ rapidly increases for $t \ge t_c$. Without the unitarity, we cannot arrive at the inequality (6.8) from (6.9).

and define the length of the interval t/n as

$$\delta t \equiv \frac{t}{\lceil \kappa t \rceil} \leq \frac{1}{\kappa}. \tag{6.13}$$

We also define $t_m \equiv m\delta t$ for $m = 0, 1, 2, \ldots n$ with $t_n = t$. Note that in each interval we can now apply the upper bound (6.9).

In the following, we connect the approximations of $\Gamma^{(l_0)}(t)$ from the first interval $[0, t_1)$ to the last interval $[t_{n-1}, t_n]$. We first approximate the time evolution $\Gamma^{(l_0)}(\delta t)$ with an l_1-local operator $\Gamma^{(l_1)}_{t_1}$. We second approximate $\Gamma^{(l_1)}_{t_1}(\delta t)$ with an l_2-local operator $\Gamma^{(l_2)}_{t_2}$. By sequentially repeating this process, we define a set of operators $\{\Gamma^{(l_m)}_{t_m}\}_{m=1}^n$ so that they may approximately satisfy

$$\|\Gamma^{(l_m)}_{t_m}(\delta t) - \Gamma^{(l_{m+1})}_{t_{m+1}}\| = 0, \tag{6.14}$$

respectively, where each of $\{l_m, \Gamma^{(l_m)}_{t_m}\}_{m=1}^n$ will be given specifically in Subsection 6.2.2 such that $l_n \leq l$. We finally obtain the approximation of $\Gamma^{(l_0)}(t)$ by the use of the set $\{\Gamma^{(l_m)}_{t_m}\}_{m=1}^n$:

$$\begin{aligned}
\|\Gamma^{(l_n)}_{t_n} - \Gamma^{(l_0)}(t)\| &= \left\| \sum_{m=1}^n \left[\Gamma^{(l_m)}_{t_m}(t_n - t_m) - \Gamma^{(l_{m-1})}_{t_{m-1}}(t_n - t_m + \delta t)\right] \right\| \\
&\leq \sum_{m=1}^n \left\|\Gamma^{(l_m)}_{t_m}(t_n - t_m) - \Gamma^{(l_{m-1})}_{t_{m-1}}(t_n - t_m + \delta t)\right\| \\
&= \sum_{m=1}^n \left\|\Gamma^{(l_m)}_{t_m} - \Gamma^{(l_{m-1})}_{t_{m-1}}(\delta t)\right\|,
\end{aligned} \tag{6.15}$$

where we utilized the equality $t_{m+1} - t_m = \delta t$ in the first line, the norm invariance during the time evolution in the third line, and we set $\Gamma^{(l_0)}_{t_0} \equiv \Gamma^{(l_0)}$. Note that we utilize the unitarity of the time evolution in the third equality.

We then prove that the following inequalities are satisfied by choosing the set $\{l_m, \Gamma^{(l_m)}_{t_m}\}_{m=1}^n$ appropriately:

$$\left\|\Gamma^{(l_m)}_{t_m} - \Gamma^{(l_{m-1})}_{t_{m-1}}(\delta t)\right\| \leq \Delta(\Delta + 1)^{m-1}\|\Gamma^{(l_0)}\| \tag{6.16}$$

for $m = 1, 2, \ldots, n$, respectively, with

$$\Delta = 4 \exp\left[-\frac{1}{\xi}\left(\frac{l}{l(t)} - l_0\right)\right]. \tag{6.17}$$

The proof is given in Subsection 6.2.2. We apply the upper bound (6.9) for the proof.

By combining the inequalities (6.15) and (6.16), we have

$$\begin{aligned}
\|\Gamma^{(l_n)}_{t_n} - \Gamma^{(l_0)}(t)\| &\leq \sum_{m=1}^n \Delta(\Delta + 1)^{m-1}\|\Gamma^{(l_0)}\| \\
&= \left[(\Delta + 1)^n - 1\right] \cdot \|\Gamma^{(l_0)}\|.
\end{aligned} \tag{6.18}$$

We can always find an operator $\Gamma^{(l_n)}_{t_n}$ such that $\|\Gamma^{(l_n)}_{t_n} - \Gamma^{(l_0)}(t)\| \leq \|\Gamma^{(l_0)}\|$ (e.g. $\Gamma^{(l_n)}_{t_n} = 0$), and hence we only have to consider the range $(\Delta + 1)^n - 1 \leq 1$ in the above inequality and obtain

$$(\Delta + 1)^n - 1 \leq 2^{1-1/n} \cdot n \cdot \Delta < 2n\Delta. \tag{6.19}$$

(Proof of (6.19): the inequality $(x+1)^n - 1 \le 1$ is satisfied for $x \le 2^{1/n} - 1$. By the use of the fact that $(x+1)^n - 1$ is the concave function for $x \ge 0$, we have

$$(x+1)^n - 1 \le nx(x_0+1)^{n-1} \tag{6.20}$$

for $0 < x \le x_0$ with x_0 a positive constant. By choosing $x_0 = 2^{1/n} - 1$, we obtain $(x+1)^n - 1 \le nx2^{(n-1)/n} \le 2nx$. $\square$)

The inequality (6.18) reduces to the inequality (6.8) by the use of the inequality (6.19) and the definitions of Δ and n, namely Eqs. (6.12) and (6.17). This completes the proof of Theorem 6.2. $\square$

6.2.1 Bound for a small-time evolution

We here prove the inequality (6.9) for a small-time evolution: in order to obtain the bound, we expand $e^{-iHt}\Gamma^{(l_0)}e^{iHt}$ by the Hadamard lemma as in Eq. (6.10), which we reproduce here:

$$e^{-iHt}\Gamma^{(l_0)}e^{iHt} = \sum_{m=0}^{\infty} \frac{(it)^m}{m!} O_m, \tag{6.21}$$

where $O_0 = \Gamma^{(l_0)}$ and $O_m = [\cdots[[\Gamma^{(l_0)}, \overbrace{H], H], \ldots], H]}^{m}$. This expansion can be terminated if the expansion converges rapidly as m increases. The termination at $m = m_0$ gives the local approximation of the operator $\Gamma^{(l_0)}(t)$ as

$$\sum_{m=0}^{m_0} \frac{(it)^m}{m!} O_m, \tag{6.22}$$

which is at most $(l_0 + m_0(k-1))$-local. In order to make the operator (6.22) less than or equal to l-local, we take

$$m_0 = \left\lfloor \frac{l - l_0}{k-1} \right\rfloor. \tag{6.23}$$

Our purpose now is to calculate the error due to the cutoff of the above expansion, namely

$$\left\| \sum_{m=m_0+1}^{\infty} \frac{(it)^m}{m!} O_m \right\| \le \sum_{m=m_0+1}^{\infty} \frac{t^m}{m!} \|O_m\|, \tag{6.24}$$

Because of the SC-condition of the Hamiltonian, we can follow the same calculation as in the Subsection 3.3.2 in Chapter 3; the inequalities (3.46) and (3.47) then give

$$\begin{aligned} \|O_m\| &\le m!\big[12g(k-1)\big]^m 2^{l_0/(k-1)} \|\Gamma^{(l_0)}\| \\ &= m!(\kappa/2)^m 2^{l_0/(k-1)} \|\Gamma^{(l_0)}\|. \end{aligned} \tag{6.25}$$

The inequalities (6.24) and (6.25) yield

$$\begin{aligned} \left\| \sum_{m=m_0+1}^{\infty} \frac{(it)^m}{m!} O_m \right\| &\le 2^{l_0/(k-1)} \|\Gamma^{(l_0)}\| \sum_{m=m_0+1}^{\infty} (\kappa t/2)^m \\ &= 2^{l_0/(k-1)} \frac{(\kappa t/2)^{m_0+1}}{1-\kappa t/2} \|\Gamma^{(ls_0)}\| \end{aligned}$$

$$\leq 2^{l_0/(k-1)} \cdot \frac{(\kappa t/2)^{\frac{l-l_0}{k-1}}}{1-\kappa t/2} \|\Gamma^{(l_0)}\| \tag{6.26}$$

for $t < 2/\kappa$, where we utilized the inequality $m_0 + 1 \geq (l - l_0)/(k-1)$ in the last inequality.

6.2.2 The proof of the inequality (6.16)

We here calculate the upper bound of each of $\{\|\Gamma_{t_m}^{(l_m)} - \Gamma_{t_{m-1}}^{(l_{m-1})}(\delta t)\|\}_{m=1}^n$. Because $\delta t \leq 1/\kappa$, we can apply the bound (6.9) to estimate the norm $\|\Gamma_{t_m}^{(l_m)} - \Gamma_{t_{m-1}}^{(l_{m-1})}(\delta t)\|$. In Subsection 6.2.2 (a), we prove that an appropriate choice of the set $\{l_m, \Gamma_{t_m}^{(l_m)}\}_{m=1}^n$ gives the following inequalities:

$$\|\Gamma_{t_m}^{(l_m)} - \Gamma_{t_{m-1}}^{(l_{m-1})}(\delta t)\| \leq \|\Gamma_{t_{m-1}}^{(l_{m-1})}\|\Delta \tag{6.27}$$

for $m = 1, 2, \ldots, n$, respectively.

After we obtain the inequality (6.27), we have to prove the following inequality:

$$\|\Gamma_{t_m}^{(l_m)}\| \leq (\Delta + 1)^m \|\Gamma^{(l_0)}\|. \tag{6.28}$$

We prove this inequality by the induction method. For $m = 1$, we have

$$\|\Gamma_{t_1}^{(l_1)}\| = \|\Gamma_{t_1}^{(l_1)} - \Gamma_{t_0}^{(l_0)} + \Gamma_{t_0}^{(l_0)}\| \leq \|\Gamma_{t_1}^{(l_1)} - \Gamma_{t_0}^{(l_0)}\| + \|\Gamma_{t_0}^{(l_0)}\| \leq (\Delta+1)\|\Gamma^{(l_0)}\|, \tag{6.29}$$

where the last inequality came from (6.27) and we utilized the definition of $\Gamma_{t_0}^{(l_0)} \equiv \Gamma^{(l_0)}$. We then assume the inequality (6.28) for $m \leq m_0$ and prove it for $m = m_0 + 1$ as follows:

$$\|\Gamma_{t_{m_0+1}}^{(l_{m_0+1})}\| = \|\Gamma_{t_{m_0+1}}^{(l_{m_0+1})} - \Gamma_{t_{m_0}}^{(l_{m_0})} + \Gamma_{t_{m_0}}^{(l_{m_0})}\| \leq (\Delta+1)\|\Gamma_{t_{m_0}}^{(l_{m_0})}\| \leq (\Delta+1)^{m_0+1}\|\Gamma^{(l_0)}\|. \tag{6.30}$$

This completes the proof of the inequality (6.28). From the inequalities (6.27) and (6.28), we prove the inequality (6.16).

6.2.2 (a) Proof of the inequality (6.27)

We first calculate $\|\Gamma_{t_m}^{(l_m)} - \Gamma_{t_{m-1}}^{(l_{m-1})}(\delta t)\|$ for some l_m and l_{m-1}. Because of the upper bound (6.9) for small-time evolutions, there exists an operator $\Gamma_{t_m}^{(l_m)}$ for $\forall \Gamma_{t_{m-1}}^{(l_{m-1})}$ which satisfies

$$\begin{aligned}\|\Gamma_{t_m}^{(l_m)} - \Gamma_{t_{m-1}}^{(l_{m-1})}(\delta t)\| &\leq 2^{\frac{l_{m-1}}{k-1}} \frac{(\kappa\delta t/2)^{\frac{l_m - l_{m-1}}{k-1}}}{1-\kappa\delta t/2} \|\Gamma_{t_{m-1}}^{(l_{m-1})}\| \\ &\leq 2^{\frac{2l_{m-1}-l_m}{k-1}+1} \|\Gamma_{t_{m-1}}^{(l_{m-1})}\|, \end{aligned} \tag{6.31}$$

where the second inequality came from $\kappa\delta t/2 \leq 1/2$ because of the definition (6.13); note that the function $x^{\frac{l_m - l_{m-1}}{k-1}}/(1-x)$ monotonically increases for $0 \leq x < 1$.

We now define a positive integer δ_l such that

$$l_m = 2l_{m-1} + \delta_l \tag{6.32}$$

for $m = 1, 2, \ldots, n$. We then obtain

$$l_n = 2^n(l_0 + \delta_l) - \delta_l. \tag{6.33}$$

Because of the condition $l_n \le l$, we have to take δ_l so that it may satisfy the inequality

$$2^n(l_0+\delta_l)-\delta_l \le l, \quad \text{or} \quad \delta_l \le \frac{l-2^n l_0}{2^n-1}. \tag{6.34}$$

Based on this inequality, we choose δ_l as $\lfloor (l-2^n l_0)/(2^n-1)\rfloor$. By combining the inequality (6.31) with the definition $\delta_l \equiv l_m - 2l_{m-1}$, we finally obtain

$$\begin{aligned}\left\|\Gamma_{t_m}^{(l_m)}-\Gamma_{t_{m-1}}^{(l_{m-1})}(\delta t)\right\| &\le 2^{-\frac{1}{k-1}\lfloor \frac{l-2^n l_0}{2^n-1}\rfloor+1}\left\|\Gamma_{t_{m-1}}^{(l_{m-1})}\right\| \\ &\le 4\|\Gamma_{(m-1)t}^{(x_{m-1})}\|\exp\left(-\log 2\frac{l/(2^n-1)-l_0}{k-1}\right).\end{aligned} \tag{6.35}$$

We here notice the equality

$$\Delta = 4\exp\left[-\frac{1}{\xi}\left(\frac{l}{l(t)}-l_0\right)\right] = 4\exp\left(-\log 2\frac{l/(2^n-1)-l_0}{k-1}\right), \tag{6.36}$$

because of the definitions of ξ, $l(t)$ and n as in Theorem 6.2 and Eq. (6.12). We thus prove the inequality (6.27) from (6.35) and (6.36).

6.3 The stability of the topological order

We here prove that the topological order is stable after the time evolution over $t=\mathcal{O}(1)$. In Subsection 6.3.1, we first introduce how to characterize the quantum topological order; we follow the same discussion as in Ref. [29]. In Subsection 6.3.2, we review the stability of the topological order after the time evolution due to short-range interacting Hamiltonians, where we can apply the Lieb-Robinson bound. In Subsection 6.3.3, we apply our new bound (6.8) to discuss the stability of the topological order after the time evolution due to k-local Hamiltonians.

6.3.1 Definition of the topological order

We here introduce the topological quantum order. The concept of the topological order is usually defined with respect to Hamiltonians rather than quantum states. However, we have several common properties which the topological ordered phases always satisfy. In Ref. [29], S. Bravyi *et al.* define that a quantum state has the quantum topological order if it has particular properties.

Slightly generalizing the definition in Ref. [29], we here define that a quantum state $|\psi\rangle$ exhibits the topological order if and only if there exists another quantum state $|\tilde{\psi}\rangle$ which satisfies

$$\langle\psi|\Gamma^{(l)}|\psi\rangle = \langle\tilde{\psi}|\Gamma^{(l)}|\tilde{\psi}\rangle \quad \text{and} \quad \langle\psi|\Gamma^{(l)}|\tilde{\psi}\rangle = 0 \tag{6.37}$$

for arbitrary l-local operators $\Gamma^{(l)}$ with $l=\mathcal{O}(N^q)$ and $0<q<1$ (in Ref. [29], they consider spatially local operator $O_{\rm loc}$ instead of l-local operators). If the quantum state satisfies the property, the coherence between the two states $|\psi\rangle$ and $|\tilde{\psi}\rangle$ can never be broken by any kinds of local operators. It is known that topologically ordered phases usually satisfy these conditions, for example the ground states of the Kitaev's toric code model [33].

In considering the stability of the topological order, we define the topological order with error (l,ϵ_l) as

$$|\langle\psi|\Gamma^{(l)}|\psi\rangle - \langle\tilde{\psi}|\Gamma^{(l)}|\tilde{\psi}\rangle| \le \epsilon_l \quad \text{and} \quad |\langle\psi|\Gamma^{(l)}|\tilde{\psi}\rangle| \le \epsilon_l, \tag{6.38}$$

where $\|\Gamma^{(l)}\| = 1$. For exactly topologically ordered states, we have $\epsilon_l = 0$ for $l \lesssim N^q$. We will see that after small-time evolution the error ϵ_l can be small sub-exponentially with respect to the system size.

6.3.2 Stability of the topological order: time evolution by short-range interacting Hamiltonians

We first discuss the time-dependence of the error (l, ϵ_l) by the short-range interacting Hamiltonians [29]. Let $(l, \epsilon_l(t))$ denote the error after the time evolution. We now consider a d-dimensional lattice system. For simplicity, we assume that the initial state $|\psi\rangle$ has the exact topological order:

$$\epsilon_l(0) = 0 \quad \text{for} \quad l \le l_0, \tag{6.39}$$

where $l_0 = \mathcal{O}(N^q)$ and $0 < q < 1$. We can prove that the error $(l, \epsilon_l(t))$ is bounded from above by

$$\epsilon_l(t) \le 2cl \cdot \exp\left(\frac{-(l_0/l)^{\frac{1}{d}} + v|t|}{\xi}\right) \quad \text{for} \quad \forall l \le l_0. \tag{6.40}$$

In order to prove the inequality (6.40), we apply the Lieb-Robinson bound and obtain the following inequalities [29]:

$$\min_{\Gamma^{(l)} \in \mathcal{U}(l)} \left(\|\Gamma^{(l)}(t) - \Gamma_t^{(l_0)}\|\right) \le cl \cdot \exp\left(\frac{-(l_0/l)^{\frac{1}{d}} + v|t|}{\xi}\right), \tag{6.41}$$

where $\|\Gamma^{(l)}\| = 1$ and c is a positive constant. Because the quantum state $|\psi\rangle$ initially has the topological order as in Eq. (6.39), we have

$$\begin{aligned}
|\langle\psi|e^{-iHt}\Gamma^{(l)}e^{iHt}|\tilde{\psi}\rangle| &\le |\langle\psi|\Gamma_t^{(l_0)}|\tilde{\psi}\rangle| + |\langle\psi|(\Gamma^{(l)}(t) - \Gamma_t^{(l_0)})|\tilde{\psi}\rangle| \\
&\le |\langle\psi|\Gamma_t^{(l_0)}|\tilde{\psi}\rangle| + cl \cdot \exp\left(\frac{-(l_0/l)^{\frac{1}{d}} + v|t|}{\xi}\right) \\
&\le cl \cdot \exp\left(\frac{-(l_0/l)^{\frac{1}{d}} + v|t|}{\xi}\right),
\end{aligned} \tag{6.42}$$

where we utilized the equality $|\langle\psi|\Gamma_t^{(l_0)}|\tilde{\psi}\rangle| \le \epsilon_{l_0}(0) = 0$. Similarly, we can obtain

$$\begin{aligned}
&|\langle\psi|e^{-iHt}\Gamma^{(l)}e^{iHt}|\psi\rangle - \langle\tilde{\psi}|e^{-iHt}\Gamma^{(l)}e^{iHt}|\tilde{\psi}\rangle| \\
\le&|\langle\psi|\Gamma_t^{(l_0)}|\psi\rangle - \langle\tilde{\psi}|\Gamma_t^{(l_0)}|\tilde{\psi}\rangle| + 2cl \cdot \exp\left(\frac{-(l_0/l)^{\frac{1}{d}} + v|t|}{\xi}\right) \\
=&2cl \cdot \exp\left(\frac{-(l_0/l)^{\frac{1}{d}} + v|t|}{\xi}\right).
\end{aligned} \tag{6.43}$$

We therefore prove the inequality (6.40) from the definition (6.38).

6.3.3 Stability of the topological order: time evolution by general k-local Hamiltonians

We can apply the same discussion as in the previous subsection when the time evolution is governed by general k-local Hamiltonians. In this case, we can obtain the bound (6.8) in Theorem 6.2 instead of the

Lieb-Robinson bound (6.41). From the similar inequalities to (6.42) and (6.43), we can obtain

$$\epsilon_l(t) \le 2\lceil \kappa t \rceil \exp\left[-\frac{1}{\xi}\left(\frac{l_0}{l(t)} - l\right)\right]. \tag{6.44}$$

If we consider the range $l \le l_0/[2l(t)]$, the error $\epsilon_l(t)$ is bounded from above by

$$\mathcal{O}(t) \cdot \exp\left(-\frac{l_0}{e^{\mathcal{O}(t)}}\right), \tag{6.45}$$

where we utilized the form of $l(t) = e^{\mathcal{O}(t)}$. This means that the error of the topological order is small exponentially with respect to the value of l_0 as long as $l \lesssim l_0 e^{-\mathcal{O}(t)}$.

6.4 Exponential concentration of distribution

We here consider a product state $|\mathrm{Prod}\rangle$ and its time evolution $e^{-iHt}|\mathrm{Prod}\rangle$. Because the time evolution maintains the locality of operators, we expect that the state $e^{-iHt}|\mathrm{Prod}\rangle$ cannot acquire macroscopic properties. From the discussion in Subsection 6.3.3, the state $e^{-iHt}|\mathrm{Prod}\rangle$ cannot have the topological order.

In this section, we search another kind of restriction on the state $e^{-iHt}|\mathrm{Prod}\rangle$; we prove the exponential concentration of spectral distribution of a k-local SC-operator A:

$$A \equiv \sum_{m=1}^{n_A} \frac{A_m^{\mathrm{c}}}{n_A} \tag{6.46}$$

with

$$A_m^{\mathrm{c}} = \sum_{|X|\le k} a_X^{(m)} \quad \text{with} \sum_{X:X\ni i} \|a_X^{(m)}\| \le g \quad \forall i. \tag{6.47}$$

Theorem 6.4. Let $\Pi^A_{[x,x')}$ as the projection operator onto the subspace of the eigenvalues of A which are in $[x,x')$. The spectrum of A in $|\mathrm{Prod}(t)\rangle$ is exponentially concentrated as

$$\|\Pi^A_{[x_m+\Delta x,\infty)}|\mathrm{Prod}(t)\rangle\| \le \mathrm{const}\cdot\exp\left(-\frac{\Delta x}{c(t)\sqrt{N}}\right), \tag{6.48}$$

where $c(t) \le \mathrm{const}\cdot l(t)\log l(t)$ and x_m is the median point such that $\Pi_{[x_m,\infty)}|\mathrm{Prod}(t)\rangle \ge 1/2$ and $\Pi_{(-\infty,x_m]}|\mathrm{Prod}(t)\rangle \ge 1/2$.

Proof of Theorem 6.4. For the proof, we prove the following inequality

$$\|\Pi^A_{(-\infty,x]}P(t)\Pi^A_{[x+\Delta x,\infty)}\| \le \mathrm{const}\cdot\exp\left(-\frac{\Delta x}{c(t)\sqrt{N}}\right), \tag{6.49}$$

where $P(t) \equiv |\mathrm{Prod}(t)\rangle\langle\mathrm{Prod}(t)|$. This inequality reduces to the inequality (6.48) by choosing $x = x_m$. Note that we have the equality

$$\begin{aligned}\|\Pi^A_{(-\infty,x_m]}P(t)\Pi^A_{[x_m+\Delta x,\infty)}\| &= \|\Pi^A_{(-\infty,x_m]}|\mathrm{Prod}(t)\rangle\|\cdot\|\langle\mathrm{Prod}(t)|\Pi^A_{[x_m+\Delta x,\infty)}\| \\ &= \frac{1}{2}\|\Pi^A_{[x_m+\Delta x,\infty)}|\mathrm{Prod}(t)\rangle\| \end{aligned} \tag{6.50}$$

In order to prove the inequality (6.49), we follow the same discussion as in the proof of Theorem 5.4.

We first define H_P as a 1-local Hamiltonian which has the state $|\mathrm{Prod}\rangle$ as its non-degenerate ground state with a unit gap:

$$H_P|\mathrm{Prod}\rangle = 0 \quad \text{and} \quad \|H_P(1-|\mathrm{Prod}\rangle\langle\mathrm{Prod}|)\| \ge 1. \tag{6.51}$$

For example, if we consider $|\mathrm{Prod}\rangle = |000\cdots 0\rangle$, the Hamiltonian H_P is given by

$$H_P(t) = \sum_{i=1}^{N}(1-P_0^{(i)}), \tag{6.52}$$

where $P_0^{(i)}$ is the projection operator of the ith spin space onto $|0\rangle\langle 0|$ for $i = 1, 2, \ldots, N$.

Then, the quantum state $|\mathrm{Prod}(t)\rangle$ is also the gapped ground state of the Hamiltonian $H_P(t)$. We, however, cannot apply the proof of Theorem 5.4 directly because $H_P(t)$ is no longer a k-local operator with $k = \mathcal{O}(1)$.

The outline of the proof is the same as that of Theorem 5.4. We first appropriate $P(t)$ by an nth order polynomial of the Hamiltonian $H_P(t)$, which we denote by $K_n[H_P(t)]$. We then obtain

$$\begin{aligned}\|\Pi^A_{(-\infty,x]}P(t)\Pi^A_{[x+\Delta x,\infty)}\| =& \|\Pi^A_{(-\infty,x]}\bigl(P(t)-K_n[H_P(t)]+K_n[H_P(t)]\bigr)\Pi^A_{[x+\Delta x,\infty)}\| \\ \le& \|P(t)-K_n[H_P(t)]\| + \|\Pi^A_{(-\infty,x]}K_n[H_P(t)]\Pi^A_{[x+\Delta x,\infty)}\|. \end{aligned} \tag{6.53}$$

If we can approximate $P(t)$ by the operator $K_n[H_P(t)]$ with small n, we can also do so by a local operator. Because local operators cannot cause a global spectral change, the term $\|\Pi^A_{(-\infty,x]}P(t)\Pi^A_{[x+\Delta x,\infty)}\|$ can be strongly bounded from above for an appropriate choice of n. Qualitatively, the first term, which describes the error of the approximation of $P(t)$, can be small as n increases, while the second term $\|\Pi^A_{(-\infty,x]}K_n[H_P(t)]\Pi^A_{[x+\Delta x,\infty)}\|$ becomes large as n increases because $K_n[H_P(t)]$ is far from a local operator for large n. We therefore choose n so that this trade-off relationship may be optimized.

We here use the nth-order Chebyshev polynomial for $K_n(x)$:

$$K_n(x) = \frac{T_n(\frac{x-1}{E_c}-1)}{T_n(\frac{-1}{E_c}-1)} \tag{6.54}$$

with

$$2E_c + 1 = \|H_P\| = N. \tag{6.55}$$

For $K_n[H_P(t)]$, we later prove the inequalities

$$\|P(t)-K_n[H_P(t)]\| \le 2e^{-2n/\sqrt{N}}, \tag{6.56}$$

$$\|\Pi^A_{(-\infty,x]}K_n[H_P(t)]\Pi^A_{[x+\Delta x,\infty)}\| \le e^{-\beta\Delta x - 2n/\sqrt{N} + c_1 n}, \tag{6.57}$$

where the parameters β and c_1 are defined by

$$\beta \equiv \frac{1}{24g}\frac{1}{\xi l(t)} = \frac{1}{\kappa}\cdot\frac{\log 2}{l(t)}, \tag{6.58}$$

$$c_1 \equiv \log\Bigl(16e^{1/\xi}\lceil\kappa t\rceil\bigl[4\xi l(t)+1\bigr]+1\Bigr) + \log\frac{4}{1-N^{-1}}. \tag{6.59}$$

We here choose $n = \lfloor \beta \Delta x / c_1 \rfloor$, which reduces the inequality (6.53) to

$$\|\Pi^A_{(-\infty,x]} P(t) \Pi^A_{[x+\Delta x,\infty)}\| \le 3e^{2/\sqrt{N}} \cdot \exp\biggl(-\frac{\Delta x}{c(t)\sqrt{N}}\biggr) \tag{6.60}$$

with

$$c(t) \equiv \frac{c_1}{2\beta} = 12g\xi l(t)\biggl[\log\Bigl(16e^{1/\xi}\lceil \kappa t\rceil\bigl[4\xi l(t)+1\bigr]+1\Bigr) + \log\frac{4}{1-N^{-1}}\biggr]. \tag{6.61}$$

This completes the proofs of the inequality (6.53) and Theorem 6.4. $\square$

6.4.1 The proof of the inequality (6.56)

The first inequality (6.56) can be derived from the basic property of the Chebyshev polynomial. The definition of the polynomial $K_n(x)$ gives

$$P(t)K_n[H_P(t)] = P(t)K_n(0) = P(t) \tag{6.62}$$

because $H_P(t)|\mathrm{Prod}(t)\rangle = 0$ and $K_n(0) = 1$. We then utilize the property of $T_n(x) \le 1$ for $|x| \le 1$ and obtain

$$\|P(t) - K_n[H_P(t)]\| = \|[1-P(t)]K_n[H_P(t)]\| = \max_{1\le x\le \|H_P\|}\bigl[K_n(x)\bigr] \le \frac{1}{T_n(\frac{-1}{E_c}-1)}, \tag{6.63}$$

where the first equality came from Eq. (6.62), while in the last inequality, we utilized the inequality $\bigl|(x-1)/E_c - 1\bigr| \le 1$ for $1 \le x \le \|H_P\|$. We then utilize the inequality (4.23) and have

$$\frac{1}{T_n(\frac{-1}{E_c}-1)} \le 2\exp\biggl(-2n\sqrt{\frac{1}{2E_c+1}}\biggr) = 2\exp\biggl(\frac{-2n}{\sqrt{N}}\biggr), \tag{6.64}$$

where the last equality came from the definition of E_c in Eq. (6.55).

6.4.2 The proof of the inequality (6.57)

We then prove the inequality (6.57). We follow the same calculation as in Subsection 5.4.5. We first obtain

$$\begin{aligned}\|\Pi^A_{(-\infty,x]}K_n[H_P(t)]\Pi^A_{[x+\Delta x,\infty)}\| &\le \|\Pi^A_{(-\infty,x]}e^{\beta A}e^{-\beta A}K_n[H_P(t)]e^{\beta A}e^{-\beta A}\Pi^A_{[x+\Delta x,\infty)}\| \\ &\le e^{-\beta\Delta x}\|\Pi^A_{(-\infty,x]}K_n\bigl[e^{-\beta A}H_P(t)e^{\beta A}\bigr]\Pi^A_{[x+\Delta x,\infty)}\| \\ &\le e^{-\beta\Delta x}\|K_n\bigl[e^{-\beta A}H_P(t)e^{\beta A}\bigr]\|,\end{aligned} \tag{6.65}$$

where β is defined in (6.58). Note that β satisfies

$$\beta \le \frac{1}{\kappa} \tag{6.66}$$

because $\log 2/l(t) \le 1$. We, in the following, calculate the upper bound of $\|K_n(e^{-\beta A}H_P(t)e^{\beta A})\|$. Because the nth order Chebyshev polynomial has the form of

$$T_n(x) = 2^{n-1}\Pi_{j=1}^n (x - x_j) \quad \text{with} \quad |x_j| \le 1, \tag{6.67}$$

we obtain the upper bound of $\|K_n(e^{-\beta A}H_P(t)e^{\beta A})\|$ as

$$\begin{aligned}\|K_n\big[e^{-\beta A}H_P(t)e^{\beta A}\big]\| &\le \frac{2^{n-1}}{T_n(\frac{-1}{E_c}-1)}\left(\frac{\|e^{-\beta A}H_P(t)e^{\beta A}\|+1}{E_c}+2\right)^n \\ &\le 2^n e^{-2n/\sqrt{N}}\left(\frac{\|e^{-\beta A}H_P(t)e^{\beta A}\|+1+2E_c}{E_c}\right)^n \\ &= e^{-2n/\sqrt{N}}\left(\frac{4\|e^{-\beta A}H_P(t)e^{\beta A}\|+4N}{N-1}\right)^n, \end{aligned} \tag{6.68}$$

where the second inequality came from (6.64) and we utilized the definition of E_c in the last equality.

In Subsection 6.4.2 (a) below, we will prove the inequality

$$\|e^{-\beta A}H_P(t)e^{\beta A}\| \le 16Ne^{1/\xi}\lceil\kappa t\rceil\big[4\xi l(t)+1\big]. \tag{6.69}$$

This reduces the inequality (6.68) to

$$\|K_n\big[e^{-\beta A}H_P(t)e^{\beta A}\big]\| \le e^{-2n/\sqrt{N}}e^{c_1 n}, \tag{6.70}$$

where c_1 is defined in (6.59). By combining the inequalities (6.65) and (6.70), we obtain the inequality (6.57).

6.4.2 (a) The derivation of the inequality (6.69)

For the calculation, we first decompose $H_P(t)$ as follows:

$$H_P(t) = \sum_{l=1}^{N-1}(H_{P,t}^{(l+1)} - H_{P,t}^{(l)}), \tag{6.71}$$

where $H_{P,t}^{(l)}$ is an appropriate l-local operator which approximates the Hamiltonian $H_P(t)$; note that we have $H_{P,t}^{(N)} = H_P(t)$. Because of Theorem 6.2, we can find $H_{P,t}^{(l)}$ such that

$$\|H_P(t) - H_{P,t}^{(l)}\| \le 8\|H_P\|\lceil\kappa t\rceil \exp\left[-\frac{1}{\xi}\Big(\frac{l}{l(t)}-1\Big)\right], \tag{6.72}$$

where we utilized that H_P is a 1-local operator. We then obtain

$$\begin{aligned}\|H_{P,t}^{(l+1)} - H_{P,t}^{(l)}\| &\le \|H_{P,t}^{(l+1)} - H_P(t)\| + \|H_{P,t}^{(l)} - H_P(t)\| \\ &\le 16\|H_P\|\lceil\kappa t\rceil \exp\left[-\frac{1}{\xi}\Big(\frac{l}{l(t)}-1\Big)\right]. \end{aligned} \tag{6.73}$$

We next calculate

$$e^{-\beta A}(H_{P,t}^{(l+1)} - H_{P,t}^{(l)})e^{\beta A} = \sum_{m=0}^{\infty}\frac{(-\beta)^m}{m!}O_m, \tag{6.74}$$

where $O_0 = H_{P,t}^{(l+1)} - H_{P,t}^{(l)}$ and $O_m = [\cdots[[O_0, \overbrace{H], H], \ldots], H]}^{m}$. Note that O_0 is now an $(l+1)$-local operator. From the same calculations as in Subsection 3.3.2, we can obtain

$$\|O_m\| \le \big[6g(k-1)\big]^m\|O_0\|\cdot r(r+1)(r+2)\cdots(r+m-1), \tag{6.75}$$

where $r \equiv (l+1)/(k-1)$ and we utilized the SC-condition of the operator A.

Because we have the bounds $r(r+1)(r+2)\cdots(r+m-1) \le (2r)^m$ for $r \ge m$ and $r(r+1)(r+2)\cdots(r+m-1) \le m!2^{r+m}$ for $r \le m$[*3], we obtain

$$
\begin{aligned}
\|e^{-\beta A}(H_{P,t}^{(l+1)} - H_{P,t}^{(l)})e^{\beta A}\| &\le \|O_0\| \sum_{m\le r} \frac{\bigl[12gr\beta(k-1)\bigr]^m}{m!} + 2^r \|O_0\| \sum_{m>r} \bigl[12g\beta(k-1)\bigr]^m \\
&\le \|O_0\| \biggl(e^{12g\beta(l+1)} + 2^r \frac{\bigl[\beta\kappa/2\bigr]^{\lceil r \rceil}}{1-\beta\kappa/2} \biggr) \\
&\le \|O_0\| \biggl[\exp\Bigl(\frac{l+1}{2\xi l(t)} \Bigr) + 2 \biggr],
\end{aligned} \tag{6.76}
$$

where we utilized the definition of β in (6.58) and the inequality (6.66).

By combining the inequalities (6.73) and (6.76), we have

$$
\|e^{-\beta A}(H_{P,t}^{(l+1)} - H_{P,t}^{(l)})e^{\beta A}\| \le 16\|H_P\| \lceil \kappa t \rceil \exp\biggl[-\frac{1}{\xi}\Bigl(\frac{l}{l(t)} - 1 \Bigr) \biggr] \biggl[\exp\Bigl(\frac{l+1}{2\xi l(t)} \Bigr) + 2 \biggr]. \tag{6.77}
$$

We finally obtain

$$
\begin{aligned}
\|e^{-\beta A} H_P(t) e^{\beta A}\| &\le \sum_{l=1}^{\infty} \|e^{-\beta A}(H_{P,t}^{(l+1)} - H_{P,t}^{(l)})e^{\beta A}\| \\
&\le 16 e^{1/\xi} \|H_P\| \lceil \kappa t \rceil \sum_{l=1}^{\infty} \biggl[\exp\Bigl(\frac{-l+1}{2\xi l(t)} \Bigr) + 2\exp\Bigl(\frac{-l}{\xi l(t)} \Bigr) \biggr] \\
&= 16 e^{1/\xi} \lceil \kappa t \rceil \|H_P\| \biggl(\frac{1}{1-e^{-1/[2\xi l(t)]}} + \frac{2}{e^{1/[\xi l(t)]} - 1} \biggr) \\
&\le 16 e^{1/\xi} \lceil \kappa t \rceil \bigl[4\xi l(t) + 1 \bigr] \|H_P\|,
\end{aligned} \tag{6.78}
$$

where we utilized the inequalities $1/(1-e^{-1/x}) \le 1+x$ and $1/(e^{1/x}-1) \le x$. We thus prove the inequality (6.69) from the equality $\|H_P\| = N$.

[*3] This inequality comes from (3.47).

Chapter 7

Summary and future works

7.1 Summary

We summarize the main implications of Chapters 3, 4, 5 and 6.[*1]

1. (Chapter 3) The k-locality of the Hamiltonian appears in the form of Theorem 3.2, where the energy excitation due to an external operator is exponentially suppressed. Almost all the results in the following chapters are based on this statement and the related Theorem 3.4 on the effective Hamiltonian.

2. (Chapter 4) We have introduced the new concept named by the local reversibility (Definition 4.2.1). We expect that the local reversibility restricts global quantum phenomena; it comes from the inference that the global quantumness cannot be recovered by local operations once it has been broken. We have also proved the local reversibility for gapped ground states in general k-local Hamiltonians. As we have shown in Section 4.3, the local reversibility can be a new indicator of the locality of the ground states. As applications of the local reversibility, we have obtained the fundamental inequality for the critical exponents and the quantitative estimation of the mean-field approximation.

3. (Chapter 5) In Chapter 4, we have also proved the exponential suppression of the macroscopic superposition. We have generalized the results in the following senses: first, we consider a low-energy states instead of the exact ground state. Second, we have generalized additive one-spin operators to the general k-local operators. Theorem 5.4 gives the complete generalization of the exponential suppression of the macroscopic superposition.

4. (Chapter 6) As a new fundamental inequality for the time-evolution, we have proved Theorem 6.2, which is on the accuracy of the operator approximation; if an operator is originally l_0-local, after time-evolution, the operator can be approximated by an $l_0 e^{\mathcal{O}(t)}$-local operator. By following the same discussion as in the case of the short-range interacting systems, we have been able to prove the stability of the topological order for the time evolution by the k-local Hamiltonians. We also consider the time-evolution of the product state, namely $|\mathrm{Prod}(t)\rangle$ with $t = \mathcal{O}(1)$. Such a state is classified into the short-range entangled (SRE) state if the time-evolution is due to a short-range Hamiltonian. In our cases, because the time-evolution is due to a k-local Hamiltonian, the state $|\mathrm{Prod}(t)\rangle$ belongs to a broader class than SRE. For this state, we have shown Theorem 6.4 on the exponential concentration of the distributions.

[*1] For the outline of each of the chapters, see Section 1.1 in Chapter 1.

7.2 Future works

We have left several open problems for future works.

1. (Chapter 3)

   (a) The most fundamental question is whether we can completely grasp the k-locality on the basis of Theorem 3.2. All of the present results rely on Theorem 3.2. Hence, if we find essentially different inequalities on the k-locality from that in Chapter 3, we may be able to use them to obtain completely new properties of the ground states.

   (b) We may be able to improve Theorem 3.2 from the form

$$\|\Pi^H_{[E',\infty)}\Gamma_L\Pi^H_{(-\infty,E]}\| \le \|\Gamma_L\|e^{-\lambda(E'-E-3g|L|)} \tag{7.1}$$

   to the form

$$\|\Pi^H_{[E',\infty)}\Gamma_L\Pi^H_{(-\infty,E]}\| \le \min\bigl(\|\Gamma_L\Pi^H_{(-\infty,E]}\|, \|\Pi^H_{[E',\infty)}\Gamma_L\|\bigr)e^{-\lambda(E'-E-3g|L|)}. \tag{7.2}$$

   In the present form (7.1), even if a state $|\psi_{(-\infty,E]}\rangle$ has the energy below E, after local operations Γ_L which satisfy $\|\Gamma_L|\psi_{(-\infty,E]}\rangle\| = 1$ and $\|\Gamma_L\| = e^{\mathcal{O}(N)}$, the state $\Gamma_L|\psi_{(-\infty,E]}\rangle$ can have an arbitrary high energy. If we can achieve this improvement, we can thereby strengthen Theorem 4.3 on the local reversibility.

   (c) Can we prove the equivalence of the extensiveness and the SC-condition (see Section 1.2 in Chapter 1)? It is important to improve Theorem 3.2 to the form in Lemma 3.3.2, namely

$$\|\Pi^H_{[E',\infty)}\Gamma^{(l)}\Pi^H_{(-\infty,E]}\| \le \|\Gamma^{(l)}\|e^{-\mathrm{const}(E'-E-\mathrm{const}\cdot l)}. \tag{7.3}$$

   As shown in Subsection 3.3.2, we cannot derive the inequality (7.3) directly from the inequality (7.1).

2. (Chapter 4)

   (a) Can we prove that the locally indistinguishable states always break the local reversibility? The locally indistinguishable states are defined as follows; if a state $|\psi\rangle$ is locally indistinguishable, there exists an orthogonal state $|\tilde{\psi}\rangle$ such that

$$\langle\psi|\Gamma_L|\psi\rangle = \langle\tilde{\psi}|\Gamma_L|\tilde{\psi}\rangle \tag{7.4}$$

   for $\forall\Gamma_L$ with $|L| = \mathcal{O}(1)$. This is a weaker expression of the topological order in Section 6.3.1. If we can prove that the state $|\psi\rangle$ is never locally reversible, we can prove a much stronger statement than that in Subsection 4.6.2,[*2] which implies the strong suggestion that no global entanglement cannot appear in the locally reversible states.

   (b) Can we obtain a tighter bound than in Theorem 4.3? The local reversibility in the ground states is roughly given by

$$\|R\Gamma_L|E_0\rangle - |E_0\rangle\| \le \frac{\|\Gamma_L\|}{|\langle E_0|\Gamma_L|E_0\rangle|}\exp\left(-k_R\sqrt{\frac{c_2\delta E}{c_1k_R+|L|}}\right) \tag{7.5}$$

   with c_1 and c_2 being constants of $\mathcal{O}(1)$. The error of the reverse operator thus behaves as $e^{-\mathcal{O}(k_R)/\sqrt{|L|}}$ for $k_R \lesssim |L|$, while it behaves as $e^{-\mathcal{O}(\sqrt{k_R})}$ for $k_R \gg |L|$. One possibility of the

[*2] In Subsection 4.6.2, we have only considered the ground states in Kitaev's toric code model.

improvement is

$$\|R\Gamma_L|E_0\rangle - |E_0\rangle\| \le \frac{\|\Gamma_L\|}{|\langle E_0|\Gamma_L|E_0\rangle|} \exp\left(-k_R\sqrt{\frac{c_2\delta E}{|L|}}\right), \tag{7.6}$$

which always gives the error as $e^{-\mathcal{O}(k_R)/\sqrt{|L|}}$. The improvement of the reverse error is deeply related to the construction of the approximate ground state projector (AGSP). If we can improve the error as in (7.6), we will be able to apply this result to obtain a non-trivial upper bound for the entanglement entropy in higher-dimensional systems.

(c) Can we utilize this result to improve Brandao's proof of the entropic area law? In Ref. [16], F. Brandao, *et al.* have proved the one-dimensional area law only from the exponential decay of bi-partite correlations. However, the upper bound of the entanglement entropy exponentially increases with the correlation length, which would mean that the bound becomes the volume law even if the correlation length is of order $\mathcal{O}(\log N)$. We may be able to improve the upper bound by combining the exponential decay of bi-partite correlations with the local reversibility.

3. (Chapter 5)
(a) Most importantly, can we improve the upper bound in Theorem 5.4 to a Gaussian form? We now obtain the upper bound in the form of

$$\|\Pi^A_{[-\infty,x)} P_{\mathrm{low}} \Pi^A_{[x+\Delta x,\infty)}\| \le e^{-\Delta x/\mathcal{O}(\sqrt{|L|})}, \tag{7.7}$$

where P_{low} denotes the low-energy space projection. We have a possibility that we can improve it as

$$\|\Pi^A_{[-\infty,x)} P_{\mathrm{low}} \Pi^A_{[x+\Delta x,\infty)}\| \le e^{-\Delta x^2/\mathcal{O}(|L|)}. \tag{7.8}$$

For example, the product state always satisfies the Gaussian inequality at least in the case where A is a one-spin additive operator $A = \sum_{i\in L} a_i$. In the context of the central limit theorem, we can indeed obtain the Gaussian bound for some quantum systems.

4. (Chapter 6)
(a) The most essential open problem is whether Theorem 6.2 holds without the SC condition for the Hamiltonian (see Subsection 1.2.4). We conjecture that we only need the extensiveness of the Hamiltonian.

(b) Let us assume that an l_0-local operator $\Gamma^{(l_0)}$ satisfies the extensiveness. Then, can we prove that the extensiveness of the l_0-local operator $\Gamma^{(l_0)}$ may be conserved after a time evolution? So far, we have only ensured that after a time evolution the operator is approximately $\left(e^{\mathcal{O}(t)} l_0\right)$-local, but we do not know detailed structures of the approximated operators. We conjecture that the extensiveness should be conserved, whereas we have not obtained any useful techniques for this problem yet.

(c) Can we apply the present results to the adiabatic continuation for the k-local Hamiltonians[*3]? We cannot directly apply pre-existing theories [38–40] for the short-range Hamiltonians to the k-local Hamiltonians; because of the equality (2.24), the adiabatic continuation operator is no longer a k-local operator with $k = \mathcal{O}(1)$. In other words, we cannot ignore the $\mathcal{O}(N)$-local terms. In fact, if we utilize the same technique in Chapter 4, we can construct an approximate adiabatic

[*3] See Section 2.4 in Chapter 2 on the adiabatic continuation.

continuation operator which is l-local with $l = \mathcal{O}(1)$. However, the norm of the operator rapidly increases with l and we cannot make efficient use of it[*4].

(d) Can we prove the local reversibility for the time-evolution of the product state, namely $|\mathrm{Prod}(t)\rangle$ with $t = \mathcal{O}(1)$? More importantly, can we find an efficient description of the state $|\mathrm{Prod}(t)\rangle$, for example, by the use of the tensor network state [27]?

(e) Recently, experimental observations of the Lieb-Robinson bound have been reported [30, 72–74]. In particular, Ref. [74] has demonstrated the Lieb-Robinson bound in long-range interacting systems. In the same experimental setup, our new bound in Theorem 6.2 will be observed. In order to observe our bound, we have to see the distribution of the observables, which should obey Theorem 6.4.

[*4]When the norm of the adiabatic continuation operator is large, the parameter evolution as in Eq. (2.22) may no longer maintain the locality.

Appendix A

Appendix for Chapter 3

A.1 The Weyl inequality

We here introduce the following Weyl inequality:

The Weyl inequality [70]. *Let A and $\tilde{A}$ be m by m Hermitian matrices, where we assume $A \geq \tilde{A}$. We denote the eigenvalues of A and $\tilde{A}$ by $\lambda_0 \leq \lambda_1 \leq x_2 \cdots$ and $\tilde{\lambda}_0 \leq \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \cdots$, respectively. We then obtain*

$$\lambda_j \geq \tilde{\lambda}_j \quad \text{for} \quad \forall j. \tag{A.1}$$

Proof of the Weyl inequality. Let P_j denote the projection operator onto the eigenspace of A in the range of $[\lambda_0, \lambda_j]$. We start from the following inequality:

$$P_j(A - \tilde{A})P_j \geq 0, \tag{A.2}$$

which comes from $A - \tilde{A} \geq 0$. Because we have $\|P_j A P_j\| \geq \|P_j \tilde{A} P_j\|$, we obtain

$$\lambda_j \geq \|P_j \tilde{A} P_j\|. \tag{A.3}$$

We therefore have to prove

$$\|P_j \tilde{A} P_j\| \geq \tilde{\lambda}_j \tag{A.4}$$

in order to obtain the inequality (A.1).

The proof of the inequality (A.4) is given as follows. We first construct a quantum state $|\psi\rangle$ by the use of the basis set $\{|\lambda_s\rangle\}_{s=0}^{j}$ as

$$|\psi\rangle = \sum_{s=0}^{j} \alpha_s |\lambda_s\rangle, \tag{A.5}$$

where $\{|\lambda_s\rangle\}_s$ are the eigenstates of A. If we can find $\{\alpha_s\}_{s=0}^{j}$ which makes $|\psi\rangle$ orthogonal to each of $\{|\tilde{\lambda}_s\rangle\}_{s=0}^{j-1}$, where $\{|\tilde{\lambda}_s\rangle\}_{s=0}^{j-1}$ are the eigenstates of $\tilde{A}$, the state $|\psi\rangle$ gives

$$\|P_j \tilde{A} P_j\| \geq \langle\psi|\tilde{A}|\psi\rangle = \langle\psi|(1 - \tilde{P}_{j-1})\tilde{A}(1 - \tilde{P}_{j-1})|\psi\rangle \geq \tilde{\lambda}_j, \tag{A.6}$$

where $\tilde{P}_{j-1}$ is the projection onto the eigenspace of $\tilde{A}$ in the range of $[\tilde{\lambda}_0, \tilde{\lambda}_{j-1}]$.

We finally have to prove the existence of $\{\alpha_s\}_{s=0}^{j}$ which gives a state $|\psi\rangle$ orthogonal to $\{|\tilde{\lambda}_s\rangle\}_{s=0}^{j-1}$. When the state $|\psi\rangle$ in Eq. (A.5) satisfies this condition, we can obtain

$$\alpha_0 \boldsymbol{v}_0 + \alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_j \boldsymbol{v}_j = 0, \tag{A.7}$$

where $\{\boldsymbol{v}_s\}_{s=0}^{j}$ are j-dimensional vectors defined by

$$\boldsymbol{v}_s \equiv \{\langle\tilde{\lambda}_0|\lambda_s\rangle, \langle\tilde{\lambda}_1|\lambda_s\rangle, \ldots, \langle\tilde{\lambda}_{j-1}|\lambda_s\rangle\}, \tag{A.8}$$

respectively. Because the vector $\boldsymbol{v}_s$ is j-dimensional, the set of states $\{\boldsymbol{v}_s\}_{s=0}^{j}$ is linearly dependent, and therefore there must exist a set of $\{\alpha_s\}_{s=0}^{j}$ which satisfies Eq. (A.7) with $\sum_{s=0}^{j} |\alpha_s|^2 \neq 0$.

This completes the proof of (A.4).

A.2 The Dyson expansion

In this section, we show the expansions of the operators $e^{-x(O_1+O_2)}$ and $e^{x(O_1+O_2)}$ [71], which give the equalities in (3.104). Let us consider the operator U_x given by $e^{-x(O_1+O_2)} \equiv e^{-xO_1}U_x$. Because of the equation

$$\frac{dU_x}{dx} = -O_2(x)U_x, \tag{A.9}$$

we have

$$U_x = 1 - \int_0^x dx_1 O_2(x_1) U_{x_1}, \tag{A.10}$$

where $O_2(x) \equiv e^{xO_1}O_2e^{-xO_1}$. By expanding this inequality sequentially, we have

$$\begin{aligned} U_x &= 1 - \int_0^x dx_1 O_2(x_1) U_{x_1} \\ &= 1 + \sum_{j=1}^{\infty} (-1)^j \int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{j-1}} dx_j O_2(x_1) O_2(x_2) \cdots O_2(x_j) U_{x_j}. \end{aligned} \tag{A.11}$$

For the operator $\tilde{U}_x$ defined by $e^{x(O_1+O_2)} \equiv \tilde{U}_x e^{xO_1}$, we also have

$$\tilde{U}_x = 1 + \int_0^{\beta} dx_1 \tilde{U}_{x_1} O_2(x_1). \tag{A.12}$$

We then similarly obtain

$$\tilde{U}_x = 1 + \sum_{j=1}^{\infty} \int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{j-1}} dx_j \tilde{U}_{x_j} O_2(x_j) O_2(x_{j-1}) \cdots O_2(x_1). \tag{A.13}$$

Appendix B

Appendix for Chapter 4

B.1 Optimality of $\sqrt{|L|}$ in the definition of the local reversibility

We here reconsider the definition of the local reversibility in Subsection 4.2.1, especially on the optimal size of k_R. We can prove that the necessary and sufficient size of k_R is $\sqrt{|L|}$ in the case of the product states. In order to prove it, we formally adopt the following form of the local reversibility:

$$\|R\Gamma_L|\psi\rangle - |\psi\rangle\| \le \frac{\|\Gamma_L\|}{|\langle\psi|\Gamma_L|\psi\rangle|} f\left(\frac{k_R}{|L|^\alpha}\right), \tag{B.1}$$

where $f(x)$ decays faster than any power-law decays. In this definition, the restoration error of the state $|\psi\rangle$ drops super-polynomially as k_R increases beyond $|L|^\alpha$. We here prove that the *minimal* size of k_R to reverse the product state $|\text{Prod}\rangle$ is given by $\sqrt{|L|}$ ($\alpha = 1/2$).

Proof of $\boldsymbol{\alpha \le 1/2}$. We have already proved it in Theorem 4.3.

Proof of $\boldsymbol{\alpha \ge 1/2}$. By following the same discussion as in Subsection 4.6.1 (a), we can prove a similar inequality to (4.54):

$$\|\Pi^A_{(-\infty,x]} P_{\text{Prod}} \Pi^A_{[x+\Delta x,\infty)}\| \le 2f\left(\frac{\lceil \Delta x/4\rceil - 1}{|L|^\alpha}\right) \quad \text{for} \quad \forall A_L, \tag{B.2}$$

where $\Delta x \ge 0$, $P_{\text{Prod}} = |\text{Prod}\rangle\langle\text{Prod}|$ and $A_L = \sum_{i\in L} a_i$. By choosing $x = x_m$ with x_m denoting the median point of the distribution, namely $\|\Pi^A_{(-\infty,x_m]}|\text{Prod}\rangle\| \ge 1/2$ and $\|\Pi^A_{[x_m,\infty)}|\text{Prod}\rangle\| \ge 1/2$, the inequality (B.2) reduces to

$$\|\Pi^A_{[x_m+\Delta x,\infty)}|\text{Prod}\rangle\| \le 4f\left(\frac{\lceil \Delta x/4\rceil - 1}{|L|^\alpha}\right) \quad \text{for} \quad \forall A_L. \tag{B.3}$$

Note that the same inequality holds for $\|\Pi^A_{(-\infty,x_m-\Delta x]}|\text{Prod}\rangle\|$.

The inequality (B.3) means that the distributions of *any* additive operators A_L are localized super-polynomially with the localization length $\mathcal{O}(|L|^\alpha)$. We therefore obtain

$$\langle(\Delta A_L)^2\rangle = \langle\text{Prod}|A_L^2|\text{Prod}\rangle - \langle\text{Prod}|A_L|\text{Prod}\rangle^2 \le \mathcal{O}(|L|^{2\alpha}) \quad \text{for} \quad \forall A_L.$$

For any product states, we can find an operator A_L such that

$$\langle(\Delta A_L)^2\rangle = \mathcal{O}(|L|). \tag{B.4}$$

Note that for an arbitrary product state we have

$$\langle(\Delta A_L)^2\rangle = \sum_{i\in L}\langle(\Delta a_i)^2\rangle. \tag{B.5}$$

For example, if we consider $|00\cdots 0\rangle$ with $\sigma^z|0\rangle = |0\rangle$, the magnetization along x-axis, namely $M_x = \sum_{i\in L}\sigma_i^x$, gives the variance of $\mathcal{O}(|L|)$. We thus prove the inequality $\alpha \geq 1/2$.

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