

学位論文

Dynamics of Scalar Condensation in the Early Universe

(初期宇宙における凝縮したスカラー場のダイナミクス)

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Dynamics of Scalar Condensation in the Early Universe

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Abstract

In this thesis, we have studied the time-evolution of the scalar condensation in the early Universe; with special emphasis on interactions between the scalar condensation and abundant particle-like excitations around it. Starting from the Kadanoff-Baym equations, which describe the evolution of one- and two-point functions, we have obtained coarse-grained equations under the “*separation of time scale*” assumption. By using the obtained equations, we have shown that these interactions can dramatically affect the dynamics of scalar condensation. In particular, we have revisited several important roles of scalar condensates in the early Universe, and shown how the characteristic features of each role are modified by these interactions.

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Chapter 1

Introduction

Scalar fields play many important roles in particle cosmology. In particular, their impacts on cosmology become significant, when they condensate homogeneously with large field values which are far from equilibrium. They start to oscillate coherently around their potential minima when the Hubble parameter becomes comparable to their mass scales. After the onset of oscillation, since they have large energy densities and tendency to dominate the Universe, the evolution of the Universe can be altered dramatically. There are several examples in which the scalar condensation plays essential roles in solving fundamental issues of our Universe, such as flatness/horizon problems, primordial density fluctuations, baryon asymmetry of our Universe (BAU), dark matter (DM) and so on.

- One of the prominent examples is the inflation [1–5], which not only solves horizon/flatness problems but also produces primordial density fluctuations observed in the cosmic microwave background (CMB) by an accelerated expansion of the Universe. Such an explosive expansion is caused by a potential energy of a scalar field, so-called *inflaton*, which slowly rolls down its potential during inflation [4, 5]. After the end of inflationary phase, the inflaton should convert its huge energy density into light particles including our Standard Model (SM) ones to create the hot Universe. This process is referred to as *reheating*.
- Though the inflaton can produce primordial density fluctuations, it is interesting to consider another possibility; a scalar field other than inflaton which acquires quantum fluctuations during inflation is responsible for primordial density perturbations [6–9]. Such a scalar field is referred to as the *curvaton*, and it has distinguishable features, such as non-Gaussianity, compared with inflaton. Eventually, the curvaton should decay into light particles so as to convert its fluctuations into the adiabatic perturbation in the radiation.
- If a scalar condensation is charged under $U(1)_{B-L}$ symmetry, then it can produce the BAU. For instance, in the *Affleck-Dine mechanism* [10, 11], the initial far from equilibrium field value of the scalar condensate, the enhanced $U(1)_{B-L}$ breaking term due to its large field value, and the initial misalignment of the phase direction of the scalar potential naturally satisfy the Sakharov’s three conditions [12]. In this case, the net BAU is produced when the scalar condensate starts to oscillate, and later it is converted to our SM particles via its decay. Another interesting case is to produce the BAU at its decay. For instance, in the case of Leptogenesis via the decay of right-handed sneutrino condensates [13–20], the Sakharov’s three conditions are fulfilled by the explicit breaking of $U(1)_{B-L}$ due to the Majorana mass term and its non-equilibrium C-/CP-violating decay.

- Also, the scalar field can be a candidate of DM. For instance, in the Peccei-Quinn (PQ) mechanism [21, 22], the condensation of the axion [23, 24], which is a pseudo NG-boson associated with the $U(1)_{\text{PQ}}$ breaking, can explain the present DM abundance [25]. In addition, the PQ scalar may cause a cosmological phase transition after the inflation. Then, the remnant of its radial component oscillation can produce hot axions, and also the phase transition might leave some detectable features [26, 27].

Importantly, from the viewpoint of particle physics, these scalar fields may be closely tied with models beyond the SM. For instance, it may be the Higgs field, which was recently discovered, flat directions in supersymmetric (SUSY) theory, SUSY breaking fields, moduli in string theory and PQ scalar fields and so on. Thus, it is of quite importance to understand the dynamics of scalar condensates and their cosmological fate in both cosmology and particle physics.

Nevertheless, the dynamics of scalar condensation in the early Universe has not been fully investigated. In most of these scenarios, the scalar field should decay into light particles (which somehow will be thermalized) before the big-bang nucleosynthesis (BBN) sets in not to overclose the Universe. Thus, an interaction term between the scalar field and light fields is required inevitably, and this very interaction makes the dynamics of scalar condensates complicated.

As an illustration, let us assume the simplest Yukawa coupling between the scalar field ϕ and light particles χ in radiation; $y\phi\bar{\chi}\chi$. Naively, one might guess that the scalar condensation decays completely, when the Hubble parameter (H), which characterizes the expansion rate of the Universe, becomes comparable to the decay rate of the scalar field (Γ_ϕ^0), which is computed in the Vacuum field theory: $H \simeq \Gamma_\phi^0 \sim y^2 m_\phi$. Here m_ϕ represents the mass of the scalar field.

However, for $y\tilde{\phi} \gg m_\phi$ with $\tilde{\phi}$ being the amplitude of the scalar field, one can see that the perturbative decay, $\phi \rightarrow \bar{\chi}\chi$, is kinematically forbidden except for $\phi \sim 0$. In this case, the notion of “ χ -particle” can become ambiguous near $\phi \sim 0$ due to the rapidly moving background scalar condensation, and hence different treatments are required. As a result, it is known that the efficient particle production can take place near $\phi \sim 0$, which is the so-called non-perturbative particle production extensively studied in the context of preheating after inflation [28, 29].

On the other hand, if there exists the background radiation, the effective potential of ϕ receives corrections because the scalar field ϕ interacts with abundant particles in radiation via χ -fields [30]. Also, owing to the interactions, the dissipation rate of the scalar condensation can significantly differ from its decay rate at Vacuum as expected [31–42]. This is because the dispersion relation of χ -particle can be modified due to interactions with radiation; it might attain the effective mass and also width. As a result, the dominant process of the energy transportation from the scalar condensation to radiation can differ from the perturbative decay of ϕ into χ -particles at Vacuum.

Although comprehensive and thorough analyses are required in order to understand the cosmological fate of scalar condensation, yet these issues have been partly tackled by separate literatures in each different context, to the best of our knowledge. Therefore, in this thesis, we investigate in detail the dynamics of scalar condensation in the early Universe; strong emphasis on the interplay between the scalar condensation and particle-like excitations around it. To treat these issues transparently, we start with the Kadanoff-Baym equations [43], which describe the time-evolution of one- and two-point correlators with systematic resummations which are often required in studying a finite density system and its time-evolution. Not to trouble with unnecessary complications, we adopt a phenomenological model [Eq. (4.2)] which is simple but, we believe that, captures essential features of realistic situations. Specifying the dy-

namical time scale, we approximate these full equations by means of “*separation of time scale*” in each situation. As a result, we obtain the coarse-grained equations which can be applicable to various cosmological situations [35, 44, 45].

Then, we study several cosmological roles of scalar condensation so as to clarify the effects of interactions with light particles by using the coarse-grained equations. Consequently, we have shown that characteristic features of these roles can be drastically modified due to interactions with light particles.

- **Reheating after Inflation** [36]: We have studied in detail the dynamics of (p)reheating after inflation, and shown that the reheating temperature, which characterizes inflation models, can be significantly changed by that of conventional estimation. We have also demonstrated that the evolution of radiation before the completion of reheating can be also modified.
- **Curvaton** [46]: We have revisited the curvaton mechanism by taking account of interactions with abundant light particles in the background plasma, and shown that the energy fraction of the curvaton at its complete decay/dissipation, which is an important parameter of the curvature perturbation, can be significantly changed. We have also derived general formulas of the scalar power spectrum and the non-linearity parameter (which characterizes the non-Gaussianity), taking account of these effects.

Organization of This Thesis

The organization of this thesis is as follows. See also the schematic figure of this thesis given in Fig. 1.1.

In Chap. 2, we review important roles of scalar condensation in the early Universe. In order to clarify the effects of interactions with light particles in later chapters, here, we summarize important features of each role with a conventional textbook argument; without the effects of interactions between the scalar condensation and light particles. Later, we will see that the important features can be dramatically modified by these effects. In Sec. 2.1, basic facts of scalar condensate as an inflaton are reviewed; with special emphasis on the reheating temperature. In Sec. 2.2, we briefly summarize the δN -formalism at first, and then characteristic features of curvaton mechanism, the scalar power spectrum and the non-linearity parameter, are derived.

In Chap. 3, we introduce the theoretical equipment, the “closed time path formalism”, in order to describe the evolution of scalar condensation which interacts with light particles. In Sec. 3.1, we see why the closed time path contour arises when one computes the evolution of expectation values. Then, in Sec. 3.2, we briefly introduce the nPI effective action technique; it provides us with a simple and systematic way of resummation scheme which is often required in order to treat the finite density system and its long time behavior. In Sec. 3.3, we derive the Kadanoff-Baym equations from 2PI effective action, which is a fundamental building block of our following discussion to study the evolution of correlators.

Chap. 4 and Chap. 5 are main parts of this thesis, based on our previous works in collaboration with Takeo Moroi, Kazunori Nakayama, and Masahiro Takimoto [35, 36, 44–46]. First, after specifying the phenomenological setup which shares essential features of scalar condensate playing important roles in the early Universe in Sec. 4.1, we obtain the coarse-grained equations to make the full equations tractable by means of “separation of time scale” in various regimes in Secs. 4.2, 4.3 and 4.4. Eventually, we obtain the evolution equations in the cosmological time scale in Sec. 4.6. Then, Chap. 5 is devoted to applying the obtained equations to various roles of scalar condensation in the early Universe. In Sec. 5.1, we study in detail the dynamics of reheating after inflation, and show that the reheating temperature can be changed significantly compared with the conventional estimation. We also discuss that the evolution of radiation before the completion of reheating can be also different from the conventional argument. In Sec. 5.2, we investigate the effects of interactions with light particles on the curvaton mechanism. We show that the scalar power spectrum and the non-linearity parameter can be significantly modified by these effects.

Chap. 6 is devoted to conclusions and discussion. Notation and conventions are summarized in Appendix A. Basic ingredients of thermal field theory which we frequently use in this thesis is introduced in Appendix B. Also we summarize minimum knowledge on the Standard Cosmology in Appendix C.

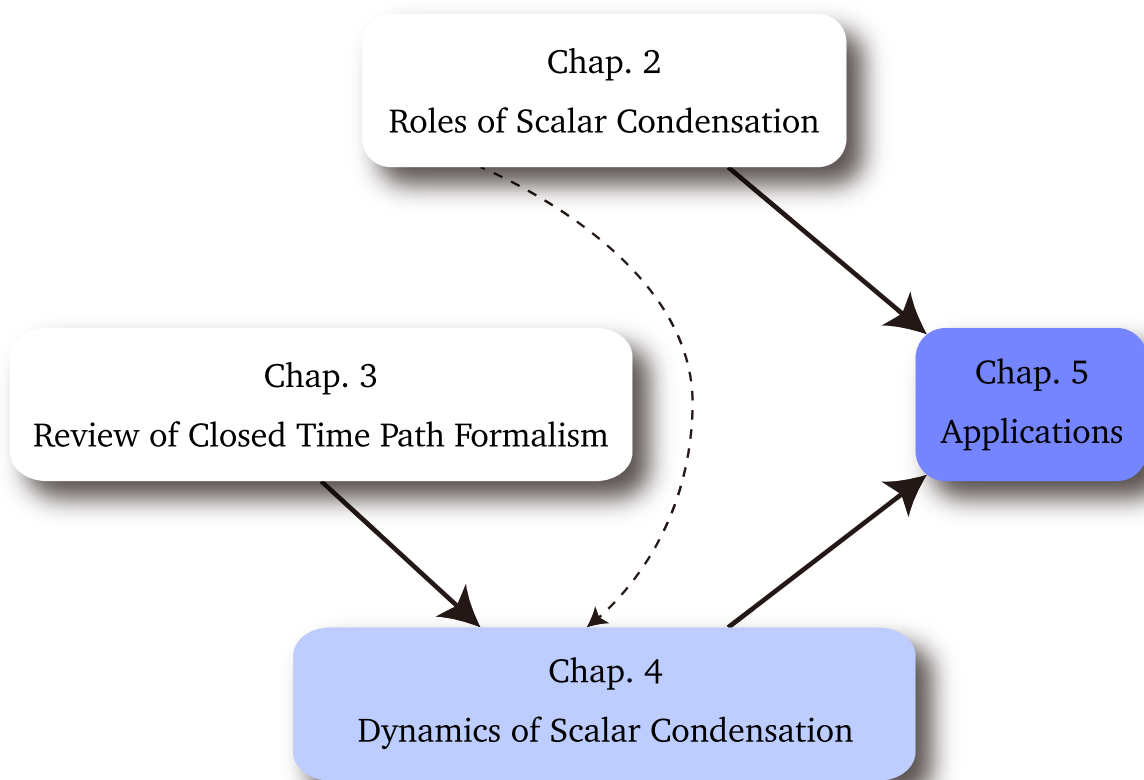


Figure 1.1: Schematic figure of interrelationship among chapters of this thesis. Chaps. 4 and 5 are main parts of this thesis.

Chapter 2

Roles of Scalar Condensation

In this chapter, several roles of scalar fields in the early Universe are reviewed briefly. In order to clarify the impacts of interactions with light fields, which are discussed in detail in later chapters, here we restrict ourselves to simple conventional arguments without seriously taking care of these effects. That is, let us consider a scalar field ϕ that obeys an equation of motion with a potential that solely depends on ϕ and with a constant dissipation term Γ_ϕ^0 evaluated as a decay rate at Vacuum:

$$\ddot{\phi} + (3H + \Gamma_\phi^0)\dot{\phi} + V'(\phi) = 0, \quad (2.1)$$

where H is the Hubble parameter and $V(\phi)$ is a scalar field potential.

Using theoretical equipment in Chap. 3, we will see in Chap. 4 that the potential and the dissipation term depend also on the temperature of background plasma T and on the field value of ϕ in general: $\Gamma_\phi(\phi; T)$, $V_{\text{eff}}(\phi; T)$. Their cosmological impacts are studied in detail in Chap. 5.

The organization of this chapter is the following: First, we review basic facts of a scalar field as an inflaton [1–5]; in particular we see how the reheating after inflation takes place if the equation of motion is given by Eq. (2.1). We also discuss why it is important to know the reheating temperature. Then, we review the curvaton mechanism [7–9], which is another source of primordial density fluctuations other than the inflaton. Especially, the scalar power spectrum and the non-linearity parameter are derived in this simple case.

The following discussion assumes basic knowledge about the Standard Cosmology. We briefly summarize minimum information in Appendix C.

2.1 Inflaton

Inflation [1–5] now has become an essential ingredient of the Standard Cosmology, which provides us with an elegant way to solve the horizon/flatness problems and also generate seeds of primordial density perturbations by an accelerated expansion of the Universe. In this section, we see how a homogeneous scalar condensation can cause the inflation; accelerated expansion phase of our Universe, and the reheating after that; energy conversion process into light particles including SM ones. For simplicity, we mainly discuss the case with the quadratic inflaton potential, $V = m_\phi^2 \phi^2/2$.

2.1.1 Slow Roll Inflation

First, let us consider the inflation phase. The basic evolution equation is nothing but Eq. (2.1) plus the Friedmann equation [See Eq. (C.3) and (C.30)]:

$$0 = \ddot{\phi} + 3H\dot{\phi} + V'(\phi), \quad (2.2)$$

$$3M_{\text{pl}}^2 H^2 = \frac{1}{2} \dot{\phi}^2 + V(\phi). \quad (2.3)$$

Here we assume that the particle creation Γ_ϕ^0 is negligible and that the energy density is dominated by the scalar field. In order to understand the condition where the accelerated expansion takes place, we consider the following equation derived from above ones:

$$\frac{\ddot{a}}{a} = H^2 - \frac{\dot{\phi}^2}{2M_{\text{pl}}^2} = \frac{1}{3M_{\text{pl}}^2} [V(\phi) - \dot{\phi}^2]. \quad (2.4)$$

Thus, as one can see, the accelerated expansion occurs if $V(\phi) \gg \dot{\phi}^2$. In order for the accelerated phase to continue sufficiently long, we impose the following condition:

$$|\ddot{\phi}| \ll |H\dot{\phi}|, |V'|. \quad (2.5)$$

At this regime, the equation of motion can be approximated with

$$3H\dot{\phi} \simeq -V'(\phi). \quad (2.6)$$

Inserting this into $V(\phi) \gg \dot{\phi}^2$, one obtains

$$\epsilon_V \equiv \frac{M_{\text{pl}}^2}{2} \left(\frac{V'}{V} \right)^2; \quad \epsilon_V \ll 1. \quad (2.7)$$

Also differentiating $H^2 \dot{\phi}^2 \sim -V'^2$ with respect to ϕ and inserting it into $V(\phi) \gg \dot{\phi}^2$, one finds

$$\eta_V \equiv M_{\text{pl}}^2 \frac{V''}{V}; \quad |\eta_V| \ll 1. \quad (2.8)$$

These two parameters ϵ_V and η_V are known as the *potential slow roll parameters*, and they are quite useful since the duration of slow-roll regime can be simply obtained from the potential shape of the inflaton. One can also define parameters for higher derivatives of the inflation potential. The inflation ends at $\max[\epsilon_V, |\eta_V|] \simeq 1$. We denote the field value at the end of inflation as ϕ_{end} .

To see how to determine the model of inflaton by cosmological observations, let us briefly explain primordial fluctuations generated during inflation and their relations with slow-roll parameters. Roughly speaking, due to the (quasi) de Sitter nature during the inflation, Vacuum fluctuations inside the horizon are continuously stretched toward the super-horizon scale. Through this process, the primordial curvature perturbation (scalar mode) and also primordial tensor perturbation (gravitational wave) are generated. Importantly, their power spectra are closely related to the slow-roll parameters. Their amplitudes are given by (See also Sec. 2.2)

$$\mathcal{P}_\zeta \simeq \left(\frac{H_*}{2\pi M_{\text{pl}}} \right)^2 \frac{1}{2\epsilon_{V*}}; \quad \mathcal{P}_h = \frac{2}{\pi^2} \frac{H_*^2}{M_{\text{pl}}^2} \rightarrow r \equiv \frac{\mathcal{P}_h}{\mathcal{P}_\zeta} \simeq 16\epsilon_{V*}, \quad (2.9)$$

and their scale dependence can be expressed as

$$n_s - 1 \equiv \frac{d \ln \mathcal{P}_\zeta}{d \ln k_*} \simeq 2\eta_{V*} - 6\epsilon_{V*}; \quad n_t \equiv \frac{d \ln \mathcal{P}_h}{d \ln k_*} \simeq -2\epsilon_{V*}, \quad (2.10)$$

for a single field slow roll inflation, where $\mathcal{P}_s [\mathcal{P}_t]$ denotes the scalar [tensor] power spectrum, and n_s and r are so-called the scalar spectral index and the tensor-to-scalar ratio respectively. Here $*$ indicates values at the horizon exit of the observed mode. One can find a consistency relation, $r = -8n_t$, for a single field slow roll inflation. It is noticeable that the scalar spectral index is almost $n_s \sim 1$, which is a remarkable feature of inflation. Importantly if the primordial density fluctuations are generated dominantly by the inflaton, then the CMB observation implies $(H_*/2\pi M_{\text{pl}})^2/2\epsilon_{V*} \simeq (5 \times 10^{-5})^2$ [47]. Moreover, if n_s and r are somehow determined in the future, then we can access the information of inflaton potential around ϕ_* . The scalar spectral index and the tensor-to-scalar ratio are already constrained by the CMB observation; $n_s = 0.9603 \pm 0.0073$ and $r < 0.11$ [48]. We can see that the observed n_s indicates the nearly scale invariant red-tilted spectrum, which strongly suggests the existence of inflation. Thus, it is quite important to predict these parameters precisely if one specifies a model of inflation.

The e-folding number which characterizes the duration of inflation is defined as

$$N(\phi) \equiv \ln \frac{a_{\text{end}}}{a} = \int_t^{t_{\text{end}}} dt H = \int_\phi^{\phi_{\text{end}}} d\phi \frac{H}{\dot{\phi}} \simeq \int_{\phi_{\text{end}}}^\phi \frac{d\phi}{M_{\text{pl}}} \frac{1}{\sqrt{2\epsilon_V}}. \quad (2.11)$$

To solve the horizon and flatness problems, the *total* number of e-folding should be larger than ~ 60 . Even if one specifies the model of inflation, one needs to know the dynamics of reheating to predict its precise value. To see this, let us relate a mode $p(t_0) = k/a_0$ with the e-folding number from its horizon exit during the inflation to the end of the inflation. Such an e-folding number is defined as

$$N_k \equiv \ln \frac{a_{\text{end}}}{a_k}; \quad \text{with } k = a_k H_k. \quad (2.12)$$

The second condition means that the observed scale with a physical momentum $p(t) = k/a(t)$ exits the horizon during the inflation at the time $p(t_k) = H_k$. To clarify its physics, we rewrite this equation as follows:

$$N_k = -\ln \left(\frac{k}{a_0 H_0} \right) + \ln \left(\frac{a_{\text{end}} H_{\text{end}}}{a_0 H_0} \right) - \ln \left(\frac{H_{\text{end}}}{H_k} \right), \quad (2.13)$$

where 0 subscript indicates present values. While the third term is $\mathcal{O}(1)$ in most inflation models, the second term strongly depends on the information about the dynamics after the inflation. Assuming that the inflaton behaves as matter after the inflation, *i.e.* quadratic potential

(See the next subsection), one obtains

$$a_{\text{end}} = a_0 \left(\frac{H_0}{H_{\text{eq}}} \right)^{3/2} \left(\frac{H_{\text{eq}}}{H_{\text{rh}}} \right)^{1/2} \left(\frac{H_{\text{rh}}}{H_{\text{end}}} \right)^{3/2}, \quad (2.14)$$

where H_{eq} and H_{rh} represent the Hubble parameter at the matter-radiation equality and at the completion of reheating. Plugging this equation to Eq. (2.13), we obtain the following form of N_k :

$$N_k \simeq 56 - \ln \left(\frac{k}{a_0 H_0} \right) + \frac{1}{3} \ln \left(\frac{T_{\text{R}}}{10^9 \text{ GeV}} \right) + \frac{1}{3} \ln \left(\frac{H_{\text{end}}}{10^{14} \text{ GeV}} \right) - \ln \left(\frac{H_{\text{end}}}{H_k} \right), \quad (2.15)$$

where T_{R} denotes the reheating temperature, which is defined as the temperature of radiation when the inflaton completely convert its energy into radiation. As one can see, the e-folding number N_k crucially depends on the reheating temperature, which can take broad range of parameters, $T_{\text{BBN}} \lesssim T_{\text{R}} \lesssim V_{\text{inf}}^{1/4}$ with T_{BBN} and V_{inf} being the temperature at the beginning of BBN and the energy scale of inflation respectively. This uncertainty motivates the commonly used range 50–60 for $k \simeq a_0 H_0$.

It is instructive to compute these parameters in the chaotic inflation [49] with a quadratic potential, $V = m_\phi^2 \phi^2 / 2$. The potential slow-roll parameters are given by

$$\epsilon_{\text{V}} = 2 \left(\frac{M_{\text{pl}}}{\phi} \right)^2; \quad \eta_{\text{V}} = 2 \left(\frac{M_{\text{pl}}}{\phi} \right)^2. \quad (2.16)$$

Hence, the inflation ends at $\phi_{\text{end}} = \sqrt{2} M_{\text{pl}}$. The e-folding number can be expressed as a function of ϕ as

$$N(\phi) \simeq \frac{1}{4M_{\text{pl}}^2} (\phi^2 - \phi_{\text{end}}^2) \leftrightarrow \phi^2(N) \simeq \phi_{\text{end}}^2 + 4NM_{\text{pl}}^2 \simeq 4NM_{\text{pl}}^2. \quad (2.17)$$

Therefore, the slow-roll parameters can be expressed in terms of the e-folding number as

$$\epsilon_{\text{V}}(N) \simeq \frac{1}{2N}; \quad \eta_{\text{V}}(N) \simeq \frac{1}{2N} \rightarrow n_s(N) \simeq 1 - \frac{2}{N}; \quad r(N) \simeq \frac{8}{N}. \quad (2.18)$$

One can predict n_s and r by inserting the e-folding number at the CMB scale: N_* . However, as we have already seen, the e-folding number N has an $\mathcal{O}(10)$ uncertainty due to the large uncertainty of the reheating temperature. So, it is essential to know when the reheating is completed in the era of precision cosmology.

2.1.2 Reheating after Inflation

Then, let us see a textbook argument of reheating dynamics, based on Eq. (2.1) [50]. Soon after the break down of the slow-roll condition, the oscillation time scale becomes much faster than the cosmic expansion, $H^2 \ll |V'/\phi|$, and the inflaton starts to oscillate around its potential minimum. Initially, we assume that the decay term is smaller than the Hubble parameter. From Eq. (2.1), one finds

$$\frac{d}{dt} \rho_\phi \equiv \frac{d}{dt} \left(\frac{1}{2} \dot{\phi}^2 + V(\phi) \right) = - (3H + \Gamma_\phi^0) \dot{\phi}^2. \quad (2.19)$$

Since this equation implies that the energy density ρ_ϕ changes slowly, $\mathcal{O}(H, \Gamma_\phi^0)$, compared with the oscillation time scale, it is convenient to take the oscillation-time average (See also Sec. 4.6). Making use of the virial theorem, one can easily show that

$$\dot{\rho}_\phi = -(3H + \Gamma_\phi^0)\rho_\phi, \quad (2.20)$$

for a quadratic potential $V = m_\phi^2 \phi^2/2$. Then, the whole evolution equations of this system are

$$\dot{\rho}_\phi = -(3H + \Gamma_\phi^0)\rho_\phi, \quad (2.21)$$

$$\dot{\rho}_{\text{rad}} = -4H\rho_{\text{rad}} + \Gamma_\phi^0\rho_\phi, \quad (2.22)$$

$$3M_{\text{pl}}^2 H^2 = \rho_\phi + \rho_{\text{rad}}, \quad (2.23)$$

where ρ_ϕ and ρ_{rad} denote the energy density of inflaton and radiation respectively. We study typical behavior of ρ_ϕ and ρ_{rad} in the following.

Due to the rapid expansion of the Universe during inflation, there is almost no radiation right after the end of inflation, t_{end} . So, the initial condition is $\rho_{\text{rad}}(t_{\text{end}}) = 0$ and $\rho_\phi(t_{\text{end}}) = V_{\text{inf}}$. First, we consider the regime $\Gamma_\phi^0 < H$. Then, Eq. (2.21) yields $\rho_\phi(t) = V_{\text{inf}} a_{\text{end}}^3 / a^3(t)$, and also Eq. (2.23) determines the evolution of scale factor $a(t) = a_{\text{end}}(t/t_{\text{end}})^{2/3}$. At that regime, the radiation obeys the following equation:

$$\rho_{\text{rad}} \simeq \Gamma_\phi^0 \int d \ln[a/a_{\text{end}}] \frac{\rho_\phi}{H} \sim \frac{\Gamma_\phi^0}{H} \rho_\phi \simeq 3M_{\text{pl}}^2 \Gamma_\phi^0 H \propto a^{-3/2}, \quad (2.24)$$

which implies $T \propto \rho_{\text{rad}}^{1/4} \propto a^{-8/3}$. In the second similarity, we have used the fact that the integrand ρ_ϕ/H is a polynomial function of a and dropped the factor. After the Hubble parameter, $H \propto a^{-3/2}$, becomes comparable to the decay term Γ_ϕ^0 , the inflaton rapidly loses its energy and decays completely, as can be seen from $\dot{\rho}_\phi \simeq -\Gamma_\phi^0 \rho_\phi$ for $\Gamma_\phi^0 > H$. Then, the produced radiation behaves as $\rho_{\text{rad}} \propto a^{-4}$, and the radiation dominant era begins. At that regime, the scale factor obeys $a \propto t^{-1/2}$ obtained from $3M_{\text{pl}}^2 H^2 = \rho_{\text{rad}}$. Therefore, the decay time which characterizes the beginning of radiation dominant era can be estimated by $H \simeq \Gamma_\phi^0$. This condition determines the temperature at that time, which is so-called the *reheating temperature*:

$$T_{\text{R}} \simeq \left(\frac{90}{\pi^2 g_*} \right)^{1/4} \sqrt{\Gamma_\phi^0 M_{\text{pl}}}, \quad (2.25)$$

where g_* denotes relativistic degrees of freedom of radiation at the temperature T_{R} . Importantly, Eq. (2.24) implies that the radiation already exists before the completion of reheating, which implies the era with the temperature $T > T_{\text{R}}$. In Sec. 5.1, we see that the conventional estimation of reheating temperature [Eq. (2.25)] can be changed dramatically by interactions with radiation which exists before the complete decay of the inflaton.

Before closing this section, we briefly comment on the compatibility of inflaton models with other mechanisms that solve several remaining problems of our Universe. Up to here, we focus on how an inflaton can solve horizon/flatness problems, produce primordial density perturbations and successfully generate the hot Universe. However, since there are remaining unsolved problems in our Universe as mentioned in the Introduction (e.g. DM, BAU, and so on), it is desirable to discuss how to construct a consistent history of the Universe by combining an inflaton model and other mechanisms which solve remaining problems. At this stage, the reheating temperature is of quite importance to discuss these issues. For instance, the (standard) thermal freeze-out DM production [50] requires that the reheating temperature should

be larger than the DM mass, the thermal Leptogenesis [51] also does that it should be larger than 10^9 GeV [52], the abundance of gravitino [53–55], axino [56, 57] and axion [58, 59] crucially depends on the reheating temperature, that of coherently oscillating moduli [60–62] and heavy particles (e.g. [63, 64]) also does since the dilution factor plays the important role, and so on. Thus, the reheating temperature and also the dynamics of reheating are essential to characterize an inflaton model.

2.2 Curvaton

In the previous section, we see that the inflaton can be the dominant source of primordial curvature perturbations, which were of super-horizon scales and almost scale invariant. Though it is a minimal scenario, one can assume that another scalar field which acquires quantum fluctuations during inflation but does not contribute to the inflation is responsible for primordial curvature perturbations [7–9]. Such a scalar field is dubbed as the curvaton. In contrast to the slow-roll inflation, a key signature of the curvaton mechanism is that it can produce large local-type non-Gaussianity which is parametrized by the non-linearity parameter f_{NL} . Since it is constrained by the CMB observation $f_{\text{NL}} = 2.7 \pm 5.8$ (68% C.L.) [65], it is important to know f_{NL} accurately so as to discuss the viability of the curvaton model.

In this section, we first summarize δN -formulas [66–69] which provide us simple ways to compute primordial curvature perturbations. And then compute the power spectrum \mathcal{P}_ζ and the non-linearity parameter f_{NL} in the simplest case where the equation of motion is given by Eq. (2.1) with a quadratic potential $V = m_\phi^2 \phi^2/2$.

2.2.1 Delta-N Formalism

In the δN -formalism, the primordial curvature perturbation ζ on a uniform density slice labeled by time t is equal to the difference in the local e-folding number with respect to an initially spatially flat slice [66–69]:

$$\zeta(t, \mathbf{x}) = N(t, \mathbf{x}) - \bar{N}(t), \quad (2.26)$$

where the local expansion on sufficiently large scales is given by the same expression as the homogeneous one

$$N(t, \mathbf{x}) = \int_{t_*}^t dt H(t, \mathbf{x}). \quad (2.27)$$

Here $N(t, \mathbf{x})$ denotes the e-folding number from the initially flat slice to the uniform density slice and \bar{N} indicates that of the background value. For each different patch \mathbf{x} on super-horizon scales, light scalar fields during inflation acquires different values $\delta\phi(\mathbf{x})$, and thus, taking the initial time as the horizon crossing, one can express ζ as

$$\zeta = N_{\phi_*} \delta\phi_* + \frac{1}{2} N_{\phi_*\phi_*} \delta\phi_* \delta\phi_* + \cdots, \quad (2.28)$$

where the ϕ subscript denotes a derivative with respect to ϕ . If there are several light scalars, then the summation is promised, like $N_{\phi_j} \delta\phi_*^j$.

Performing the Fourier transform

$$\zeta_k = \int d^3x e^{-ik \cdot \mathbf{x}} \zeta(\mathbf{x}), \quad (2.29)$$

we can write down the two point function as

$$\langle \zeta_{k_1} \zeta_{k_2} \rangle \equiv (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2) P_\zeta(k_1). \quad (2.30)$$

Here we define the power spectrum

$$\mathcal{P}_\zeta(k) \equiv \frac{k^3}{2\pi^2} P_\zeta(k) \quad (2.31)$$

so that the variance of ζ in the real space becomes $\int_0^\infty d \ln k \mathcal{P}_\zeta(k)$. By means of the δN -formula [Eq. (2.28)], the power spectrum can be expressed as

$$P_\zeta(k) = N_{\phi_*}^2 P_{\delta\phi}(k); \quad \langle \delta\phi_{k_1} \delta\phi_{k_2} \rangle = (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2) P_{\delta\phi}(k), \quad (2.32)$$

at the leading order in fluctuations of scalar fields. In a de Sitter space with the Hubble parameter H , light fields ($m_\phi \ll H$) acquire following quantum fluctuations on super-horizon scales [See also Eq. (C.39)]:

$$\langle \delta\phi_{k_1} \delta\phi_{k_2} \rangle = (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2) \frac{H^2}{2k^3}. \quad (2.33)$$

Thus, the power spectrum can be expressed as

$$\mathcal{P}_\zeta(k) = \left(\frac{H_*}{2\pi} \right)^2 N_{\phi_*}^2 \quad (2.34)$$

Here note that H_* indicates the Hubble parameter at the horizon crossing $a_* H_* = k$, because the Hubble parameter gradually changes in the inflationary phase.

Also, Eq. (2.33) implies that the tensor mode, which obeys the same action as a massless scalar field in the FLRW background up to a normalization factor $M_{\text{pl}}/2$, also acquires the following fluctuations:

$$P_h(k) = 2 \times \frac{4}{M_{\text{pl}}^2} P_{\delta\phi}(k), \quad (2.35)$$

where the factor 2 comes from two polarizations and $4/M_{\text{pl}}^2$ from the normalization factor. Thus, the power spectrum for the tensor perturbation is given by

$$\mathcal{P}_h(k) = \frac{2}{\pi^2} \frac{H_*^2}{M_{\text{pl}}^2}. \quad (2.36)$$

Then, we move on to the three point function which imprints the non-Gaussianity. The bispectrum is defined as

$$\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle \equiv (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) B_\zeta(k_1, k_2, k_3), \quad (2.37)$$

where the leading order bispectrum can be expressed as

$$B_\zeta(k_1, k_2, k_3) = N_{\phi_*}^2 N_{\phi_*\phi_*} \left[P_{\delta\phi}(k_1) P_{\delta\phi}(k_2) + P_{\delta\phi}(k_2) P_{\delta\phi}(k_3) + P_{\delta\phi}(k_3) P_{\delta\phi}(k_1) \right]. \quad (2.38)$$

Defining the non-linearity parameter f_{NL} by

$$B_\zeta(k_1, k_2, k_3) = \frac{6}{5} f_{\text{NL}} \left[P_\zeta(k_1) P_\zeta(k_2) + P_\zeta(k_2) P_\zeta(k_3) + P_\zeta(k_3) P_\zeta(k_1) \right], \quad (2.39)$$

we obtain the following expression of the non-linearity parameter:

$$\frac{6}{5}f_{\text{NL}} = \frac{N_{\phi_*\phi_*}}{N_{\phi_*}^2}. \quad (2.40)$$

It is instructive to apply obtained formulas to a single field slow-roll inflation discussed in the previous section. Let us move back to Eq. (2.11), which relates the e-folding number with the field value. Note that the e-folding number after the inflation does not depend on ϕ because the end of inflation is determined by the same ϕ_{end} for each patch. Differentiating Eq. (2.11) with respect to ϕ , we obtain

$$N_{\phi_*} = -\frac{H_*}{\dot{\phi}_*} \simeq \frac{1}{M_{\text{pl}}\sqrt{2\epsilon_{V*}}}; \quad N_{\phi_*\phi_*} \simeq N_{\phi_*}^2 [2\epsilon_{V*} - \eta_{V*}]. \quad (2.41)$$

Therefore, the power spectrum of curvature perturbation and the non-linearity parameter can be expressed as

$$\mathcal{P}_\zeta \simeq \left(\frac{H_*}{2\pi M_{\text{pl}}} \right)^2 \frac{1}{2\epsilon_{V*}}; \quad \frac{6}{5}f_{\text{NL}} \simeq 2\epsilon_{V*} - \eta_{V*}, \quad (2.42)$$

in terms of the slow-roll parameters. One can clearly see that the non-Gaussianity is suppressed by the slow-roll parameters.

2.2.2 Curvaton Paradigm

Now we are in a position to discuss the curvature perturbation in the curvaton mechanism [7–9]. Basically, the idea of the curvaton mechanism is the following; first the curvaton which remains light during inflation acquires super-horizon fluctuations, then it starts to oscillate around its potential minimum, and finally it decays into radiation and produces adiabatic perturbations in radiation. To be explicit, let consider a scenario where the curvaton energy density is negligible at least before the completion of reheating by the inflaton and the curvaton decays after that. We also assume that the curvaton obeys Eq. (2.1) with $V = m_\phi^2 \phi^2/2$, and that the inflaton contribution to the curvature perturbation is negligible for simplicity.

Let us compute the curvaton contribution to the e-folding number N . Before the onset of oscillation, the curvaton field is almost frozen to its initial value. Hence, we expect that the initial amplitude of curvaton at the beginning of oscillation, ϕ_i , is almost the same as that of horizon exit, $\phi_i \simeq \phi_*$, and do not distinguish them. Since the curvature perturbation is assumed to be negligible, we can take the initial slice to be the inflaton decay surface; $H = \Gamma_I$ with Γ_I being the decay rate of inflaton. The final uniform density slice is taken to be the decay surface of curvaton; $\bar{H} = \Gamma_\phi$. The energy density of curvaton can be expressed as^{♦1}

$$\rho_\phi^{(\text{dec})}(\mathbf{x}) = \rho_\phi^{(\text{ini})}(\mathbf{x}) \left(\frac{\Gamma_I}{H_{\text{os}}} \right)^2 e^{-3N(\mathbf{x})}; \quad N(\mathbf{x}) = \ln \frac{a_{\text{dec}}(\mathbf{x})}{a_{\text{reh}}}, \quad (2.43)$$

where $\rho_\phi^{(\text{ini/dec})}$ is the energy density of curvaton at the onset of oscillation/decay, H_{os} is the Hubble parameter at the onset of oscillation, and $a_{\text{reh/dec}}$ represents the scale factor at the reheating/curvaton-decay. That of radiation is given by

$$\rho_{\text{rad}}^{(\text{dec})}(\mathbf{x}) = \rho_{\text{rad}}^{(\text{reh})} e^{-4N(\mathbf{x})}. \quad (2.44)$$

^{♦1} Here we adopt the expression for $H_{\text{os}} > \Gamma_I$ for later usage (See Sec. 5.2). For $H_{\text{os}} < \Gamma_I$, Γ_I/H_{os} should be replaced with the identity.

On the final uniform density surface, the total energy density of the Universe should satisfy

$$3M_{\text{pl}}^2 \Gamma_\phi^2 = \rho_\phi^{(\text{dec})}(\mathbf{x}) + \rho_{\text{rad}}^{(\text{dec})}(\mathbf{x}) = \rho_\phi^{(\text{ini})}(\mathbf{x}) \left(\frac{\Gamma_I}{H_{\text{os}}} \right)^2 e^{-3N(\mathbf{x})} + \rho_{\text{rad}}^{(\text{reh})} e^{-4N(\mathbf{x})}. \quad (2.45)$$

Note that the left-hand-side does not depend on ϕ_i . Differentiating this equation with respect to ϕ_i , one finds

$$N_{\phi_i} = r \frac{2}{3\phi_i}; \quad N_{\phi_i \phi_i} = \frac{2r}{\phi_i^2} (3 - 4r - 2r^2); \quad r \equiv \frac{3\tilde{R}}{3\tilde{R} + 4}; \quad \tilde{R} \equiv \frac{\bar{\rho}_\phi^{(\text{dec})}}{\bar{\rho}_{\text{rad}}^{(\text{dec})}}. \quad (2.46)$$

Here we have used $\rho_\phi^{(\text{ini})} = m_\phi^2 \phi_i^2 / 2$. Thus, the power spectrum for the curvature perturbation and the non-linearity parameter are given by

$$\mathcal{P}_\zeta = \left(\frac{H_*}{2\pi\phi_i} \frac{2r}{3} \right)^2; \quad \frac{6}{5} f_{\text{NL}} = \frac{3}{2r} - 2 - r. \quad (2.47)$$

In contrast to the inflaton case, if the energy fraction of the curvaton at the decay is small $\tilde{R} < 1$, then the non-linearity parameter can be enhanced, $f_{\text{NL}} \sim 5/(3\tilde{R})$.

Up to here, we assume that the inflaton contribution to the curvature perturbation is negligible. For completeness, let us briefly summarize formulas in the mixed scenario where both the inflaton and the curvaton yield sizable curvature perturbation [70, 71]. Assuming that there are no correlations between the inflaton and the curvaton fluctuations, the curvature perturbation can be expanded as

$$\zeta = N_{\phi_*} \delta\phi_* + N_{I_*} \delta I_* + \frac{1}{2} (N_{\phi_* \phi_*} \delta\phi_*^2 + N_{I_* I_*} \delta I_*^2) + \dots, \quad (2.48)$$

where I denotes the inflaton field. This equation implies the following expressions of the power spectrum and the non-linearity parameter:

$$\mathcal{P}_\zeta = (1 + \mathcal{E}) \mathcal{P}_\zeta^I, \quad (2.49)$$

$$\frac{6}{5} f_{\text{NL}} = \frac{N_{I_*}^2 N_{I_* I_*} + N_{\phi_*}^2 N_{\phi_* \phi_*}}{(N_{I_*}^2 + N_{\phi_*}^2)^2} \simeq \left(\frac{\mathcal{E}}{1 + \mathcal{E}} \right)^2 \left(\frac{3}{2r} - 2 - r \right), \quad (2.50)$$

where

$$\mathcal{P}_\zeta^I = \left(\frac{H_*}{2\pi M_{\text{pl}}} \right)^2 \frac{1}{2\epsilon_{V_*}}; \quad \mathcal{E} = \left(\frac{2r M_{\text{pl}}}{3\phi_i} \right)^2 2\epsilon_{V_*}. \quad (2.51)$$

Here we have neglected $N_{I_* I_*}$ since it is suppressed by the slow-roll parameters. One can see that the non-linearity parameter becomes suppressed if the inflaton contribution becomes large, as expected.

Finally, we comment on the scalar spectral index n_s . By using the ratio \mathcal{E} , one can express the scalar spectral index as [71].

$$n_s - 1 \simeq M_{\text{pl}}^2 \frac{V''(\phi_*)}{3H_*^2} - 2\epsilon_{V_*} - \frac{4\epsilon_{V_*} - 2\eta_{V_*}}{1 + \mathcal{E}}. \quad (2.52)$$

For $\mathcal{E} \rightarrow 0$, Eq. (2.10) is obtained as expected. On the other hand, if the curvaton dominates the curvature perturbation $\mathcal{E} \gg 1$, then the spectral index becomes $n_s \simeq 1 - 2\epsilon_{V_*} + M_{\text{pl}}^2 V''/(3H_*^2)$. Hence, the observed red-tilted spectrum can be expressed by a large field inflation model with $\epsilon_{V_*} = \mathcal{O}(0.01)$ or a slightly steep tachyonic curvaton potential.

Chapter 3

Review of Closed Time Path Formalism

The aim of this chapter is to introduce theoretical equipment [72–74] in order to describe the evolution of scalar condensation which interacts with light fields.

To treat the evolution of expectation values of operators systematically as an initial value problem, it is convenient to employ the “closed time path formalism” also known as “in-in formalism” [75–77]; rather than “in-out formalism” which is useful in computing scattering amplitudes between in and out on-shell asymptotic states. In addition, since there exists background medium in general, resummations are sometimes required in contrast to the perturbation around vacuum. Hence we also have to know a systematic and consistent resummation scheme which respects symmetries of an underlying theory. First, let us see why the closed time path contour [72, 73, 77] arises when one considers the evolution of expectation values. Second, a systematic way to truncate the Schwinger-Dyson hierarchy, which is an infinite series of coupled (evolution) equations of correlators, is introduced. It is so-called n-Particle Irreducible (nPI) effective action technique that provides us with a consistent and simple resummation scheme [78–81]. Then, as a stationary solution of 2PI effective action the Kadanoff-Baym equations [43] are derived.

In later chapters, regarding the obtained equations as a starting point, we will use some approximations applicable to cosmological situations in consideration, so as to make the obtained equations tractable. By doing so, we believe that the meaning and limitation of approximations are clarified, and also the relation with traditional arguments reviewed in Chap. 2 becomes transparent.

3.1 Closed Time Path

3.1.1 Density Matrix

Let $\hat{\rho}$ be a density matrix which characterizes a system, namely it gives maps of all the observables to probability functions: $\hat{\rho} : \hat{O} \mapsto \mathcal{P}_\rho(o)$ where o represents a measured value of an observable \hat{O} and $\mathcal{P}(o)$ stands for a probability of measuring o . For a state $\hat{\rho}'$, if there exists an observable \hat{O} that gives a different probability function $\mathcal{P}_{\rho'}(o)$, then $\hat{\rho}'$ is a different state from $\hat{\rho}$. For all the observables \hat{O} , if the probability function can be universally decomposed into a weighted sum of different states, $\mathcal{P}_\rho(o) = x\mathcal{P}_{\rho_1}(o) + (1-x)\mathcal{P}_{\rho_2}(o)$ with $0 < x < 1$, then such a state $\hat{\rho}$ is called a mixed state. If this is not the case, then it is called a pure state.

To make this map concrete, let us consider a projection operator $\hat{P}_o = |o\rangle\langle o|$ where $|o\rangle$ denotes an eigenvector of \hat{O} with an eigenvalue o . Then, the probability function is given by $\mathcal{P}_\rho(o) = \text{Tr}[\hat{P}_o \hat{\rho}]$ where (i) $\hat{\rho}$ is a Hermitian operator. Since the probability should be positive for all the states, (ii) $\langle \psi | \hat{\rho} | \psi \rangle \geq 0$ is required for an arbitrary $|\psi\rangle$. Also the sum of probability functions for all the exclusive possibilities should be one: (iii) $1 = \sum_o \mathcal{P}_\rho(o) = \text{Tr}[\hat{\rho}]$. Formally, a density matrix is an operator satisfying these three properties (i)–(iii). Note that the pure state is characterized by $\hat{\rho}^2 = \hat{\rho}$. By construction, the mean value $\langle o \rangle$ is given by $\langle o \rangle = \sum_o \mathcal{P}_\rho(o)o = \text{Tr}[\hat{O} \hat{\rho}]$ for instance. We sometimes write down the ensemble average as $\langle \bullet \rangle \equiv \text{Tr}[\hat{\rho} \bullet]$.

One can express the density matrix as

$$\hat{\rho} = \sum_{\psi} \omega_{\psi} |\psi\rangle\langle\psi|; \text{ with } \omega_{\psi} \geq 0, \sum_{\psi} \omega_{\psi} = 1. \quad (3.1)$$

Note that this expression is not always unique because there exists different representations due to the superposition principle: $\hat{\rho} = \sum_{\psi'} \omega_{\psi'} |\psi'\rangle\langle\psi'|$ with $\sqrt{\omega_{\psi'}} |\psi'\rangle = \sum_{\psi} U_{\psi'\psi} \sqrt{\omega_{\psi}} |\psi\rangle$ and U is a unitary matrix. This fact implies that one cannot disentangle quantum and statistical fluctuations because it depends on the choice of density matrix representation.

Throughout this thesis, we adopt the Heisenberg picture where operators evolve according to the Heisenberg equation, $i\partial_t \hat{O}(t) = [\hat{O}(t), \hat{H}]$ with \hat{H} being the Hamiltonian, but the state $\hat{\rho}$ does not. Note that, in this case, the eigenvector of \hat{O} also depends on time: $\hat{O}(t)|o; t\rangle = o|o; t\rangle$ where $i\partial_t |o; t\rangle = -\hat{H}|o; t\rangle$.

3.1.2 Closed Time Path Integral

Then let us see how to express correlators in the path integral language. To see why it is convenient to employ the closed time path contour, let us first consider an expectation value $\langle \hat{O} \rangle$ where O is a function of fundamental field $\hat{\phi}$, like $O(\phi)$. In terms of a time-evolution operator which obeys

$$i\partial_t \hat{\mathcal{U}}(t) = \hat{H} \hat{\mathcal{U}}(t); \quad \hat{\mathcal{U}}(0) = \hat{1}, \quad (3.2)$$

it can be expressed as

$$\langle \hat{O}(x) \rangle = \text{Tr}[\hat{\rho} \hat{\mathcal{U}}^\dagger(t) \hat{O}(0, \mathbf{x}) \hat{\mathcal{U}}(t)]. \quad (3.3)$$

As mentioned previously, the eigenvector of $\hat{\phi}(x)$, $|\phi; t\rangle$, depends on time: $i\partial_t |\phi; t\rangle = -\hat{H}|\phi; t\rangle$ with $\hat{\phi}(x)|\phi; t\rangle = \phi(\mathbf{x})|\phi; t\rangle$. Hence one finds

$$\langle \phi_f; t | \phi_i; 0 \rangle = \langle \phi_f | \hat{\mathcal{U}}(t) | \phi_i \rangle = \int_{\phi_i}^{\phi_f} \mathcal{D}\phi e^{iS[\phi]}, \quad (3.4)$$

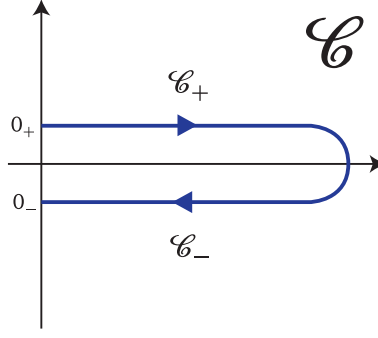


Figure 3.1: The “Closed Time Path” contour, $\mathcal{C} = \mathcal{C}_+ + \mathcal{C}_-$, is shown in the Blue line.

where the boundaries of path integral are given by $\phi_i(\mathbf{x})$ and $\phi_f(\mathbf{x})$. Here and hereafter we suppress $t = 0$ for brevity as done in the second equality of Eq. (3.4). In the path integral language, Eq. (3.3) can be expressed as

$$\langle \hat{O} \rangle = \int d\varphi_+ d\varphi_- \langle \varphi_+ | \hat{\rho} | \varphi_- \rangle \int d\varphi_f O(\varphi_f) \int_{\varphi_+}^{\varphi_f} \mathcal{D}\varphi e^{iS[\varphi]} \int_{\varphi_-}^{\varphi_f} \mathcal{D}\varphi e^{-iS[\varphi]} \quad (3.5)$$

This suggests the following generating functional:

$$Z[J] \equiv e^{iW[J]} \equiv \int d\varphi_+ d\varphi_- \langle \varphi_+ | \hat{\rho} | \varphi_- \rangle \int d\varphi_f \int_{\varphi_+}^{\varphi_f} \mathcal{D}\varphi e^{iS[\varphi] + iJ \cdot \varphi} \int_{\varphi_-}^{\varphi_f} \mathcal{D}\varphi e^{-iS[\varphi] - iJ \cdot \varphi}, \quad (3.6)$$

where we have used the shorthanded notation, $J \cdot \varphi = \int d^4x J(x) \varphi(x)$ and dropped the state dependence of the generating functional, $Z[J](= Z[J; \rho])$. Differentiating W with respect to J , one can obtain the mean value of $O(\hat{\phi})$.

The generating functional, Eq. (3.6), can be written down conveniently if one introduces the closed time path contour [72, 73, 77] shown in Fig. 3.1:

$$Z[J] = \int d\varphi_+ d\varphi_- \langle \varphi_+ | \hat{\rho} | \varphi_- \rangle \int_{\varphi_+}^{\varphi_-} \mathcal{D}\varphi \exp \left(i \int_{\mathcal{C}} d^4x \mathcal{L}(x) + i \int_{\mathcal{C}} d^4x J(x) \varphi(x) \right). \quad (3.7)$$

If one defines the contour \mathcal{C} ordering $T_{\mathcal{C}}$, then it can be also expressed as

$$Z[J] = \text{Tr} \left[\hat{\rho} T_{\mathcal{C}} \exp \left(i \int_C d^4x J(x) \hat{\phi}(x) \right) \right]. \quad (3.8)$$

3.1.3 Green Functions on the Closed Time Path

To study the dynamics of a given theory, it may be more convenient to consider Green functions rather than the generating functional itself. As in the case of ordinary in-out formalism, connected Green functions can be obtained from

$$G^{(n)}(x_1, \dots, x_n) = \frac{i \delta W[J]}{i^n \delta J(x_1) \cdots \delta J(x_n)} \Big|_{J=0} = \langle T_{\mathcal{C}} \hat{\phi}(x_1) \cdots \hat{\phi}(x_n) \rangle_{\text{con}}. \quad (3.9)$$

The vertex function, which is an amputated Green function that consists of one-particle irreducible (1PI) graphs, is defined as

$$\prod_{i=1}^n \left(\int_C d^4y_i G(x_i, y_i) \right) i\Gamma(y_1, \dots, y_n) \equiv G_{\text{prop}}^{(n)}(x_1, \dots, x_n), \quad (3.10)$$

where the subscript prop means that the Green function contains only 1PI diagrams.

In particular, let us study the properties of two-point function (See also Appendix A.3 where we summarize basic properties of propagators), which is so-called the Schwinger-Keldysh propagator defined by

$$G(x, y) = \frac{i\delta W[J]}{i^2\delta J(x)\delta J(y)} \Big|_{J=0} = \langle T_{\mathcal{C}} \hat{\phi}(x) \hat{\phi}(y) \rangle_{\text{con}}. \quad (3.11)$$

Here a possible internal degree of freedom is implicit, for instance, if we have multi fields $\hat{\phi}_i$, then the argument x should be replaced with a set $\{x, i\}$. Here and hereafter, we omit the subscript “con” for brevity. Generally there are two independent basic propagators:

$$G_{>}(x, y) \equiv \langle \hat{\phi}(x) \hat{\phi}(y) \rangle \quad (3.12)$$

$$G_{<}(x, y) \equiv (-)^{|\phi|} \langle \hat{\phi}(y) \hat{\phi}(x) \rangle, \quad (3.13)$$

where $|\phi| = 0, 1$ for the bosonic/fermionic field ϕ . By construction, the Schwinger-Keldysh propagator can be expressed as

$$G(x, y) = \theta_{\mathcal{C}}(x_0, y_0) G_{>}(x, y) + \theta_{\mathcal{C}}(y_0, x_0) G_{<}(x, y), \quad (3.14)$$

where $\theta_{\mathcal{C}}$ represents the step function defined on the contour \mathcal{C} . Depending on a choice of time arguments, the Schwinger-Keldysh propagator reads

$$G(x, y) = \begin{cases} G_{\text{Fyn}}(x, y) & \text{for } x^0 \in \mathcal{C}_+, y^0 \in \mathcal{C}_+ \\ G_{>}(x, y) & \text{for } x^0 \in \mathcal{C}_-, y^0 \in \mathcal{C}_+ \\ G_{<}(x, y) & \text{for } x^0 \in \mathcal{C}_+, y^0 \in \mathcal{C}_- \\ G_{\text{Dys}}(x, y) & \text{for } x^0 \in \mathcal{C}_-, y^0 \in \mathcal{C}_- \end{cases}. \quad (3.15)$$

Here the time ordered and anti time ordered propagators, which are so-called Feynman propagator and Dyson propagator respectively, are defined by

$$\text{Feynman Propagator : } G_{\text{Fyn}}(x, y) \equiv \theta(x_0 - y_0) G_{>}(x, y) + \theta(y_0 - x_0) G_{<}(x, y), \quad (3.16)$$

$$\text{Dyson Propagator : } G_{\text{Dys}}(x, y) \equiv \theta(y_0 - x_0) G_{>}(x, y) + \theta(x_0 - y_0) G_{<}(x, y). \quad (3.17)$$

As you know, the former one is frequently used in computing scattering amplitudes in the “in-out” formalism. For later convenience, we define the following propagators:^{♦1}

$$\text{Hadamard Propagator : } G_{\text{H}}(x, y) \equiv \langle [\hat{\phi}(x), \hat{\phi}(y)]_+ \rangle = G_{>}(x, y) + G_{<}(x, y), \quad (3.19)$$

$$\text{Jordan Propagator : } G_{\text{J}}(x, y) \equiv \langle [\hat{\phi}(x), \hat{\phi}(y)]_- \rangle = G_{>}(x, y) - G_{<}(x, y), \quad (3.20)$$

where $[\bullet, \bullet]_{\pm}$ represents a commutator/anti-commutator respectively, which is defined as

$$[A, B]_{\pm} \equiv AB \pm (-)^{|A||B|} BA. \quad (3.21)$$

The Hadamard/Jordan propagator is also known as statistical/spectral function in some literature. As explained in Sec. 3.3 and also can be seen in the next subsection, the Hadamard propagator encodes the occupation number of quasi-particle excitations and the Jordan propagator does the spectrum of the theory. We sometimes use the retarded and advanced propagators:

$$G_{\text{ret}}(x, y) = i\theta(x_0 - y_0) G_{\text{J}}(x, y); \quad G_{\text{adv}}(x, y) = -i\theta(y_0 - x_0) G_{\text{J}}(x, y). \quad (3.22)$$

^{♦1} In some literature, e.g. [74], the following convention of statistical/spectral function is used

$$F(x, y) \equiv \frac{1}{2} G_{\text{H}}(x, y); \quad \rho(x, y) \equiv i G_{\text{J}}(x, y). \quad (3.18)$$

We basically adopt the convention used in [73, 82].

3.1.4 Green Functions on the Thermal Path

In this subsection, we briefly explain special properties of Green functions in the case of thermal equilibrium. See also Appendix B for basic ingredients of thermal field theory.

If the system can be regarded as thermal equilibrium state, we have several special properties of Green functions because the density matrix commutes with the translation operator,

$$0 = [\hat{\rho}_{\text{eq}}, \hat{H}] = [\hat{\rho}_{\text{eq}}, \hat{P}], \quad (3.23)$$

and can be written by the Gibbs state which can be interpreted as a complex time-evolution operator:

$$\hat{\rho}_{\text{eq}} = \frac{e^{-\beta \hat{H}}}{\text{Tr}[e^{-\beta \hat{H}}]}. \quad (3.24)$$

Here we omit chemical potentials since they can be neglected in most cases from cosmological point of view (that for Baryon/Lepton number is much smaller than the background temperature $\mu/T \sim 10^{-10}$).

Thanks to the translational invariance, Eq. (3.23), all the Green functions do not depend on the constant space-time shift: $G_{\bullet}(x_1, \dots, x_n) = G_{\bullet}(x_1 + c, \dots, x_n + c)$. Hence, it is convenient to perform the Fourier transform. In particular, a two point function only depends on the space-time difference:

$$G_{\bullet}(x, y) = G_{\bullet}(x - y) \xrightarrow{\text{Fourier tr. w.r.t. } x - y} G_{\bullet}(P). \quad (3.25)$$

In addition, since the density matrix can be regarded as a complex time-evolution operator, $\hat{\rho} \propto e^{-i(-i\beta\hat{H})}$, there is a relation between the two basic propagators:

$$G_{>}(x^0, \mathbf{x}) = (-)^{|\phi|} G_{<}(x^0 + i\beta, \mathbf{x}) \xrightarrow{\text{Fourier tr.}} G_{>}(P) = (-)^{|\phi|} e^{\beta p^0} G_{<}(P), \quad (3.26)$$

which is the so-called ‘‘Kubo-Martin-Schwinger’’ (KMS) relation [83, 84]. Thus, all the propagators can be expressed in terms of a single propagator. We choose the Jordan propagator as this role and call its Fourier transform the spectral density, which is defined as^{♦2}

$$\rho(P) \equiv G_J(P). \quad (3.27)$$

By using it, one can express the other propagators:

$$G_{>}(P) = (1 \pm f_{\text{B/F}}(p_0)) \rho(P), \quad (3.28)$$

$$G_{<}(P) = \pm f_{\text{B/F}}(p_0) \rho(P), \quad (3.29)$$

$$G_{\text{Fyn}}(P) = \int \frac{dk_0}{2\pi} \frac{\rho(k_0, \mathbf{p})}{ik_0 - ip_0 + \epsilon} \pm f_{\text{B/F}}(p_0) \rho(P), \quad (3.30)$$

$$G_{\text{Dys}}(P) = \int \frac{dk_0}{2\pi} \frac{\rho(k_0, \mathbf{p})}{ip_0 - ik_0 + \epsilon} \pm f_{\text{B/F}}(p_0) \rho(P), \quad (3.31)$$

$$G_{\text{H}}(P) = (1 \pm 2f_{\text{B/F}}(p_0)) \rho(P), \quad (3.32)$$

for $|\phi| = 0, 1$ respectively. We define the Bose-Einstein/Fermi-Dirac distribution as $f_{\text{B/F}}(p_0)$. Here one can clearly see that the Hadamard propagator encodes the occupation number of each quasi-particle excitations since the spectral density represents the spectrum of the theory.

^{♦2} ρ should not be confused with the density matrix $\hat{\rho}$.

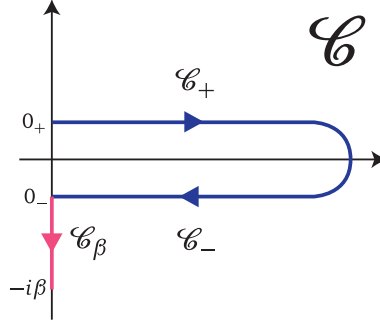


Figure 3.2: The “Thermal Path” is shown in the **Blue** line $\mathcal{C}_+ + \mathcal{C}_-$, plus the **Red** line \mathcal{C}_β . The latter **Red** one corresponds to the “Matsubara Contour”.

It is instructive to see the path integral representation so as to clarify the special feature of thermal equilibrium system. For this purpose, let us move back to the generating functional. For a thermal equilibrium system, the density matrix kernel $\langle \varphi_+ | \hat{\rho}_{\text{eq}} | \varphi_- \rangle$ can be written down explicitly by means of path-integral because the density matrix can be regarded as a complex time-evolution operator:

$$\langle \varphi_+ | \hat{\rho}_{\text{eq}} | \varphi_- \rangle = \int_{\varphi_-}^{\varphi_+} \mathcal{D}\varphi \exp \left(i \int_{\mathcal{C}_\beta} d^4x \mathcal{L}(x) \right), \quad (3.33)$$

where \mathcal{C}_β denotes the Matsubara contour [85], which starts from 0_- and goes straight down to $-i\beta$. See also Fig. 3.2. Hence, it is convenient to define the generating functional as follows:

$$Z_\beta[J] \equiv \text{Tr} \left[\hat{\rho} T_{\mathcal{C}_+ + \mathcal{C}_\beta} \exp \left(i \int_{\mathcal{C}_+ + \mathcal{C}_\beta} d^4x J(x) \hat{\phi}(x) \right) \right] \quad (3.34)$$

$$= \int_{[\text{anti-}] \text{periodic}} \mathcal{D}\varphi \exp \left(i \int_{\mathcal{C}_+ + \mathcal{C}_\beta} d^4x \mathcal{L}(x) + i \int_{\mathcal{C}_+ + \mathcal{C}_\beta} d^4x J(x) \varphi(x) \right), \quad (3.35)$$

where the path-integral is performed under the (anti-)periodic boundary condition, $\varphi(0_+, \mathbf{x}) = (-)^{|\phi|} \varphi(-i\beta, \mathbf{x})$ for $|\phi| = [1], 0$.^{♦3} There are two well-known conventions of thermal field theory. If the source term solely lie on the contour \mathcal{C} , then all the Green functions have real time arguments, which is the so-called “real time formalism” [75–77]. On the other hand, if the source term only lie on the contour \mathcal{C}_β , then all the Green functions have imaginary time arguments, which is the so-called “imaginary time formalism” [85]. In the following, we adopt the real time formalism. See Appendix B for the relation between them.

Finally, let us summarize properties of the Schwinger-Keldysh propagator. By using the KMS relation, one can rewrite it as

$$G(x, y) = \left\langle T_{\mathcal{C}_+ + \mathcal{C}_\beta} \hat{\phi}(x) \hat{\phi}(y) \right\rangle = \int \frac{d^4P}{(2\pi)^4} e^{-iP \cdot (x-y)} \left[\theta_{\mathcal{C}_+ + \mathcal{C}_\beta}(x_0, y_0) \pm f_{\text{B/F}}(p_0) \right] \rho(P), \quad (3.36)$$

for $|\phi| = 0, 1$. For the real time formalism, the Schwinger-Keldysh propagator encodes the

^{♦3} To derive the anti-periodic boundary condition, we have to carefully treat the path-integral expression of Eq. (3.33) and also the trace formula of Grassmann numbers: $\text{Tr}[\hat{O}] = \int d\eta d\eta^\dagger e^{\eta^\dagger \eta} \langle \eta | \hat{O} | \eta \rangle$ where $\hat{\eta} | \eta \rangle = \eta | \eta \rangle$, $\langle \eta | \hat{\eta}^\dagger = \langle \eta | \eta^\dagger$, $[\hat{\eta}, \hat{\eta}^\dagger]_+ = 1$ and $[\hat{\eta}, \hat{\eta}]_+ = [\hat{\eta}^\dagger, \hat{\eta}^\dagger]_+ = 0$. See for instance [86].

following ones as can be seen from Eq. (3.15)

$$G^{(\text{RTF})}(x) = \begin{cases} \int \frac{d^4 p}{(2\pi)^4} e^{-iP \cdot x} \left[\int \frac{dk_0}{2\pi} \frac{\rho(k_0, \mathbf{p})}{ik_0 - ip_0 + \epsilon} \pm f_{\text{B/F}}(p_0) \rho(P) \right] & \text{for } x^0 \in \mathcal{C}_+, y^0 \in \mathcal{C}_+ \\ \int \frac{d^4 p}{(2\pi)^4} e^{-iP \cdot x} \left[1 \pm f_{\text{B/F}}(p_0) \right] \rho(P) & \text{for } x^0 \in \mathcal{C}_-, y^0 \in \mathcal{C}_+ \\ \int \frac{d^4 p}{(2\pi)^4} e^{-iP \cdot x} \pm f_{\text{B/F}}(p_0) \rho(P) & \text{for } x^0 \in \mathcal{C}_+, y^0 \in \mathcal{C}_- \\ \int \frac{d^4 p}{(2\pi)^4} e^{-iP \cdot x} \left[\int \frac{dk_0}{2\pi} \frac{\rho(k_0, \mathbf{p})}{ip_0 - ik_0 + \epsilon} \pm f_{\text{B/F}}(p_0) \rho(P) \right] & \text{for } x^0 \in \mathcal{C}_-, y^0 \in \mathcal{C}_- \end{cases}, \quad (3.37)$$

for $|\phi| = 0, 1$.

3.2 Effective Actions

In this section, the effective action techniques are introduced as a tool of resummation. Before going to discuss resummation for a dynamical and finite density system, let us first review the 1PI (one-particle irreducible) effective action in zero-temperature field theory, and clarify its implicit resummation.

3.2.1 1PI Effective Action

The 1PI effective action is defined as the Legendre transform of the generating functional W :

$$\Gamma[\phi] \equiv W[J] - J \cdot \phi, \quad (3.38)$$

where

$$\phi(x; J) \equiv \frac{\delta W[J]}{\delta J(x)} = \frac{\langle T_{\mathcal{C}} \hat{\phi}(x) \exp(iJ \cdot \hat{\phi}) \rangle}{\langle T_{\mathcal{C}} \exp(iJ \cdot \hat{\phi}) \rangle}. \quad (3.39)$$

Here we have used a shorthanded notation, $J \cdot \phi = \int_{\mathcal{C}} d^4 x J(x) \phi(x)$, and the collective notation, that is, if we have several fields $\phi_i(x)$ then the summation with respect to i is implicit. As one can see, $\phi(x; J)$ given in Eq. (3.39) means the expectation value of $\hat{\phi}$ in the presence of the external field J .

The inverse Legendre transform gives the equation of motion including quantum effects:^{♦4}

$$-J(x) = \frac{\delta \Gamma[\phi]}{\delta \phi(x; J)}. \quad (3.40)$$

In particular, for vanishing external field J , this gives the equation of motion for $\langle \hat{\phi}(x) \rangle$ in the original theory (without the external source J). If we consider an equilibrium system, *e.g.*, the vacuum state $\hat{\rho} = |0\rangle\langle 0|$, then this equation of motion determines the expectation value of $\hat{\phi}$ at equilibrium:

$$0 = \frac{\partial V_{\text{eff}}(\phi)}{\partial \phi}; \quad \text{with } V_{\text{eff}}(\phi) = -\frac{\Gamma[\phi]}{\text{Vol}(\mathbf{R}^{1,3})}. \quad (3.41)$$

Here we have used the fact that the expectation value of ϕ for a translation invariant system does not depend on the space time x , and hence one can factor out the space time volume $\text{Vol}(\mathbf{R}^{1,3})$ from the effective action Γ .

^{♦4} Here the functional derivative is taken from the right.

In order to clarify its implicit resummation nature, let us rewrite the 1PI effective action by using the background field method:

$$\begin{aligned} \exp(i\Gamma[\phi]) &= \int d\varphi_+ d\varphi_- \langle \varphi_+ | \hat{\rho} | \varphi_- \rangle \int_{\varphi_-}^{\varphi_+} \mathcal{D}\varphi \exp \left[i \int_{\mathcal{C}} d^4x \mathcal{L}(\varphi) - i \frac{\delta\Gamma}{\delta\phi} \cdot (\varphi - \phi) \right] \\ &= \int d\varphi_+ d\varphi_- \langle \varphi_+ + \phi | \hat{\rho} | \varphi_- + \phi \rangle \int_{\varphi_-}^{\varphi_+} \mathcal{D}\varphi \exp \left[i \int_{\mathcal{C}} d^4x \mathcal{L}(\varphi + \phi) - i \frac{\delta\Gamma}{\delta\phi} \cdot \varphi \right]. \end{aligned} \quad (3.42)$$

$$(3.43)$$

Expanding the Lagrangian around ϕ ; $\mathcal{L}(\varphi + \phi) = \mathcal{L}(\phi) + \mathcal{L}'(\phi)\varphi + \mathcal{L}''(\phi)\varphi^2/2 + \dots$, one obtains

$$\Gamma[\phi] = S[\phi] + \frac{i}{2} \ln \det G_F^{-1}[\phi] + \Gamma_1[\phi], \quad (3.44)$$

where

$$\begin{aligned} \exp(i\Gamma_1[\phi]) &\equiv [\det G_F]^{-1/2} \int d\varphi_+ d\varphi_- \langle \varphi_+ + \phi | \hat{\rho} | \varphi_- + \phi \rangle \\ &\int_{\varphi_-}^{\varphi_+} \mathcal{D}\varphi \exp \left[-\frac{1}{2} \varphi \cdot G_F^{-1} \cdot \varphi + i S_{\text{int}}[\varphi; \phi] + i \left(\frac{\delta S}{\delta\phi} - \frac{\delta\Gamma}{\delta\phi} \right) \cdot \varphi \right] \end{aligned} \quad (3.45)$$

Here we have defined

$$G_F^{-1}(x, y) \equiv -i \frac{\delta^2 S[\phi]}{\delta\phi(x) \delta\phi(y)}, \quad (3.46)$$

$$\mathcal{L}_{\text{int}}(\varphi; \phi) \equiv \sum_{n \geq 3} \mathcal{L}^{(n)}(\phi) \frac{\varphi^n}{n!}; \quad S_{\text{int}}[\varphi; \phi] = \int_{\mathcal{C}} d^4x \mathcal{L}_{\text{int}}(\varphi; \phi). \quad (3.47)$$

Note that the Green function G_F depends on the background field ϕ . The last term on the exponent of Eq. (3.45), $\delta(S - \Gamma)/\delta\phi \cdot \varphi$, implies that all the one-particle reducible diagram, which is defined as a diagram that can be separated into disconnected diagrams if one cuts one internal line, does not contribute. Thus all one has to do is to sum up one-particle irreducible bubbles. The second term in Eq. (3.44) corresponds to the one-loop 1PI bubble and the last term contains ones at higher order loops. To be concrete, let us consider a ϕ^4 theory, where the tree level potential is given by

$$V(\phi) = \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4. \quad (3.48)$$

Then Γ_1 is given by the sum of following 1PI diagrams in terms of the coupling λ expansion:

$$\Gamma_1[\phi] = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots \quad (3.49)$$

Here the blob denotes the three point interaction induced by the background field ϕ and thick lines represent the ϕ dependent propagator, G_F .

The equation of motion [Eq. (3.40)] can be rewritten by means of Eq. (3.44):

$$0 = \frac{\delta S[\phi]}{\delta\phi} + \frac{i}{2} \frac{\delta G_F^{-1}}{\delta\phi} \cdot G_F + \frac{\Gamma_1[\phi]}{\delta\phi}. \quad (3.50)$$

Importantly, one obtains the exact propagator by a second differentiation of $\Gamma[\phi]$:

$$G^{-1}(x, y) = -i \frac{\delta^2 \Gamma[\phi]}{\delta \phi(x) \delta \phi(y)} \quad (3.51)$$

$$\begin{aligned} &= G_F^{-1}(x, y) - \frac{i}{2} \left[\frac{\delta^4 S}{\delta \phi^4} \cdot G_F \right](x, y) - \frac{i^2}{2} \left[\frac{\delta^3 S}{\delta \phi^3} \cdot G_F \cdot \frac{\delta^3 S}{\delta \phi^3} \cdot G_F \right](x, y) \\ &\quad - i \frac{\delta^2 \Gamma_1[\phi]}{\delta \phi(x) \delta \phi(y)}. \end{aligned} \quad (3.52)$$

The quantum correction to the classical propagator G_F comes from the last three terms and hence these contributions are called the self energy:

$$\Pi(x, y) \equiv -\frac{i}{2} \left[\frac{\delta^4 S}{\delta \phi^4} \cdot G_F \right](x, y) - \frac{i^2}{2} \left[\frac{\delta^3 S}{\delta \phi^3} \cdot G_F \cdot \frac{\delta^3 S}{\delta \phi^3} \cdot G_F \right](x, y) - i \frac{\delta \Gamma_1[\phi]}{\delta \phi(x) \delta \phi(y)}. \quad (3.53)$$

The first two terms correspond to the one-loop result and the last term contains higher loops, and both consist of 1PI diagrams.

Now we are in a position to see the resummation nature of 1PI effective action as mentioned at the beginning of this section. For this purpose, let us compare the propagator for $\phi = 0$ with one for $\phi \neq 0$. By definition, the propagator for $\phi = 0$, G_0 , can be expressed as

$$G_F^{-1} = G_0^{-1} + i [V''(\phi) - V''(0)] \delta_{\mathcal{C}}(x, y), \quad (3.54)$$

with $\delta_{\mathcal{C}}$ being a delta function defined on the contour \mathcal{C} . Hence, G_F is formally recovered by the infinite resummation of G_0 :

$$G_F = G_0 + \{-i G_0 [V''(\phi) - V''(0)]\} \cdot G_F \quad (3.55)$$

$$\begin{aligned} &= G_0 + \{-i G_0 [V''(\phi) - V''(0)]\} \cdot G_0 \\ &\quad + \{-i G_0 [V''(\phi) - V''(0)]\} \cdot \{-i G_0 [V''(\phi) - V''(0)]\} \cdot G_0 + \dots \end{aligned} \quad (3.56)$$

This fact implies that if one computes Feynman diagrams around $\phi = 0$ by means of G_0 and truncates the perturbative expansion at some order in *e.g.* the coupling λ , then the correct perturbative expansion around $\phi \neq 0$ cannot be fully recovered, especially at its infrared scale $\mu \ll \lambda|\phi|$. The advantage of 1PI effective action is that, for any expectation values of ϕ , the perturbative expansion is controlled at least $V''(\phi) > 0$ since the resummation is automatic. In fact, to discuss a spontaneous symmetry breaking with ϕ being an order parameter, the 1PI effective action plays important roles since the nonzero expectation value of ϕ completely modifies the particle-like excitation (and also the interaction). Roughly speaking, the 1PI effective action systematically resums effects caused by the condensation of one-point function $\langle \hat{\phi} \rangle$.

3.2.2 2PI Effective Action

As we have seen above, the condensation of one-point function requires the infinite resummation from the viewpoint of theory at $\phi = 0$ since the particle-like excitation is completely modified by the background field. In a finite density medium, as you might guess, not only the condensation of background field, one-point function ϕ , but also the finite number excitation of particle-like excitations, two-point function G , itself can change its spectrum due to interactions among them. Therefore some generalization of 1PI effective action is expected. Let us see how to achieve it in the following.

For this purpose, let us slightly generalize the 1PI effective action by introducing a bi-local source $K(x, y)$:

$$e^{iW[J, K]} \equiv \text{Tr} \left[\hat{\rho} T_{\mathcal{C}} \exp \left(iJ \cdot \hat{\phi} + \frac{i}{2} \hat{\phi} \cdot K \cdot \hat{\phi} \right) \right] \quad (3.57)$$

$$= \int d\varphi_+ d\varphi_- \langle \varphi_+ | \hat{\rho} | \varphi_- \rangle \int_{\varphi_-}^{\varphi_+} \mathcal{D}\varphi \exp \left(iS[\varphi] + iJ \cdot \varphi + \frac{i}{2} \varphi \cdot K \cdot \varphi \right). \quad (3.58)$$

The 2PI effective action is defined as the double Legendre transform of $W[J, K]$ [81]:

$$\Gamma[\phi, G] \equiv W[J, K] - J \cdot \phi - K \cdot \frac{1}{2} (\phi \phi + G), \quad (3.59)$$

where

$$\phi(x; J, K) \equiv \frac{\delta W[J, K]}{\delta J(x)} = \frac{\langle T_{\mathcal{C}} \hat{\phi}(x) \exp(iJ \cdot \hat{\phi} + \frac{i}{2} K \cdot \hat{\phi}^2) \rangle}{\langle T_{\mathcal{C}} \exp(iJ \cdot \hat{\phi} + \frac{i}{2} K \cdot \hat{\phi}^2) \rangle}, \quad (3.60)$$

$$G(x, y; J, K) \equiv 2 \frac{\delta W[J, K]}{\delta K(x, y)} - \phi(x; J, K) \phi(y; J, K). \quad (3.61)$$

Here we have explicitly written down the external source dependence. The inverse Legendre transform gives the quantum equations of motion

$$-J(x) - [K \cdot \phi](x) = \frac{\delta \Gamma[\phi, G]}{\delta \phi(x; J, K)}, \quad (3.62)$$

$$-\frac{1}{2} K(x, y) = \frac{\delta \Gamma[\phi, G]}{\delta G(x, y; J, K)}. \quad (3.63)$$

For vanishing external sources $J, K = 0$, this set of equations describes the dynamics of the original theory with quantum effects.

In the path integral language, the 2PI effective action can be expressed as

$$\begin{aligned} \exp(i\Gamma[\phi, G]) &= \int d\varphi_+ d\varphi_- \langle \varphi_+ | \hat{\rho} | \varphi_- \rangle \int_{\varphi_-}^{\varphi_+} \mathcal{D}\varphi \\ &\quad \exp \left[i \int_{\mathcal{C}} d^4x \mathcal{L}(\varphi) - i \frac{\delta \Gamma}{\delta \phi} \cdot (\varphi - \phi) - i \frac{\delta \Gamma}{\delta G} \cdot [(\varphi - \phi)(\varphi - \phi) - G] \right] \end{aligned} \quad (3.64)$$

$$\begin{aligned} &= \int d\varphi_+ d\varphi_- \langle \varphi_+ + \phi | \hat{\rho} | \varphi_- + \phi \rangle \int_{\varphi_-}^{\varphi_+} \mathcal{D}\varphi \\ &\quad \exp \left[i \int_{\mathcal{C}} d^4x \mathcal{L}(\varphi + \phi) - i \frac{\delta \Gamma}{\delta \phi} \cdot \varphi - i \frac{\delta \Gamma}{\delta G} \cdot (\varphi \varphi - G) \right] \end{aligned} \quad (3.65)$$

Here note that if one exactly solves the equation of motion given in Eq. (3.61) with vanishing an external source $K = 0$ and inserts the solution into the 2PI effective action, then the 2PI effective action becomes equivalent to the 1PI effective action, *i.e.*,

$$\Gamma[\phi] = \Gamma[\phi, G_{\text{sol}}[\phi]]; \quad 0 = \left. \frac{\delta \Gamma}{\delta G} \right|_{G=G_{\text{sol}}[\phi]}. \quad (3.66)$$

As we will see soon, a truncation of perturbative expansions produces crucial differences.

As in the case of 1PI effective action, performing the Taylor expansion around ϕ , one obtains the following form of 2PI effective action:

$$\Gamma[\phi, G] = S[\phi] + \frac{i}{2} G_F^{-1} \cdot G + \frac{i}{2} \ln \det G^{-1} - \frac{i}{2} G^{-1} \cdot G + \Gamma_2[\phi, G], \quad (3.67)$$

where

$$\begin{aligned} \exp(i\Gamma_2[\phi, G]) &\equiv [\det G_F]^{-1/2} \int d\varphi_+ d\varphi_- \langle \varphi_+ + \phi | \hat{\rho} | \varphi_- + \phi \rangle \\ &\int_{\varphi_-}^{\varphi_+} \mathcal{D}\varphi \exp \left[-\frac{1}{2} \varphi \cdot G^{-1} \cdot \varphi + iS_{\text{int}}[\varphi; \phi] \right. \\ &\left. + i \left(\frac{\delta S}{\delta \phi} - \frac{\delta \Gamma}{\delta \phi} \right) \cdot \varphi + \frac{1}{2} \left(G^{-1} - G_F^{-1} + 2i \frac{\delta \Gamma}{\delta G} \right) \cdot (\varphi \varphi - G) \right]. \end{aligned} \quad (3.68)$$

Since the last two terms on the exponent, $\delta(S - \Gamma)/\delta \phi \cdot \varphi$ and $(G^{-1} - G_F^{-1} + 2i\delta\Gamma/\delta G) \cdot (\varphi \varphi - G)$, verify that two particle reducible diagrams, which are defined as diagrams that can be separated into disconnected diagrams if one cuts one or two internal lines, does not contribute to Γ_2 . Therefore one can compute Γ_2 perturbatively by summing all the 2PI bubbles. One can express the equations of motion [Eqs. (3.60) and (3.61)] with vanishing external sources by using Eq. (3.67):

$$0 = \frac{\delta S[\phi]}{\delta \phi} + \frac{i}{2} \frac{\delta G_F^{-1}}{\delta \phi} \cdot G + \frac{\delta \Gamma_2[\phi, G]}{\delta \phi}, \quad (3.69)$$

$$0 = G^{-1} - G_F^{-1} + 2i \frac{\delta \Gamma_2[\phi, G]}{\delta G}. \quad (3.70)$$

As one can see, the self energy Π is given by

$$\Pi[\phi, G] = -2i \frac{\delta \Gamma_2[\phi, G]}{\delta G}. \quad (3.71)$$

Here it is instructive to compare Eq. (3.71) with Eq. (3.53). On the one hand, the self energy computed in terms of the 1PI effective action [Eq. (3.53)] only contains 1PI diagrams. On the other hand, expanding G around G_F and inserting it to Γ_2 , one can recover the diagrammatic expression of Eq. (3.53). Hence, Γ_2 cannot contain two (or one) particle reducible diagram since otherwise $\delta\Gamma_2/\delta G$ involves one particle reducible (disconnected) diagrams, which contradicts the fact that Eq. (3.53) is made up of 1PI diagrams. From this observation, one can also see that Γ_2 only contains 2PI diagrams.

Finally, we see how the 2PI effective action resums effects from finite density particle-like excitations. To be explicit, let us again consider the ϕ^4 theory. In this case, Γ_2 is given by the sum of following 2PI bubbles in terms of the coupling λ expansion:

$$\Gamma_2[\phi, G] = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots \quad (3.72)$$

Here the red thick line represents the full propagator G in contrast to G_F . Note that if higher order diagrams are taken into account, the diagrammatic expression of Γ_2 [Eq. (3.72)] completely differs from that of Γ_1 [Eq. (3.49)]. Moreover, the above diagrammatic expression coincides

with Γ_1 at this order, but the meaning is different, that is, the thick line in Eq. (3.72) stands for the exact propagator not G_F . For simplicity, we consider a state where the mean field vanishes $\phi = 0$. The functional derivative of Γ_2 with respect to G reads

$$\frac{\delta \Gamma_2[G]}{\delta G} = \text{tadpole diagram} + \text{sunset diagram} + \dots \quad (3.73)$$

The first tadpole diagram of Eq. (3.73) yields a self energy proportional to a delta function, and it gives a dominant finite density correction to the mass term:

$$\Pi_{\text{loc}}(x) = \frac{\lambda}{2} G(x, x); \quad \Pi[G](x, y) = i \Pi_{\text{loc}}(x) \delta_{\mathcal{G}}(x, y) + \Pi_{\text{non-loc}}(x, y). \quad (3.74)$$

For instance, in the thermal equilibrium case, the finite density contribution to Π_{loc} is so-called the *thermal mass*. It is encoded in

$$\Pi_{\text{loc}} = \frac{\lambda}{2} \int_p \left[\frac{1}{2} + f_B(p_0) \right] \rho(P) = \frac{\lambda}{2} \int_p \frac{1}{\omega_p} \left[\frac{1}{2} + f_B(\omega_p) \right] + \dots, \quad (3.75)$$

where \dots represents higher order terms in λ . Here we have used the thermal propagator given in Eq. (3.32), and inserted a dominant contribution of spectral density which comes from a particle-like excitation (See also Sec. B.3). The first term is the zero-temperature contribution and is renormalized by the mass counter term. The second term which depends on the Bose-Einstein distribution function gives the finite density correction. For the temperature larger than the mass term $T \gg m$, it can be expanded as [30]

$$\frac{\lambda}{2} \int_p \frac{f_B(\omega_p)}{\omega_p} = \lambda \left[\frac{T^2}{24} - \frac{mT}{8\pi} + \mathcal{O}(m^2 \ln m^2 / T^2) \right] = \frac{\lambda T^2}{24} + \dots \quad (3.76)$$

Note that though the numerical factor depends on the details of model, its typical scale λT^2 is rather generic for dimension less couplings. Therefore, when the typical scale of physics, e.g. mass m and momentum p , is smaller than $\sqrt{\lambda} T$, then the correct perturbative expansion cannot be arrived from that of vacuum if one truncates the expansion. To see this, let us move back to the equation of motion for the propagator:

$$G = G_F - G_F \cdot \Pi \cdot G = G_F - G_F \cdot \Pi \cdot G_F + (-)^2 G_F \cdot \Pi \cdot G_F \cdot \Pi \cdot G_F + \dots \quad (3.77)$$

$$\begin{aligned} \xrightarrow{\text{Fourier tr.}} \frac{1}{i(m^2 - P^2) + \Pi(P)} &= \frac{1}{i(m^2 - P^2)} - \frac{1}{i(m^2 - P^2)} \frac{\Pi(P)}{i(m^2 - P^2)} + \dots, \\ &= \frac{1}{i(m^2 - P^2)} - \frac{1}{i(m^2 - P^2)} \frac{\lambda T^2 / 24}{(m^2 - P^2)} + \dots \end{aligned} \quad (3.78)$$

As one can see, around $(m^2 - P^2) \sim \lambda T^2$, this expansion does not converge, which implies that one should resum the thermal mass contribution to capture the correct behavior of this theory at the infrared.

The second sunset diagram of Eq. (3.73) contains an imaginary part contribution at the leading order in the coupling λ , since it has cutting diagrams. When one considers time-evolution of a quasi-particle excitation in the plasma, this imaginary part plays an important role. Roughly speaking, such a damping effect Γ appears with the combination of Γt in the evolution equation. Though Γ is higher-order in the coupling λ , the combination Γt grows after

$t \gtrsim 1/\Gamma$ and eventually the perturbation breaks down. It is known as the *secular term* (See [87] for instance). In addition, if one studies the equations of motion in the non-equilibrium case as an initial value problem (See also in Sec. 3.3), it is shown that the 2PI effective action provides us with a systematic resummation scheme to cope with the secular term in terms of numerical studies of simple toy models [74, 88–92]. Therefore if one is interested in the physics where the long time behavior becomes important, one should take account of not only the real part but the imaginary part correction to the self energy.

3.2.3 Equivalence Hierarchy

As we have already seen, the truncation of perturbative expansions makes the difference among nPI effective actions. Let us briefly discuss this issue in this subsection. To make our discussion general, we consider the nPI effective action, which is defined as

$$\begin{aligned} \Gamma[\phi, G, G_3, \dots, G_n] = & W[J_1, \dots, J_n] - J_1 \cdot \phi - \frac{1}{2} J_2 \cdot (\phi \phi + G) \\ & - \frac{1}{6} J_3 \cdot (G_3 + 3G\phi + \phi \phi \phi) - \dots, \end{aligned} \quad (3.79)$$

where

$$e^{iW[J_1, \dots, J_n]} \equiv \text{Tr} \left[\hat{\rho} T_{\mathcal{C}} \exp \left(i \sum_{m=1}^n \frac{1}{m!} J_m \cdot \hat{\phi}^m \right) \right]. \quad (3.80)$$

The equations of motion are given by

$$0 = \frac{\delta \Gamma}{\delta \phi} = \frac{\delta \Gamma}{\delta G} = \frac{\delta \Gamma}{\delta G_3} = \dots = \frac{\delta \Gamma}{\delta G_n}. \quad (3.81)$$

First of all, if one were to compute Γ and solve these equations of motion exactly, then all the nPI effective actions fall into the same results:

$$\Gamma[\phi] = \Gamma[\phi, G_{\text{sol}}[\phi]] = \Gamma[\phi, G_{\text{sol}}[\phi], G_{3,\text{sol}}[\phi]] = \dots. \quad (3.82)$$

This fact can be easily seen, for instance, in Eq. (3.65) for the 2PI case, since the 2PI effective action becomes exactly the same as 1PI one if one inserts $\delta \Gamma / \delta G = 0$.

However, the above relation is broken down if one truncates the perturbative expansion at some finite order. In other words, this is the reason why the higher PI effective action makes sense. If one specifies the perturbation parameter, then it is known that there exists an “*Equivalence Hierarchy*”. For the loop expansion, the Equivalence Hierarchy is given by [93]:

$$\begin{aligned} \Gamma^{(1\text{-loop})}[\phi] &= \Gamma^{(1\text{-loop})}[\phi, G_{\text{sol}}] = \dots, \\ \Gamma^{(2\text{-loop})}[\phi] &\neq \Gamma^{(2\text{-loop})}[\phi, G] = \Gamma^{(2\text{-loop})}[\phi, G, G_{3,\text{sol}}] = \dots, \\ \Gamma^{(3\text{-loop})}[\phi] &\neq \Gamma^{(3\text{-loop})}[\phi, G] \neq \Gamma^{(3\text{-loop})}[\phi, G, G_3] = \Gamma^{(3\text{-loop})}[\phi, G, G_3, G_{4,\text{sol}}] = \dots. \end{aligned} \quad (3.83)$$

As one can see, this hierarchy implies that the nPI effective action is sufficient if one truncates the perturbative expansion at the n-loop.

3.3 Kadanoff-Baym Equations

In this section, we derive the Kadanoff-Baym equations [43] by rewriting the equation of motion for the propagator [Eq. (3.70)] in terms of the Hadamard and Jordan propagators given in Eqs. (3.19) and (3.20). By doing so, the physical interpretation of evolution equations becomes clearer as we will see. Since the 2PI effective action self-consistently resums effects caused by the dynamical finite density background, the Kadanoff-Baym equations provide us with the fundamental building blocks to study the dynamics of quantum fields with finite density.^{♦5} In the following, we briefly derive the Kadanoff-Baym equations and then the relation between the initial condition of these equations and the density matrix is discussed. We also comment on the case where a system is close to thermal equilibrium.

3.3.1 Kadanoff-Baym Equations

To make our discussion concrete, let us consider a scalar field theory unless otherwise stated. First of all, let us recall the equation of motion for the propagator [Eq. (3.70)],

$$[(G_F^{-1} + \Pi) \cdot G](x, y) = \delta_{\mathcal{C}}(x, y), \quad (3.84)$$

with

$$G_F^{-1}(x, y) = i[\square_x + V''(\phi)]\delta_{\mathcal{C}}(x, y). \quad (3.85)$$

The Schwinger-Keldysh propagator can be written in terms of the Hadamard and Jordan propagator as

$$G(x, y) = \frac{1}{2} [G_H(x, y) + \text{sgn}_{\mathcal{C}}(x_0, y_0) G_J(x, y)], \quad (3.86)$$

where the sign function $\text{sgn}_{\mathcal{C}}$ is defined on the closed-time-path contour \mathcal{C} . By using this relation, let us rewrite Eq. (3.84) as equations for the Hadamard and Jordan propagators. For this purpose, the following formulas are useful:

$$\partial_{x_0}^2 [\text{sgn}_{\mathcal{C}}(x_0, y_0) f(x)] = \text{sgn}_{\mathcal{C}}(x_0, y_0) \partial_{x_0}^2 f(x) + 2\delta_{\mathcal{C}}(x_0, y_0) \partial_{x_0} f(x), \quad (3.87)$$

$$\begin{aligned} [A \cdot B](x, y) &= \frac{1}{4} [A_H \cdot B_H + A_H \cdot \text{sgn}_{\mathcal{C}} B_J + \text{sgn}_{\mathcal{C}} A_J \cdot B_H + \text{sgn}_{\mathcal{C}} A_J \cdot \text{sgn}_{\mathcal{C}} B_J] \\ &= \frac{1}{2} \left[- \int_0^{y_0} d^4 z A_H(x, z) B_J(z, y) + \int_0^{x_0} d^4 z A_J(x, z) B_H(z, y) \right. \\ &\quad \left. + \text{sgn}_{\mathcal{C}}(x_0, y_0) \int_{y_0}^{x_0} d^4 z A_J(x, z) B_J(z, y) \right]. \end{aligned} \quad (3.88)$$

Plugging these formulas to the left-hand-side of Eq. (3.84), one obtains

$$[G_F^{-1} \cdot G](x, y) = \frac{1}{2} [G_F^{-1} \cdot G_H + \text{sgn}_{\mathcal{C}} G_F^{-1} \cdot G_J](x, y) + \delta_{\mathcal{C}}(x, y), \quad (3.89)$$

$$\begin{aligned} [\Pi \cdot G](x, y) &= \frac{1}{2} \left[- \int_0^{y_0} d^4 z \Pi_H(x, z) G_J(z, y) + \int_0^{x_0} d^4 z \Pi_J(x, z) G_H(z, y) \right. \\ &\quad \left. + \text{sgn}_{\mathcal{C}}(x_0, y_0) \int_{y_0}^{x_0} d^4 z \Pi_J(x, z) G_J(z, y) \right]. \end{aligned} \quad (3.90)$$

^{♦5} Implicitly, we assume that the 2PI effective action is sufficient in the sense of the equivalence hierarchy, that is, we do not consider higher order terms in some expansion parameter which require the higher PI effective action.

Here we have used the flexible notation; on the left hand side of Eq. (3.89), G_F^{-1} stands for the one defined on the closed-time-path contour, $G_F^{-1}(x, y) = i[\square_x + V''(\phi)]\delta_\phi(x, y)$, but on the right hand side, it stands for $G_F^{-1}(x, y) = i[\square_x + V''(\phi)]\delta(x - y)$. We do not use different symbols for notational simplicity since it is obvious from the context.

Then, Eq. (3.84) can be expressed in terms of $G_{H/J}$ as coupled integro-differential equations, which are so-called Kadanoff-Baym equations [43]:

$$[G_F^{-1} \cdot G_J](x, y) = - \int_{y_0}^{x_0} d^4z \Pi_J(x, z) G_J(z, y), \quad (3.91)$$

$$[G_F^{-1} \cdot G_H](x, y) = - \int_0^{x_0} d^4z \Pi_J(x, z) G_H(z, y) + \int_0^{y_0} d^4z \Pi_H(x, z) G_J(z, y). \quad (3.92)$$

Note that we do not need the equations where the derivative operator acting on the second variable y because we have $G_{J/H}(x, y) = \mp G_{J/H}(y, x)$. Here again possible internal degrees of freedom are implicit: $x \rightarrow \{x, i\}$ for multiple scalar fields ϕ_i . Roughly speaking, if the quasi-particle description is valid, the first equation describes the evolution of the spectrum of each particle-like excitation and the second one encodes the evolution of its number density. These relations may become clearer after we see the subsequent discussion in Sec. 3.3.4 where we derive the Boltzmann equation from the Kadanoff-Baym equations. If the expectation value of ϕ is also dynamical, the complete set of equations is given by above two Eqs. together with

$$0 = \frac{\delta S[\phi]}{\delta \phi} + \frac{i}{2} \frac{\delta G_F^{-1}}{\delta \phi} \cdot G + \frac{\delta \Gamma_2[\phi, G]}{\delta \phi}. \quad (3.93)$$

Note that since Γ_2 (and hence $\Pi \propto \delta \Gamma_2 / \delta G$) depends both on ϕ and G , these equations are non-linear in general.

Similarly, the Kadanoff-Baym equations for fermionic fields can be obtained from the Schwinger-Dyson equations:

$$[(S_F^{-1} + \Sigma) \cdot S](x, y) = \delta_\phi(x, y), \quad (3.94)$$

where S is the Schwinger-Keldysh propagator for fermions $S(x, y) \equiv \langle T_\phi \hat{\psi}(x) \hat{\bar{\psi}}(y) \rangle$, its free propagator is defined as

$$S_{\chi, F}^{-1}(x, y) = i \left[-i \not{\partial}_x - \frac{\partial^2 \mathcal{L}}{\partial_{\bar{\psi}} \partial_{\psi}} \right] \delta_\phi(x, y), \quad (3.95)$$

and Σ represents the self energy which is defined as

$$\Sigma(x, y) \equiv i \frac{\delta \Gamma_2}{\delta S(y, x)}. \quad (3.96)$$

By using the following formula

$$\partial_{x_0} \gamma^0 [\text{sgn}_\phi(x_0, y_0) S_J(x, y)] = 2\delta_\phi(x, y) + \text{sgn}_\phi(x_0, y_0) \gamma^0 \partial_{x_0} S_J(x, y), \quad (3.97)$$

we obtain the Kadanoff-Baym equations:

$$[S_F^{-1} \cdot S_J](x, y) = - \int_{y_0}^{x_0} d^4z \Sigma_J(x, z) S_J(z, y), \quad (3.98)$$

$$[S_F^{-1} \cdot S_H](x, y) = - \int_0^{x_0} d^4z \Sigma_J(x, z) S_H(z, y) + \int_0^{y_0} d^4z \Sigma_H(x, z) S_J(z, y). \quad (3.99)$$

3.3.2 Initial Condition

The obtained Eqs. (3.91), (3.92) and (3.93) [(3.98) and (3.99)] are second-order differential equations, and hence we have to specify their initial condition; their values and first order derivatives at the initial time $t = 0$. The initial condition of the mean field can be expressed as

$$\phi(0, \mathbf{x}) = \langle \hat{\phi}(0, \mathbf{x}) \rangle; \quad \dot{\phi}(0, \mathbf{x}) = \langle \dot{\hat{\phi}}(0, \mathbf{x}) \rangle. \quad (3.100)$$

And the initial condition of the Hadamard propagator also relates with the density matrix as

$$G_H(x, y)|_{x_0, y_0=0} = 2[\langle \hat{\phi}(0, \mathbf{x}) \hat{\phi}(0, \mathbf{y}) \rangle - \phi(0, \mathbf{x}) \phi(0, \mathbf{y})], \quad (3.101)$$

$$\partial_{x_0} G_H(x, y)|_{x_0, y_0=0} = \langle \dot{\hat{\phi}}(0, \mathbf{x}) \hat{\phi}(0, \mathbf{y}) + \hat{\phi}(0, \mathbf{y}) \dot{\hat{\phi}}(0, \mathbf{x}) \rangle - 2\dot{\phi}(0, \mathbf{x}) \phi(0, \mathbf{y}), \quad (3.102)$$

$$\partial_{x_0} \partial_{y_0} G_H(x, y)|_{x_0, y_0=0} = 2[\langle \dot{\hat{\phi}}(0, \mathbf{x}) \dot{\hat{\phi}}(0, \mathbf{y}) \rangle - \dot{\phi}(0, \mathbf{x}) \dot{\phi}(0, \mathbf{y})]. \quad (3.103)$$

On the other hand, the initial condition of the Jordan propagator does not depend on the density matrix because it is completely determined by the canonical commutation relation:

$$G_J(x, y)|_{x_0, y_0=0} = \langle [\hat{\phi}(0, \mathbf{x}), \hat{\phi}(0, \mathbf{y})] \rangle = 0, \quad (3.104)$$

$$\partial_{x_0} \partial_{y_0} G_J(x, y)|_{x_0, y_0=0} = \langle [\dot{\hat{\phi}}(0, \mathbf{x}), \dot{\hat{\phi}}(0, \mathbf{y})] \rangle = 0, \quad (3.105)$$

$$\partial_{x_0} G_J(x, y)|_{x_0, y_0=0} = -\partial_{y_0} G_J(x, y)|_{x_0, y_0=0} = \langle [\dot{\hat{\phi}}(0, \mathbf{x}), \hat{\phi}(0, \mathbf{y})] \rangle = -i\delta(\mathbf{x} - \mathbf{y}). \quad (3.106)$$

Since there are no other degrees of freedom in the Kadanoff-Baym equations, it is implicitly assumed that the initial condition of higher order correlators should be reduced to that of one and two point Green functions. Therefore, there is an implicit assumption on the (initial) density matrix. In fact, this set of initial conditions indicate the so-called Gaussian density matrix, which roughly implies that a set of free quasi-particles under a mean field is prepared at the initial time. To see this clearly, it is instructive to explicitly relate these initial conditions with the density matrix.

For this purpose, let us first move back to the density matrix kernel in the path integral. Following Refs. [73, 94], we parametrize the density matrix as

$$\langle \varphi_+ | \hat{\rho} | \varphi_- \rangle \equiv \exp(\mathcal{F}[\varphi]). \quad (3.107)$$

For a general density matrix, the functional $\mathcal{F}[\varphi]$ is an arbitrary functional of φ and it may be Taylor expanded as [94]

$$\mathcal{F}[\varphi] = \sum_{n=0} \frac{1}{n!} \alpha_n \cdot \varphi^n \quad (3.108)$$

$$= \alpha_0 + \int_{\mathcal{C}} d^4x \alpha_1(x) \varphi(x) + \frac{1}{2} \int_{\mathcal{C}} d^4x d^4y \alpha_2(x, y) \varphi(x) \varphi(y) + \dots, \quad (3.109)$$

where we have used the shorthanded notation in the first equality. α_0 is not physical because it is determined by the normalization condition. Since the density matrix kernel only depends on the φ_{\pm} at the initial time surface, the functions α_n with $n \geq 1$ can be expressed as

$$\alpha_n(x_1, \dots, x_n) = \sum_{\{\epsilon_i = \pm\}} \alpha_n^{\epsilon_1 \dots \epsilon_n}(x_1, \dots, x_n) \delta_{\mathcal{C}_{\epsilon_1}}(x_1^0) \dots \delta_{\mathcal{C}_{\epsilon_n}}(x_n^0), \quad (3.110)$$

where $\delta_{\mathcal{C}_\pm}$ stands for the delta function defined on the contour \mathcal{C}_\pm . Performing the time-integral on the contour \mathcal{C} , one obtains the following form of the functional \mathcal{F}

$$\mathcal{F}[\varphi] = \alpha_0 + \int d^3x \sum_{\epsilon_1=\pm} \alpha_1^{\epsilon_1}(\mathbf{x}) \varphi_{\epsilon_1}(\mathbf{x}) + \frac{1}{2} \int d^3x d^3y \sum_{\epsilon_1, \epsilon_2=\pm} \alpha_2^{\epsilon_1 \epsilon_2}(\mathbf{x}, \mathbf{y}) \varphi_{\epsilon_1}(\mathbf{x}) \varphi_{\epsilon_2}(\mathbf{y}) + \dots \quad (3.111)$$

Note that, due to the hermiticity of the density matrix, $\langle \varphi_+ | \hat{\rho} | \varphi_- \rangle^* = \langle \varphi_- | \hat{\rho} | \varphi_+ \rangle$, all the functions α_n are not independent:^{♦6}

$$[i \alpha_n^{\epsilon_1 \dots \epsilon_n}(\mathbf{x}_1, \dots, \mathbf{x}_n)]^* = i \alpha_n^{(-\epsilon_1) \dots (-\epsilon_n)}(\mathbf{x}_1, \dots, \mathbf{x}_n). \quad (3.112)$$

In addition, depending on the property of the system, the functions α_n might be further restricted. For instance, if the system is spatially homogeneous $[\hat{P}, \hat{\rho}] = 0$, then the following equality holds: $\langle \varphi_+ | \hat{\rho} | \varphi_- \rangle = \langle \varphi_+ | e^{-i\hat{P} \cdot \mathbf{c}} \hat{\rho} e^{i\hat{P} \cdot \mathbf{c}} | \varphi_- \rangle$ for an arbitrary \mathbf{c} . Recalling that $e^{i\hat{P} \cdot \mathbf{c}}$ acts as a spatial translation operator, $\hat{\phi}(t, \mathbf{x}) e^{i\hat{P} \cdot \mathbf{c}} | \varphi \rangle = \varphi(\mathbf{x} + \mathbf{c}) e^{i\hat{P} \cdot \mathbf{c}} | \varphi \rangle$, one finds

$$\alpha_n^{\epsilon_1 \dots \epsilon_n}(\mathbf{x}_1, \dots, \mathbf{x}_n) = \alpha_n^{\epsilon_1 \dots \epsilon_n}(\mathbf{x}_1 + \mathbf{c}, \dots, \mathbf{x}_n + \mathbf{c}) \quad \text{for an arbitrary } \mathbf{c}, \quad (3.113)$$

which implies

$$\alpha_n^{\epsilon_1 \dots \epsilon_n}(\mathbf{x}_1, \dots, \mathbf{x}_n) = \int \frac{d^3k_1}{(2\pi)^3} \dots \frac{d^3k_n}{(2\pi)^3} e^{i(k_1 \cdot \mathbf{x}_1 + \dots + k_n \cdot \mathbf{x}_n)} (2\pi)^3 \delta(\mathbf{k}_1 + \dots + \mathbf{k}_n) \times \alpha_n^{\epsilon_1 \dots \epsilon_n}(\mathbf{k}_1, \dots, \mathbf{k}_n). \quad (3.114)$$

Thus, for a spatially homogeneous system, the functional \mathcal{F} can be expressed as

$$\mathcal{F}[\varphi] = \alpha_0 + \sum_{\epsilon_1=\pm} \alpha_1^{\epsilon_1} \varphi_{\epsilon_1}(\mathbf{0}) + \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \sum_{\epsilon_1, \epsilon_2=\pm} \alpha_2^{\epsilon_1 \epsilon_2}(\mathbf{k}) \varphi_{\epsilon_1}(\mathbf{k}) \varphi_{\epsilon_2}(-\mathbf{k}) + \dots, \quad (3.115)$$

where the hermiticity of the density matrix relates the functions α as

$$\alpha_1^+ = -\alpha_1^{-*}, \quad (3.116)$$

$$\alpha_2^{++}(\mathbf{k}) = -\alpha_2^{-*}(-\mathbf{k}), \quad (3.117)$$

$$\alpha_2^{+-}(\mathbf{k}) = -\alpha_2^{-+*}(-\mathbf{k}), \quad (3.118)$$

...

Let us explicitly see all the initial conditions Eqs. (3.100)–(3.103) are completely determined by $\alpha_1^{\epsilon_1}$ and $\alpha_2^{\epsilon_1 \epsilon_2}$. For simplicity, we consider one real scalar field in the following discussion. Then, the number of independent functions is five since we also have $\alpha_2^{+-}(\mathbf{k}) = \alpha_2^{-+}(-\mathbf{k})$. Using these five independent functions $\alpha_1^{\epsilon_1}$, $\alpha_2^{\epsilon_1 \epsilon_2}$, the initial conditions can be expressed as

$$\phi(0) = i\xi(0)(\alpha_1^+ + \alpha_1^-), \quad (3.119)$$

$$\dot{\phi}(0) = \frac{1}{2}[(\alpha_1^+ - \alpha_1^-) + 2i\eta(0)\xi(0)(\alpha_1^+ + \alpha_1^-)], \quad (3.120)$$

$$G_H(x_0, y_0; \mathbf{k})|_{x_0, y_0=0} = \frac{2i}{\alpha_2^{++}(\mathbf{k}) + 2\alpha_2^{+-}(\mathbf{k}) + \alpha_2^{--}(\mathbf{k})} \equiv 2\xi^2(\mathbf{k}), \quad (3.121)$$

$$\partial_{x_0} G_H(x_0, y_0; \mathbf{k})|_{x_0, y_0=0} = (\alpha_2^{++}(\mathbf{k}) - \alpha_2^{--}(\mathbf{k})) \xi^2(\mathbf{k}) \equiv 2\eta(\mathbf{k})\xi(\mathbf{k}), \quad (3.122)$$

$$\begin{aligned} \partial_{x_0} \partial_{y_0} G_H(x_0, y_0; \mathbf{k})|_{x_0, y_0=0} &= 2 \left[\eta^2(\mathbf{k}) + \frac{i}{4} (-\alpha_2^{++}(\mathbf{k}) - \alpha_2^{--}(\mathbf{k}) + 2\alpha_2^{+-}(\mathbf{k})) \right] \\ &\equiv 2 \left[\eta^2(\mathbf{k}) + \frac{\sigma^2(\mathbf{k})}{4\xi^2(\mathbf{k})} \right]. \end{aligned} \quad (3.123)$$

^{♦6} Here we have adopted the notation where all the scalar fields are decomposed into real fields: e.g. for one complex scalar field theory, we use two real fields.

Note that the hermiticity of the density matrix ensures that all the above values are correctly real: $\alpha_1^+ + \alpha_1^- = 2i\Im\alpha_1^+$, $\alpha_1^+ - \alpha_1^- = 2\Re\alpha_1^+$, $\alpha_2^{++}(\mathbf{k}) + \alpha_2^{--}(\mathbf{k}) = 2i\Im\alpha_2^{++}(\mathbf{k})$, $\alpha_2^{+-}(\mathbf{k}) = i\Im\alpha_2^{+-}(\mathbf{k})$, $\alpha_2^{++}(\mathbf{k}) - \alpha_2^{--}(\mathbf{k}) = 2\Re\alpha_2^{++}(\mathbf{k})$. Therefore, the Kadanoff-Baym equations derived from the 2PI effective action is sufficient, if the initial density matrix can be well approximated by the quadratic terms in the field ϕ , which is the so-called Gaussian density matrix.

If this is not the case, we have to slightly generalize the computation of 2PI effective action, as discussed in Ref. [94]. To see the idea briefly, let us move back to the generating functional in the language of the path-integral [Eq. (3.58)]:

$$e^{iW[J,K]} = \int d\varphi_+ d\varphi_- \exp(\mathcal{F}_G[\varphi] + \mathcal{F}_{NG}[\varphi]) \int_{\varphi_+}^{\varphi_-} \mathcal{D}\varphi \exp\left(iS[\varphi] + iJ \cdot \varphi + \frac{i}{2} \varphi \cdot K \cdot \varphi\right), \quad (3.124)$$

where we have decomposed the functional of the initial density matrix into the Gaussian part and the non-Gaussian part: $\mathcal{F} = \mathcal{F}_G + \mathcal{F}_{NG}$. Regarding the sum of the tree level action $S[\varphi]$ and the non-Gaussian part $\mathcal{F}_{NG}[\varphi]$ as an effective tree level action, one can formally reproduce the 2PI effective action along with the discussion we have made. Thus, the same Kadanoff-Baym equations can be obtained formally, but the self energy contains the information of the initial non-Gaussianity. See Ref. [94] for details, and here we stop our discussion on the non-Gaussian initial condition.

3.3.3 Thermal Fixed Point

As one can guess, since the Kadanoff-Baym equations describe the dynamics of a system, thermal propagators should be a static solution of these equations. In the following, let us see this feature explicitly.

The density matrix, $\hat{\rho}_{\text{eq}}$, commutes with translation operators \hat{H} and \hat{P} , and hence the one point function should be constant, $\phi(x) = \phi$. So, we only have to care about the equations for propagators given in Eqs. (3.91) and (3.92) [(3.98) and (3.99)]. In particular, the latter equation is relevant to the dynamics, since it depends on the initial condition. On the other hand, the former equation does not contains information of a non-equilibrium initial condition directly, which becomes relevant through the dynamics of G_H in the self energy. Aside from a term which leads to the same equation as Eq. (3.91), one finds that the remnant of Eq. (3.92) becomes^{♦7}

$$-\Pi_J(P) G_H(P) + \Pi_H(P) \rho(P). \quad (3.125)$$

Here we have used the fact that one can take the initial time to the infinity past and that the equation depends solely on the difference $x - y$ because the density matrix commutes with \hat{H} . The KMS relation implies $A_H(P) = (1 \pm 2f_{B/F}(p_0))A_J(P)$, and thus the right-hand-side of Eq. (3.92) vanishes, which yields the consistent result that there is no dynamics in thermal equilibrium. In other words, the Kadanoff-Baym equations has a fixed point, a space-time translation invariant solution, which corresponds to thermal equilibrium.

3.3.4 Boltzmann equation from Kadanoff-Baym equation

Taking the free field limit, one can clearly see that the Hadamard propagator encodes the number density and the Jordan propagator does the spectrum. If we consider a system which

^{♦7} A similar relation also holds for fermions.

is not equilibrium (even in the lower order correlators) and turn on interactions, then the number density and the spectrum start to evolve. Hence, as one can guess, the Kadanoff-Baym equations which describe the dynamics of one and two point functions include the so-called Boltzmann equation in a certain limit.

Hence, in this section, we give a brief sketch of how to derive the Boltzmann equation from the Kadanoff-Baym equations. There are several derivations in the literature [43, 72–74, 95–103], for instance. In this subsection, we follow the derivation with an emphasis on WKB approximation given in Ref. [104] for later convenience.

In the following, we consider a spatially homogenous system, and thus it is convenient to perform the Fourier transform with respect to the spatial difference, $\mathbf{x} - \mathbf{y}$. Then Kadanoff-Baym equations can be expressed as:

$$[\partial_t^2 + \mathbf{p}^2 + M^2(t)] G_J(t, t'; \mathbf{p}) = +i \int_{t'}^t d\tau \Pi_J(t, \tau; \mathbf{p}) G_J(\tau, t'; \mathbf{p}), \quad (3.126)$$

$$\begin{aligned} [\partial_t^2 + \mathbf{p}^2 + M^2(t)] G_H(t, t'; \mathbf{p}) = & +i \int_{t_{\text{ini}}}^t d\tau \Pi_J(t, \tau; \mathbf{p}) G_H(\tau, t'; \mathbf{p}) \\ & -i \int_{t_{\text{ini}}}^{t'} d\tau \Pi_H(t, \tau; \mathbf{p}) G_J(\tau, t'; \mathbf{p}), \end{aligned} \quad (3.127)$$

where we have extracted the possible local contribution to the self energy as $\Pi[G](x, y) = i\Pi_{\text{loc}}(x)\delta_{\mathcal{G}}(x, y) + \Pi_{\text{non-loc}}(x, y)$, and define the “mass” term as

$$M^2(t) \equiv V''(\phi(t)) + \Pi_{\text{loc}}(t; \mathbf{0}). \quad (3.128)$$

Here we have suppressed “non-loc” subscript for brevity and recovered t_{ini} for later convenience. We define the partial transform with respect to relative time as

$$\Pi_{\bullet}(t; \omega, \mathbf{p}) \equiv -i \int_0^{\infty} d\tau e^{i\omega\tau} \Pi_{\bullet}(t, t - \tau; \mathbf{p}), \quad (3.129)$$

for $\bullet = J, H$. Notice that this integration is dominated by $\tau \lesssim \tau_{\text{int}}$ with τ_{int} being a typical damping time scale of self energies,[♦] and that $\Pi_J(t; \omega, \mathbf{p})$ is nothing but the retarded self energy. Basically, we make use of the following assumptions to derive the Boltzmann-like equations.

(i) Quasi-particle spectrum We assume that the spectrum is dominated by particle-like excitations which have the following dispersion relation:

$$\Omega_p(t) \simeq \sqrt{M^2(t) + \mathbf{p}^2 + \Re \Pi_{\text{ret}}(t; \Omega_p, \mathbf{p})}; \quad \Gamma_p(t) \equiv -\frac{\Im \Pi_{\text{ret}}(t; \Omega_p, \mathbf{p})}{\Omega_p(t)}. \quad (3.130)$$

Here we implicitly assume the narrow width approximation; $\Omega_p \gg \Gamma_p$, which is typically satisfied for a weakly coupling system.

(ii) Separation of time scale To employ the WKB method, we assume that the following separation of time scale:

$$\dot{\Gamma}_p / \Gamma_p, \dot{\Omega}_p / \Omega_p \ll \Omega_p, \tau_{\text{int}}^{-1} \quad (3.131)$$

[♦] Roughly speaking, it can be regarded as the duration of each interaction, which may be typically related with the de Broglie wave length [104].

which is nothing but the adiabaticity condition (See Sec. 4.3.1). Roughly speaking, it implies that the background dynamics is so slow that the quasi-particle can see it as almost static.

(iii) Negligence of initial time Finally, we take the initial time to the remote past, $t_{\text{ini}} \rightarrow -\infty$, which implies that the finite time effects can be neglected.

Under these assumptions, one can obtain the WKB solution of the Jordan propagator [104]:

$$G_J(t, t'; \mathbf{p}) \simeq -i \frac{\sin \left[\int_{t'}^t d\tau \Omega_p(\tau) \right] e^{-\frac{1}{2} \left| \int_{t'}^t d\tau \Gamma_p(\tau) \right|}}{\sqrt{\Omega_p(t) \Omega_p(t')}} \quad (3.132)$$

Then, we move to the Hadamard propagator. Generally, the Hadamard propagator can be expressed by the following form:

$$G_H(t, t'; \mathbf{p}) = G_H^{(\text{hom})}(t, t'; \mathbf{p}) + \int_{t_{\text{ini}}}^t d\tau \int_{t_{\text{ini}}}^{t'} d\tau' G_J(t, \tau; \mathbf{p}) \Pi_H(\tau, \tau'; \mathbf{p}) G_J(\tau', t'; \mathbf{p}). \quad (3.133)$$

Here $G_H^{(\text{hom})}$ is the homogeneous solution:

$$\left[\partial_t^2 + \mathbf{p}^2 + M^2(t) \right] G_H^{(\text{hom})}(t, t'; \mathbf{p}) - i \int_{t_{\text{ini}}}^t d\tau \Pi_J(t, \tau; \mathbf{p}) G_H^{(\text{hom})}(\tau, t'; \mathbf{p}) = 0, \quad (3.134)$$

which can be written down in terms of the Jordan propagator and the initial condition of G_H [103]. Since we send the initial time to the remote past later, the initial condition dependence is suppressed. Hence, let us concentrate on the latter term, which is so called the memory integral. Inserting the WKB solution of Jordan propagator, one can obtain the WKB solution of the Hadamard propagator up to $\mathcal{O}(\Gamma_p/\Omega_p, \Gamma_p/\tau_{\text{int}}^{-1})$ [104]:

$$G_H(t, t'; \mathbf{p}) \simeq \frac{\cos \left[\int_{t'}^t d\tau \Omega_p(\tau) \right] e^{-\frac{1}{2} \left| \int_{t'}^t d\tau \Gamma_p(\tau) \right|}}{\sqrt{\Omega_p(t) \Omega_p(t')}} [1 + 2f(\mathbf{p}; t)], \quad (3.135)$$

with

$$\partial_t f(\mathbf{p}; t) = \Gamma_{<,p}(t) [1 + f(\mathbf{p}; t)] - \Gamma_{>,p}(t) f(\mathbf{p}; t). \quad (3.136)$$

Here we have defined the gain/loss term in the collision term as

$$\Gamma_{< />,p}(t) \equiv \frac{\Pi_{< />}(t; \Omega_p, \mathbf{p})}{2\Omega_p(t)}, \quad (3.137)$$

which implies

$$\Gamma_p(t) = \Gamma_{>,p}(t) - \Gamma_{<,p}(t). \quad (3.138)$$

It is noticeable that the obtained result is consistent with the following definition of the distribution function of quasi-particle excitations [100, 103, 105, 106]:

$$f(\mathbf{p}; t) = \frac{1}{4\Omega_p(t)} \left[\partial_t \partial_{t'} + \Omega_p^2(t) \right] G_H(t, t'; \mathbf{p}) \Big|_{t' \rightarrow t} - \frac{1}{2}, \quad (3.139)$$

up to $\mathcal{O}(\dot{\Omega}_p/\Omega_p^2, \dot{\Gamma}_p/\Gamma_p^2, \Gamma_p/\Omega_p)$. Here and hereafter we adopt this definition of distribution function since it directly relates with the energy momentum tensor. See also Sec. 4.3.1. Its generalization to fermion case is straightforward.

For later use, let us express the derivative of distribution function in terms of the Hadamard propagator. We follow the arguments given in Ref. [107]. From Eq. (3.139), one immediately obtains

$$\partial_t f(\mathbf{p}; t) = \frac{\dot{\Omega}_p}{4\Omega_p} \left[\Omega_p - \frac{1}{\Omega_p} \partial_t \partial_{t'} \right] G_H(t, t'; \mathbf{p}) \Big|_{t \rightarrow t'} \quad (3.140)$$

$$+ \frac{1}{4\Omega_p} \left[\partial_t^2 \partial_{t'} + \partial_t \partial_{t'}^2 + \Omega_p^2 (\partial_t + \partial_{t'}) \right] G_H(t, t'; \mathbf{p}) \Big|_{t \rightarrow t'}. \quad (3.141)$$

The first term corresponds to the particle production due to the change of the mass term and the second term imprints ordinary collision term due to interactions with the fixed mass term; one can show that this term reproduces the ordinary collision term of Boltzmann equations under the assumptions (i)–(iii). (See for instance [74]). Hence let us concentrate on the first term. Note that we have the following formulas:

$$[\partial_{t_1} + i\Omega_p] G_H(t_1, t_2; \mathbf{p}) \Big|_{t_1 \rightarrow t} = \int_{-\infty}^t dt_1 e^{i\Omega_p(t-t_1)} [\partial_{t_1}^2 + \Omega_p^2] G_H(t_1, t_2; \mathbf{p}), \quad (3.142)$$

$$[\partial_{t_2} - i\Omega_p] G_H(t_1, t_2; \mathbf{p}) \Big|_{t_2 \rightarrow t} = \int_{-\infty}^t dt_2 e^{-i\Omega_p(t-t_2)} [\partial_{t_2}^2 + \Omega_p^2] G_H(t_1, t_2; \mathbf{p}), \quad (3.143)$$

$$[\partial_t^2 + \Omega_p^2] [\partial_{t'}^2 + \Omega_p^2] G_H(t, t'; \mathbf{p}) = \Pi_H(t, t'; \mathbf{p}) + \dots, \quad (3.144)$$

where \dots represents higher order terms in the coupling. By using these formulas, one finds^{♦9}

$$\text{Eq. (3.140)} = -\Im \left\{ \frac{\dot{\Omega}_p}{2\Omega_p} \int_{-\infty}^t dt_2 e^{-i\Omega_p(t-t_2)} [M^2(t) - M^2(t_2)] G_H(t, t_2; \mathbf{p}) \right\}. \quad (3.145)$$

Later we will see that this term actually represents the energy transportation from the scalar condensation to radiation due to the adiabatic mass change in Sec. 4.5.

^{♦9} In the case of complex Φ -field, here we have assumed that there is no net U(1) number associated with $\Phi \rightarrow e^{i\theta} \Phi$.

Chapter 4

Dynamics of Scalar Condensation

Now we are in a position to discuss the time-evolution of scalar condensates in the early Universe by using theoretical equipment introduced in the previous chapter. Unless the scalar condensates contribute to today's energy component of our Universe, they should eventually decay into light particles before the BBN takes place. It implies that the scalar field inevitably couples to light fields and this very coupling makes the dynamics of the scalar condensate complicated. In a general far from equilibrium situation, it is quite difficult to follow the dynamics except for limited simple examples [108–113]. Instead of following all the dynamics with a simple toy model, in the following, we assume the “*separation of time scale*”, that is, the dynamical time scale of a scalar field is assumed to be much slower than the typical interaction time scale of other “fast” fields. Then, we can obtain the coarse-grained equations which are more tractable by assuming the “fast” fields remain close to thermal equilibrium, which is simply characterized by a temperature T .

In a word, the aim of this chapter is to understand the theoretical origin of the coarse-grained equation:

$$\ddot{\phi} + [3H + \Gamma_{\phi}(\phi; T)]\dot{\phi} + \frac{\partial V_{\text{eff}}(\phi; T)}{\partial \phi} = 0, \quad (4.1)$$

in particular, $V_{\text{eff}}(\phi; T)$ and $\Gamma_{\phi}(\phi; T)$. Note that the effects of background plasma are simply characterized by T under our assumption. Though this assumption poses restrictions to our treatment, there are still many examples of cosmological situations to which the obtained equations can be applied. Due to this correction, we will see in the Chap. 5 that the simplified arguments in Chap. 2 should be modified and the dynamics of the early Universe can be altered significantly.

The organization of this chapter is the following. First, in Sec. 4.1, let us clarify our setup; it captures essential features of scalar condensates which play important roles in the early Universe. Second, in Sec. 4.2, we discuss when the scalar condensate starts to oscillate. Third, in Sec. 4.3, we study the case where the oscillating scalar field acts as a non-adiabatic background for fields which couple to the oscillating scalar. Fourth, in Sec. 4.4, we focus on the case where all the relevant fields other than the oscillating scalar can be regarded to remain close to thermal equilibrium. Finally, in Sec. 4.6, we perform further coarse-graining, *oscillation-time average*, to obtain the evolution equation with the cosmological time scale H^{-1} . For the practical usage, one may skip to Sec. 4.6 after reading Sec. 4.1; there the relevant equations which we use in the following chapter are summarized.

This chapter is mainly based on our previous works [35, 44, 45].

4.1 Preliminaries

4.1.1 Setup

As explained repeatedly, the interaction between the scalar field and other fields is required in order to convert the energy of the oscillating scalar into radiation. In addition, these coupled fields should interact with radiation; via for instance SM gauge interaction. To capture these essential feature but not to trouble with unnecessary complications, let us consider the following phenomenological setup in the following discussion:

$$\mathcal{L} = \mathcal{L}_{\text{kin}} - \frac{1}{2}m_\phi^2\phi^2 - \lambda^2\phi^2|\tilde{\chi}|^2 - y\phi\tilde{\chi}\chi + \mathcal{L}_{\text{others}}, \quad (4.2)$$

where λ and y are coupling constants assumed to be perturbative, \mathcal{L}_{kin} denotes canonical kinetic terms and $\mathcal{L}_{\text{others}}$ represents the other fields except for ϕ including gauge bosons (also interactions among them). Note that the scalar ϕ is assumed to interact with other light fields only through $\lambda^2\phi^2|\chi|^2$ and $y\tilde{\chi}\chi$. For notational simplicity, we sometimes denote these couplings, y and λ , as $\tilde{\lambda}$ collectively. A complex scalar $\tilde{\chi}$ and a Dirac fermion χ are assumed to be charged under some gauge group (e.g. SM gauge group) and lighter than ϕ in the vacuum and their bare masses are neglected in the following. Through this gauge interaction, the χ -fields contact with other light fields in radiation. Hereafter g represents the gauge coupling constant, and we sometimes use the fine structure constant, $\alpha \equiv g^2/(4\pi)$. For simplicity, we mainly concentrate on the case where the coupling between ϕ and χ is relatively smaller than the gauge coupling constant, $\lambda, y < \alpha$ (See also discussion in Sec. 4.4). A schematic figure of this Lagrangian is shown in Fig. 4.1.

One might wonder that if we write down the action respecting the gauge symmetry at a renormalizable level, then one must have following terms including the scalar field ϕ : a three point interaction with the complex scalar $\Lambda\phi|\tilde{\chi}|^2$ and a four point self interaction $\lambda_\phi^2\phi^4/4!$. In particular, the latter term should be generated radiatively by the χ loops [Eq. (4.5)]^{†1} even if one discards it at the tree level. These terms cause another complications to the dynamics; Including the three point interaction induces a possible instability of $\tilde{\chi}$ field and also cause a shift of the ϕ 's potential minimum, and the four point self interaction may make the ϕ -condensate inhomogeneous by the non-perturbative ϕ -particle production with a highly non-thermal distribution. The latter effect is discussed separately in Sec. 4.3.5.

For this aspect, it is instructive to consider a SUSY theory, as a well motivated candidate of new physics, because it restricts the couplings due to the symmetry. Let us consider a superpotential of the form, $W = \lambda\phi\tilde{\chi}\chi$. Then the Lagrangian reads

$$\mathcal{L} = \mathcal{L}_{\text{kin}} - (\lambda\phi\chi\tilde{\chi} + \text{h.c.}) - \lambda^2|\phi|^2(|\tilde{\chi}|^2 + |\tilde{\tilde{\chi}}|^2) - V_{\text{SB}} + \mathcal{L}_{\text{others}}, \quad (4.3)$$

with the soft SUSY breaking term being

$$V_{\text{SB}} = m_\phi^2|\phi|^2 + m_\chi^2|\tilde{\chi}|^2 + m_{\tilde{\chi}}^2|\tilde{\tilde{\chi}}|^2 + (A_\chi\phi\tilde{\chi}\tilde{\tilde{\chi}} + \text{h.c.}), \quad (4.4)$$

where $\chi(\tilde{\chi})/\tilde{\chi}(\tilde{\tilde{\chi}})$ are fermionic/scalar components of each chiral superfield and $m_\phi, m_\chi, m_{\tilde{\chi}}, A_\chi$ are SUSY breaking parameters of same orders of magnitude. Here note that χ and $\tilde{\chi}$ are Weyl fermions. As one can see, the Coleman-Weinberg (CW) correction [114] get suppressed, which is given by

$$V_{\text{CW}} = \sum_F \epsilon_F \frac{m_\chi^4(\phi)}{64\pi^2} \left[\ln \frac{m_\chi^2(\phi)}{\mu^2} - \frac{3}{2} \right], \quad (4.5)$$

^{†1} For brevity, we sometimes use χ for both the fermion χ and boson $\tilde{\chi}$ unless we need to distinguish them.

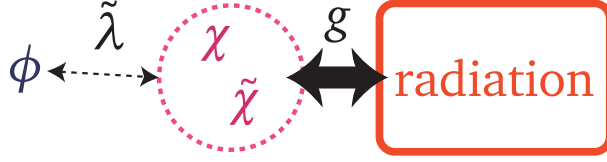


Figure 4.1: Schematic figure of our phenomenological setup.

where $\epsilon_F = +1$ for a real scalar boson $\tilde{\chi}$ and $\epsilon_F = -2$ for a (Weyl) fermion χ , since both the Yukawa coupling and the quartic coupling are given by λ . And there remain small logarithmic corrections due to the SUSY breaking effects. In this case, the motion of scalar condensate becomes elliptical in general since it is a complex scalar field. However, if the A -term contribution to the scalar potential is small, one can approximate the dynamics of complex scalar field with its one-dimensional radial component. After all, one can reduce the relevant Lagrangian as

$$\mathcal{L} = \mathcal{L}_{\text{kin}} - \frac{1}{2}m_\phi^2\phi^2 - (\lambda\phi\chi\tilde{\chi} + \text{h.c.}) - \lambda^2\phi(|\tilde{\chi}|^2 + |\chi|^2) + \mathcal{L}_{\text{others}}, \quad (4.6)$$

where the coupling λ is redefined as $\lambda/\sqrt{2} \rightarrow \lambda$, and the A -term contribution is neglected since its effect is smaller than $\lambda^2|\tilde{\chi}|^2\phi^2$ for a large expectation value of scalar condensate which we are interested in. As one can see, the setup given in Eq. (4.2) includes this class of models.

In Eq. (4.2), we assume that the potential origin of ϕ coincides with the enhanced symmetry point of χ -fields, though there are no a priori reasons. We comment on this issue in the following. If it is different from the enhanced symmetry point, then Eq. (4.2) should become

$$\mathcal{L} = \mathcal{L}_{\text{kin}} - \frac{1}{2}m_\phi^2(\phi - v)^2 - \lambda^2\phi^2|\tilde{\chi}|^2 - y\phi\tilde{\chi}\chi + \mathcal{L}_{\text{others}}. \quad (4.7)$$

First of all, if the typical scale of dynamics in consideration, *e.g.* temperature of background plasma, is much larger than the VEV, $\tilde{\lambda}v \ll T$, then the dynamics may be well approximated with Eq. (4.2). Hence, let us consider the case with a sizable VEV; $\tilde{\lambda}v \gg T, m_\phi$. In this case, after the possible efficient χ -particle production at the preheating stage (See Sec. 4.3), the effective Lagrangian can be obtained from integrating out heavy χ -fields with masses $\sim \tilde{\lambda}v$:

$$\mathcal{L} = \mathcal{L}_{\text{kin}} - \frac{1}{2}m_\phi^2\phi^2 - A\frac{\phi}{v}F^{a\mu\nu}F_{\mu\nu}^a + \mathcal{L}_{\text{others}}, \quad (4.8)$$

where A is a model dependent factor and $F_{\mu\nu}^a$ represents the field strength of gauge field under which χ -fields are charged [See also Eq. (4.140)]. The effects of background plasma in this case can be also inferred from the discussion given in Sec. 4.4.

Here we only write down renormalizable terms, but our following analyses can be applied to more general forms of the scalar potential for ϕ . For instance, it is easy to include the following higher dimensional terms:

$$V_{\text{higher}}(\phi) = -\frac{c}{2}H^2\phi^2 + \frac{\kappa\phi^n}{nM_{\text{pl}}^{n-4}}, \quad (4.9)$$

where $n \geq 4$ is an integer, κ is a coupling constant, H denotes the Hubble parameter and c is an $\mathcal{O}(1)$ constant. This is the typical form in the case of MSSM (Minimal Supersymmetric Standard Model) flat direction when the A -term contribution is small [11]. Also higher dimensional terms which induce decays of ϕ into radiation can be included.

Finally, let us briefly comment on the case where ϕ is also charged under the SM gauge group. This is the case, for instance, where ϕ is the SM Higgs field, ϕ is the MSSM flat

direction and so on. In this case, the scalar ϕ also couples to the SM gauge bosons, *e.g.* $\phi^2 A_\mu^2$, $\phi^2 B_\mu^2$ where A_μ and B_μ represent SM gauge bosons of $SU(2)_W$ and $U(1)_Y$ respectively. Unless the SM gauge group is completely broken down, we expect that (some parts of) our analyses are useful even in this case. However, note that we should be careful in studying this case because the basic assumption that other fields can remain close to thermal equilibrium might be threatened since the SM gauge group is broken down by the scalar condensation (If it is completely broken down, our following analyses may not be used).

4.1.2 Kadanoff-Baym Equations

To derive the coarse-grained equation, let us move back to the Kadanoff-Baym equations of this system as a starting point of our discussion. We will not explicitly write down the effects of cosmic expansion in computing the 2PI effective action, rather consider its effect separately as an adiabatic background.^{♦2} Also, for notational simplicity, we will not write down contributions from other light fields except for χ explicitly unless otherwise stated. The 2PI effective action of Eq. (4.2) is given by [44, 116]

$$\begin{aligned}\Gamma[\phi, G_\phi, G_\chi, S_\chi] = & S[\phi] + \frac{i}{2} \text{Tr} G_{\phi, \text{F}}^{-1} \cdot G_\phi - \frac{i}{2} \text{Tr} \ln G_\phi - \frac{i}{2} \text{Tr} G_\phi^{-1} \cdot G_\phi \\ & + i \text{Tr} G_{\chi, \text{F}}^{-1} \cdot G_\chi - i \text{Tr} \ln G_\chi - i \text{Tr} G_\chi^{-1} \cdot G_\chi \\ & - i \text{Tr} S_{\chi, \text{F}}^{-1} \cdot S_\chi + i \text{Tr} \ln S_\chi + i \text{Tr} S_\chi^{-1} \cdot S_\chi \\ & + \Gamma_2[\phi, G_\phi, G_\chi, S_\chi],\end{aligned}\tag{4.10}$$

where G_\bullet denotes Schwinger-Keldysh propagators for scalar fields, and S_\bullet is one for a fermion field. Here the Schwinger-Keldysh propagators are defined as

$$G_\phi(x, y) \equiv \langle T_\epsilon \hat{\phi}(x) \hat{\phi}(y) \rangle, \tag{4.11}$$

$$G_\chi(x, y) \equiv \langle T_\epsilon \hat{\chi}(x) \hat{\chi}^*(y) \rangle, \tag{4.12}$$

$$S_\chi(x, y) \equiv \langle T_\epsilon \hat{\chi}(x) \hat{\chi}(y) \rangle. \tag{4.13}$$

The free propagators are given by

$$G_{\phi, \text{F}}^{-1}(x, y) = i [\square_x + m_\phi^2] \delta_\epsilon(x, y), \tag{4.14}$$

$$G_{\chi, \text{F}}^{-1}(x, y) = i [\square_x + \lambda^2 \phi^2(x)] \delta_\epsilon(x, y), \tag{4.15}$$

$$S_{\chi, \text{F}}^{-1}(x, y) = i [-i \not{\partial}_x + y \phi(x)] \delta_\epsilon(x, y). \tag{4.16}$$

^{♦2} We can start with the Schwinger-Dyson Eqs. on the closed time path contour in terms of conformal time [115], and then reduce the obtained set of equations making use of the fact that the cosmic expansion is adiabatic. See also Appendix C.3.

Here and hereafter possible gauge indices are suppressed for notational simplicity unless otherwise stated. Performing a small coupling expansion, one can compute Γ_2 perturbatively^{♦3}

$$i\Gamma_2[\phi, G_\phi, G_\chi, S_\chi] = -i\lambda^2 \int_{\mathcal{C}} d^4x G_\phi(x, x) G_\chi(x, x) \quad (4.17)$$

$$+ \frac{1}{2}(-2i\lambda^2)^2 \int_{\mathcal{C}} d^4x d^4y \phi(x) G_\phi(x, y) G_\chi(x, y) G_\chi(y, x) \phi(y) \quad (4.18)$$

$$+ 2\frac{1}{2}(-i\lambda^2)^2 \int_{\mathcal{C}} d^4x d^4y G_\phi(x, y)^2 G_\chi(x, y) G_\chi(y, x) \quad (4.19)$$

$$- \frac{(-iy)^2}{2} \int_{\mathcal{C}} d^4x d^4y G_\phi(x, y) \text{Tr}[S_\chi(x, y) S_\chi(y, x)] \quad (4.20)$$

+

The equations of motion can be obtained from the stationary condition of the 2PI effective action. Since we are interested in a spatially homogeneous system, it is convenient to perform the Fourier transform with respect to the spatial difference: $A(x_0, y_0, \mathbf{x}-\mathbf{y}) \xrightarrow{\text{Fourier tr.}} A(x_0, y_0; \mathbf{p})$. Defining the self energy as^{♦4}

$$\Pi^\phi(x, y) \equiv -2i \frac{\delta \Gamma_2}{\delta G_\phi(x, y)}; \quad \Pi^\chi(x, y) \equiv -i \frac{\delta \Gamma_2}{\delta G_\chi(y, x)}; \quad \Sigma^\chi(x, y) \equiv i \frac{\delta \Gamma_2}{\delta S_\chi(y, x)}, \quad (4.21)$$

one can write down the Schwinger-Dyson equations on the close time path contour as

$$\left[(G_F^{\chi/\phi^{-1}} + \Pi^{\chi/\phi}) \cdot G^{\chi/\phi} \right](x, y) = \delta_{\mathcal{C}}(x, y); \quad \left[(S_F^{\chi^{-1}} + \Sigma^\chi) \cdot S^\chi \right](x, y) = \delta_{\mathcal{C}}(x, y). \quad (4.22)$$

Then, the Kadanoff-Baym equations of the propagators can be obtained as follows:

$$\left[\partial_t^2 + \mathbf{p}^2 + M_{\tilde{\chi}/\phi}^2(t) \right] G_J^{\chi/\phi}(t, t'; \mathbf{p}) = +i \int_{t'}^t d\tau \Pi_J^{\chi/\phi}(t, \tau; \mathbf{p}) G_J^{\chi/\phi}(\tau, t'; \mathbf{p}), \quad (4.23)$$

$$\begin{aligned} \left[\partial_t^2 + \mathbf{p}^2 + M_{\tilde{\chi}/\phi}^2(t) \right] G_H^{\chi/\phi}(t, t'; \mathbf{p}) &= +i \int_0^t d\tau \Pi_J^{\chi/\phi}(t, \tau; \mathbf{p}) G_H^{\chi/\phi}(\tau, t'; \mathbf{p}) \\ &- i \int_0^{t'} d\tau \Pi_H^{\chi/\phi}(t, \tau; \mathbf{p}) G_J^{\chi/\phi}(\tau, t'; \mathbf{p}), \end{aligned} \quad (4.24)$$

$$\left[(-i\gamma^0 \partial_t + \boldsymbol{\gamma} \cdot \mathbf{p} + M_\chi) S_J^\chi \right](t, t'; \mathbf{p}) = +i \int_{t'}^t d\tau \Sigma_J^\chi(t, \tau; \mathbf{p}) S_J^\chi(\tau, t'; \mathbf{p}), \quad (4.25)$$

$$\begin{aligned} \left[(-i\gamma^0 \partial_t + \boldsymbol{\gamma} \cdot \mathbf{p} + M_\chi) S_H^\chi \right](t, t'; \mathbf{p}) &= +i \int_0^t d\tau \Sigma_J^\chi(t, \tau; \mathbf{p}) S_H^\chi(\tau, t'; \mathbf{p}) \\ &- i \int_0^{t'} d\tau \Sigma_H^\chi(t, \tau; \mathbf{p}) S_J^\chi(\tau, t'; \mathbf{p}), \end{aligned} \quad (4.26)$$

^{♦3} Note that the smallness of couplings λ and y alone is not enough to justify the above perturbation. Also $\lambda^2 \phi$ expansion should be controlled because we are interested in a large field value of ϕ . Since the effective mass $\lambda^2 \phi^2$ is completely resummed, χ becomes heavy at a large field value of ϕ . Then, their number density is quite suppressed under our assumption, that is, χ has thermal contact with other light particles and can quickly reduce their number. Thus, contributions from a large ϕ are expected to be suppressed for $\lambda|\phi| > T$.

^{♦4} Note that the Schwinger-Keldysh propagator for a real field satisfies $G(x, y) = \theta_{\mathcal{C}}(x_0, y_0) \langle \hat{\phi}(x) \hat{\phi}(y) \rangle + \theta_{\mathcal{C}}(y_0, x_0) \langle \hat{\phi}(y) \hat{\phi}(x) \rangle = G(y, x)$, but this is not the case with the complex scalar field and the fermion field. Thus, one has to be careful in differentiating the 2PI effective action.

where the effective masses are defined as

$$M_\phi^2(t) = m_\phi^2 + \lambda^2 \int_p G_H^\chi(t, t; \mathbf{p}), \quad (4.27)$$

$$M_{\tilde{\chi}}^2(t) = \lambda^2 \left[\phi^2(t) + \frac{1}{2} \int_p G_H^\phi(t, t; \mathbf{p}) \right] + m_{\chi, \text{th}}^2, \quad (4.28)$$

$$M_\chi(t, t') = y\phi(t)\delta(t - t') + \Sigma_{\text{th}}(t, t'; \mathbf{p}). \quad (4.29)$$

Here we have explicitly pulled out dominant corrections to dispersion relations caused by the presence of other light fields in thermal plasma. For the scalars $\tilde{\chi}$, such corrections can be roughly parametrized by the mass correction to the quadratic dispersion relation as $m_{\chi, \text{th}} \sim g^2 T^2$ with g being the typical coupling constant which connects $\tilde{\chi}$ with the thermal plasma. For the fermions χ , such corrections are involved due to the non-trivial chiral structure even at the thermal equilibrium [117] (See also Sec. 4.2), and hence we simply write down such corrections as Σ_{th} . Yet, for our rough arguments given in the following chapters, such complicated structures are not so important, rather its typical order is essential, which is the same as the scalar case, $\Sigma_{\text{th}} \sim gT$. Aside from contributions from other light fields in the plasma, the self energies of ϕ, χ can be obtained from the definition [Eqs. (4.21)]:^{♦5}

$$\Pi_J^\phi(x, y) \supset + 2\lambda^4 \phi(x) [G_J^\chi(x, y)G_H^\chi(y, x) - G_H^\chi(x, y)G_J^\chi(y, x)] \phi(y) \quad (4.30)$$

$$+ \lambda^4 [G_J^\chi(x, y)G_H^\chi(y, x) - G_H^\chi(x, y)G_J^\chi(y, x)] G_H^\phi(x, y) \\ + \lambda^4 [G_H^\chi(x, y)G_H^\chi(y, x) - G_J^\chi(x, y)G_J^\chi(y, x)] G_J^\phi(x, y) \quad (4.31)$$

$$- \frac{y^2}{2} \text{Tr} [S_J^\chi(x, y)S_H^\chi(y, x) - S_H^\chi(x, y)S_J^\chi(y, x)], \quad (4.32)$$

$$\Pi_H^\phi(x, y) \supset + 2\lambda^4 \phi(x) [G_H^\chi(x, y)G_H^\chi(y, x) - G_J^\chi(x, y)G_J^\chi(y, x)] \phi(y) \quad (4.33)$$

$$+ \lambda^4 [G_J^\chi(x, y)G_H^\chi(y, x) - G_H^\chi(x, y)G_J^\chi(y, x)] G_J^\phi(x, y) \\ + \lambda^4 [G_H^\chi(x, y)G_H^\chi(y, x) - G_J^\chi(x, y)G_J^\chi(y, x)] G_H^\phi(x, y) \quad (4.34)$$

$$- \frac{y^2}{2} \text{Tr} [S_H^\chi(x, y)S_H^\chi(y, x) - S_J^\chi(x, y)S_J^\chi(y, x)], \quad (4.35)$$

$$\Pi_J^\chi(x, y) \supset + 2\lambda^4 \phi(x) [G_J^\phi(x, y)G_H^\chi(x, y) + G_H^\phi(x, y)G_J^\chi(x, y)] \phi(y) \quad (4.36)$$

$$+ \lambda^4 G_J^\phi(x, y)G_H^\phi(x, y)G_H^\chi(x, y) + \frac{\lambda^4}{2} [G_H^{\phi^2}(x, y) + G_J^{\phi^2}(x, y)] G_J^\chi(x, y), \quad (4.37)$$

$$\Pi_H^\chi(x, y) \supset + 2\lambda^4 \phi(x) [G_H^\phi(x, y)G_H^\chi(x, y) + G_J^\phi(x, y)G_J^\chi(x, y)] \phi(y) \quad (4.38)$$

$$+ \lambda^4 G_J^\phi(x, y)G_H^\phi(x, y)G_J^\chi(x, y) + \frac{\lambda^4}{2} [G_H^{\phi^2}(x, y) + G_J^{\phi^2}(x, y)] G_H^\chi(x, y), \quad (4.39)$$

$$\Sigma_H(x, y) \supset \frac{y^2}{2} [G_H^\phi(x, y)S_H^\chi(x, y) + G_J^\phi(x, y)S_J^\chi(x, y)], \quad (4.40)$$

$$\Sigma_J(x, y) \supset \frac{y^2}{2} [G_J^\phi(x, y)S_H^\chi(x, y) + G_H^\phi(x, y)S_J^\chi(x, y)] \quad (4.41)$$

^{♦5} \supset indicates that we have omitted contributions from other light fields except for χ in radiation.

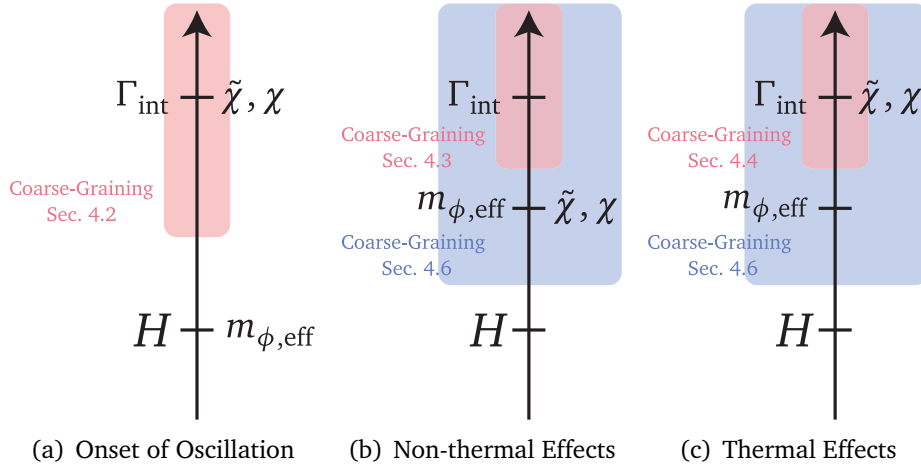


Figure 4.2: Rough sketch of coarse-graining scales for each case which we will discuss. Here Γ_{int} denotes the typical interaction time scale among particles in the thermal plasma, $m_{\phi,\text{eff}}$ is the effective mass of the ϕ -condensation (See Sec. 4.2.3), and the symbol $\tilde{\chi}, \chi$ means the dynamical time scale of $\tilde{\chi}, \chi$ -fields. Note that here we only write down the case discussed in Sec. 4.4.1 as Fig. 4.2(c) for simplicity, but the condition $m_{\phi,\text{eff}} \ll \Gamma_{\text{int}}$ is not mandatory if the oscillation amplitude is small enough $\tilde{\lambda}\tilde{\phi} \ll gT$. See also the discussion at the beginning of Sec. 4.4.2.

Since we are interested in particle-like excitations which have much larger energy than the Hubble parameter, we will treat the cosmic expansion separately as an adiabatic expansion; it only make momenta of each quasi-particles and the temperature red-shifted.

The equation of motion for the mean field can be obtained from $0 = \delta\Gamma/\delta\phi = \delta S/\delta\phi + \delta\Delta\Gamma/\delta\phi$ with $\Delta\Gamma$ being the correction to the tree level action, and it reads

$$0 = \left[\partial_t^2 + 3H\partial_t + M_\phi^2(t) \right] \phi(t) - \frac{\gamma}{2} \int_p \text{Tr} [S_H^\chi(t, t; \mathbf{p})] - i \int_{t_{\text{ini}}}^t d\tau \Pi_J(t, \tau) \phi(\tau), \quad (4.42)$$

where

$$\begin{aligned} & \lambda^4 \left[G_H^\chi(x, y) G_H^\chi(y, x) G_J^\phi(x, y) - G_J^\chi(x, y) G_J^\chi(y, x) G_J^\phi(x, y) \right. \\ & \quad \left. + G_J^\chi(x, y) G_H^\chi(y, x) G_H^\phi(x, y) - G_H^\chi(x, y) G_J^\chi(y, x) G_H^\phi(x, y) \right] \\ & \xrightarrow{\text{Fourier tr. w.r.t. } \mathbf{x}-\mathbf{y}} \Pi_J(x_0, y_0; \mathbf{p}); \quad \Pi_J(t, t') \equiv \Pi_J(t, t'; \mathbf{0}). \end{aligned} \quad (4.43)$$

Note that a peculiar term $\text{Tr} S_H^\chi$ vanishes for $\phi = 0$ due to the chiral symmetry and hence it finally yields a term proportional to ϕ as we will see in the subsequent section. Here the adiabatic expansion of the Universe is taken into account explicitly as a friction term in the equation of motion.

In the following sections, we make use of coarse-graining to obtain more tractable equations in various regimes. There are three stages; (a) the scalar condensate starts to oscillate coherently against the expansion of the Universe, (b) the oscillating scalar condensate acts as a non-adiabatic background for the coupled χ -fields, (c) the oscillating scalar condensate can be regarded as an adiabatic background due to the rapid interactions among light particles. Fig. 4.2 shows a rough sketch of coarse-graining scales in each regime. Eventually, we will obtain coarse-grained equations which describe the evolution in the cosmological time scale $\sim H^{-1}$. In Sec. 4.2, we discuss the time when the scalar condensate starts to oscillate around its effective potential minimum [(a)]. Then, in Secs. 4.3 [(b)] and 4.4 [(c)], we study the dynamics of scalar condensate with neglecting the cosmic expansion because the Hubble parameter

soon becomes much smaller than the effective mass of scalar field after the onset of oscillation. Finally, we make use of further coarse-graining in order to obtain the evolution equation in the cosmological time scale (H^{-1}) in Sec. 4.6.

4.2 Beginning of Oscillation

In this section, we discuss the time when the scalar field starts to oscillate against the expansion of the Universe. Throughout this section, we assume that the ambient plasma is produced via the reheating (e.g. by the inflaton) before the onset of oscillation. Hence, in the case of the inflaton, for instance, the following discussion is not applicable because there is no plasma right after the end of inflation.^{♦6} It plays essential roles in the scalar condensates other than inflaton; such as curvaton [46], axion [45, 118, 119], Affleck-Dine field [120–122] and so on.

As can be seen from Eq. (4.42), the scalar condensation starts to oscillate when the Hubble parameter becomes comparable to the typical dynamical scale (e.g. $m_\phi \sim H$ if the vacuum potential dominates the force). Before this time, the plasma has enough time to attain thermal equilibrium in the presence of the constant homogeneous background ϕ . This implies that the Kadanoff-Baym equations for light fields including χ are close to the thermal fixed point and one can formally obtain their solutions as thermal propagators in the presence of ϕ : $G_\chi^{\text{th}}(P; \phi)$ and $S_\chi^{\text{th}}(P; \phi)$. Thus, the coarse-graining scale is nothing but interactions in the thermal plasma (See Fig. 4.2).

Typically, there are two regimes depending on the expectation value of ϕ ; a large field value regime, $\tilde{\lambda}|\phi| > T$, and a small field value regime, $\tilde{\lambda}|\phi| < T$. In the large field value regime, there are no χ -particles in the thermal plasma because of the Boltzmann suppression by the large effective mass $\tilde{\lambda}|\phi| > T$. In the small field value regime, χ -particles are thermally populated since their effective mass is smaller than the temperature $\tilde{\lambda}|\phi| < T$. Thus, the underlying physics is different and we discuss these two regimes separately in the following.

4.2.1 Thermal Mass

In the small field value regime $\tilde{\lambda}|\phi| < T$, the χ -particles are in the thermal plasma, that is, Eqs. (4.23)–(4.26) are close to the thermal fixed point. Thus, the only non-trivial equation is that of the Jordan propagator since it is related with the Hadamard propagator via the KMS relation. It is convenient to rewrite the equation by means of the retarded and advanced propagator in obtaining the formal solution of Jordan propagator:

$$\left[\partial_t^2 + \mathbf{p}^2 + M_\chi^2 \right] G_{\text{ret/adv}}^\chi(t; \mathbf{p}) + \int_{-\infty}^{\infty} d\tau \Pi_{\text{ret/adv}}^\chi(t - \tau; \mathbf{p}) G_{\text{ret/adv}}^\chi(\tau; \mathbf{p}) = \delta(t), \quad (4.44)$$

$$\left[-i\gamma^0 \partial_t + \boldsymbol{\gamma} \cdot \mathbf{p} + \lambda\phi \right] S_{\text{ret/adv}}^\chi(t; \mathbf{p}) + \int_{-\infty}^{\infty} d\tau \Sigma_{\text{ret/adv}}^\chi(t - \tau; \mathbf{p}) S_{\text{ret/adv}}^\chi(\tau; \mathbf{p}) = \delta(t), \quad (4.45)$$

where the retarded and advanced self energies are defined as $A_{\text{ret/adv}}(t, t') = \mp i\theta(\pm t \mp t')A_J(t, t')$ with $A = \Pi, \Sigma$. It is noticeable that the retarded and advanced self energies can be expressed as

$$\left\{ \begin{array}{l} \Pi_{\text{ret/adv}}^\chi(p_0, \mathbf{p}) \\ \Sigma_{\text{ret/adv}}^\chi(p_0, \mathbf{p}) \end{array} \right\} = \text{PV} \int \frac{dk_0}{2\pi} \frac{1}{p_0 - k_0} \left\{ \begin{array}{l} \Pi_J^\chi(k_0, \mathbf{p}) \\ \Sigma_J^\chi(k_0, \mathbf{p}) \end{array} \right\} \mp \frac{i}{2} \left\{ \begin{array}{l} \Pi_J^\chi(p_0, \mathbf{p}) \\ \Sigma_J^\chi(p_0, \mathbf{p}) \end{array} \right\}. \quad (4.46)$$

^{♦6} The warm inflation [37] is an exception.

Here we have restored the contribution from the thermal plasma Σ_{th} , which gives the dominant correction to the dispersion relation, into Σ^χ for notational simplicity. Performing the Fourier transform, one can obtain the formal expression of spectral densities, which is given by $\rho_\bullet = -iG_{\text{ret}}^\bullet + iG_{\text{adv}}^\bullet$:

$$\rho_{\tilde{\chi}}(P) = \frac{1}{i} \left[\frac{1}{M_{\tilde{\chi}}^2 - P^2 + \Pi_{\text{ret}}^\chi(P)} - \frac{1}{M_{\tilde{\chi}}^2 - P^2 + \Pi_{\text{adv}}^\chi(P)} \right], \quad (4.47)$$

$$\rho_\chi(P) = \frac{1}{i} \left[\frac{1}{\lambda\phi - \not{P} + \Sigma_{\text{ret}}^\chi(P)} - \frac{1}{\lambda\phi - \not{P} + \Sigma_{\text{adv}}^\chi(P)} \right]. \quad (4.48)$$

If the spectral density is well concentrated around the pole, one may approximate it with the Breit-Wigner form (See also Appendix B.3). For bosons, the spectral density can be expressed as

$$\rho_{\tilde{\chi}}(P) \simeq Z_p \frac{2p_0\Gamma_p}{[p_0^2 - \Omega_p^2]^2 + [p_0\Gamma_p]^2} + \rho_{\tilde{\chi}}^{(\text{cont})}(P), \quad (4.49)$$

where $\Omega_p = \sqrt{\lambda^2\phi^2 + \mathbf{p}^2 + \Re\Pi_{\text{ret}}^\chi(\Omega_p, \mathbf{p})}$, $\Gamma_p = -Z_p \Im\Pi_{\text{ret}}^\chi(\Omega_p, \mathbf{p})/\Omega_p$, and the wave functional renormalization is given by

$$Z_p = \left[1 - \frac{1}{2\Omega_p} \frac{\partial \Re\Pi_{\text{ret}}^\chi(\omega, \mathbf{p})}{\partial \omega} \Big|_{\omega=\Omega_p} \right]^{-1}. \quad (4.50)$$

The latter part $\rho_{\tilde{\chi}}^{(\text{cont})}$ gives the continuum spectrum. In the following, we approximate that the pole contribution is dominated by the quasi-particle pole; $Z_p \simeq 1$.

For fermions, there is one more complication due to the chiral symmetry with vanishing ϕ . Using the Kramers-Kronig relation given in Eq. (4.46), one can rewrite the retarded/advanced self energy as $\Sigma_{\text{ret/adv}}^\chi = \Sigma_J^\chi \mp i\Sigma_J^\chi/2$. In Vacuum, the form of self energy is strongly constrained by the Lorentz and chiral symmetry as you know. In the presence of thermal plasma, the plasma introduces a special frame, *i.e.* its rest frame. In a general frame, the thermal plasma has a four velocity u^μ with $u^2 = 1$, and hence the self energy can be expressed as [117]

$$\Sigma^\chi = -A^R \not{p} - B^R \not{u} - y\phi C^R; \quad \frac{\Sigma_J^\chi}{2} = A^I \not{p} + B^I \not{u} + y\phi C^I, \quad (4.51)$$

where A^\bullet, B^\bullet and C^\bullet are Lorentz invariant functions. Here and hereafter, we take the rest frame of thermal plasma, and then they become

$$\Sigma^\chi = -A^R \not{p} - B^R \gamma_0 - y\phi C^R; \quad \frac{\Sigma_J^\chi}{2} = A^I \not{p} + B^I \gamma_0 + y\phi C^I. \quad (4.52)$$

To see its pole structure, let us consider two regimes; (i) $y|\phi| \ll gT$ and (ii) $y|\phi| \gg gT$. If one specifies constituents of the plasma and interactions, one can compute these thermal corrections perturbatively. Here we simply explain their typical behavior.

In the first case (i), the Dirac mass from the background field is smaller than the typical size of thermal corrections, and hence the self energy can be approximated with

$$\Sigma^\chi \simeq -A^R \not{p} - B^R \gamma_0; \quad \frac{\Sigma_J^\chi}{2} \simeq A^I \not{p} + B^I \gamma_0. \quad (4.53)$$

Then, the spectral density has the following form [123]

$$\rho_\chi(P) \simeq \sum_{s=\pm} \frac{Z_p^s}{2} \left[\frac{\Gamma_p^s}{[p_0 - \Omega_p^s]^2 + \Gamma_p^{s2}/4} (\gamma_0 - \hat{\mathbf{p}} \cdot \boldsymbol{\gamma}) + \frac{\Gamma_p^s}{[p_0 + \Omega_p^s]^2 + \Gamma_p^{s2}/4} (\gamma_0 + \hat{\mathbf{p}} \cdot \boldsymbol{\gamma}) \right] + \rho_\chi^{(\text{cont})}(P), \quad (4.54)$$

where the dispersion relation is determined by

$$\Omega_p^\pm = \pm p - A_p^R(\Omega_p^\pm) (\Omega_p^\pm \mp p) - B_p^R(\Omega_p^\pm); \quad \Gamma_p^\pm/2 = Z_p^\pm [A_p^I(\Omega_p^\pm) (\Omega_p^\pm \mp p) + B_p^I(\Omega_p^\pm)], \quad (4.55)$$

and the wave functional renormalization is given by

$$Z_p^\pm = \left(1 + \frac{\partial}{\partial \omega} [A_p^R(\omega)(\omega \mp p) + B_p^R(\omega)] \right) \Big|_{\omega=\Omega_p^\pm}^{-1}. \quad (4.56)$$

The plus contribution represents the ordinary particle-like excitation corrected by the thermal plasma. In addition, one can see that there is another minus contribution. It stands for the collective excitation in the presence of thermal bath, which is the so-called “plasmino” [117]. The plasmino contribution becomes important when its momentum is soft, $p \lesssim gT$, and it decouples for a larger momentum, $p \gtrsim gT$. This behavior is imprinted in the wave function renormalization, Z_p^- . In the following discussion, the plasmino contribution is not so important since the typical momentum is dominated by $p \sim T$ in most cases. Thus, we assume that the spectral density is dominated by the ordinary particle-like excitation; $Z_p^+ \simeq 1$, $Z_p^- \simeq 0$. The dispersion relation is not given by the ordinary quadratic one as you can see, but its typical behavior can be well approximated by $\Omega_p^+ \simeq \sqrt{p^2 + m_{\chi,\text{as}}^2}$ with $m_{\chi,\text{as}}$ being the so-called asymptotic mass, $m_{\chi,\text{as}} \sim gT$. See for instance Ref. [86] for concrete examples. In the following, we do not worry about the difference between the screening mass and the asymptotic mass since we are interested in its typical value not precise one. Hence, we simply write the asymptotic mass as $m_{\chi,\text{th}}$ for brevity.

Though this is the leading contribution, the next leading term proportional to $y\phi$ is important for later discussion. This is because the fermion contribution $\text{Tr} S_H^\chi$ in Eq. (4.42) vanishes for a chiral fermion. The leading term that contributes to $\text{Tr} S_H^\chi$ can be expressed as

$$\begin{aligned} \rho_\chi(P)|_\phi &\simeq -\frac{y\phi}{i} \left[\frac{1}{(-p_0 + \Omega_p^+ - i\Gamma_p^+/2)(-p_0 - \Omega_p^+ - i\Gamma_p^+/2)} - (\text{c.c.}) \right] \\ &\simeq y\phi \frac{2p_0\Gamma_p^+}{[p_0^2 - \Omega_p^{+2}]^2 + [p_0\Gamma_p^+]^2}. \end{aligned} \quad (4.57)$$

Here we have neglected the plasmino contribution since it is sub-leading.

In the second case (ii), the plasmino contribution decouples owing to a large $y\phi$, and the pole is dominated by the Dirac mass term $y\phi$ with a thermal correction; but the correction is smaller than $y\phi$ because $y|\phi| \gg gT$. Thus, the spectral density can be approximated with

$$\rho_\chi(P) \simeq (y\phi + \not{p}) \frac{2p_0\Gamma_p}{[p_0^2 - \omega_p^2]^2 + [p_0\Gamma_p]^2} + \rho_\chi^{(\text{cont})}(P), \quad (4.58)$$

where the dispersion relation is given by

$$\omega_p = \sqrt{\lambda^2 \phi^2 + \mathbf{p}^2}; \quad \Gamma_p/2 = \frac{1}{\omega_p} \left[\lambda^2 \phi^2 \left(A_p^I(\omega_p) + C_p^I(\omega_p) \right) + B_p^I(\omega_p) \omega_p \right]. \quad (4.59)$$

Now we are in a position to discuss effects of thermally populated χ quasi-particles on the mean field ϕ . Let us focus on the leading order contribution in the coupling λ and y in the following, which is encoded in M_ϕ^2 and $y \text{Tr}[S_H^\chi]$ in Eq. (4.42). The contributions from $\tilde{\chi}$ -bosons are imprinted in

$$\begin{aligned} M_\phi^2 &\supset \lambda^2 \int_p G_H^\chi(t, t; \mathbf{p}) = \lambda^2 \int_p [1 + 2f_B(p_0)] \rho_{\tilde{\chi}}(P) \\ &= N_{\tilde{\chi}} \lambda^2 \int_p \frac{1}{\Omega_p} [1 + 2f_B(\Omega_p)] + \dots, \end{aligned} \quad (4.60)$$

where $N_{\tilde{\chi}}$ is the number of $\tilde{\chi}$ fields normalized by one complex scalar. In the first equality, we have used the KMS relation, and in the second equality, we have approximated the spectral density with the Breit-Wigner form given in Eq. (4.49). Recalling the discussion around Eq. (3.76), one obtains the following expression for $T \gg \lambda|\phi|$:

$$M_\phi^2 = m_\phi^2 + N_{\tilde{\chi}} \frac{\lambda^2 T^2}{6} + \dots, \quad (4.61)$$

at the leading order in λ . Here we have omitted the Vacuum contribution; which leads to the CW potential [Eq. (4.5)] apart from the infinite term canceled by the counter term [30, 124]. This is nothing but the *thermal mass* of the ϕ -field which stems from thermally populated $\tilde{\chi}$ quasi-particles. Note that for a large field value $\lambda|\phi| \gg T$, this contribution is Boltzmann suppressed as can be seen from Eq. (4.60). Similarly, the contribution from χ -fermions is given by

$$\begin{aligned} -\frac{y}{2} \int_p \text{Tr}[S_H(t, t; \mathbf{p})] &= -\frac{y}{2} \int_p [1 - 2f_F(p_0)] \text{Tr}[\rho_\chi(P)] \\ &= -N_\chi y^2 \phi \int_p \frac{2}{\Omega_p} [1 - 2f_F(\Omega_p)] + \dots, \end{aligned} \quad (4.62)$$

where N_χ is the number of χ fields normalized by one Dirac fermion. Note that the result is the same in both regimes: Eqs. (4.57) and (4.58). Using the high temperature expansion $y|\phi| \ll T$, one obtains the following form [30]:

$$\phi \times N_\chi \frac{y^2 T^2}{6} + \dots, \quad (4.63)$$

at the leading order in y . Note again that this contribution is Boltzmann suppressed in the large field value regime $y|\phi| \gg T$.

To sum up, in the small field value regime, the ϕ -condensation feels the force from thermally populated χ quasi-particles. At the leading order in the coupling λ and y , it is nothing but the thermal mass:

$$m_{\phi, \text{th}}^2 \equiv N_{\tilde{\chi}} \times \frac{\lambda^2 T^2}{6} + N_\chi \times \frac{y^2 T^2}{6}, \quad (4.64)$$

for $\tilde{\lambda}|\phi| \ll T$. This implies the following approximated form of the effective potential:

$$V_{\text{th-mass}}(\phi) \equiv \theta(T - \lambda|\phi|)N_{\tilde{\chi}}\frac{\lambda^2 T^2}{12}\phi^2 + \theta(T - y|\phi|)N_{\chi}\frac{y^2 T^2}{12}\phi^2. \quad (4.65)$$

Finally, we would like to comment on the simpler derivation of the above effective potential. In the above derivation, we stick to Eqs. (4.25), (4.26) and (4.42) to demonstrate that these equations successfully describe various situations (as we will see in the following). However, there is a straightforward way to derive it since the χ -fields are close to thermal equilibrium. In this case, inserting the formal solution of propagators $G_{\chi}(\phi)$ and $S_{\chi}(\phi)$, which depend on the background ϕ , into the 2PI effective action, one obtains the ordinary 1PI effective action. This is because we have the following identity, $\Gamma[\phi] = \Gamma[\phi, \{G_{\bullet}(\phi)\}]$ with $\{G_{\bullet}(\phi)\}$ being a set of propagators. Precisely speaking, since we have truncated the perturbative expansion, the above identity does not hold in general. So, the correct statement is that, at the one-loop order, both descriptions fall into the same result as can be seen from the equivalence hierarchy. Hence, seeing that the background field ϕ does not depend on the space-time, all one has to do is to compute the effective potential, which is given by $V_{\text{eff}}(\phi) = -\Gamma[\phi]/\text{Vol}(\mathbf{R}^{1,3})$. Then, we can derive Eq. (4.65) as explicitly done in Ref. [30] for instance.

4.2.2 Thermal Log

In the large field value regime $\tilde{\lambda}|\phi| > T$, there are no χ -particles in the thermal plasma due to the Boltzmann suppression caused by the large effective mass. Hence, one might think that the ϕ -condensate is free from the thermal plasma because the χ -particles which directly couple to ϕ are absent. However, this naive guess is not true due to the quantum nature of field theory; that is, the property of thermal plasma depends on the effective mass of fields which are already decoupled, through the running coupling constant. Since the running coupling constant depends on ϕ as $g(T; \phi)$, thermodynamic functions also depend on ϕ . Through this effect, the ϕ -condensation feels the existence of background thermal plasma.

In the following, we take the second way to discuss the effects of thermal plasma as demonstrated at the last paragraph in the previous section; that is, compute the effective potential. Since there are almost no χ -particles in the thermal plasma, one can safely integrate out χ -fields to discuss thermal effects. Then, the running coupling constant at the temperature scale is given by

$$\begin{aligned} \frac{1}{g^2(T; \phi)} &= \frac{1}{g^2(\Lambda)} - \frac{1}{16\pi^2} \left[\frac{11}{3}T(\text{Ad}) - \frac{2}{3}T(F) - \frac{1}{3}T(S) \right] \ln \frac{\Lambda^2}{(\lambda \text{ or } y)^2 \phi^2} \\ &\quad - \frac{1}{16\pi^2} \left[\frac{11}{3}T(\text{Ad}) - \frac{2}{3}T(F') - \frac{1}{3}T(S') \right] \ln \frac{(\lambda \text{ or } y)^2 \phi^2}{T^2}, \end{aligned} \quad (4.66)$$

where $T(\mathbf{r})$ denotes one-half of the Dynkin index of the representation \mathbf{r} , which is defined as $\text{Tr}[t^a(\mathbf{r})t^b(\mathbf{r})] = T(\mathbf{r})\delta^{ab}$, and $T(F/S)$ denote the sum of $T(\mathbf{r})$ for all the Weyl fermions/complex scalars. The prime on F/S stands for the sum except for χ -fields which directly couple to ϕ . Thus, the ϕ -dependence can be extracted as

$$\frac{1}{g^2(T; \phi)} - \frac{1}{g^2(T)} = \frac{1}{16\pi^2} \left[N_{\chi} \frac{4}{3}T(\mathbf{r}_F) \ln \frac{y^2 \phi^2}{T^2} + N_{\tilde{\chi}} \frac{1}{3}T(\mathbf{r}_S) \ln \frac{\lambda^2 \phi^2}{T^2} \right], \quad (4.67)$$

where N_{χ} is the number of Dirac fermions and $N_{\tilde{\chi}}$ is the number of complex scalars. Here $g(T)$ stands for the running coupling constant with $\phi = 0$. Note that this result implies the

following operator:

$$-\frac{1}{4} \left[\frac{1}{g^2(T)} + \frac{1}{16\pi^2} \left(N_{\chi} \frac{4}{3} T(\mathbf{r}_F) \ln \frac{y^2 \phi^2}{T^2} + N_{\tilde{\chi}} \frac{1}{3} T(\mathbf{r}_S) \ln \frac{\lambda^2 \phi^2}{T^2} \right) \right] F_{\mu\nu}^a F^{a\mu\nu}, \quad (4.68)$$

with $F^{a\mu\nu}$ being the field strength. The ϕ -condensation couples with the background plasma via this operator, which is renormalized by the running coupling constant $g(T; \phi)$. Hence, one can obtain the effective potential easily by replacing the coupling $g(T)$ with $g(T; \phi)$ in the free energy of thermal plasma. Recalling that the free energy of thermal plasma has a contribution proportional to $g^2 T^4$, one finds that the free energy contains the following ϕ -dependent part, which is the so-called *thermal log* potential [121]:

$$V_{\text{th-log}}(\phi) = \theta(\lambda|\phi| - T) a_{L,\tilde{\chi}} \alpha^2(T) T^4 \ln \left[\frac{\lambda^2 \phi^2}{T^2} \right] + \theta(y|\phi| - T) a_{L,\chi} \alpha^2(T) T^4 \ln \left[\frac{y^2 \phi^2}{T^2} \right], \quad (4.69)$$

where $a_{L,\bullet}$ is a model dependent order one constant. In the following analysis, we take $a_{L,\bullet} = 1$ for simplicity.

4.2.3 Onset of Oscillation

Before deriving the condition for the onset of oscillation, let us first summarize the effective potential of ϕ -condensation derived in the previous section. There are four terms; the tree level potential, the CW potential, the thermal mass potential and the thermal log potential:

$$V_{\text{eff}}(\phi; T) = \frac{1}{2} m_{\phi}^2 \phi^2 + V_{\text{CW}}(\phi) + V_{\text{th-mass}}(\phi; T) + V_{\text{th-log}}(\phi; T), \quad (4.70)$$

where

$$V_{\text{CW}}(\phi) = N_{\tilde{\chi}} \frac{\lambda^4 \phi^4}{32\pi^2} \left[\ln \frac{\lambda^2 \phi^2}{\mu^2} - \frac{3}{2} \right] - N_{\chi} \frac{y^4 \phi^4}{16\pi^2} \left[\ln \frac{y^2 \phi^2}{\mu^2} - \frac{3}{2} \right], \quad (4.71)$$

$$V_{\text{th-mass}}(\phi; T) = \theta(T - \lambda|\phi|) N_{\tilde{\chi}} \frac{\lambda^2 T^2}{12} \phi^2 + \theta(T - y|\phi|) N_{\chi} \frac{y^2 T^2}{12} \phi^2, \quad (4.72)$$

$$V_{\text{th-log}}(\phi; T) = \theta(\lambda|\phi| - T) \alpha^2(T) T^4 \ln \left[\frac{\lambda^2 \phi^2}{T^2} \right] + \theta(y|\phi| - T) \alpha^2(T) T^4 \ln \left[\frac{y^2 \phi^2}{T^2} \right]. \quad (4.73)$$

Here and hereafter, to avoid a possible instability caused by the CW potential, we assume $\lambda \gtrsim y$.

The ϕ -condensation starts to oscillate when the force from the effective potential becomes comparable to the Hubble parameter:

$$H_{\text{os}} = m_{\phi, \text{eff}} \equiv \sqrt{2 \frac{\partial V_{\text{eff}}}{\partial \phi^2}} \quad (4.74)$$

$$\simeq \max \left[m_{\phi}, \lambda^2 \phi_i, \lambda(y) T \text{ for } \lambda(y) \phi_i < T, \frac{\alpha T^2}{\phi_i^2} \text{ for } \lambda(y) \phi_i > T \right], \quad (4.75)$$

with the initial amplitude being ϕ_i . Here we have assumed $\lambda \gtrsim y$. In the second line, we have omitted model dependent order one constants to capture its essential feature. As one can

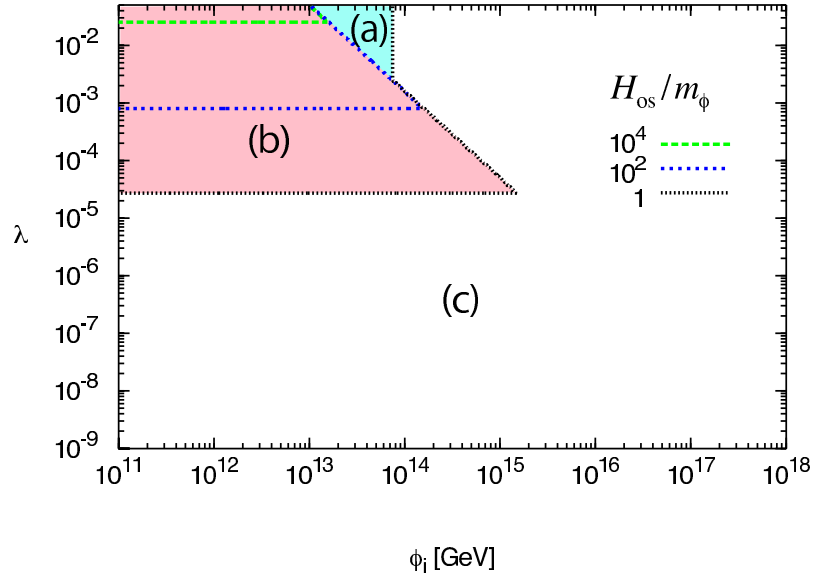
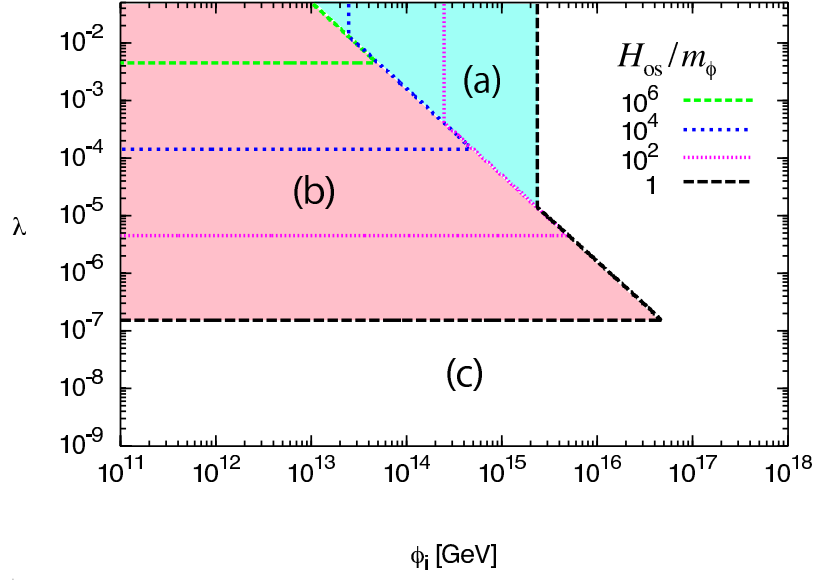


Figure 4.3: **Top Panel:** Contours of H_{os}/m_ϕ for $\alpha = 0.05$, $T_R = 10^9$ GeV and $m_\phi = 1$ TeV. **Bottom Panel:** Same as top panel, but for $m_\phi = 10^3$ TeV.

see, if the CW potential is present, it soon dominates the effective potential for a large field value. Hence, to focus on the feature of thermal effects, let us consider the case where the CW potential is suppressed $y \sim \lambda$, e.g. SUSY case. Then, the temperature T_{os} at the beginning of oscillation for each cases is summarized as follows [35, 45]:

- The ϕ -condensation begins to oscillate with *thermal log* potential if

$$\phi_i < \alpha T_R \sqrt{\frac{M_{\text{pl}}}{m_\phi}} \quad \text{and} \quad \phi_i > (T_R/\lambda)^{2/3} (\alpha M_{\text{pl}})^{1/3}. \quad (4.76)$$

The temperature at the beginning of oscillation is given by

$$T_{\text{os}} = \sqrt{\frac{\alpha M_{\text{pl}}}{\phi_i}} T_R. \quad (4.77)$$

In addition to above two inequalities, the condition $T_{\text{os}} > T_R$ should be met since otherwise the scalar oscillates with the zero-temperature mass term.

- The ϕ -condensation begins to oscillate with *thermal mass* if

$$\begin{cases} \lambda > \left[\frac{m_\phi^3}{T_R^2 M_{\text{pl}}} \right]^{1/4} & \text{and} \quad \phi_i < (T_R/\lambda)^{2/3} (\alpha M_{\text{pl}})^{1/3} & \text{for } \lambda M_{\text{pl}} > T_R \\ \lambda > \sqrt{\frac{m_\phi}{M_{\text{pl}}}} & & \text{for } \lambda M_{\text{pl}} < T_R \end{cases} \quad (4.78)$$

The temperature at the onset of oscillation is given by

$$T_{\text{os}} = \begin{cases} [\lambda T_R^2 M_{\text{pl}}]^{1/3} & \text{for } \lambda M_{\text{pl}} > T_R \\ \lambda M_{\text{pl}} & \text{for } \lambda M_{\text{pl}} < T_R \end{cases} \quad (4.79)$$

- Otherwise, the ϕ -condensation begins to oscillate with *zero temperature mass*, and the temperature is given by

$$T_{\text{os}} = \begin{cases} [m_\phi M_{\text{pl}} T_R^2]^{1/4} & \text{for } m_\phi M_{\text{pl}} > T_R^2 \\ \sqrt{m_\phi M_{\text{pl}}} & \text{for } m_\phi M_{\text{pl}} < T_R^2. \end{cases} \quad (4.80)$$

Here we have assumed that the reheating takes place in the conventional way, which is already explained in Sec. 2.1. Also, note that we assume that χ particles are absent initially in the thermal log case. Otherwise, ϕ would feel correction to the effective potential. This assumption might break down if heavy χ particles are substantially produced by the direct decay of inflaton or the inflaton preheating process. Whether this occurs or not depends on the inflation model, and hence we simply assume the absence of χ in the thermal log case.

The Hubble parameter at the beginning of oscillation, H_{os} , is given by

$$H_{\text{os}} \simeq \begin{cases} \alpha^2 T_R^2 M_P / \phi_i^2 & \text{for thermal log,} \\ (\lambda^4 T_R^2 M_P)^{1/3} & \text{for thermal mass,} \\ m_\phi & \text{for zero temperature mass,} \end{cases} \quad (4.81)$$

where we have omitted the case with $\lambda M_{\text{pl}} < T_R$ since this inequality is violated in most cases. Fig. 4.3 shows contours of H_{os}/m_ϕ on (ϕ_i, λ) -plane for $\alpha = 0.05$, $T_R = 10^9$ GeV and $m_\phi = 1$ TeV (top) and for $m_\phi = 10^3$ TeV (bottom). Regions (a)–(c) correspond to the cases; (a) thermal log, (b) thermal mass, (c) zero temperature mass. Note that it is possible that even if ϕ begins to oscillate with zero-temperature mass term, thermal effects become dominant thereafter. Conversely, the zero-temperature term eventually becomes dominant even if thermal effects are important at the beginning of oscillation. These facts make the scalar dynamics quite rich.

4.3 Non-thermal Effects

After the onset of oscillation, the scalar field oscillates around its effective potential minimum,^{♦7} and hence the coupled χ -particles have a time dependent dispersion relation. In this case, it is known that the non-perturbative χ -particle production can take place due to the breakdown of the adiabaticity condition for χ -fields [28, 29]. Hence, the χ -propagators are also dynamical and we have to study the evolution of ϕ and G_χ/S_χ at least simultaneously. Since the oscillation time scale soon becomes larger than the Hubble parameter, $m_{\phi, \text{eff}} \gg H$, the expansion of the Universe can be neglected in the oscillation-time scale. We take into account the cosmic expansion later as an adiabatic expansion in Sec. 4.6.

4.3.1 Non-perturbative Particle Production

To illustrate essential features, let us consider the following set of equations at first [74], dropping the self energy contributions in Eqs. (4.23)–(4.26) and (4.42):^{♦8}

$$0 = [\partial_t^2 + m_\phi^2] \phi(t), \quad (4.82)$$

$$0 = [\partial_t^2 + \mathbf{p}^2 + m_{\chi, \text{th}}^2 + \lambda^2 \phi^2(t)] G_{J/H}^\chi(t, t'; \mathbf{p}), \quad (4.83)$$

$$0 = [-i\gamma^0 \partial_t + \boldsymbol{\gamma} \cdot \hat{\mathbf{p}} \Omega_p + y \phi(t)] S_{J/H}^\chi(t, t'; \mathbf{p}), \quad (4.84)$$

where we have approximated the thermal correction to the fermion dispersion relation as $\Omega_p = \sqrt{\mathbf{p}^2 + m_{\chi, \text{as}}^2}$ and neglected the plasmino contribution. The applicability of these approximated equations is discussed later. Since we neglect the finite density correction including the back-reaction to the oscillating scalar, the first equation can be solved easily:

$$\phi(t) = \tilde{\phi} \cos[m_\phi t], \quad (4.85)$$

where $\tilde{\phi}$ denotes the amplitude of ϕ . Then, let us move to the equation of motion for propagators. In the following, we mainly concentrate on Eq. (4.83) since the essential feature of Eq. (4.84) is the same except for its statistics [125–127]. Initially, the $\tilde{\chi}$ -particles are assumed to be absent, so the initial condition for G_H^χ is given by

$$G_H^\chi(t, t'; \mathbf{p})|_{t, t'=0} = \frac{1}{\Omega_{\chi, p}(0)}, \quad (4.86)$$

$$\partial_t \partial_{t'} G_H^\chi(t, t'; \mathbf{p})|_{t, t'=0} = \Omega_{\chi, p}(0), \quad (4.87)$$

$$\partial_t G_H^\chi(t, t'; \mathbf{p})|_{t, t'=0} = \partial_{t'} G_H^\chi(t, t'; \mathbf{p})|_{t, t'=0} = 0, \quad (4.88)$$

^{♦7} It is not necessary to coincide with the effective potential derived in the previous section due to the efficient χ -particle production as we will see.

^{♦8} Here we have assumed that the background plasma can remain close to thermal equilibrium. The applicability of this treatment is discussed later.

where

$$\Omega_{\chi,p}(t) \equiv \sqrt{\mathbf{p}^2 + m_{\chi,\text{th}}^2 + \lambda^2 \phi^2(t)}. \quad (4.89)$$

Note that G_J^χ satisfies the canonical commutation relations:

$$G_J^\chi(t, t'; \mathbf{p}) \Big|_{t'=t} = \partial_t \partial_{t'} G_J^\chi(t, t'; \mathbf{p}) \Big|_{t'=t} = 0, \quad (4.90)$$

$$\partial_{t'} G_J^\chi(t, t'; \mathbf{p}) \Big|_{t'=t} = -\partial_t G_J^\chi(t, t'; \mathbf{p}) \Big|_{t'=t} = i. \quad (4.91)$$

Then we obtain the following factorized solution:

$$G_H^\chi(t, t'; \mathbf{p}) = [f_p(t) f_p^*(t') + f_p^*(t) f_p(t')], \quad (4.92)$$

$$G_J^\chi(t, t'; \mathbf{p}) = [f_p(t) f_p^*(t') - f_p^*(t) f_p(t')], \quad (4.93)$$

where the equation of motion for each mode is given by

$$0 = [\partial_t^2 + \mathbf{p}^2 + m_{\chi,\text{th}}^2 + \lambda^2 \phi^2(t)] f_p(t), \quad (4.94)$$

with the initial condition being

$$f_p(0) = \frac{1}{\sqrt{2\Omega_{\chi,p}(0)}}, \quad \dot{f}_p(0) = -i \sqrt{\frac{\Omega_{\chi,p}(0)}{2}}. \quad (4.95)$$

As one can see, this equation is nothing but the *Mathieu equation* [128].

In the following, let us focus on the feature of this equation. Though the property of Eq. (4.94) is well described as the stability/instability chart, given for instance in Ref. [128], it is instructive to consider its approximated behavior so as to capture its physics. The WKB method is useful for this purpose. One obtains the following WKB solution,

$$f_p(t) \simeq \frac{e^{-i \int^t dt' \Omega_{\chi,p}(t')}}{\sqrt{2\Omega_{\chi,p}(t)}}, \quad (4.96)$$

if the adiabaticity is maintained,

$$\left| \frac{\dot{\Omega}_{\chi,p}}{\Omega_{\chi,p}^2} \right| \ll 1. \quad (4.97)$$

The physical interpretation is that, if the background ϕ -condensation moves slowly, then the χ -particles can be regarded as ones with an effective mass, $m_{\chi,\text{eff}} \equiv \sqrt{m_{\chi,\text{th}}^2 + \lambda^2 \phi^2(t)}$. For $\lambda \tilde{\phi} \ll m_\phi$, the oscillating scalar is expected to decay/annihilate into χ -particles perturbatively (See Sec. 4.3.4 for effects of Bose/Fermi statistics; so-called *narrow resonance* [28, 129, 130]).

Then we concentrate on the case with $\lambda \tilde{\phi} \gg m_\phi$, where the perturbative decay/annihilation of ϕ into χ -particles are forbidden except for the region near $\phi \sim 0$. Around the origin of the effective potential, χ -fields become lighter, and as a result the background ϕ -condensate might not be regarded as an adiabatic background. In fact, Eq. (4.97) implies that the adiabaticity is broken down if the following inequality is met:

$$\lambda \tilde{\phi} \gg \frac{m_{\chi,\text{th}}^2}{m_\phi}. \quad (4.98)$$

Together with $\lambda\tilde{\phi} \gg m_\phi$, the condition can be expressed as [35, 131]

$$k_* \gg \max[m_{\chi,\text{th}}, m_\phi]; \quad k_*^2 \equiv \lambda \dot{\phi}|_{\phi=0} \sim \lambda\tilde{\phi}m_\phi, \quad (4.99)$$

where k_* represents the typical scale of non-perturbative particle production. In other words, the former inequality verifies that this process can excite quasi-particles interacting with radiation. The non-adiabatic region near $\phi \sim 0$ can be estimated as

$$|\phi(t)| < \phi_{\text{NP}} \equiv \left[\frac{m_\phi \tilde{\phi}}{\lambda} \right]^{1/2}. \quad (4.100)$$

Basically, these equations [Eqs. (4.99) and (4.100)] are the same as the fermion case if one replaces λ with y , and hence we do not distinguish them in the following unless otherwise stated. Also note that even if the scalar condensation does not oscillate with the quadratic potential, one may use these equations by replacing the mass m_ϕ with the effective mass $m_{\phi,\text{eff}}$. Inside this region, the notion of χ -particles become ambiguous due to the rapidly moving background field ϕ . Notice that the perturbative decay cannot occur outside this region because we have $\lambda\phi_{\text{NP}} \sim k_* \gg m_\phi$. Nevertheless, the mode function f_p grows rapidly around $\phi \sim 0$ and the χ -particles are produced efficiently as a consequence, if the condition Eq. (4.99) is met. After the first passage of the non-adiabatic region $|\phi| < \phi_{\text{NP}}$, the mode function becomes the following form outside the non-adiabatic region [28, 29]:

$$f_p(t) \simeq \alpha_p \frac{e^{-i \int^t dt' \Omega_{\chi,p}(t')}}{\sqrt{2\Omega_{\chi,p}(t)}} + \beta_p \frac{e^{i \int^t dt' \Omega_{\chi,p}(t')}}{\sqrt{2\Omega_{\chi,p}(t)}}, \quad (4.101)$$

where α_p and β_p are the Bogolyubov coefficients satisfying $|\alpha_p|^2 - |\beta_p|^2 = 1$.^{♦9} For our purpose, the important point is that $|\beta_k|^2 \simeq e^{-\pi k^2/k_*^2}$. The distribution function of particles are given by $f_\bullet(\mathbf{p}; t) = [\partial_t \partial_{t'} + \Omega_p^2] G_H^\bullet(t, t'; \mathbf{p}) / (4\Omega_p^2)|_{t' \rightarrow t} - 1/2$ [Eq. (3.139)] outside the non-adiabatic region, $\phi_{\text{NP}} \ll |\phi(t)|$ [100, 103, 105, 106].^{♦10} Then, the number density of χ -particles is given by

$$n_\chi = \int_k f_\chi(\mathbf{k}) \simeq 2N_{\text{d.o.f.}} \times \int_k |\beta_k|^2 \sim N_{\text{d.o.f.}} \times \frac{k_*^3}{4\pi^3}, \quad (4.102)$$

where $N_{\text{d.o.f.}}$ is normalized by one complex scalar or one chiral fermion. As one can see from Eq. (4.99), this non-perturbative particle production is suppressed if the thermal effects are efficient; $k_* \lesssim m_{\chi,\text{th}}$, which implies that this process cannot excite quasi-particles.

Before going into details, we would like to discuss the applicability of Eqs. (4.82)–(4.84). In these equations, we only take into account thermal masses of χ -fields, but completely neglect (a) dissipative effects caused by interactions between the χ -fields and the thermal plasma, (b) corrections to the dispersion relation due to the non-perturbatively produced χ -particles, and (c) back-reaction to the ϕ -condensation, which are imprinted in the self energies. Since the issues (b) and (c) are related with the property of χ -fields as we will discuss later, here we examine whether or not the effect (a) can disturb the non-perturbative production. Though it is rather subtle to estimate the dissipative effects inside the non-adiabatic region since we cannot define χ particles, nevertheless we may roughly evaluate it as follows. If $\Gamma_{\text{int}} \delta t_{\text{NP}} \ll 1$ is satisfied with $\delta t_{\text{NP}} \sim k_*^{-1}$ being the time scale which ϕ takes to pass the non-adiabatic region

^{♦9} $|\alpha_p|^2 + |\beta_p|^2 = 1$ for fermions.

^{♦10} There are some ambiguities on the definition of particle number in terms of Green function [105].

and Γ_{int} being the typical interaction rate of χ with the background plasma, then we expect that the dissipation cannot disturb the non-perturbative production. In most cases we expect $\Gamma_{\text{int}} < m_{\chi, \text{th}} \sim gT$ and hence Eq. (4.99) implies $\Gamma_{\text{int}} \delta t_{\text{NP}} < m_{\chi, \text{th}}/k_* \ll 1$. Therefore, dissipative effects are sub-leading compared with effects from thermal masses.

Now we are in a position to discuss the subsequent oscillation after the first passage of $\phi \sim 0$. The subsequent evolution; second, third, \dots passages of non-adiabatic region, crucially depends on the property of χ -fields: whether or not χ -particles are stable. Let us assume that the χ -fields can decay into other light particles with a rate being $\Gamma_{\chi, \text{dec}} \sim \epsilon^2 g^2 m_{\chi, \text{eff}}(|\phi(t)|)$, which is imprinted in the χ 's self energy. Note that since the decay is dominated at the outside of the non-adiabatic region, the concept of χ -particle decay with the effective mass $m_{\chi, \text{eff}}$ is justified a posteriori. In this case, the WKB solution outside the non-adiabatic region may be given by

$$f_p(t) \simeq \alpha_p(t) \frac{e^{-i \int^t dt' [\Omega_{\chi, p}(t') - i\Gamma_{\chi, \text{dec}}(t')/2]}}{\sqrt{2\Omega_{\chi, p}(t)}} + \beta_p(t) \frac{e^{i \int^t dt' [\Omega_{\chi, p}(t') + i\Gamma_{\chi, \text{dec}}(t')/2]}}{\sqrt{2\Omega_{\chi, p}(t)}}, \quad (4.103)$$

after the first passage. Hence, the χ -particles decay dominantly at $1 \sim \Gamma_{\chi, \text{dec}}(t_{\text{dec}})t_{\text{dec}} \sim \epsilon^2 g^2 \lambda \tilde{\phi} m_\phi t_{\text{dec}}^2$ which implies that the decay time measured from the first passage is given by $t_{\text{dec}} \sim [\epsilon^2 g^2 \lambda \tilde{\phi} m_\phi]^{-1/2}$. As one can see, the non-perturbatively produced χ -particles dominantly decay at $\phi(t_{\text{dec}}) \sim [\tilde{\phi} m_\phi / (\epsilon^2 g^2 \lambda)]^{1/2} \gg \phi_{\text{NP}}$ which is outside of the non-adiabatic region. Also, note that one can show that χ -particles decay after they become non-relativistic, $t_{\text{dec}} \gg t_{\text{NR}}$ with $m_\phi \sim \lambda \phi(t_{\text{NR}}) \sim \lambda \tilde{\phi} m_\phi t_{\text{NR}}$, by using the non-adiabatic condition [Eq. (4.98)].

The subsequent oscillation crucially depends on χ 's property, $\epsilon^2 g^2 \lambda \tilde{\phi} \lesseqgtr m_\phi$, and thus we classify each case in the following. See a schematic figure given in Fig. 4.4.

4.3.2 Instant Preheating

If the adiabaticity is broken down [Eq. (4.99)] and the decay rate is so large

$$\epsilon^2 g^2 \lambda \tilde{\phi} \gg m_\phi \quad \text{and} \quad \lambda \tilde{\phi} m_\phi \gg m_{\chi, \text{th}}^2 \sim g^2 T^2, \quad (4.104)$$

then the non-perturbatively produced χ -particles can decay completely well before the ϕ moves back to its origin [132, 133].^{♦11} See also Fig. 4.4. Therefore, the fractional energy density which ϕ loses in one oscillation is given by

$$\delta_\phi \equiv \frac{\delta \rho_\phi}{\rho_\phi} \simeq N_{\text{d.o.f.}} \times \frac{\lambda^2}{2\pi^3 |\epsilon g|}. \quad (4.105)$$

The dissipation rate of the ϕ -condensation is estimated as [35]

$$\Gamma_\phi^{\text{NP}} \sim \frac{1}{\pi} \delta_\phi m_\phi \sim N_{\text{d.o.f.}} \times \frac{\lambda^2 m_\phi}{2\pi^4 |\epsilon g|}. \quad (4.106)$$

In deriving the condition Eq. (4.98), we have assumed that the background thermal plasma can remain close to thermal equilibrium. To clarify the applicability of Eq. (4.98), let us discuss the typical thermalization time scale of light particles produced via the decay of χ -particles. Since

^{♦11} If the coupling is somehow fine tuned to $\phi(t_{\text{dec}}) \simeq \tilde{\phi}$, then the energy conversion rate from the scalar condensation to radiation becomes quite large [132]. Here we do not assume such a tuning as can be seen from $\epsilon^2 g^2 \lambda \tilde{\phi} \gg m_\phi$.

the χ -particles decay into light particles dominantly at $t_{\text{dec}} \sim [\epsilon^2 g^2 \lambda \tilde{\phi} m_\phi]^{-1/2}$, their typical momenta are given by $K \sim m_{\chi, \text{eff}}(t_{\text{dec}}) \sim k_*/|\epsilon g|$, which is larger than the temperature of thermal plasma when the non-perturbative production takes place: $K \gg T/\epsilon \gtrsim T$. Such high energy particles lose their energies via multiple splittings due to interactions with the thermal plasma. Its time scale can be estimated as $t_{\text{split}} \sim (\alpha^2 T)^{-1} \sqrt{K/T}$ [36, 134–136]. Hence, the background plasma can remain in thermal equilibrium if $t_{\text{split}} \ll m_\phi^{-1}$. Through this process, the thermal plasma is heated and eventually it terminates when the condition Eq. (4.98) is saturated, $k_* \sim m_{\chi, \text{th}} \sim gT$.

4.3.3 Broad Resonance

On the other hand, for $\epsilon^2 g^2 \lambda \tilde{\phi} \ll m_\phi$ or stable χ -particles, the χ -particles might remain in the same distribution until the ϕ -condensation moves back to its potential origin again. This regime corresponds to (See Fig. 4.4)

$$\frac{m_\phi}{\epsilon^2 g^2} \gg \lambda \tilde{\phi} \gg \max \left[m_\phi, \frac{m_{\chi, \text{th}}^2}{m_\phi} \right]. \quad (4.107)$$

If this is the case, Bose/Fermi-statistics becomes important due to the previously produced particles. In particular, the bosons are produced explosively due to the induced emission as ϕ crosses the non-adiabatic region and they significantly affect the dynamics, but the effects of fermion is sub-dominant since its number density is soon saturated due to the Pauli suppression. The distribution function of bosons grows exponentially $f_{\tilde{\chi}} \propto e^{\mu t}$. Here μ is the Floquet index characterizing the growth rate; which is roughly given by $\mu \sim m_{\phi, \text{eff}}$. Though it depends on the model of interactions, we assume that the $\tilde{\chi}$ -field has a self interaction in the following. Since the $\tilde{\chi}$ -particles are produced explosively, they affect their own dispersion relations via the self interaction [(b)] and also the dynamics of ϕ -condensation [(c)]. Even though the χ -particles become heavy after the passage of the non-adiabatic region, the χ -particles cannot decay efficiently into other particles in this case. Hence, these heavy particles produce the linear potential

$$V_{\text{linear}}^{(\text{broad})}(\phi) \simeq N_{\tilde{\chi}} \lambda^2 \phi^2 \int_k \frac{1}{\Omega_k} f_{\tilde{\chi}}(k) \simeq N_{\tilde{\chi}} \lambda |\phi| n_{\tilde{\chi}} \quad \text{for } |\phi| > \phi_{\text{NP}}. \quad (4.108)$$

Also, the self interaction of $\tilde{\chi}$ induces the following effective mass of $\tilde{\chi}$ -fields:^{♦12}

$$\lambda_\chi^2 \int_p \frac{1}{\Omega_p} f_{\tilde{\chi}}(p) \sim \lambda_\chi^2 \frac{n_{\tilde{\chi}}}{k_*}, \quad (4.109)$$

where λ_χ denotes the coupling of $\tilde{\chi}$'s self interaction, $\lambda_\chi^2 |\tilde{\chi}|^4$, which is assumed to be the same order of magnitude as g for simplicity, $\lambda_\chi \sim g$. Therefore, the condition for non-perturbative particle production becomes

$$k_*^2 \gg \max \left[\bar{m}_{\phi, \text{eff}}^2, m_{\chi, \text{th}}^2, g^2 \frac{n_{\tilde{\chi}}}{k_*} \right]; \quad k_* \sim [\lambda \tilde{\phi} \bar{m}_{\phi, \text{eff}}], \quad (4.110)$$

where the effective mass for the ϕ -condensation is give by

$$\bar{m}_{\phi, \text{eff}}^2 \sim \max \left[m_\phi^2, \lambda \frac{n_{\tilde{\chi}}}{\tilde{\phi}} \right]. \quad (4.111)$$

^{♦12} Strictly speaking, this expression holds for $|\phi| \lesssim \phi_{\text{NP}}$, and hence the concept of $\tilde{\chi}$ -particle is not well defined.

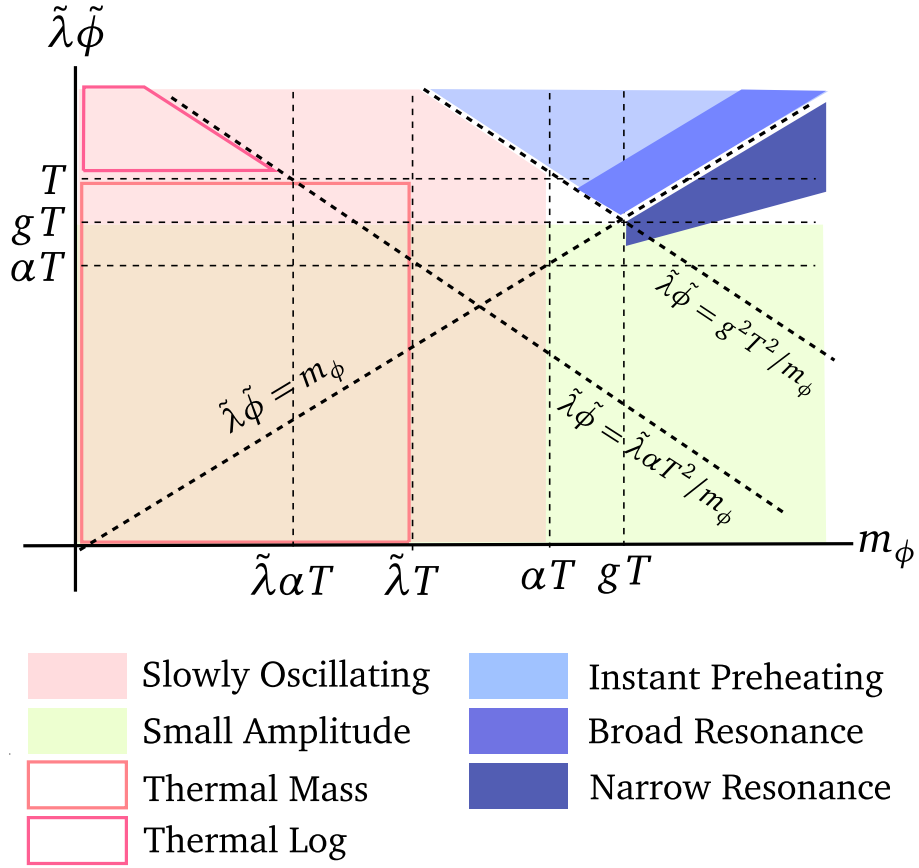


Figure 4.4: Rough sketch of classification of the scalar oscillation in $(m_\phi, \tilde{\lambda}\tilde{\phi})$ -plane. **Slowly Oscillating:** Sec. 4.4.1, **Small Amplitude:** Sec. 4.4.2, **Instant Preheating:** Sec. 4.3.2, **Broad Resonance:** Sec. 4.3.3, **Narrow Resonance:** Sec. 4.3.4.

Eventually, the parametric resonance stage finishes when the inequality is saturated. This condition can be rewritten as^{♦13}

$$\bar{m}_{\phi, \text{eff}} \sim \min \left[k_*, \frac{\lambda^2}{g^2} k_* \right] \equiv \frac{k_*}{c}. \quad (4.112)$$

After the saturation, the typical value of $\tilde{\chi}$'s distribution function reaches $f_{\tilde{\chi}} \sim 1/g^2$. Thus, their energy is transferred into radiation efficiently via (multiple-)annihilations. Through this process, the ϕ -condensate is expected to lose its energy [45]. Still the produced particles sharply concentrate on the low momentum regime compared with thermal distribution, and hence they cascade towards high momentum regime. Though it is difficult to study its dynamics in a realistic setup, in some simple toy models, it is known that the system enters the *turbulent regime* which may be understood as an approximate fixed point solution of Kadanoff-Baym equations due to large fluctuations (large number density), $G_H \gg G_J$ ($f(p) \gg 1$) [110, 112, 137, 138]. See also Sec. 4.3.5.

4.3.4 Narrow Resonance

In this subsection, we comment on the effect of narrow resonance which could happen in some parameter ranges. We follow the arguments in Refs. [28, 129, 130, 139, 140]. The narrow

^{♦13} We consider the case $\lambda \lesssim g^2$ for simplicity, and hence $c = g^2/\lambda^2$.

resonance takes place for $q \ll 1$ and $m_\phi \gg m_{\chi,\text{th}}$, where q characterizes the resonance band. In this subsection, we concentrate on this case. Hereafter we assume $\epsilon \sim 1$ for simplicity.

Let us consider the first instability band for the bosons $\tilde{\chi}$ at $k \simeq m_\phi$, where k is the physical wavenumber of $\tilde{\chi}$ in the Fourier mode. The width of the instability band is given by $\Delta k/k \sim q$, and the growth rate of $\tilde{\chi}$ is given by $\sim qm_\phi$, where $q \equiv \lambda^2 \tilde{\phi}^2/m_\phi^2 \ll 1$. This is understood as the perturbative annihilation of ϕ combined with the induced emission effect. The perturbative annihilation rate of ϕ is given by $\Gamma_\phi \sim \lambda^4 \tilde{\phi}^2/m_\phi$ and the phase space density of $\tilde{\chi}$ is given by $f_{\tilde{\chi}}(k) \sim n_{\tilde{\chi}}/(k^2 \Delta k)$ peaked around $k \simeq m_\phi$. Thus the evolution of the number density is governed by $\dot{n}_{\tilde{\chi}} \sim \Gamma_\phi n_\phi f_{\tilde{\chi}} \sim qm_\phi n_{\tilde{\chi}}$. This gives $n_{\tilde{\chi}} \propto \exp(qm_\phi t)$. On the other hand, in the case of fermions χ , the production rate is soon suppressed due to the Pauli suppression.

In order for the resonance to occur, the momentum distribution of $\tilde{\chi}$ must not be disturbed in a time interval of $(qm_\phi)^{-1}$. The sources for termination of the resonance are the $\tilde{\chi}$'s interaction with the thermal plasma and the decay of $\tilde{\chi}$. Thus we need $qm_\phi \gg \max[\Gamma_{\text{damp}}(\sim \alpha T), \epsilon^2 g^2 \lambda \tilde{\phi}]$ for the resonance. See also Fig. 4.4. Another source is the Hubble expansion, which redshifts the physical momentum of $\tilde{\chi}$. The time required for removing $\tilde{\chi}$ -particles from the resonance band $k \sim m(1 \pm q)$ is $\Delta t_H \sim q/H$. During this time interval, the growth of $\tilde{\chi}$ number density is at most $\sim \exp(qm_\phi \Delta t_H) \sim \exp(q^2 m_\phi/H)$. Therefore, we also need $q^2 m_\phi \gg H$ for the efficient resonance. If these two conditions are satisfied, the $\tilde{\chi}$ number density exponentially grows due to the narrow resonance effect. Fig. 4.4 shows the parameter region where the narrow resonance can occur.

If it happens, the end of the exponential growth may be caused by the self-interaction of $\tilde{\chi}$. For example, the rate of the self-annihilation process $\tilde{\chi} \tilde{\chi} \rightarrow g g$ (gauge bosons) is estimated as $\Gamma_{\tilde{\chi}\tilde{\chi} \rightarrow gg} \sim \alpha^2 n_{\tilde{\chi}}/m_\phi^2$. If this becomes equal to qm_ϕ , the resonance stops. It happens at $n_{\tilde{\chi}} \sim (\lambda/\alpha)^2 n_\phi$. Therefore, for $\lambda < \alpha$, we have $\rho_{\tilde{\chi}} < \rho_\phi$ at the end of resonance and hence it does not drastically affect the dynamics of ϕ field. (It is same order of the energy loss rate at the preheating stage just before the narrow resonance regime.) The evolution of ϕ field after the end of the resonance should be solved in a way described in the following sections and the results are not much affected.

4.3.5 Non-perturbative Production via Self Interaction

Aside from the quartic (Yukawa) interaction $\lambda^2 \phi^2 |\chi|^2$ ($y \bar{\psi} \psi$), there is a possible source that drives the ϕ condensation towards a higher momentum, that is, the four point self interaction of ϕ . The main aim of this subsection is to clarify its typical time scale and compare it with that of quartic/Yukawa interactions discussed in the previous and next sections.

For this purpose, let us first discuss the dynamics of the self interaction alone with neglecting the interactions with other fields. In this case, the potential of scalar field is dominated by

$$V = \lambda^4 \phi^4. \quad (4.113)$$

Here we consider a potential motivated by the CW potential. The following discussion closely follows Appendix of [46].

In the following discussion, we assume that ϕ has an initially large amplitude; $\lambda^2 \tilde{\phi} \gg m_\phi$. The effects of a four point self interaction are well studied for example in [137, 138] numerically and we simply summarize their results for the sake of completeness. According to [138], the typical scale is written as

$$Q = \lambda(\rho_\phi)^{1/4}, \quad (4.114)$$

which is the same as k_* at first. In this case, ϕ -particles are produced non-perturbatively at the crossing of $\phi \sim 0$. In addition, produced ϕ -particles remain the same distribution at their

production and hence the parametric resonance takes place. Since the Floquet index, which characterizes the exponent of distribution function $f_\phi \propto e^{\mu t}$, is roughly given by $\mu \sim Q$ in this case, the non-perturbative production of ϕ particles occurs during $t \lesssim Q \log \lambda^{-4}$, and then energy density of them becomes compatible to that of condensation at $t_{\text{NP}} \sim Q \log \lambda^{-4}$. It implies that the condition of non-perturbative production is violated by the effective mass of ϕ -particles: $k_*^2 \sim m_{\phi, \text{eff}}^2 \sim \lambda^4 k_*^2 e^{\mu t}$. At that time, the amplitude of ϕ is changed by factor not order.

After the end of parametric resonance, the system enters the turbulent regime [110, 112, 137, 138] due to the large value of distribution function $f_\phi \sim 1/\lambda^4$. There, the distribution function obeys the self-similar evolution. The distribution function of high momentum modes with $p \gg Q$ and that of low momentum modes with $p \ll Q$ evolve in different ways. This phenomenon is dubbed as the dual cascade [138] and characterized by different exponents κ of distribution function, $f(p) \propto (Q/p)^\kappa$.

- For low momentum modes ($p \ll Q$), an inverse particle cascade toward the infrared momentum takes place, which is driven by the number conserving interactions among the soft sector ($p \ll Q$).^{♦14} The exponent is given by $\kappa_M = 4/3$ for $f(p) \lesssim \mathcal{O}(1/\lambda^4)$. In terms of perturbative kinetic picture in λ , the stationary particle flow driven by the four point interaction implies this exponent [137]. For ultra-soft modes $f(p) \gtrsim \mathcal{O}(1/\lambda^4)$, the exponent is turned out to be more stronger: $\kappa_S = 4$ [138]. This is because the perturbative expansion in λ is broken down and we have to consider many other processes non-perturbatively. In the case of $N \geq 2$ with $O(N)$ -symmetric scalar field theory, in terms of the $1/N$ expansion, it is shown that the anomalous exponent $\kappa_S = 4$ can be understood as the consequence of momentum dependent effective coupling $\lambda_{\text{eff}}(p) \sim p^2$ [138].
- For high momentum modes ($p \gg Q$), an energy cascade toward UV regime takes place, which is driven by the effective three point interaction of hard modes: $\phi(\text{hard}) + \phi(\text{hard}) \rightarrow \phi(\text{soft}) + \phi(\text{hard})$. It is characterized by the Kolmogorov exponent $\kappa_H = 3/2$ and the stationary energy flow towards UV implies this exponent [137]. The distribution function obeys the following self-similar evolution: $f(t, p) = (Qt)^\alpha f_s((Qt)^\beta p)$ with $\alpha = 4\beta$ and $\beta = 1/(2n - 1)$ for a n -point interaction. Here f_s represents the stationary scaling solution of effective Boltzmann equations [137]. Hence one finds $(\alpha, \beta) = (4/5, 1/5)$ in this case. The maximum momentum $p_{\text{max}} \equiv Q(Qt)^\beta$ which has dominant energy density grows higher and higher.

After the higher modes arrive at $f \sim 1$, the quantum effects become important and lead to thermal equilibrium. This fact implies that the turbulent regime ends at

$$t_{\text{quant}} \sim Q^{-1} \lambda^{-5} = \lambda^{-6} (\rho_\phi)^{-1/4}. \quad (4.115)$$

Eventually, the maximum momentum reaches the “would-be” temperature of this system; $p_{\text{max}} \sim \rho_\phi^{1/4}$.

Then, let us compare the time scale t_{quant} and that of the quartic/Yukawa interaction. Though precise arguments may require complicated numerical simulations, here we simply consider these effects separately and discuss the sufficient condition where the effects of the

^{♦14} Though the exact zero mode might decay by a power law with $\phi_0(t) \sim Q(Qt)^{-1/3}$ [137], the particles are still condensed in low momenta regime and phenomenological consequence is not clear. Therefore, we simply regard such a condensation below $p \ll Q$ as a zero mode effectively in the following. In addition, even if the effective zero mode may decay with this power law, the change of exponent from $\tilde{\phi} \propto a^{-1}$ can be neglected practically in our case since the scalar disappears before this difference becomes significant.

four point self interaction can be safely neglected.^{♦15} For this purpose, we focus on the case with $\lambda^2 \tilde{\phi} \gg \tilde{\lambda} T$ so as to maximize the effects of the four point self interaction.

In our case, since the coupling with the χ -fields is larger than the four point self interaction, the non-perturbative production of χ -particles takes place at first. As discussed in the previous sections (Secs. 4.3.2 and 4.3.3), if the χ -particles are (almost) stable, the parametric resonance takes place and then the whole system may enter the turbulent regime, which may require numerical studies. Here we consider the other case where the χ -particles can decay promptly. In this case, high energy particles are efficiently produced by the decay of “heavy” χ -particles in contrast to the turbulent case, and they are thermalized by their own interactions. The non-perturbative production of χ -particles terminates at $g^2 T^2 \sim k_*^2 \sim \lambda^3 \tilde{\phi}^2$ with T being the temperature of produced light particles. This condition indicates that the soft modes ($p < Q$) of ϕ -particles always oscillate much slower than the typical time scale of interactions between the χ -field and the thermal plasma, $Q \sim \lambda^2 \tilde{\phi} \lesssim (\lambda \alpha)^{1/2} T \ll \Gamma_{\text{int}}$. It implies that the χ -particles see the soft modes of ϕ -fields as slowly moving homogeneous condensates. And also the turbulent evolution toward the UV regime is much slower than the typical interaction time scale of particles in thermal bath, $t_{\text{quant}}^{-1} \sim (\lambda^5 \alpha)^{1/2} T \ll \Gamma_{\text{int}}$ (for $\alpha > \lambda$).

Finally, let us compare the time scale of cascades of ϕ -particles towards the UV regime caused by the background plasma with the four point self interaction. One can show the following inequality, $\Gamma_{\phi}^{\text{eff}} t_{\text{quant}} \gg \lambda^4 T t_{\text{quant}} \gg 1$, with $\Gamma_{\phi}^{\text{eff}}$ being the dissipation of the ϕ -condensation caused by interactions with the background plasma. In the first inequality, we have used the dissipation rate $\Gamma_{\phi}^{\text{eff}}$ which is shown in Sec. 4.6.2. Its intuitive meaning is that the scatterings $\phi \chi \rightarrow \phi \chi$ occur more frequently than the four point self interaction. Therefore, the ϕ -particles produced by the four-point interaction is at most accumulated in the infrared regime, $p \ll \Gamma_{\text{int}}$, and their evolution toward the UV regime is dominated by the interaction with the thermal plasma. And hence the non-perturbative production due to the CW potential can be safely neglected.

4.4 Thermal Effects

In the previous section, we have seen that the oscillating ϕ -condensation produces χ -particles with a highly non-thermal distribution, and hence we have to follow the evolution equations of the mean field ϕ and propagators $G_{\chi/\phi}$, S_{χ} at least, while the other light fields remain close to thermal equilibrium due to the separation of time scale. However, there are particular cases where the equations become further simple: we do not have to track the evolution of χ -propagators in contrast to the case studied in Sec. 4.3. If the oscillation of the scalar field is so slow that even χ -particles can regard the scalar condensation as a static background, one can reduce the full set of equations to the coarse-grained equations with assuming that low order correlators of fast χ -fields can be approximated with the thermal ones (Sec. 4.4.1). Or, if the amplitude of oscillating scalar is smaller than the thermal mass of χ , $\lambda \tilde{\phi} \ll m_{\chi, \text{th}}$, one can compute the thermal corrections by simply assuming that the background plasma including χ -particles remains in thermal equilibrium (Sec. 4.4.2). In other words, the following section is devoted to study the case where the oscillating scalar condensate can be regarded as an adiabatic background for χ -particles, while the non-adiabatic case is studied in the previous section.

Let us discuss effects of the background thermal plasma in these two cases in the following. This section closely follows our previous works [35, 44]. See also [31–34, 37–40].

^{♦15} If the time scales of two effects become comparable, our estimation is not reliable.

4.4.1 Slowly Oscillating Scalar

In this subsection, we consider the case where the time-evolution of ϕ -condensation is so slow that, at least, lower order correlators of χ -fields can remain close to thermal equilibrium with the background ϕ . Such a slowly oscillating scalar condensation was studied in the pioneering works of Refs. [141, 142]. Hence, in the following, we concentrate on the regime where the adiabaticity of χ -fields is not broken down, $k_* \lesssim \max[m_\phi, m_{\chi,\text{th}}]$, and the oscillation time scale of ϕ is much slower than the typical damping time scale of each quasi-particle excitations, $m_{\phi,\text{eff}} \ll \Gamma_{\text{damp}} \sim \alpha T$ (See e.g. [143]). See Fig. 4.4 for clarity. Roughly speaking, what we will do in the following is that: (I) Obtain the approximate solutions of propagators with the form $G_\chi = G_{\chi,\text{th}}|_\phi + \delta G_\chi$ ($S_\chi = S_{\chi,\text{th}}|_\phi + \delta S_\chi$) where $G_{\chi,\text{th}}|_\phi$ ($S_{\chi,\text{th}}|_\phi$) represents the thermal one with the background ϕ , and δG (δS) is caused by the moving ϕ and hence it is roughly proportional to $\dot{\phi}$; (II) Insert them into the equation of motion for the mean field ϕ [Eq. (4.42)] and obtain the coarse-grained equation. The following arguments are complete contrasts to those given in the previous Sec. 4.3, where the broken down of adiabaticity plays the key role

Before going into details, in order to clarify the above assumptions, we have to derive the condition for χ -particles to be thermally populated when the ϕ -condensation passes through its origin; $|\phi| \lesssim \phi_c \equiv T/\tilde{\lambda}$ where χ -particles become light.^{♦16} The time scale δt which the ϕ -condensate takes to pass through $|\phi| \lesssim \phi_c$ can be estimated as $\delta t \sim \phi_c/(m_{\phi,\text{eff}}\tilde{\phi}) \sim T/k_*^2$. The abundance of χ -particles which are produced during this time interval is given by

$$n_\chi \sim n_{\text{th}}^2 \langle \sigma_{\text{prod}} |v| \rangle \delta t \sim \max[\epsilon^2, \alpha] \alpha \frac{T^2}{k_*^2} T^3 \gtrsim \max[\epsilon^2, \alpha] T^3, \quad (4.116)$$

where we consider two production mechanisms of χ : inverse decay $\langle \sigma_{\text{prod}} |v| \rangle \sim \epsilon^2 g^2/T^2$ and annihilation $\langle \sigma_{\text{prod}} |v| \rangle \sim g^4/T^2$. In the last inequality, we have used the adiabatic condition: $k_* \lesssim m_{\chi,\text{rh}}$. If ϵ is not suppressed, $\epsilon \sim 1$, then the χ -particles can always attain the thermal abundance.

On the other hand, if ϵ is extremely suppressed (e.g. stable χ), there are two cases depending on the ratio: $k_*^2/(g^4 T^2)$. For $k_*^2/(g^4 T^2) > 1$, χ -particles do not reach thermal equilibrium during δt and hence their annihilation is neglected after the passage of $|\phi| \simeq \phi_c$. For $k_*^2/(g^4 T^2) < 1$, χ -particles are thermally populated and their pair annihilation takes place after the ϕ -condensation climbs up the potential around $|\phi| \sim \phi_c$. As a result, the number density of χ -particles after the passage of $|\phi| \lesssim \phi_c$ is given by

$$n_\chi^{(\text{stb})} \sim d^{(\text{stb})} T^3, \quad (4.117)$$

where $d^{(\text{stb})}$ encodes the model dependent numerical factor; $d^{(\text{stb})} \sim g^4 T^2/k_*^2$ for $k_*^2/g^4 T^2 > 1$, and $d^{(\text{stb})} < 1$ for $k_*^2/(g^4 T^2) < 1$.^{♦17} In both cases; $k_*^2/(g^4 T^2) \gtrsim 1$, the produced χ -particles cannot reduce their number within the time scale of ϕ 's oscillation period, and hence the linear potential discussed around Eq. (4.108) dominates:

$$V_{\text{linear}}^{(\text{stb})}(\phi) \sim N_{\text{d.o.f.}} \tilde{\lambda} |\phi| n_\chi^{(\text{stb})} \quad \text{for } \tilde{\lambda} |\phi| > T. \quad (4.118)$$

^{♦16} Be careful that the amplitude of oscillation $\tilde{\phi}$ can be much larger $\tilde{\phi} \gg \phi_c$.

^{♦17} Here we do not explicitly write down the form of $d^{(\text{stb})}$ for $k_*^2/(g^4 T^2) < 1$ since there are many cases depending on which term dominates the potential. For instance, if the scalar field oscillates with the thermal log potential, the number density of χ -particles reduce to $n_\chi \sim 1/(\langle \sigma |v| \rangle \delta t_{\text{log}}) \sim (\tilde{\lambda}/\alpha^3) T^3$ with $\delta t_{\text{log}} \sim \alpha/(\tilde{\lambda} T)$, which implies $d^{(\text{stb})} \sim \min[1, \tilde{\lambda}/\alpha^3]$. Note that it takes δt_{log} to reach the maximum value of the linear potential $\sim \tilde{\lambda} T^3 \phi_{\text{max}} \sim \alpha^2 T^4$.

Obviously, in this case, χ -fields do not remain close to thermal equilibrium during the course of ϕ 's oscillation since they do not have enough time to reduce their number densities for $\lambda|\phi| > T$ within the oscillation period. Thus, in the following, we concentrate on the first case: $\epsilon \sim 1$.

• **Small Field Value Regime:** First, we study the situation where the ϕ -condensation passes through $|\phi| \lesssim \phi_c$.^{◆18} Let us again move back to Eqs. (4.23)–(4.26) and (4.42) as a starting point of our discussion. Since the motion of the ϕ -condensate is slow, one can approximate Eq. (4.22) around a time \bar{t} with $\phi(\bar{t}) = \bar{\phi}$:

$$G_\chi(t, t'; \mathbf{p}) = G_{\chi, \text{th}}(t, t'; \mathbf{p})|_{\bar{\phi}} - i \int_{\mathcal{C}} d\tau d\tau' G_{\chi, \text{th}}(t, \tau; \mathbf{p})|_{\bar{\phi}} D_{\bar{\chi}}(\tau, \tau'; \mathbf{p})|_{\bar{\phi}} G_\chi(\tau', t'; \mathbf{p}) \quad (4.119)$$

$$= G_{\chi, \text{th}}(t, t'; \mathbf{p})|_{\bar{\phi}} - 2i\lambda^2 \bar{\phi} \int_{\mathcal{C}} d\tau G_{\chi, \text{th}}(t, \tau; \mathbf{p})|_{\bar{\phi}} \delta\phi(\tau; \bar{t}) G_{\chi, \text{th}}(\tau, t'; \mathbf{p})|_{\bar{\phi}} \quad (4.120)$$

$+\dots,$

where $D_{\bar{\chi}}(\tau, \tau'; \mathbf{p})|_{\bar{\phi}} \equiv \lambda^2 [2\bar{\phi} \delta\phi(\tau; \bar{t}) + \delta\phi^2(\tau; \bar{t})] \delta_{\mathcal{C}}(\tau, \tau') - i [\Pi_\chi - \Pi_{\chi, \text{th}}](\tau, \tau'; \mathbf{p})|_{\bar{\phi}}$, and $\delta\phi(\tau; \bar{t}) \equiv \phi(\tau) - \bar{\phi}$. Here $G_{\chi, \text{th}}|_{\bar{\phi}}$ is the thermal propagators with $\phi(\bar{t}) = \bar{\phi}$ and we implicitly assume that the average of time-arguments in the Green function is near \bar{t} : $(t+t')/2 \simeq \bar{t}$. In the second equality, we have neglected contributions in D from higher orders in $\delta\phi$ and also from the self energy of χ . Note that in estimating the dissipation rate of slowly oscillating scalar, the latter self energy contribution can be comparable to the result from $\lambda^2 \bar{\phi} \delta\phi$, and the resultant dissipation rate can change by several factors [40, 144]. Nevertheless, we roughly estimate the dissipation factor with dropping contributions from the self energy; rather, concentrate on the order of magnitude estimation and its qualitative behavior. Similarly, one can obtain the approximated solution of fermionic χ -fields around a time \bar{t} as

$$S_\chi(t, t'; \mathbf{p}) = S_{\chi, \text{th}}(t, t'; \mathbf{p})|_{\bar{\phi}} - i \int_{\mathcal{C}} d\tau d\tau' S_{\chi, \text{th}}(t, \tau; \mathbf{p})|_{\bar{\phi}} D_\chi(\tau, \tau'; \mathbf{p})|_{\bar{\phi}} S_\chi(\tau', t'; \mathbf{p}) \quad (4.121)$$

$$= S_{\chi, \text{th}}(t, t'; \mathbf{p})|_{\bar{\phi}} - i y \int_{\mathcal{C}} d\tau S_{\chi, \text{th}}(t, \tau; \mathbf{p})|_{\bar{\phi}} \delta\phi(\tau; \bar{t}) S_{\chi, \text{th}}(\tau, t'; \mathbf{p})|_{\bar{\phi}} \quad (4.122)$$

$+\dots,$

where $D_\chi(\tau, \tau'; \mathbf{p})|_{\bar{\phi}} = y \delta\phi(\tau; \bar{t}) \delta_{\mathcal{C}}(\tau, \tau') - i [\Sigma_\chi - \Sigma_{\chi, \text{th}}](\tau, \tau'; \mathbf{p})|_{\bar{\phi}}$.

By inserting these approximated solutions to Eq. (4.42), we can obtain the coarse-grained equation for the mean field ϕ . The effective mass term $M_\phi^2(t)$ can be approximated with^{◆19}

$$M_\phi^2(\bar{t}) = m_\phi^2 + \lambda^2 \int_{\mathbf{p}} G_{\text{H}}^{\chi, \text{th}}(\bar{t}, \bar{t}; \mathbf{p})|_{\bar{\phi}} \quad (4.123)$$

$$- 4i\lambda^4 \bar{\phi} \int_{\mathcal{C}} d\tau \int_{\mathbf{p}} G_{\chi, \text{th}}(\bar{t}, \tau; \mathbf{p})|_{\bar{\phi}} \delta\phi(\tau; \bar{t}) G_{\chi, \text{th}}(\tau, \bar{t}; \mathbf{p})|_{\bar{\phi}} + \dots \quad (4.124)$$

^{◆18} See the footnote ◆16.

^{◆19} Contributions from thermal log (See the next subsection) and possible Coleman-Weinberg potentials may be imprinted in Eq. (4.123) for $\lambda|\phi| > T$.

For $\lambda|\phi| < T$, the former term [Eq. (4.123)] encodes the thermal mass of ϕ from the χ -particles $m_{\phi,\text{th}}$ [Eq. (4.65)] as expected:

$$m_{\phi,\text{th}}^2 \Big|_{\tilde{\chi}} = N_{\tilde{\chi}} \times \lambda^2 \int_p \frac{f_B(|\mathbf{p}|)}{|\mathbf{p}|} = N_{\tilde{\chi}} \times \frac{\lambda^2 T^2}{12}, \quad (4.125)$$

at the leading order in high temperature expansion. The latter term [Eq. (4.124)] encodes the friction term due to the abundant χ particles, so let us concentrate on this term:

$$-4i\lambda^4 \bar{\phi} \int_{t_{\text{ini}}}^{\bar{t}} d\tau \delta\phi(\tau; \bar{t}) \int_p \left[G_{>}^{\chi,\text{th}}(\bar{t}, \tau; \mathbf{p})|_{\bar{\phi}} G_{<}^{\chi,\text{th}}(\tau, \bar{t}; \mathbf{p})|_{\bar{\phi}} - G_{<}^{\chi,\text{th}}(\bar{t}, \tau; \mathbf{p})|_{\bar{\phi}} G_{>}^{\chi,\text{th}}(\tau, \bar{t}; \mathbf{p})|_{\bar{\phi}} \right].$$

Hereafter, we adopt the following approximations; $\delta\phi(\tau; \bar{t}) = \dot{\phi}(\tau - \bar{t}) + \dots$ and $t_{\text{ini}} \rightarrow -\infty$, since the motion of ϕ is assumed to be so slow compared with the typical damping time scale of particle excitations in the thermal plasma (See also footnote [◆20](#)). Then, the friction term of ϕ reads

$$\begin{aligned} \Gamma_{\phi}^{(3\text{p}, \text{slow}), \tilde{\chi}} &= -4i\lambda^4 \phi(\bar{t})^2 \int_{-\infty}^0 d\tau \tau \int_{\omega_1, \omega_2, \mathbf{p}} e^{-i(\omega_1 - \omega_2)\tau} [f_B(\omega_1) - f_B(\omega_2)] \rho_{\tilde{\chi},\text{th}}(\omega_1, \mathbf{p})|_{\bar{\phi}} \rho_{\tilde{\chi},\text{th}}(\omega_2, \mathbf{p})|_{\bar{\phi}} \\ &= \frac{2\lambda^4 \phi^2(\bar{t})}{T} \int_{\omega, \mathbf{p}} [-f_B(\omega)f_B(-\omega)] \rho_{\tilde{\chi},\text{th}}(\omega, \mathbf{p})|_{\bar{\phi}} \rho_{\tilde{\chi},\text{th}}(\omega, \mathbf{p})|_{\bar{\phi}}. \end{aligned} \quad (4.126)$$

Here we have used the KMS relations; $G_{>}^{\chi,\text{th}}(P) = [1 + f_B(p^0)]\rho_{\tilde{\chi},\text{th}}(P)$, $G_{<}^{\chi,\text{th}}(P) = f_B(p^0)\rho_{\tilde{\chi},\text{th}}(P)$. Assuming the Breit-Wigner form for the spectral density of χ quasi-particles as in Eq. (4.49) and roughly approximating the thermal width as $\Gamma \sim \alpha T^2/\omega$, one can estimate the dissipation rate of ϕ -condensate as

$$\Gamma_{\phi}^{(3\text{p}, \text{slow}), \tilde{\chi}} \sim N_{\tilde{\chi}} \times \frac{\lambda^4 \phi^2(\bar{t})}{\pi^2 \alpha T}. \quad (4.127)$$

For fermionic χ -fields, similar computations can be done as the bosonic $\tilde{\chi}$ -fields. Inserting the approximated solutions to $\int_p \text{Tr}[S_H^{\chi}(t, t; \mathbf{p})]$, one obtains

$$-\frac{y}{2} \int_p \text{Tr}[S_H^{\chi}(t, t; \mathbf{p})] = -\frac{y}{2} \int_p \text{Tr}[S_H^{\chi,\text{th}}(t, t; \mathbf{p})]|_{\bar{\phi}} \quad (4.128)$$

$$+ iy^2 \int_{\mathcal{C}} d\tau \int_p \text{Tr}[S_{\chi,\text{th}}(t, \tau; \mathbf{p})|_{\bar{\phi}} \delta\phi(\tau; \bar{t}) S_{\chi,\text{th}}(\tau, t; \mathbf{p})|_{\bar{\phi}}] \quad (4.129)$$

$$+ \dots$$

The first term on the right hand side is nothing but the thermal mass from abundant χ -particles as computed in Eq. (4.62):

$$m_{\phi,\text{th}}^2 \Big|_{\chi} = N_{\chi} \times \frac{y^2 T^2}{6}, \quad (4.130)$$

for $y|\phi| < T$. The second term [Eq. (4.129)] imprints the dissipation rate of ϕ -condensation due to the abundant χ -particles and it can be expressed as

$$iy^2 \int_{t_{\text{ini}}}^{\bar{t}} d\tau \delta\phi(\tau; \bar{t}) \int_p \text{Tr}[S_{>}^{\chi,\text{th}}(\bar{t}, \tau; \mathbf{p})|_{\bar{\phi}} S_{<}^{\chi,\text{th}}(\tau, \bar{t}; \mathbf{p})|_{\bar{\phi}} - S_{<}^{\chi,\text{th}}(\bar{t}, \tau; \mathbf{p})|_{\bar{\phi}} S_{>}^{\chi,\text{th}}(\tau, \bar{t}; \mathbf{p})|_{\bar{\phi}}]. \quad (4.131)$$

Assuming that the ϕ -condensation moves sufficiently slow; $\delta\phi(\bar{t}; \tau) = \dot{\phi}(\tau - \bar{t}) + \dots$ and $t_{\text{ini}} \rightarrow -\infty$, we obtain the dissipation rate of ϕ -condensation:

$$\begin{aligned}\Gamma_{\phi}^{(\text{slow}),\chi} &= iy^2 \int_{-\infty}^0 d\tau \tau \int_{\omega_1, \omega_2, \mathbf{p}} e^{i(\omega_1 - \omega_2)\tau} [f_F(\omega_1) - f_F(\omega_2)] \text{Tr}[\rho_{\chi, \text{th}}(\omega_1, \mathbf{p})|_{\bar{\phi}} \rho_{\chi, \text{th}}(\omega_2, \mathbf{p})|_{\bar{\phi}}] \\ &= \frac{y^2}{2T} \int_{\omega, \mathbf{p}} [f_F(\omega) f_F(-\omega)] \text{Tr}[\rho_{\chi, \text{th}}(\omega, \mathbf{p})|_{\bar{\phi}} \rho_{\chi, \text{th}}(\omega, \mathbf{p})|_{\bar{\phi}}].\end{aligned}\quad (4.132)$$

Approximating the spectral densities with the Breit-Wigner forms given in Eqs. (4.54) and (4.58), one can obtain the dissipation rate of ϕ -condensation caused by abundant χ -fermions:

$$\Gamma_{\phi}^{(\text{slow}),\chi} \sim N_{\chi} \times \begin{cases} \frac{y^2 \alpha T}{4\pi^2} & \text{for } y|\phi| \ll m_{\chi, \text{th}} \sim gT \\ \frac{2y^4 \phi^2(\bar{t})}{\pi^2 \alpha T} & \text{for } m_{\chi, \text{th}} \sim gT \ll y|\phi| \ll T \end{cases}, \quad (4.133)$$

where we have roughly approximated the thermal width of fermions as $\Gamma \sim \alpha T$ (See for instance [143]).

Interestingly, these ϕ -dependent dissipation terms given in Eqs. (4.127) and (4.133) can be understood as the energy transportation caused by the adiabatic mass change of χ -particles because their effective mass depends on ϕ . This picture is explicitly demonstrated in Sec. 4.5. Note that this is a complete contrast to the dissipation rates discussed in the previous Sec. 4.3 where the broken down of adiabaticity plays the key role.

Then, let us evaluate the non-local term, $-i\Pi_J \cdot \phi$. Again we assume that the dynamics of ϕ is sufficiently slow $\delta\phi(\bar{t}; \tau) = \dot{\phi}(\tau - \bar{t}) + \dots$ and $t_{\text{ini}} \rightarrow -\infty$, and hence the non-local term can be approximated with

$$-i \int_{-\infty}^{\bar{t}} d\tau \Pi_J(\bar{t}, \tau) \phi(\tau) \supset -i \int_{-\infty}^{\bar{t}} d\tau \Pi_J(\bar{t} - \tau; \bar{t})|_{\bar{\phi}} \delta\phi(\tau; \bar{t}) \simeq \dot{\phi}(\bar{t}) \left[\frac{\Pi_J(\omega; \bar{t})|_{\bar{\phi}}}{2\omega} \right]_{\omega \rightarrow 0} \quad (4.134)$$

where the self energy is approximated by the gradient-expansion: $\Pi_J(\bar{t}, \tau) = \Pi_J(\bar{t} - \tau; [\bar{t} + \tau]/2) = \Pi_J(\bar{t} - \tau; \bar{t}) + \dots$.^{♦20} Thus, the dissipation rate can be expressed as

$$\Gamma_{\phi}^{(4p),\tilde{\chi}} = \left[\frac{\Pi_J(\omega; \bar{t})|_{\bar{\phi}}}{2\omega} \right]_{\omega \rightarrow 0}. \quad (4.135)$$

Since there are no ϕ particles initially, let us assume that the propagators of ϕ can be approximated with the vacuum one. Then, one finds

$$\begin{aligned}\Pi_J(\omega; \bar{t})|_{\bar{\phi}} &= \lambda^4 \int_{K, Q, L} [(1 + 2f_{B, k_0})(1 + 2f_{B, q_0}) + 1 + 2(1 + 2f_{B, k_0}) \text{sgn}(l_0)] \\ &\quad \rho_{\chi, \text{th}}(K)|_{\bar{\phi}} \rho_{\chi, \text{th}}(Q)|_{\bar{\phi}} \rho_{\phi, \text{vac}}(L)\end{aligned}\quad (4.136)$$

where

$$\int_{K, Q, L} = \int \frac{d^4 K}{(2\pi)^4} \frac{d^4 Q}{(2\pi)^4} \frac{d^4 L}{(2\pi)^4} (2\pi)^4 \delta(-\omega + k_0 + q_0 + l_0) \delta(\mathbf{k} + \mathbf{q} + \mathbf{l}). \quad (4.137)$$

^{♦20} Note that the correlation of two distinct time in the self energy is expected to decay much faster than the motion of oscillating scalar since the scalar oscillates much slower than the typical interaction in thermal plasma.

As a result, the dissipation rate reads

$$\Gamma_{\phi}^{(4p, \text{slow}), \tilde{\chi}} \sim N_{\tilde{\chi}} \times \frac{\lambda^4 T^3}{12\pi m_{\chi, \text{th}}^2}. \quad (4.138)$$

Before closing the discussion on the small field value regime, it is instructive to reconsider the condition $m_{\text{eff}} \ll \Gamma_{\text{damp}} \sim \alpha T$. If the ϕ -condensate oscillates with the thermal mass potential, then this inequality poses the following requirement $\lambda, y < \alpha$. Also, one can show that the non-perturbative production does not take place in the thermal potential under $\lambda, y < \alpha$ [See discussion below Eq. (4.199)]. Hence, the thermal mass potential derived in Sec. 4.2 can be safely applied to the oscillating scalar field for $\lambda, y < \alpha$

• **Large Field Value Regime:** Second, let us consider the large field value regime $|\phi| \gtrsim \phi_c$, where the χ -particles are absent due to the Boltzmann-suppression. In this case, we adopt the different strategy to obtain the coarse-grained equation since it is easier as we have seen in Sec. 4.2. We start with the effective operator given in Eq. (4.68)

$$-\frac{1}{4} \left[\frac{1}{g^2(T)} + \frac{1}{16\pi^2} \left(N_{\chi} \frac{4}{3} T(\mathbf{r}_F) \ln \frac{y^2 \phi^2}{T^2} + N_{\tilde{\chi}} \frac{1}{3} T(\mathbf{r}_S) \ln \frac{\lambda^2 \phi^2}{T^2} \right) \right] F_{\mu\nu}^a F^{a\mu\nu}, \quad (4.139)$$

which is obtained by integrating out heavy χ -fields. Differentiating with respect to ϕ , one obtains the following operator for $\delta\phi(\tau; \bar{t}) = \phi(t) - \bar{\phi}$:

$$-\frac{1}{16\pi^2} \left[N_{\chi} \frac{2}{3} T(\mathbf{r}_F) + N_{\tilde{\chi}} \frac{1}{6} T(\mathbf{r}_S) \right] \frac{\delta\phi(t; \bar{t})}{\bar{\phi}} F_{\mu\nu}^a F^{a\mu\nu} \equiv -\frac{\delta\phi(t; \bar{t})}{\bar{\phi}} \frac{T_{\text{sum}}}{16\pi^2} F_{\mu\nu}^a F^{a\mu\nu}. \quad (4.140)$$

Basically, the equation of motion for the ϕ -condensation is obtained from $0 = \delta\Gamma/\delta\phi$. Again assuming that the motion of ϕ is much slower than the typical interaction time scale in the thermal plasma, one can approximate that all the other light fields remain close to thermal equilibrium and these fields see the ϕ -condensation as an almost static background, $\Gamma[\phi, \{G_{\bullet}|\bar{\phi}\}]$ with $\{G_{\bullet}|\bar{\phi}\}$ being a set of propagators which are close to thermal equilibrium in the presence of the constant background $\bar{\phi}$. In this case, we can evaluate the effective action:

$$\begin{aligned} \frac{1}{\text{vol}(\mathbf{R}^3)} \frac{\delta\Gamma}{\delta\phi} \Big|_{\bar{\phi} + \delta\phi} &= -\ddot{\bar{\phi}} - 3H\dot{\bar{\phi}} - \frac{\partial V_{\text{eff}}(\bar{\phi})}{\partial \bar{\phi}} \\ &+ \frac{1}{\text{vol}(\mathbf{R}^3)} \sum_{n=1} \frac{1}{n!} \int dt_1 \cdots dt_n \frac{\delta^{n+1}\Gamma}{\delta\phi(\bar{t})\delta\phi(t_1)\cdots\delta\phi(t_n)} \Big|_{\phi=\bar{\phi}} \delta\phi(t_1) \cdots \delta\phi(t_n) \\ &= -\ddot{\bar{\phi}} - 3H\dot{\bar{\phi}} - \frac{\partial V_{\text{eff}}(\bar{\phi})}{\partial \bar{\phi}} - \int d\tau \Pi_{\text{ret}}^{\phi}(\bar{t} - \tau; \mathbf{0})|_{\bar{\phi}} \delta\phi(\tau; \bar{t}) + \cdots \end{aligned} \quad (4.141)$$

at the first order in $\delta\phi$. The first term in Eq. (4.141) is nothing but the effective potential caused by the thermal plasma in the presence of $\bar{\phi}$, which is computed in Sec. 4.2.^{♦21}

Before going to discuss the dissipation rate of ϕ -condensate, it is instructive to verify the underlying assumption of our discussion: the oscillation time scale of ϕ is much slower than that of typical interactions in the thermal plasma. If the effective potential is dominated by the zero temperature mass term, then this assumption simply constraints the range of applicability of our discussion, that is, $m_{\phi} \lll T$. In the thermal mass case, the assumption nothing but

^{♦21} It is noticeable that the discussion in the previous case (small field value regime) can be reproduced by this strategy.

implies the coupling hierarchy $\tilde{\lambda} \ll \alpha$. However, if it is dominated by the thermal log potential, the oscillation time scale is determined by the parameters in the thermal plasma, $\alpha T^2/|\phi|$ for $\phi \gg \phi_c$. Even in this case, one can show that it is much slower than the typical interaction time scale: $\alpha T^2/|\phi| \ll \alpha \tilde{\lambda} T \ll \alpha^2 T$. Thus, under our assumption, we can safely use the thermal potential in the equation of motion.

In our case, $\delta\phi$ couples with the thermal plasma via the effective operator given in Eq. (4.140), and hence the self energy is given by

$$\Pi_{\text{ret}}^\phi(x)|_{\bar{\phi}} = -i\theta(x_0) \Pi_J^\phi(x)|_{\bar{\phi}} \equiv -i\theta(x_0) \left(\frac{T_{\text{sum}}}{16\pi^2\bar{\phi}} \right)^2 \langle [FF(x), FF(0)] \rangle, \quad (4.142)$$

where the ensemble average $\langle \dots \rangle$ is taken with the thermal one, and the summation over Lorentz and gauge indices are promised, $FF = F_{\mu\nu}^a F^{a\mu\nu}$. As in the previous case (small field value regime), one can approximate the self energy as follows, since the typical interaction time scale in the thermal plasma is much faster than the motion of ϕ -condensate:

$$\begin{aligned} \int d\tau \Pi_{\text{ret}}^\phi(\bar{t} - \tau, \mathbf{0})|_{\bar{\phi}} \delta\phi(\tau; \bar{t}) &= -\dot{\phi} \frac{\partial}{\partial i\omega} \int d\tau e^{i\omega\tau} \Pi_{\text{ret}}^\phi(\tau, \mathbf{0})|_{\bar{\phi}} \Big|_{\omega \rightarrow 0} + \dots \\ &= \dot{\phi} \left[-\lim_{\omega \rightarrow 0} \frac{\Im \Pi_{\text{ret}}^\phi(\omega, \mathbf{0})|_{\bar{\phi}}}{\omega} \right]. \end{aligned} \quad (4.143)$$

In the second line, we have used the fact that $\Re/\Im \Pi_{\text{ret}}(\omega, \mathbf{0})$ is an even/odd function in ω . This equation implies that the dissipation rate of ϕ -condensation is given by

$$\Gamma_\phi^{(\text{lf})} = \lim_{\omega \rightarrow 0} \frac{\Pi_J^\phi(\omega, \mathbf{0})|_{\bar{\phi}}}{2\omega}, \quad (4.144)$$

where we have used the Kramers-Kronig relation.

As mentioned in the previous discussion, it is known that one should perform technical resummations in order to obtain Γ_ϕ at the *complete leading order* in the coupling in the case of vanishing external four momentum. Since the complete resummation is beyond the scope of this thesis, we restrict ourselves to order of magnitude estimations in the previous case. However, fortunately, in the large field value regime, the complete result can be extracted from that given in a different context; as a bulk viscosity of non-Abelian plasma:

$$\zeta = \frac{1}{9} \lim_{\omega \rightarrow 0} \frac{1}{2\omega} \int d^4x e^{i\omega x_0} \langle [T^\mu{}_\mu(x), T^\nu{}_\nu(0)] \rangle, \quad (4.145)$$

where

$$T^\mu{}_\mu \simeq -\frac{b_0}{2} F^{a\mu\nu} F_{\mu\nu}^a. \quad (4.146)$$

Here we have assumed that all the light fields in the thermal plasma are massless, and b_0 is the coefficient of the beta function; $b_0 \equiv \beta(g)/g^3$ with $\beta(g) \equiv \mu \partial g / \partial \mu$. In Ref. [145], the bulk viscosity ζ for QCD-like theory is determined at the complete leading order by means of the effective kinetic theory developed in Ref. [146, 147]. Here we simply quote their result:²²

$$\zeta \simeq \frac{b_0^2 g^4 T^3}{4 \ln(1/\alpha)}. \quad (4.147)$$

²² In our case, there might be scalar fields charged under the gauge group, for instance in SUSY theory. But, we expect that the qualitative behavior is the same since the order of magnitude estimation is still valid in terms of the one-loop diagram [148].

As can be seen from Eqs. (4.145)–(4.147), the dissipation rate $\Gamma_\phi^{(\text{lf})}$ can be expressed as [41, 42]

$$\begin{aligned}\Gamma_\phi^{(\text{lf})} &\simeq \left(\frac{T_{\text{sum}}}{16\pi^2}\right)^2 \frac{36\zeta}{b_0^2 \bar{\phi}^2} \simeq \left(\frac{T_{\text{sum}}}{16\pi^2}\right)^2 \frac{(12\pi\alpha)^2}{\ln \alpha^{-1}} \frac{T^3}{\bar{\phi}^2} \\ &\equiv \frac{b\alpha^2 T^3}{\bar{\phi}^2},\end{aligned}\tag{4.148}$$

typically $b \sim 10^{-3}$.

Before closing this subsection, let us briefly comment on the case where the ϕ -condensation oscillates around a sizable VEV [Eq. (4.8)]. Obviously, the dissipation rate of the ϕ -condensation is given by [41, 42, 45, 118, 119]:

$$\Gamma_\phi^{(\text{vev})} \simeq \frac{b\alpha^2 T^3}{v^2}.\tag{4.149}$$

This effect plays important roles in the dynamics after the symmetry breaking; for instance, dynamics after the thermal inflation, dynamics of PQ-scalars after the $U(1)_{\text{PQ}}$ breaking [45, 119] and so on.

4.4.2 Oscillating Scalar with Small Amplitude

Then, let us study the case where the amplitude $\tilde{\phi}$ is smaller than the thermal mass of χ , $\lambda\tilde{\phi} \ll m_{\chi,\text{th}} \sim gT$, but not necessarily the oscillation period of ϕ -condensate is slower than the typical damping time scale of each quasi-particle excitations, $m_\phi \gtrsim \Gamma_{\text{damp}} \sim \alpha T$, in contrast to the previous case. In this case, we expect the background plasma including χ -particles remains close to thermal equilibrium during the course of ϕ 's oscillation for the following reasons. First, the ϕ -dependent mass term of χ can be safely neglected. Second, the energy transportation time scale from the oscillating scalar ϕ to thermal plasma is much slower than the typical interaction time scale of thermal plasma in our case. Third, the broad resonance does not occur in this case since $\lambda\tilde{\phi} \ll m_{\chi,\text{th}}$ violates the condition for non-perturbative particle production, Eq. (4.99). The narrow resonance also does not occur in our cases. This is because at least $m_{\phi,\text{eff}} > m_{\chi,\text{th}}$ is required for the narrow resonance to take place, and in addition the growth rate of narrow resonance should be larger than the decay and dissipation rate of χ [139, 140], $\lambda^2 \tilde{\phi}^2 / m_\phi \gg \max[\alpha T, \epsilon^2 \alpha \lambda \tilde{\phi}]$, in order for the induced emission to be efficient. These conditions are unlikely to be satisfied in most cases of our interest. See Fig. 4.4. (See also the discussion in Sec. 4.3.) Finally, radiation has huge degrees of freedom. Therefore, one can calculate thermal corrections to the oscillating scalar field by simply assuming that the background plasma including χ particles can remain in thermal equilibrium [31, 33].

First, we evaluate the effective mass term $M_\phi^2(t)$. The following arguments are essentially equivalent to those in Ref. [31]. Since the amplitude is small compared to the thermal mass of χ , the approximate solution of χ 's propagator can be obtained in the similar way as in the previous case:

$$\begin{aligned}G_\chi(t, t'; \mathbf{p}) &= G_{\chi,\text{th}}(t, t'; \mathbf{p})|_{\bar{\phi}=0} - i \int_{\mathcal{C}} d\tau d\tau' G_{\chi,\text{th}}(t, \tau; \mathbf{p})|_{\bar{\phi}=0} D'_\chi(\tau, \tau'; \mathbf{p}) G_\chi(\tau, t'; \mathbf{p}) \\ &= G_{\chi,\text{th}}(t, t'; \mathbf{p})|_{\bar{\phi}=0} - i\lambda^2 \int_{\mathcal{C}} d\tau G_{\chi,\text{th}}(t, \tau; \mathbf{p})|_{\bar{\phi}=0} \phi^2(\tau) G_{\chi,\text{th}}(\tau, t'; \mathbf{p})|_{\bar{\phi}=0} + \cdots,\end{aligned}\tag{4.150}$$

where $D'_{\tilde{\chi}}(\tau, \tau'; \mathbf{p}) \equiv \lambda^2 \phi^2(\tau) \delta_{\varphi}(\tau, \tau') - i[\Pi_{\chi} - \Pi_{\chi, \text{th}}](\tau, \tau'; \mathbf{p})|_{\tilde{\phi}=0}$. In the following, we drop the subscript $\tilde{\phi} = 0$ for brevity. Aside from the leading contribution to the thermal mass of ϕ , $\lambda^2 T^2$, this term leads to the following contribution:

$$\begin{aligned}
& -2i\lambda^4 \int_{-\infty}^{\bar{t}} d\tau \phi^2(\tau) \int_{\mathbf{p}} [G_{>}^{\chi, \text{th}}(\bar{t}, \tau; \mathbf{p}) G_{<}^{\chi, \text{th}}(\tau, \bar{t}; \mathbf{p}) - G_{<}^{\chi, \text{th}}(\bar{t}, \tau; \mathbf{p}) G_{>}^{\chi, \text{th}}(\tau, \bar{t}; \mathbf{p})] \\
& = -2i\lambda^4 \int_0^{\infty} d\tau \phi^2(\bar{t} - \tau) \frac{\partial}{\partial \tau} C_{\tilde{\chi}}(\tau) \\
& = +2i\lambda^4 \phi^2(\bar{t}) C_{\tilde{\chi}}(0) - 4i\lambda^4 \int_0^{\infty} d\tau \phi(\bar{t} - \tau) \dot{\phi}(\bar{t} - \tau) \int_{\omega} e^{-i\omega\tau} C_{\tilde{\chi}}(\omega)
\end{aligned} \tag{4.151}$$

where

$$\frac{\partial}{\partial \tau} C_{\tilde{\chi}}(\tau) \equiv \int_{\mathbf{p}} [G_{>}^{\chi, \text{th}}(\tau; \mathbf{p}) G_{<}^{\chi, \text{th}}(-\tau; \mathbf{p}) - G_{<}^{\chi, \text{th}}(\tau; \mathbf{p}) G_{>}^{\chi, \text{th}}(-\tau; \mathbf{p})]. \tag{4.152}$$

This implies

$$-i\omega C_{\tilde{\chi}}(\omega) = \int d\tau e^{i\omega\tau} \int_{\mathbf{p}} [G_{>}^{\chi, \text{th}}(\tau; \mathbf{p}) G_{<}^{\chi, \text{th}}(-\tau; \mathbf{p}) - G_{<}^{\chi, \text{th}}(\tau; \mathbf{p}) G_{>}^{\chi, \text{th}}(-\tau; \mathbf{p})]. \tag{4.153}$$

Here note that we take the initial time t_{ini} to the infinity past as in the previous case because the background plasma can remain close to thermal equilibrium much longer than the oscillation time scale of ϕ -condensate. However, we cannot approximate $\phi(\tau)$ by its Taylor expansion because the oscillation period is much faster than the typical damping time scale of thermal propagators in this case.

The first term of Eq. (4.151) represents a higher order correction to the thermal mass of ϕ in the coupling λ . Hence, let us concentrate on the latter term of Eq. (4.151) that encodes the dissipation rate. Inserting $\phi(\bar{t} - \tau) = \phi(\bar{t}) \cos[m_{\phi, \text{eff}}\tau] - [\dot{\phi}(\bar{t})/m_{\phi, \text{eff}}] \sin[m_{\phi, \text{eff}}\tau]$, one finds

$$\begin{aligned}
& -4i\lambda^4 \int_0^{\infty} d\tau \int_{\omega} e^{-i\omega\tau} C_{\tilde{\chi}}(\omega) \left[\phi(\bar{t}) \dot{\phi}(\bar{t}) \cos(2m_{\phi, \text{eff}}\tau) - \frac{1}{2m_{\phi, \text{eff}}} (\dot{\phi}^2(\bar{t}) - m_{\phi, \text{eff}}^2 \phi^2(\bar{t})) \right] \\
& \equiv +C_{\tilde{\chi}}^{(1)} \phi(\bar{t}) \dot{\phi}(\bar{t}) + C_{\tilde{\chi}}^{(2)} \frac{1}{2m_{\phi, \text{eff}}} (\dot{\phi}^2(\bar{t}) - m_{\phi, \text{eff}}^2 \phi^2(\bar{t})).
\end{aligned} \tag{4.154}$$

Since the second term vanishes if we consider the oscillation time averaged evolution equation of ϕ 's energy density, we concentrate on the first term $C_{\tilde{\chi}}^{(1)}$ that leads to the dissipation of oscillating scalar. By definition, we have

$$\begin{aligned}
C_{\tilde{\chi}}^{(1)} & = -4i\lambda^4 \int_0^{\infty} d\tau \int_{\omega} e^{-i\omega\tau} C_{\tilde{\chi}}(\omega) \cos(2m_{\phi, \text{eff}}\tau) \\
& = \lambda^4 \frac{1}{m_{\phi, \text{eff}}} \int d\tau e^{i2m_{\phi, \text{eff}}\tau} \int_{\mathbf{p}} [G_{>}^{\chi, \text{th}}(\tau; \mathbf{p}) G_{<}^{\chi, \text{th}}(-\tau; \mathbf{p}) - G_{<}^{\chi, \text{th}}(\tau; \mathbf{p}) G_{>}^{\chi, \text{th}}(-\tau; \mathbf{p})] \\
& = 2\lambda^4 \left[\frac{1}{\omega} \int_{\mathbf{p}} [1 + f_{\text{B}}(p_0) + f_{\text{B}}(\omega - p_0)] \rho_{\tilde{\chi}, \text{th}}(p_0, \mathbf{p}) \rho_{\tilde{\chi}, \text{th}}(\omega - p_0, \mathbf{p}) \right]_{\omega=2m_{\phi, \text{eff}}} .
\end{aligned} \tag{4.155}$$

Therefore, the dissipation rate can be expressed as

$$\Gamma_{\phi}^{(3p, \text{small}), \tilde{\chi}} = 2\lambda^4 \bar{\phi}^2 \left[\frac{1}{\omega} \int_p [1 + f_B(p_0) + f_B(\omega - p_0)] \rho_{\tilde{\chi}, \text{th}}(p_0, \mathbf{p}) \rho_{\tilde{\chi}, \text{th}}(\omega - p_0, \mathbf{p}) \right]_{\omega=2m_{\phi, \text{eff}}} . \quad (4.156)$$

Taking the vanishing effective mass limit $m_{\phi, \text{eff}} \rightarrow 0$, one can obtain Eq. (4.126) consistently. In contrast to the former case [Eq. (4.127)], the effective mass of oscillating scalar can be larger than the thermal mass of χ , $m_{\phi, \text{eff}} > m_{\chi, \text{th}}$. Then, the perturbative decay (annihilation) of ϕ into two $\tilde{\chi}$ quasi-particles is kinematically allowed and the dissipation factor is given by

$$\Gamma_{\phi}^{(3p, \text{small}), \tilde{\chi}} = N_{\tilde{\chi}} \times \frac{\lambda^4 \bar{\phi}^2}{8\pi m_{\phi, \text{eff}}} \sqrt{1 - \frac{m_{\chi, \text{th}}^2}{m_{\phi, \text{eff}}^2}} [1 + 2f_B(m_{\phi, \text{eff}})] \theta(m_{\phi, \text{eff}} - m_{\chi, \text{th}}), \quad (4.157)$$

for $\lambda \tilde{\phi} < m_{\chi, \text{th}} < m_{\phi, \text{eff}}$.

Next, let us evaluate the fermion contribution $\text{Tr}[S_H]$. As in the boson case, one finds the following approximate solution for χ -fermions:

$$\begin{aligned} S_{\chi}(t, t'; \mathbf{p}) &= S_{\chi, \text{th}}(t, t'; \mathbf{p})|_{\bar{\phi}=0} - i \int_{\mathcal{C}} d\tau d\tau' S_{\chi, \text{th}}(t, \tau; \mathbf{p})|_{\bar{\phi}=0} D'_{\chi}(\tau, \tau'; \mathbf{p}) S_{\chi}(\tau', t'; \mathbf{p}) \\ &= S_{\chi, \text{th}}(t, t'; \mathbf{p})|_{\bar{\phi}=0} - iy \int_{\mathcal{C}} d\tau S_{\chi, \text{th}}(t, \tau; \mathbf{p})|_{\bar{\phi}=0} \phi(\tau) S_{\chi, \text{th}}(\tau, t'; \mathbf{p})|_{\bar{\phi}=0} + \dots, \end{aligned} \quad (4.158)$$

where $D'_{\chi}(\tau, \tau'; \mathbf{p}) = y\phi(\tau)\delta_{\mathcal{C}}(\tau, \tau') - i[\Sigma_{\chi} - \Sigma_{\chi, \text{th}}](\tau, \tau'; \mathbf{p})|_{\bar{\phi}=0}$. Since the first term of Eq. (4.158) vanishes in $\text{Tr}[S_H]$, the following term remains

$$iy^2 \int_0^{\infty} d\tau \phi(\bar{t} - \tau) \int_p \text{Tr}[S_{>}^{\chi, \text{th}}(\tau; \mathbf{p}) S_{<}^{\chi, \text{th}}(-\tau; \mathbf{p}) - S_{<}^{\chi, \text{th}}(\tau; \mathbf{p}) S_{>}^{\chi, \text{th}}(-\tau; \mathbf{p})]. \quad (4.159)$$

Inserting $\phi(\bar{t} - \tau) = \phi(\bar{t}) \cos[m_{\phi, \text{eff}} \tau] - [\dot{\phi}(\bar{t})/m_{\phi, \text{eff}}] \sin[m_{\phi, \text{eff}} \tau]$, we obtain the following term which contributes to the dissipation rate of ϕ -condensation:

$$\begin{aligned} & -\dot{\phi} \frac{y^2}{2m_{\phi, \text{eff}}} \int d\tau e^{im_{\phi, \text{eff}} \tau} \text{Tr}[S_{>}^{\chi, \text{th}}(\tau; \mathbf{p}) S_{<}^{\chi, \text{th}}(-\tau; \mathbf{p}) - S_{<}^{\chi, \text{th}}(\tau; \mathbf{p}) S_{>}^{\chi, \text{th}}(-\tau; \mathbf{p})] \\ &= \dot{\phi} y^2 \left[\frac{1}{2\omega} \int_Q [f_F(q_0) - f_F(q_0 + \omega)] \text{Tr}[\rho_{\chi, \text{th}}(q_0, \mathbf{q}) \rho_{\chi, \text{th}}(\omega + q_0, \mathbf{q})] \right]_{\omega=m_{\phi, \text{eff}}}, \end{aligned} \quad (4.160)$$

where we have used the fact that the function, $\text{Tr}[S_{>}^{\chi, \text{th}}(\tau; \mathbf{p}) S_{<}^{\chi, \text{th}}(-\tau; \mathbf{p}) - (> \leftrightarrow <)]$, is an odd function in τ . Note that the contribution proportional to $\bar{\phi}$ leads to nothing but the thermal mass of ϕ due to the abundant χ -particles. The dissipation rate is obtained as

$$\Gamma_{\phi}^{(\text{small}), \chi} = y^2 \left[\frac{1}{2\omega} \int_Q [f_F(q_0) - f_F(q_0 + \omega)] \text{Tr}[\rho_{\chi, \text{th}}(q_0, \mathbf{q}) \rho_{\chi, \text{th}}(\omega + q_0, \mathbf{q})] \right]_{\omega=m_{\phi, \text{eff}}}. \quad (4.161)$$

The former result [Eq. (4.132)] is recovered by taking $m_{\phi, \text{eff}} \rightarrow 0$. If the effective mass of oscillating scalar is larger than the thermal mass (asymptotic mass) of χ , $m_{\phi, \text{eff}} \gg m_{\chi, \text{th}}$, then the perturbative decay of ϕ into χ quasi-particles is kinematically allowed. Its rate is given by

$$\Gamma_{\phi}^{(\text{small}), \chi} = \frac{y^2 m_{\phi, \text{eff}}}{8\pi} \sqrt{1 - \frac{4m_{\chi, \text{th}}^2}{m_{\phi, \text{eff}}^2}} [1 - f_F(m_{\phi, \text{eff}}/2)] \theta(m_{\phi, \text{eff}} - 2m_{\chi, \text{th}}), \quad (4.162)$$

for $y\tilde{\phi} < m_{\chi,\text{th}} \ll m_{\phi,\text{eff}}/2$. Here we have omitted the plasmino contribution. This is because emitted χ quasi-particles have larger momenta than its thermal mass due to the large effective mass of ϕ , $m_{\phi,\text{eff}} \gg m_{\chi,\text{th}}$, and hence the plasmino contribution is decoupled. However, near the threshold $m_{\phi,\text{eff}}/2 \sim m_{\chi,\text{th}}$, the plasmino contribution becomes comparable. In the following discussion, we do not care about precise behavior of ϕ 's dissipation rate near the threshold, rather concentrate on its qualitative behavior and order of magnitude estimation not to trouble with unnecessary complications. For this purpose, we expect that Eq. (4.162) is sufficient.

Finally, let us evaluate the non-local term, $-i\Pi_J \cdot \phi$. A similar computation yields

$$-i \int_0^\infty d\tau \Pi_J(\tau; \bar{t}) \phi(\bar{t} - \tau) \supset i \frac{\dot{\phi}(\bar{t})}{m_{\phi,\text{eff}}} \int_0^\infty d\tau \Pi_J(\tau; \bar{t}) \sin(m_{\phi,\text{eff}}\tau) = \dot{\phi}(\bar{t}) \left[\frac{\Pi_J(\omega; \bar{t})}{2\omega} \right]_{\omega=m_{\phi,\text{eff}}}. \quad (4.163)$$

Here we have extracted a term that contributes to the dissipation factor. Therefore, the dissipation rate is given by

$$\Gamma_\phi^{(4\text{p, small}), \tilde{\chi}} = \left[\frac{\Pi_J(\omega; \bar{t})}{2\omega} \right]_{\omega=m_{\phi,\text{eff}}}. \quad (4.164)$$

Obviously one can obtain the former result [Eq. (4.135)] by taking $m_{\phi,\text{eff}} \rightarrow 0$. In the former case $m_{\phi,\text{eff}} \ll \Gamma_{\text{damp}} \sim \alpha T$, the phase space for the scattering $\phi\chi \rightarrow \phi\chi$ is quite suppressed since the final particles are almost collinear in the rest frame of thermal plasma. For comparison, let us evaluate the dissipation rate in the case of $\alpha T \ll m_{\phi,\text{eff}} \ll T$. In this case, one finds

$$\begin{aligned} \Gamma_\phi^{(4\text{p, small}), \tilde{\chi}} &\sim N_{\tilde{\chi}} \frac{\lambda^4}{2m_{\phi,\text{eff}}} \frac{2}{(2\pi)^3} \int d\Omega_k f_B(\Omega_k) \int^{\Omega_k} d\Omega_q \\ &\simeq N_{\tilde{\chi}} \times \frac{\lambda^4 T^2}{48\pi m_{\phi,\text{eff}}}, \end{aligned} \quad (4.165)$$

where we have assumed that ϕ -particles are not as abundant as thermal ones.

4.5 Boltzmann Equation

Up to here, we concentrate on the scalar condensation; in particular its energy transportation to radiation. In this section, we briefly discuss results obtained in the previous section from the viewpoint of evolution of propagators for completeness. In particular, we see that the energy transportation can be interpreted as the particle production. As extensively studied previously (See *e.g.* Refs. [43, 73, 74, 95, 98, 100–104, 106, 149]), to make the problem more tractable, the Boltzmann-like equations are frequently derived from the Kadanoff-Baym equations under several assumptions as explained in Sec. 3.3.4: (i) Quasi-particle spectrum, (ii) Separation of time scales, (iii) Negligence of finite time effects $t_{\text{ini}} \rightarrow -\infty$. From the Kadanoff-Baym equations, one can derive the Boltzmann-like equation for ϕ - and χ -particles under the above assumptions and the adiabatic expansion of the Universe (See also Sec. 3.3.4):

$$\partial_t f_\bullet(\mathbf{p}; t) - H\mathbf{p} \cdot \partial_{\mathbf{p}} f_\bullet(\mathbf{p}; t) = \mathcal{C}[f_\bullet]. \quad (4.166)$$

First, we discuss the dissipation term given in Eq. (4.127) [(4.133)], which will be proved to play dominant roles in the large amplitude regime $\tilde{\lambda}\tilde{\phi} \gg T$ of a slowly oscillating scalar

condensate in Sec. 4.6.2. We soon recognize that these terms can be interpreted as particle production events due to the adiabatic change of the mass term. Let us start with Eq. (3.145):

$$\mathcal{C}^{(\text{mass})}[f_\chi] = -\Im \left\{ \frac{\lambda^4 \phi(t) \dot{\phi}(t)}{2\Omega_p^2} \int_{-\infty}^t dt_2 e^{-i\Omega_p(t-t_2)} [\phi^2(t) - \phi^2(t_2)] G_H^\chi(t, t_2; \mathbf{p}) \right\}. \quad (4.167)$$

Again assuming that the scalar field moves slowly so that the integrand can be approximated with the Taylor expansion $\phi^2(t) - \phi^2(t_2) = 2\phi(t)\dot{\phi}(t)(t-t_2)\cdots$, then we obtain

$$\mathcal{C}^{(\text{mass})}[f_\chi] \simeq -\frac{2\lambda^4 \phi^2(t) \dot{\phi}^2(t)}{\Omega_p^3(t) \Gamma_p(t) T} [f_B(-\Omega_p(t)) f_B(\Omega_p(t))]. \quad (4.168)$$

Here we have assumed that the Hadamard propagator of χ -field remains close to thermal one [See also Eq. (4.120)] and that the spectral density is well approximated with the Breit-Wigner form [Eq. (4.49)]. Hence, radiation (including χ -particles for $\tilde{\lambda}|\phi| < T$) obtains the following energy density:

$$\dot{\rho}_{\text{rad}}^{(\text{mass})} = \int_p \Omega_p \mathcal{C}^{(\text{mass})}[f_\chi] = \dot{\phi}^2 \frac{2\lambda^4 \phi^2}{T} \int_p [-f_B(-\Omega_p) f_B(\Omega_p)] \frac{1}{\Gamma_p \Omega_p^2} \quad (4.169)$$

$$= \Gamma_\phi^{(3p, \text{slow}), \tilde{\chi}} \dot{\phi}^2. \quad (4.170)$$

As we will see later in Eq. (4.179) in Sec. 4.6.1, the energy transportation rate from the scalar condensation to radiation coincides with $\Gamma_\phi \dot{\phi}^2$, and thus the dissipation $\Gamma_\phi^{(3p, \text{slow}), \tilde{\chi}}$ can be regarded as the particle production associated with the adiabatic mass shift. It is straightforward to perform the same computation in the fermionic case. Also, for the small amplitude case, one can perform the same computation.

Then, let us move to the dissipation term given in Eq. (4.138). Since it becomes important in the small amplitude regime, $\lambda\tilde{\phi} \ll gT$, we focus on this case in the following. See also Sec. 4.6.2. Technically, within this regime, the background plasma including χ particles can be regarded as thermal bath and the above assumptions are likely to be satisfied. Practically, the thermalization of ϕ -particles become important when the amplitude of oscillating scalar becomes smaller than the temperature of background plasma $\tilde{\phi} < T$, and also the oscillation period is slow enough $m_\phi < \alpha T$. (See Sec. 4.6.2 and also Sec. 5.1.2.) As can be seen from Eq. (4.138), this dissipation may be understood as the following process $\phi^{(c)}\chi \rightarrow \phi^{(p)}\chi$ with $\phi^{(c)}$ and $\phi^{(p)}$ being the ϕ -condensation and -particle respectively. Thus, the ϕ -particle production is expected to be the dominant source of the dissipation,^{♦23} and we concentrate on the collision term of ϕ -particles in the following. The collision term due to interactions with the fixed mass term can be expressed as

$$\mathcal{C}[f_p^\phi] = \frac{i}{2\Omega_p} \int_{-\infty}^t d\tau [\Pi_J^\phi(t, \tau; \mathbf{p}) \partial_t G_H^\phi(\tau, t; \mathbf{p}) - \Pi_H^\phi(t, \tau; \mathbf{p}) \partial_t G_J^\phi(\tau, t; \mathbf{p})]. \quad (4.171)$$

^{♦23} The ϕ -condensate with a small amplitude can be understood as ϕ -particles at rest.

Plugging Eqs. (4.30)–(4.35), one finds

$$\begin{aligned} \mathcal{C}^{(4\text{pt})}[f_\phi] &= 4N_{\tilde{\chi}}\lambda^4 \int_{l,q,k} (2\pi)^4 \delta(\mathbf{p}-\mathbf{l}-\mathbf{q}-\mathbf{k}) \frac{1}{2\Omega_p^\phi 2\Omega_l^\phi 2\Omega_q^\chi 2\Omega_k^\chi} \\ &\quad \left\{ 2 \left[(1+f_p^\phi) f_l^\phi (1+f_q^\chi) f_k^\chi - f_p^\phi (1+f_l^\phi) f_q^\chi (1+f_k^\chi) \right] \delta(\Omega_p^\phi - \Omega_l^\phi + \Omega_q^\chi - \Omega_k^\chi) \right. \\ &\quad \left. + \left[(1+f_p^\phi)(1+f_l^\phi) f_q^\chi f_k^\chi - f_p^\phi f_l^\phi (1+f_q^\chi)(1+f_k^\chi) \right] \delta(\Omega_p^\phi + \Omega_l^\phi - \Omega_q^\chi - \Omega_k^\chi) \right\}, \end{aligned} \quad (4.172)$$

$$\begin{aligned} \mathcal{C}^{(3\text{pt})}[f_\phi] &= 2N_{\tilde{\chi}}\lambda^4 \phi^2(t) \int_{q,k} (2\pi)^4 \delta(\mathbf{p}-\mathbf{q}-\mathbf{k}) \frac{1}{2\Omega_p^\phi 2\Omega_q^\chi 2\Omega_k^\chi} \\ &\quad \left\{ 2 \left[(1+f_p^\phi)(1+f_q^\chi) f_k^\chi - f_p^\phi f_q^\chi (1+f_k^\chi) \right] \delta(-M_\phi + \Omega_p^\phi + \Omega_q^\chi - \Omega_k^\chi) \right. \\ &\quad \left. - \left[f_p^\phi (1+f_q^\chi)(1+f_k^\chi) - (1+f_p^\phi) f_q^\chi f_k^\chi \right] \delta(M_\phi + \Omega_p^\phi - \Omega_q^\chi - \Omega_k^\chi) \right\}. \end{aligned} \quad (4.173)$$

Here we have dropped the contribution proportional to $\phi\dot{\phi}$ which vanishes with the oscillation time average, and also omitted interactions with χ -fermions since they do not directly contribute to the energy transportation from the scalar condensate to radiation.^{♦24} Taking the oscillation time average, one can show

$$\begin{aligned} \frac{\dot{\rho}_{\text{rad}}^{(3\text{pt})}}{\rho_\phi} &= \frac{1}{\rho_\phi} \int_p \Omega_p^\phi \mathcal{C}^{3\text{pt}}[f_\phi] \\ &\simeq \frac{4N_{\tilde{\chi}}\lambda^4}{m_\phi} \int_{p,k,l} (2\pi)^4 \delta(\mathbf{p}-\mathbf{q}-\mathbf{k}) \frac{1}{2\Omega_p^\phi 2\Omega_q^\chi 2\Omega_k^\chi} \\ &\quad \left\{ 2 \left[(1+f_p^\phi)(1+f_q^\chi) f_k^\chi - f_p^\phi f_q^\chi (1+f_k^\chi) \right] \delta(-M_\phi + \Omega_p^\phi + \Omega_q^\chi - \Omega_k^\chi) \right. \\ &\quad \left. - \left[f_p^\phi (1+f_q^\chi)(1+f_k^\chi) - (1+f_p^\phi) f_q^\chi f_k^\chi \right] \delta(M_\phi + \Omega_p^\phi - \Omega_q^\chi - \Omega_k^\chi) \right\} \\ &= \Gamma_\phi^{(4\text{p},\text{slow}),\tilde{\chi}} \end{aligned} \quad (4.174)$$

One can see that the dissipation term $\Gamma_\phi^{(4\text{p},\text{slow}),\tilde{\chi}}$ is dominated by the scattering process $\phi^{(c)}\chi \rightarrow \phi^{(p)}\chi$.

4.6 Evolution of the Universe

In the previous Secs. 4.3 and 4.4, we have derived the coarse-grained equation for the ϕ -condensate in various regimes by utilizing the *separation of time scale* and specifying the dynamical degree of freedom. Starting from the evolution equations for propagators [Eqs. (4.23)–(4.26)] and for the mean field [Eq. (4.42)] that is given by

$$0 = \left[\partial_t^2 + 3H\partial_t + M_\phi^2(t) \right] \phi(t) - \frac{y}{2} \int_p \text{Tr} \left[S_H^\chi(t, t; \mathbf{p}) \right] - i \int_{t_{\text{ini}}}^t d\tau \Pi_J(t, \tau) \phi(\tau), \quad (4.176)$$

^{♦24} Precisely speaking, scatterings with χ -fermions involving gauge bosons can play an important role so that the ϕ -particles which are still concentrated on the infrared momentum cascade towards the UV regime comparable to the temperature of background plasma. See also Sec. 5.1.2.

we have obtained the following coarse-grained evolution equation for the mean field,

$$\ddot{\phi} + [3H + \Gamma(\phi; T)]\dot{\phi} + \frac{\partial V_{\text{eff}}(\phi; T)}{\partial \phi} = 0, \quad (4.177)$$

which is more tractable than Eq. (4.176). The key assumption in driving this equation is the “*separation of time scale*” under which the behavior of propagators can be obtained approximately as discussed in Secs. 4.3 and 4.4. The dissipation rate $\Gamma(\phi; T)$ is given in Eqs. (4.106), (4.127), (4.133), (4.138) and (4.148); and the effective potential $V_{\text{eff}}(\phi; T)$ is in Eqs. (4.70)–(4.73).

Actually, however, after the onset of oscillation, we are interested in the evolution of the Universe whose typical time scale is given by H^{-1} , rather than the oscillation of ϕ -condensate whose time scale is $m_{\phi, \text{eff}}$ with $m_{\phi, \text{eff}} \gg H$. Thus, it is convenient to perform further coarse-graining, that is, taking the oscillation-time average of physical quantity which varies as slow as the cosmic expansion. In the case of oscillating scalar, the *physical quantity* is nothing but its energy density, which is roughly given by^{♦25}

$$\rho_{\phi} \simeq \frac{m_{\phi, \text{eff}}^2(\tilde{\phi}; T)\tilde{\phi}^2}{2}, \quad (4.178)$$

where $m_{\phi, \text{eff}}(\tilde{\phi}; T)$ represents the typical effective mass of ϕ at the amplitude $\tilde{\phi}$ and the temperature T . Hence, in the following, we derive the equation how the amplitude $\tilde{\phi}$ evolves in the cosmological time scale.

4.6.1 Oscillation-time Average

The aim of this subsection is to derive the equation of motion for the amplitude $\tilde{\phi}$ by utilizing the oscillation-time average. Let us start with the coarse-grained equation of motion:

$$\ddot{\phi} + [3H(T) + \Gamma_{\phi}(\phi; T)]\dot{\phi} + \frac{\partial V_{\text{eff}}(\phi; T)}{\partial \phi} = 0. \quad (4.179)$$

From this, we immediately find the following relation between the kinetic energy $K = \dot{\phi}^2/2$ and the effective potential energy V_{eff} :

$$\frac{d}{dt}(K + V_{\text{eff}}) = -(6H + 2\Gamma_{\phi})K + \frac{\dot{T}}{T} \left(T \frac{\partial V_{\text{eff}}}{\partial T} \right). \quad (4.180)$$

This is an exact relation. It implies that $K + V_{\text{eff}}$ changes slowly $\mathcal{O}(H, \Gamma_{\phi})$, and approximately conserves with time. Therefore, we can take the oscillation-time average of this equation:

$$\frac{d}{dt} \langle \overline{K + V_{\text{eff}}} \rangle = -6H \langle \overline{K} \rangle - 2 \langle \overline{\Gamma_{\phi} K} \rangle + \frac{\dot{T}}{T} \left\langle \overline{T \frac{\partial V_{\text{eff}}}{\partial T}} \right\rangle, \quad (4.181)$$

where $\langle \overline{\cdots} \rangle$ stands for the oscillation average. Then we make use of the virial theorem. By noting that $2K = d(\phi \dot{\phi})/dt - \phi \ddot{\phi}$, we obtain

$$\langle \overline{2K} \rangle = \left\langle \overline{\phi \frac{\partial V_{\text{eff}}}{\partial \phi}} \right\rangle, \quad (4.182)$$

^{♦25} Strictly speaking, there is subtlety on the definition of energy density of ϕ and energy transportation from ϕ to radiation in the case of oscillation with thermal potential. However, in this case, the energy density of ϕ is at most that of one degree of freedom in thermal bath $\sim T^4$. Hence, it is merely a small change of g_* and can be neglected practically within an accuracy of our estimation.

discarding $\mathcal{O}(H, \Gamma_\phi)$ terms. Also, we define the oscillation averaged dissipation rate as follows:

$$\Gamma_\phi^{\text{eff}}(\tilde{\phi}; T) \equiv \frac{\langle \overline{\Gamma_\phi \dot{\phi}^2} \rangle}{\langle \overline{\dot{\phi}^2} \rangle}. \quad (4.183)$$

Thus, we obtain the time-averaged evolution equation for the ϕ -condensation [35]:

$$\frac{d}{dt} \left\langle \phi \frac{\partial V_{\text{eff}}}{\partial \phi} + 2V_{\text{eff}} \right\rangle = - (6H + 2\Gamma_\phi^{\text{eff}}) \left\langle \phi \frac{\partial V_{\text{eff}}}{\partial \phi} \right\rangle + 2 \frac{\dot{T}}{T} \left\langle T \frac{\partial V_{\text{eff}}}{\partial T} \right\rangle. \quad (4.184)$$

From this equation, we can derive the evolution equation of the amplitude $\tilde{\phi}$ by inserting the concrete form of the effective potential. Let us see some concrete examples in the following.

• **Zero-temperature Polynomial Potential:** Suppose that the oscillation is dominated by the following vacuum potential, $V_{\text{eff}} \propto |\phi|^n$. Plugging it into Eq. (4.184), one obtains

$$\frac{d}{dt} \langle |\phi|^n \rangle = - \frac{2n}{n+2} (3H + \Gamma_\phi^{\text{eff}}) \langle |\phi|^n \rangle, \quad (4.185)$$

which implies the following equation:

$$\frac{d}{dt} \ln \tilde{\phi} = - \frac{2}{n+2} (3H + \Gamma_\phi^{\text{eff}}). \quad (4.186)$$

Thus, one finds

$$\tilde{\phi} \propto a^{-6/(n+2)} \quad \text{for } \Gamma_\phi^{\text{eff}} \ll H. \quad (4.187)$$

• **Thermal Mass:** Next, suppose that the oscillation is dominated by the thermal mass, $V_{\text{eff}} \propto T^2 \phi^2$. Plugging it into Eq. (4.184), one obtains

$$\frac{d}{dt} \langle \phi^2 \rangle = - \left(3H + \Gamma_\phi^{\text{eff}} + \frac{\dot{T}}{T} \right) \langle \phi^2 \rangle, \quad (4.188)$$

which implies the following equation:

$$\frac{d}{dt} \ln \tilde{\phi} = - \frac{1}{2} \left(3H + \frac{\dot{T}}{T} + \Gamma_\phi^{\text{eff}} \right). \quad (4.189)$$

Thus, one finds

$$\tilde{\phi} \propto a^{-3/2} T^{-1/2} \quad \text{for } \Gamma_\phi^{\text{eff}} \ll H. \quad (4.190)$$

Note that the evolution of temperature crucially depends on whether or not the Universe is dominated by the inflaton (*c.f.* moduli) which reheats the Universe: $T \propto a^{-n}$ with $n = 1[3/8]$ for the radiation [inflaton] dominant era. Here we assume the conventional reheating with a quadratic inflaton potential (See also Secs. 2.1 and 5.1).

• **Thermal Log:** Finally, suppose that the oscillation is dominated by the thermal log, $V_{\text{eff}} \propto T^4 \ln(\phi^2/T^2)$. Plugging it into Eq. (4.184), one finds

$$\frac{d}{dt} \left\langle T^4 (1 + \ln \phi^2/T^2) \right\rangle = - (6H + 2\Gamma_\phi^{\text{eff}}) T^4 + \frac{4\dot{T}}{T} \left\langle T^4 \ln \phi^2/T^2 \right\rangle - \frac{2\dot{T}}{T} T^4. \quad (4.191)$$

Since the left-hand-side can be written as $4(\dot{T}/T)\langle\overline{T^4(1+\ln\phi^2)}\rangle + T^4 d\ln\tilde{\phi}^2/dt - 2T^4\dot{T}/T$, this equation implies

$$\frac{d}{dt}\ln\tilde{\phi} = -\left(3H + \Gamma_{\phi}^{\text{eff}} + \frac{2\dot{T}}{T}\right). \quad (4.192)$$

Thus, one finds

$$\tilde{\phi} \propto a^{-3}T^{-2} \text{ for } \Gamma_{\phi}^{\text{eff}} \ll H. \quad (4.193)$$

Again, note that $T \propto a^{-n}$ with $n = 1[3/8]$ for the radiation [inflaton] dominant era.

Before closing this subsection, let us summarize the obtained equations in a convenient form:

$$\frac{d}{dt}\ln\tilde{\phi} = -\frac{1}{n_1}\left(3H + n_2\frac{\dot{T}}{T} + \Gamma_{\phi}^{\text{eff}}\right), \quad (4.194)$$

where

$$(n_1, n_2) = \begin{cases} \left(\frac{n+2}{2}, 0\right) & \text{for vacuum potential with } V \propto |\phi|^n, \\ (2, 1) & \text{for thermal mass,} \\ (1, 2) & \text{for thermal log,} \end{cases} \quad (4.195)$$

and

$$\frac{\dot{T}}{T} = \begin{cases} -H & \text{for radiation dominant era,} \\ -\frac{3}{8}H & \text{for inflaton dominant era.} \end{cases} \quad (4.196)$$

Here note again that we have assumed the conventional reheating with a quadratic inflaton potential. As one can see, until the effective dissipation rate becomes comparable to the Hubble parameter, $\Gamma_{\phi}^{\text{eff}} \simeq H$, the amplitude obeys the following scaling:

$$\tilde{\phi} \propto a^{-3/n_1}T^{-n_2/n_1}. \quad (4.197)$$

In the next subsection, we evaluate the effective dissipation rate by using the results in Secs. 4.3 and 4.4.

4.6.2 Effective Dissipation Rate

In this subsection, we evaluate the effective dissipation rate, which is defined as

$$\Gamma_{\phi}^{\text{eff}}(\tilde{\phi}; T) \equiv \frac{\langle\overline{\Gamma_{\phi}(\phi; T)\dot{\phi}^2}\rangle}{\dot{\phi}^2}. \quad (4.198)$$

In a word, the aim of this subsection is to obtain the effective dissipation rate as a function of $\tilde{\phi}$ and T . In the following discussion, we drop numerical factors for brevity basically since it might depend on models, and let us perform order of magnitude estimation. The following discussion is based on Refs. [36, 44, 46]. See Fig. 4.4 for clarity.

• **Dissipation by Non-perturbative Particle Production:** As discussed in Sec. 4.3, the non-perturbative production takes place when the following condition [Eq. (4.99)] is fulfilled

$$k_* \gg \max[m_{\chi,\text{th}}, m_{\phi,\text{eff}}]; \quad k_*^2 \equiv \tilde{\lambda} \dot{\phi} \big|_{\phi=0}. \quad (4.199)$$

Here it is noticeable that the non-perturbative production does not take place if ϕ oscillates dominantly with the thermal potential. This is because the typical scale is at most $k_*^2 \sim \tilde{\lambda} T^2$ in this case, and hence it is smaller than the thermal mass, $k_*^2 \sim \lambda T^2 < m_{\chi,\text{th}}^2 \sim g^2 T^2$ because we assume $\lambda, y < \alpha$ (See also Sec. 4.4).

If the broad resonance takes place, the system enters the turbulent regime and complicated numerical studies might be required. So, we mainly focus on the instant preheating regime:

$$\epsilon^2 g^2 (\lambda \text{ or } y) \tilde{\phi} \gg m_\phi. \quad (4.200)$$

with $\epsilon \sim 1$. Therefore, aside from the CW potential, the following region in m_ϕ - $\tilde{\phi}$ plane is relevant in the following discussion:

$$\text{NP-production: } \epsilon^2 g^2 (\lambda \text{ or } y) \tilde{\phi} \gg m_\phi \text{ and } (\lambda \text{ or } y) \tilde{\phi} \gg \frac{g^2 T^2}{m_\phi}. \quad (4.201)$$

Inside this region, the dissipation rate [Eq. (4.106)] does not depend on both ϕ and T , and hence it coincides with the oscillation-time averaged one:

$$\Gamma_\phi^{\text{eff}} = \Gamma_\phi^{\text{NP}} \sim N_{\text{d.o.f.}} \times \frac{\tilde{\lambda}^2 m_\phi}{2\pi^4 |\epsilon g|} \text{ for NP-production,} \quad (4.202)$$

where $N_{\text{d.o.f.}}$ stands for the degree of freedom normalized by one complex scalar or one chiral fermion.

• **Dissipation by thermally populated χ :** If the ϕ -condensate passes through $|\phi| < \phi_c$, the χ -particles are thermally populated (for $\epsilon \sim 1$) as discussed at the beginning of Sec. 4.4.1. Basically, the following region in m_ϕ - $\tilde{\phi}$ plane is relevant in the following discussion (See Sec. 4.4):

$$\text{Thermal } \chi\text{-fields: } \left[m_{\phi,\text{eff}} \ll \alpha T \text{ and } \tilde{\lambda} \tilde{\phi} \gg \frac{g^2 T^2}{m_\phi} \right] \text{ or } \tilde{\lambda} \tilde{\phi} \ll m_{\chi,\text{th}} \sim g T. \quad (4.203)$$

Let us discuss these regions in turn in the following.

For $k_* \ll m_{\chi,\text{th}}$ and $m_{\phi,\text{eff}} \ll \alpha T$, the oscillating scalar can be regarded as a slowly moving object in the fast interacting particles of thermal plasma, as explained in Sec. 4.4.1. Taking the oscillation-time average of results obtained in Sec. 4.4.1, one can estimate the effective dissipation rate as follows. The effective dissipation rate caused by thermally populated $\tilde{\chi}$ -bosons is given by

$$\Gamma_\phi^{\text{eff, slow}, \tilde{\chi}} \sim N_{\tilde{\chi}} \begin{cases} \frac{\lambda T^2}{\alpha \tilde{\phi}} \eta & \text{for } T/\lambda \ll \tilde{\phi}, \\ \frac{\lambda^4 \tilde{\phi}^2}{\alpha T} & \text{for } T < \tilde{\phi} \ll T/\lambda, \\ \frac{\lambda^4 T}{\alpha} & \text{for } \tilde{\phi} < T, \end{cases} \quad (4.204)$$

with $\eta = \alpha$ if ϕ oscillates dominantly with the thermal log potential, otherwise $\eta = 1$. And that cause by χ -fermions is estimated as

$$\Gamma_{\phi}^{\text{eff,slow},\chi} \sim N_{\chi} \begin{cases} \frac{yT^2}{\alpha\tilde{\phi}}\eta & \text{for } T/y \ll \tilde{\phi}, \\ \frac{y^4\tilde{\phi}^2}{\alpha T} & \text{for } \frac{m_{\chi,\text{th}}}{y} \sim \frac{gT}{y} < \tilde{\phi} \ll T/y, \\ y^2\alpha T & \text{for } \tilde{\phi} < \frac{m_{\chi,\text{th}}}{y} \sim \frac{gT}{y}, \end{cases} \quad (4.205)$$

with the same definition of η .^{♦26}

On the other hand, for $\tilde{\lambda}\tilde{\phi} \ll m_{\chi,\text{th}}$, we can safely assume that the χ -particles are in the thermal plasma since the amplitude of oscillating scalar can be neglected, as discussed in Sec. 4.4.2. The effective dissipation rate from abundant $\tilde{\chi}$ -particles is given by

$$\Gamma_{\phi}^{\text{eff,small},\tilde{\chi}} \sim N_{\chi} \begin{cases} \frac{\lambda^4\tilde{\phi}^2}{\alpha T} & \text{for } \sqrt{\frac{\alpha T}{m_{\phi,\text{eff}}}} T < \tilde{\phi} < \frac{m_{\chi,\text{th}}}{\lambda} \sim \frac{gT}{\lambda} \\ \frac{\lambda^4 T^2}{m_{\phi,\text{eff}}} & \text{for } \tilde{\phi} < \sqrt{\frac{\alpha T}{m_{\phi,\text{eff}}}} T \end{cases} \quad \text{with } \alpha T \ll m_{\phi,\text{eff}} < m_{\chi,\text{th}} \sim gT; \quad (4.206)$$

$$\Gamma_{\phi}^{\text{eff,small},\tilde{\chi}} \sim N_{\chi} \begin{cases} \frac{\lambda^4\tilde{\phi}^2}{m_{\phi,\text{eff}}} & \text{for } T < \tilde{\phi} < \frac{m_{\chi,\text{th}}}{\lambda} \sim \frac{gT}{\lambda} \\ \frac{\lambda^4 T^2}{m_{\phi,\text{eff}}} & \text{for } \tilde{\phi} < T \end{cases} \quad \text{with } m_{\chi,\text{th}} \sim gT < m_{\phi,\text{eff}} < T; \quad (4.207)$$

$$\Gamma_{\phi}^{\text{eff,small},\tilde{\chi}} \sim N_{\chi} \frac{\lambda^4\tilde{\phi}^2}{m_{\phi,\text{eff}}} \quad \text{with } T < m_{\phi,\text{eff}}. \quad (4.208)$$

And that caused by thermally populated χ -particles is the following

$$\Gamma_{\phi}^{\text{eff,small},\chi} \sim N_{\chi} \begin{cases} y^2\alpha T & \text{for } m_{\phi,\text{eff}} < 2m_{\chi,\text{th}} \sim gT, \\ y^2m_{\phi,\text{eff}} & \text{for } 2m_{\chi,\text{th}} \sim gT < m_{\phi,\text{eff}}. \end{cases} \quad (4.209)$$

Here we do not repeat the results in the case with $m_{\phi,\text{eff}} \ll \alpha_{\text{th},\chi} T$ for brevity. Note that we have also dropped the Bose-enhancement/Pauli-suppression factor for simplicity.

• **Dissipation by dimension-five operator:** Even though the χ -particles are absent in the thermal plasma due to the large field value $\tilde{\lambda}|\phi| \gg T$, the ϕ -condensation interacts with the thermal plasma via the dimension-five operator as explained in Sec. 4.4. Basically, the derivation implicitly assumes that the ϕ -condensate should move slowly compared with the typical

^{♦26} Strictly speaking, even though the non-perturbative production takes place at the very origin of the potential, the background plasma may remain close to thermal equilibrium for $\epsilon^2 g^2 (\lambda \text{ or } y) \tilde{\phi} \gg m_{\phi}$. In this case, the above effective dissipation rate cannot be used unless the non-adiabatic region becomes smaller than the threshold value, $\phi_{\text{NP}} < \phi_c$. See also footnote ^{♦27}.

interaction time scale of background plasma and also that the χ particles are absent in the thermal plasma due to the large field value. Note that even if the non-perturbative production is efficient at the very origin of the potential, the produced particles immediately decay into radiation for $\epsilon^2 g^2 (\lambda \text{ or } y) \tilde{\phi} \gg m_\phi$. Hence, even in this case, the ϕ -condensate dissipates its energy via the dimension-five operator though it tends to be sub-dominant compared with Eq. (4.202). Here, roughly estimating the damping time scale as αT , one finds the relevant region in m_ϕ - $\tilde{\phi}$ plane:

$$\text{Dimension-5: } \tilde{\lambda} \tilde{\phi} > T \text{ and } \alpha T > m_\phi. \quad (4.210)$$

The effective dissipation rate can be expressed as^{♦27}

$$\Gamma_\phi^{\text{eff, large}} \sim \eta b \alpha^2 \frac{\tilde{\lambda} T^2}{\tilde{\phi}}, \quad (4.211)$$

typically $b \sim 10^{-3}$.

• **The Missed Region:** There are two missed regions in m_ϕ - $\tilde{\phi}$ plane in our simple calculation. The first one corresponds to the broad resonance because its dynamics can be highly non-thermal and our strategy does not work as already mentioned:^{♦28}

$$m_\phi \ll \tilde{\lambda} \tilde{\phi} \ll \frac{m_\phi}{\epsilon^2 g^2} \text{ and } \tilde{\lambda} \tilde{\phi} \gg \frac{g^2 T^2}{m_\phi}. \quad (4.212)$$

Next, since we consider the typical regimes where the dynamics become simple, unfortunately the following small regime is not covered:

$$\alpha T < m_\phi < g T \text{ and } g T < \tilde{\lambda} \tilde{\phi} < \frac{g^2 T^2}{m_\phi}. \quad (4.213)$$

Inside this regime, we simply interpolate the “Thermal χ -fields” regime as a rough approximation.

♦27 If the non-perturbative production takes place around the potential origin, then the effective dissipation rate is given by Eq. (4.211) times the following factor ϕ_c/ϕ_{NP} for $\phi_{\text{NP}} > \phi_c$.

♦28 Still, it might be possible to say something with a quite rough estimation even in this case [45].

Chapter 5

Applications

As we have shown in the previous chapter in detail, the equation of motion for the ϕ -condensation is changed by the interaction with the light χ -fields as

$$\ddot{\phi} + [3H + \Gamma(\phi; T)]\dot{\phi} + \frac{\partial V_{\text{eff}}(\phi; T)}{\partial \phi} = 0. \quad (5.1)$$

In particular, we have obtained the cosmological evolution of $\tilde{\phi}$ by taking the oscillation-time average,

$$\frac{d}{dt} \ln \tilde{\phi} = -\frac{1}{n_1} \left(3H + n_2 \frac{\dot{T}}{T} + \Gamma_{\phi}^{\text{eff}} \right). \quad (5.2)$$

Here the two parameters depend on which term dominates the effective potential:

$$(n_1, n_2) = \begin{cases} \left(\frac{n+2}{2}, 0 \right) & \text{for vacuum potential with } V \propto |\phi|^n, \\ (2, 1) & \text{for thermal mass,} \\ (1, 2) & \text{for thermal log.} \end{cases} \quad (5.3)$$

And also note that the effective dissipation rate $\Gamma_{\phi}^{\text{eff}}$ depends on $\tilde{\phi}$ and T . Though our treatment is based on some assumptions to make the equations tractable, we have demonstrated that it has a wide range of applicability in the m_{ϕ} - $\tilde{\phi}$ plane. It is obvious that the cosmological fate of the ϕ -condensation can be significantly modified from that estimated in Vacuum. This is because the beginning time of oscillation is modified due to the thermal potential $H_{\text{os}}^2 \simeq 2\partial V_{\text{eff}}/\partial \phi^2$, the scaling solution of $\tilde{\phi}$ for $H > \Gamma_{\phi}^{\text{eff}}$ is changed by the interaction with the background plasma as $\tilde{\phi} \propto a^{-3/n_1} T^{-n_2/n_1}$ and the decay time of the ϕ -condensate is also changed from that estimated in Vacuum since it is determined by $H \simeq \Gamma_{\phi}^{\text{eff}}$. The remaining task is to study how these effects influence the thermal history of the Universe.

Thus, the aim of this chapter is to clarify the impacts of Eq. (5.2) on the evolution of the Universe. For this purpose, let us reconsider the following well-motivated roles of scalar condensates in the early Universe, inflaton and curvaton, and see how these effects change essential features of these models. In Sec. 5.1, we study reheating after inflation, and in particular show that the reheating temperature, which characterizes the important feature of the inflaton model, is significantly modified from the conventional argument. We also discuss the detailed dynamics before the completion of reheating. In Sec. 5.2, we derive formulas for the curvature perturbation by taking all these effects into account.

This chapter is based on our previous works [36, 46]

5.1 Reheating after Inflation

In this section, we investigate the reheating dynamics after the inflation, *i.e.* ϕ represents the inflaton throughout this section. The reheating temperature is an important parameter of inflation because it characterizes the efficiency of Lepto/Baryogenesis, the abundance of unwanted relics (e.g. moduli, gravitino), and also the spectral index of inflation as explained in Sec. 2.1. Thus, the reheating temperature is essential to discuss a compatibility of an inflation model with thermal history of our Universe.

The main aim of this section is to clarify how the background plasma changes the conventional estimation of reheating temperature [Eq. (2.25)]:

$$T_R^{(\text{cnv})} \simeq \left(\frac{90}{\pi^2 g_*} \right)^{1/4} \sqrt{\Gamma_\phi^0 M_{\text{pl}}}, \quad (5.4)$$

where g_* denote the relativistic degrees of freedom at the temperature $T_R^{(\text{cnv})}$ and Γ_ϕ^0 represents the inflaton decay rate evaluated at the Vacuum. Correspondingly, as one can guess from the discussion around Eq. (2.24), we see that the evolution of radiation before the completion of reheating is also altered. Recall that, as already explained in Sec. 2.1.2, the Universe is often already filled with plasma before the completion of reheating.

Before going into details, we summarize related previous works on this aspect. In Ref. [150], it was argued that large thermal masses of χ -fields prevent the inflaton decay and the temperature of plasma cannot be as high as the inflaton mass (divided by a coupling constant). This is not correct statement because, roughly speaking, the inflaton can lose its energy even if it is oscillating very slowly as we have already shown in Sec. 4.4. In the small amplitude case $\tilde{\lambda}\tilde{\phi} \ll m_{\chi, \text{th}}$, this aspect is correctly pointed out in Refs. [32, 33] (for $\Gamma_{\chi, \text{damp}} \ll m_\phi$). For $m_\phi \ll \Gamma_{\chi, \text{damp}}$, the dissipation rate of slowly moving inflaton is discussed in the context of the warm inflation [37–40], and the relation between the moduli dissipation rate and the bulk viscosity is pointed out in the context of moduli dynamics [41, 42]. We successfully treat all these regimes in a single framework as shown in detail so far. In addition, we also address the issues how such a high temperature plasma with $T \gg m_\phi$ is produced before the completion of reheating in the same framework. As a result, we can predict the reheating temperature and the reheating dynamics of the universe in a wide range of parameters for the inflaton model whose essential feature falls into Eq. (4.2).

We first explain how the reheating dynamics takes place in a time order, referring to results derived in the previous sections. Then, the numerical results are shown in some typical parameters. Finally, we summarize our findings. This section is based on our previous work [36] and also [44, 151].

In the following discussion, we mainly consider two limits, $\lambda \sim y$ and $\lambda \gg y \sim 0$, to avoid unnecessary complexities. Basic evolution equations are given by

$$\dot{\rho}_\phi = -\frac{n}{n_1} [3H + \Gamma_\phi^{\text{eff}}] \rho_\phi, \quad (5.5)$$

$$\dot{\rho}_{\text{rad}} = -4H\rho_{\text{rad}} + \frac{2n}{n_1} \Gamma_\phi^{\text{eff}} \rho_\phi, \quad (5.6)$$

$$3M_{\text{pl}}^2 H^2 = \rho_\phi + \rho_{\text{rad}}, \quad (5.7)$$

with $n_1 = (n + 2)/2$ for $V \propto |\phi|^n$. For simplicity, we consider the quadratic potential and neglect the CW potential. Hence, $n, n_1 = 2$. We assume that the parametric resonance does not take place, which implies that the following condition should hold during the preheating stage, $\epsilon^2 g^2 \tilde{\lambda} \tilde{\phi} \gg m_\phi$ with $\epsilon \sim 1$. Here note that the thermal potential is not so important

in the study of reheating dynamics, since the inflaton energy density is already subdominant when it dominates the effective potential, $\rho_{\phi,\text{th}} \sim T^4 < \rho_{\text{rad}}$.

5.1.1 Preheating

After the era of inflation, the inflaton ϕ starts to oscillate coherently around its potential minimum with a large initial amplitude ϕ_i . Since the χ -fields couple to the inflaton, its effective mass term depends on the field value of ϕ , and the dispersion relation of the χ -fields depends on time in the inflaton oscillation regime. Due to the rapid expansion of inflation, there are almost no particles in the Universe at first, and hence the adiabaticity condition given in Eq. (4.99) is broken down if $\tilde{\lambda}\tilde{\phi} \gg m_\phi$. In the following, we concentrate on the case where the adiabaticity is broken down initially $\tilde{\lambda}\phi_i \gg m_\phi$.

Hence, the χ -particles can be produced copiously via the so-called non-perturbative particle production [28, 29]. In the following, we consider the case where the χ -fields are not stable since there is no reason to protect them in general.^{♦1} If they can decay into other light particles efficiently before the oscillating inflaton moves back to its potential origin [132], $\epsilon^2 g^2 \tilde{\lambda}\tilde{\phi} \gg m_\phi$ with $\epsilon \sim 1$, the inflaton loses its energy with the rate given in Eq. (4.202) [35, 131]:

$$\Gamma_\phi^{\text{eff}} = \Gamma_\phi^{\text{NP}} \sim c_0 \times \frac{\tilde{\lambda}^2 m_\phi}{2\pi^4 |\epsilon g|}, \quad (5.8)$$

where we write down the model dependent factor as c_0 . We concentrate on the case where the condition $g^2 \tilde{\lambda}\tilde{\phi} \gg m_\phi$ holds during the preheating stage to avoid the possible turbulent regime which requires different treatments [110, 137]. Through this process, radiation is gradually produced, and eventually the non-perturbative production shuts off when the temperature of background plasma becomes^{♦2}

$$T_{\text{pR}}^2 \sim \frac{\tilde{\lambda}\tilde{\phi}_{\text{pR}} m_\phi}{g^2}, \quad (5.9)$$

which indicates the following relation:

$$\rho_{\text{rad}}|_{T_{\text{pR}}} \sim \frac{\pi^2 g_* \tilde{\lambda}^2}{30 g^4} \rho_\phi|_{\text{pR}} \ll \rho_\phi|_{\text{pR}}, \quad (5.10)$$

for $\alpha \gg \tilde{\lambda}$. Otherwise, $\alpha \lesssim \tilde{\lambda}$, the inflaton loses an order one fraction of its energy via this process and immediately evaporates due to frequent interactions with the produced thermal plasma.

If the Hubble parameter becomes comparable to the dissipation rate before the end of preheating stage, then the reheating temperature is given by

$$T_{\text{R}} \sim \left(\frac{90}{\pi^2 g_*} \right)^{1/4} \left(\frac{c_0}{2\pi^4 |\epsilon g|} \right)^{1/2} \sqrt{\tilde{\lambda}^2 m_\phi M_{\text{pl}}}. \quad (5.11)$$

^{♦1} One of exceptions is dark matter for instance.

^{♦2} Here we assume that the background plasma can be well approximated with the thermal one. As discussed below Eq. (4.106), this implies the following condition $m_\phi^{-1} > t_{\text{split}} \sim (\alpha^2 T)^{-1} \sqrt{K/T}$ with $K \sim m_{\chi,\text{eff}}(t_{\text{dec}}) \sim k_*/|\epsilon g|$. We also check this condition in our numerical calculation.

5.1.2 Reheating

After the termination of non-perturbative production, the remaining inflaton condensate dissipates its energy via frequent interactions with abundant particles in the thermal plasma. Under our assumption $g^2\tilde{\lambda}\tilde{\phi} \gg m_\phi$ during the preheating stage, we have the following inequality right after the end of preheating

$$\tilde{\lambda}\tilde{\phi}_{\text{pR}} \sim \frac{g^2 T_{\text{pR}}^2}{m_\phi} \gg T_{\text{pR}}. \quad (5.12)$$

Note that the assumption $g^2\tilde{\lambda}\tilde{\phi} \gg m_\phi$ indicates the oscillation of inflaton is quite slow compared with the typical damping rate of quasi-particles in the thermal plasma

$$g^2 T_{\text{pR}} \sim \sqrt{g^2\tilde{\lambda}\tilde{\phi}m_\phi} \gg m_\phi. \quad (5.13)$$

Hence, the χ -particles are not yet in the thermal plasma and the effective dissipation rate is given by

$$\Gamma_\phi^{\text{eff}}(\tilde{\phi}; T) \simeq c \frac{\tilde{\lambda} T^2}{\alpha \tilde{\phi}} + b \frac{\alpha^2 \tilde{\lambda} T^2}{\tilde{\phi}} \equiv c_1 \frac{\tilde{\lambda} T^2}{\alpha \tilde{\phi}}, \quad (5.14)$$

until the amplitude decreases as small as the temperature, $\tilde{\lambda}\tilde{\phi} \sim T$. Here b and c denote model-dependent factors and their typical values are $b \sim 10^{-3}$ and $c \sim 10^{-1}$ (See Sec. 4.4).^{♦3} At this regime, the evolution of radiation can be roughly approximated with

$$\rho_{\text{rad}} \sim \frac{\Gamma_\phi^{\text{eff}}}{H} \rho_\phi \sim \sqrt{3} c_1 M_{\text{pl}} \frac{\tilde{\lambda} T^2}{\alpha \tilde{\phi}} \rho_\phi^{1/2} \sim \sqrt{3} c_1 \frac{\tilde{\lambda}}{\alpha} m_\phi M_{\text{pl}} T^2 \leftrightarrow T \sim c_1^{1/2} \left(\frac{30\sqrt{3}}{\pi^2 g_* \alpha} \right)^{1/2} \sqrt{\tilde{\lambda} m_\phi M_{\text{pl}}}. \quad (5.15)$$

Interestingly, the temperature of background plasma is constant at this regime. If the effective dissipation rate becomes comparable to the Hubble parameter during this regime, then the reheating temperature is given by Eq. (5.15).

After the amplitude of inflaton decreases as small as the temperature $\tilde{\lambda}\tilde{\phi} \sim T$, the χ -particles are thermally populated. Practically, the dissipation coefficient proportional to the amplitude squared, $\Gamma_\phi^{\text{eff}} \propto \tilde{\phi}^2$, is not so important because it decreases faster than the Hubble parameter which is proportional to the amplitude, $H \propto \tilde{\phi}$.^{♦4} Hence, this term alone cannot dissipate the energy of inflaton condensate completely. Let us concentrate on the smaller amplitude regime in the following. At this regime, the effective dissipation rate can be expressed as

$$\Gamma_\phi^{\text{eff}}(\tilde{\phi}; T) \simeq c_2 \begin{cases} \lambda^4 T / \alpha & \text{for } \tilde{\phi} < T \\ y^2 \alpha T & \text{for } \tilde{\phi} < gT/y \end{cases} \quad (5.16)$$

with $m_\phi \ll \alpha T$. Here c_2 is a model-dependent parameter, typically $c_2 \sim 10^{-1}$. The radiation approximately follows the following equation:

$$\rho_{\text{rad}} \sim 3\Gamma_\phi^{\text{eff}} M_{\text{pl}}^2 H \leftrightarrow T \sim c_2^{1/3} \left(\frac{90}{\pi^2 g_*} \right)^{1/3} (M_{\text{pl}}^2 H)^{1/3} \begin{cases} (\lambda^4 / \alpha)^{1/3} & \text{for } \lambda \gg y \sim 0 \\ (y^2 \alpha)^{1/3} & \text{for } \lambda \sim y \end{cases}, \quad (5.17)$$

^{♦3} As mentioned below Eq. (4.120) and the footnote ♦22 [also below Eq. (4.144)], the parameter also have theoretical uncertainties due to the rough estimation, which results in $c_1^{1/2}$ uncertainty of reheating temperature as we will see soon. Still, the qualitative behavior is correct.

^{♦4} Here we assume that the vacuum potential is given by the quadratic one.

which implies that $T \propto a^{-1/2}$. If the dissipation rate becomes comparable to the Hubble parameter at this regime, then the reheating temperature is given by

$$T_R \sim c_2 \left(\frac{90}{\pi^2 g_*} \right)^{1/2} M_{\text{pl}} \begin{cases} \lambda^4/\alpha & \text{for } \lambda \gg y \sim 0 \\ y^2 \alpha & \text{for } \lambda \sim y \end{cases}. \quad (5.18)$$

In the first case $\lambda \gg y \sim 0$, the inflaton dominantly loses its energy via the scattering $\phi^{(c)}\chi \rightarrow \phi^{(p)}\chi$ with $\phi^{(c)}$ and $\phi^{(p)}$ being the ϕ -condensate and ϕ -particle respectively. One might wonder that the distribution of ϕ -particles is still sharply dominated by the IR-regime. So, let us roughly estimate the time scale, δt_{UV} , which the ϕ 's distribution takes to evolve towards the UV-regime T via interactions with the thermal plasma. First, the typical gain of ϕ 's momentum in each scatter $\phi^{(p)}\chi \rightarrow \phi^{(p)}\chi$, denoted by δp , is given by $\delta p \sim E_\phi/g^2$ as long as $E_\phi \lesssim gT$. Thus the typical energy of ϕ just after the dissipation of ϕ -condensate is given by $\sim m_\phi/g^2$. Next, the typical scattering rate for $m_\phi < \alpha T$ is $\Gamma_{\text{scat}} \sim \lambda^4 T/\alpha$. Then the momentum of ϕ grows in a time scale of $\delta t_{\text{UV}} \sim g^2/\Gamma_{\text{scat}}$. It is comparable to the Hubble time scale at the completion of dissipation. Hence the distribution is expected to soon cascade towards T , and the inflaton ϕ participates in the thermal plasma.

The subsequent evolution has a crucial difference between $\lambda \sim y$ and $\lambda \gg y \sim 0$, so we discuss them separately. First, let us consider the simple case where the Yukawa coupling is sizable $\lambda \sim y$. In this case, the dissipation rate is given by the ordinary decay rate if the temperature decreases as small as $2m_{\chi,\text{th}} \sim gT < m_\phi$:

$$\Gamma_\phi^{\text{eff}} \sim c_2 y^2 m_\phi. \quad (5.19)$$

Thus the evolution of radiation is exactly the same as Eq. (2.24) and the reheating temperature is given by the conventional one (2.25). Second, we move to the other case where the Yukawa coupling is suppressed $\lambda \gg y \sim 0$. In this case, the dissipation rate becomes proportional to the temperature squared, $\Gamma_\phi^{\text{eff}} \propto T^2$, when the temperature decreases as $m_\phi > \alpha T$. Hence it cannot reach the Hubble parameter, if the coupling is smaller than the critical value [44]:

$$\lambda > \lambda_c \sim \left(\frac{m_\phi}{M_{\text{pl}}} \right)^{1/4}. \quad (5.20)$$

Roughly speaking, this inequality can be understood as follows: in order for the inflaton to participate in the thermal plasma, at least the scattering process, $\phi^{(c)}\chi \leftrightarrow \phi^{(p)}\chi$, should be efficient; $\lambda^4 T^3/(m_\phi T) > H \sim T^2/M_{\text{pl}}$.

We comment on the difference between two cases, $\lambda \sim y$ and $\lambda \gg y \sim 0$, after the inflaton condensate dissipates its energy completely via interactions with the thermal plasma. In the former case $\lambda \sim y$, one might wonder that inflaton particles are eventually thermally decoupled and reheat the Universe again via their late-time decay. However, such an event hardly happens: once the thermal dissipation rate becomes larger than the Hubble parameter, the Hubble parameter cannot exceed the dissipation rate after that. This is because, after the dissipation of inflaton condensate, the effective dissipation rate of inflaton particles behaves as

$$\Gamma_\phi^{\text{eff}} \sim c_2 \begin{cases} y^2 \alpha T & \text{for } \alpha T < m_\phi < gT \\ y^2 m_f & \text{for } gT < m_\phi \end{cases}, \quad (5.21)$$

while the Hubble parameter is proportional to T^2 . Note that the renormalizable interaction between the inflaton and radiation is essential for this consequence. See discussion at the final paragraph of this subsection.

However, in the latter case $\lambda \gg y \sim 0$, inflaton particles are (almost) stable, and hence they can reduce number densities only through annihilations. Therefore, they are eventually decoupled from the thermal plasma, and behaves as matter with a fixed number after that. As one can guess, the inflaton itself can be a candidate of WIMP (weakly interacting massive particle) DM if it is singlet under $U(1)_{\text{em}}$ and $SU(3)_c$. For instance, the singlet extension of SM can simultaneously explain the primordial inflation and the present DM abundance [151–153].

Up to here, we stick to the setup Eq. (4.2) and concentrate on the dissipation that comes from the Yukawa and quartic interactions imprinted in Eq. (4.2). Before closing this section, we briefly comment on the case where the dissipation term has a rather general form:

$$\Gamma_\phi^{\text{eff}}(\tilde{\phi}; T) = A \frac{\tilde{\phi}^l T^m}{2^{l/2} m_\phi^{l+m-1}} \quad \text{for } m_\phi \ll T, \quad (5.22)$$

with A being a dimensionless factor that does not depend on $\tilde{\phi}$ and T . See also Ref. [154]. In this case, the energy density of radiation before the completion of reheating follows

$$\rho_{\text{rad}} \sim \frac{\Gamma_\phi^{\text{eff}}}{H} \rho_\phi \simeq A \frac{\rho_\phi^{1+l/2} T^m}{H m_\phi^{2l+m-1}} \simeq A (3M_{\text{pl}}^2)^{1+l/2} H^{1+l} \frac{T^m}{m_\phi^{2l+m-1}}, \quad (5.23)$$

which implies

$$T \sim \left[3^{1+l/2} A \left(\frac{30}{\pi^2 g_*} \right) \frac{M_{\text{pl}}^{l+2} H^{l+1}}{m_\phi^{2l+3}} \right]^{\frac{1}{4-m}} m_\phi \propto a^{-\frac{3}{2} \frac{l+1}{4-m}}. \quad (5.24)$$

One can see that the evolution of radiation can be significantly modified from the conventional scenario $(l, m) = (0, 0)$. Note that, for $l > 1$, the dissipation factor may decrease faster than the Hubble parameter and hence the inflaton cannot completely lose its energy completely by this term alone. In addition, also note that, for $m > 2$, even if the inflaton participates in the thermal plasma once, the Hubble parameter soon becomes larger than the interaction rate between the inflaton and radiation because $H \propto T^2/M_{\text{pl}}$ but $\Gamma_\phi^{\text{eff}} \propto T^m$. Thus, the inflaton particles might dominate the Universe again. If this is the case, the reheating temperature may be determined by the perturbative decay rate at the regime $m_\phi \gg T$ where the dissipation rate is expected to have different behavior, like $\Gamma_\phi^{\text{eff}} \propto m_\phi$. Therefore, if the inflaton oscillates with a sizable VEV, then the final reheating might be determined by its perturbative decay.

5.1.3 Numerical Result

In this subsection, we calculate the evolution of inflaton/plasma system after inflation. We numerically solve the differential Eqs. (5.5)–(5.7) to study the dynamics of inflaton/plasma system, using the effective dissipation summarized in Sec. 4.6.2. The gauge coupling constant is assumed to be $\alpha = 0.05$, and concentrate on the sizable Yukawa case $y \sim \lambda$ because the evolution is basically the same except for its final behavior as explained in the previous subsection. We closely follow Ref. [36] in the following.

The top panel is computed with $(m_\phi, \tilde{\lambda}, \phi_i) = (1 \text{ TeV}, 10^{-3}, 10^{18} \text{ GeV})$. First, the radiation with high temperature, $T \sim 10^8 \text{ GeV}$, is produced via the preheating. The condition for non-perturbative production [Eq. (4.199)] is soon saturated, since the amplitude scales as $\tilde{\phi} \propto a^{-3/2}$ where a is the scale factor of the Universe while the temperature scales as $T \propto a^{-3/8}$, and consequently the non-perturbative production shuts off. Then, as can be seen from the plateau of ρ_{rad} around $H \sim 5 \times 10^{-1} - 10^{-2} \text{ GeV}$, the temperature of thermal plasma becomes nearly

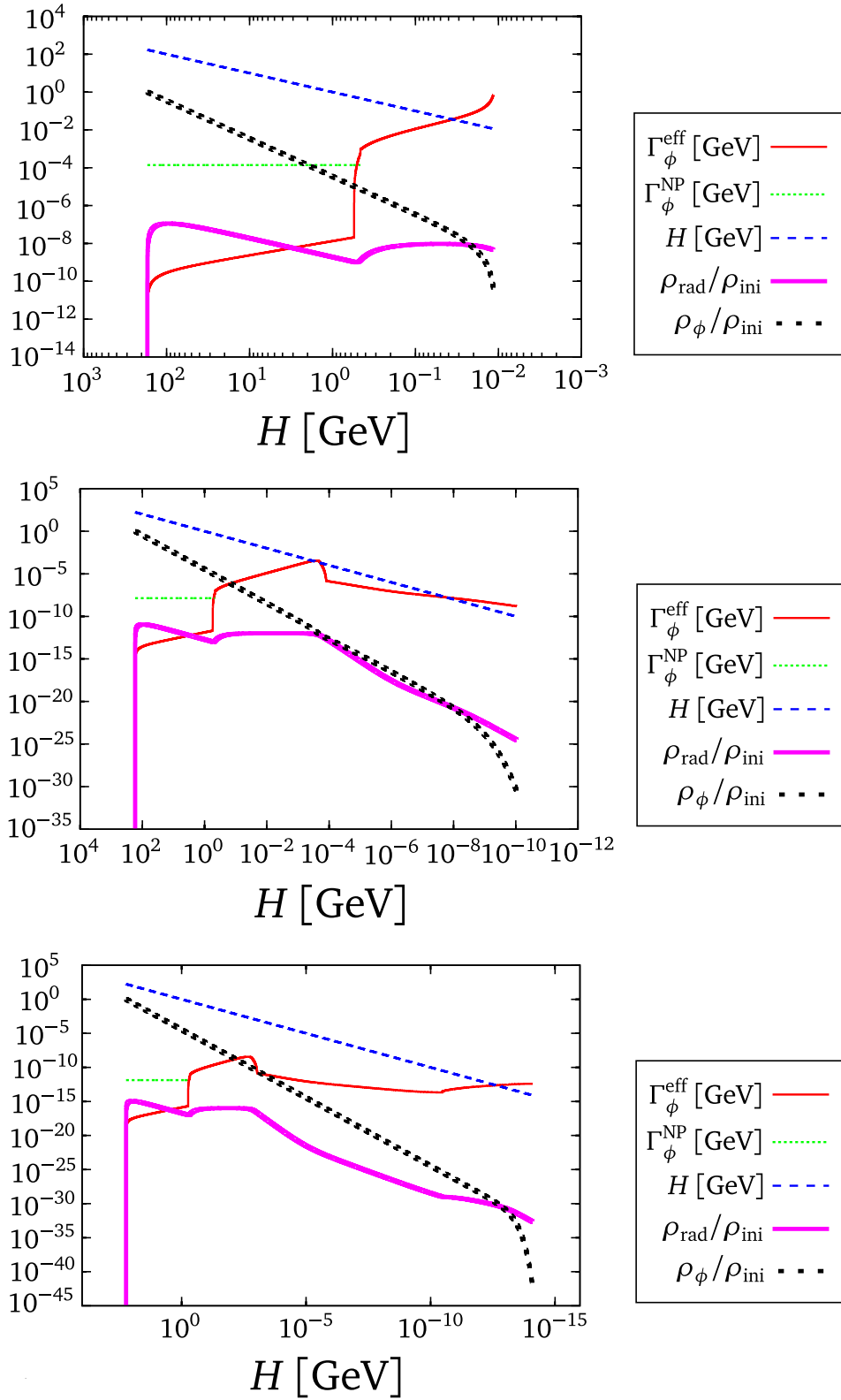


Figure 5.1: The evolution of various quantities as a function of Hubble scale H : the effective dissipation rate except for non-perturbative particle production Γ_ϕ^{eff} [Red thin solid], one for non-perturbative particle production Γ_ϕ^{NP} [Green thin dotted], the energy density of radiation ρ_{rad} [Magenta thick solid] and inflaton ρ_ϕ [Black thick dashed] normalized by an initial energy density ρ_{ini} . **Top:** $(m_\phi, \tilde{\lambda}, \phi_i) = (1 \text{ TeV}, 10^{-3}, 10^{18} \text{ GeV})$, **Middle:** $(m_\phi, \tilde{\lambda}, \phi_i) = (1 \text{ TeV}, 10^{-5}, 10^{18} \text{ GeV})$, **Bottom:** $(m_\phi, \tilde{\lambda}, \phi_i) = (1 \text{ TeV}, 10^{-7}, 10^{18} \text{ GeV})$.

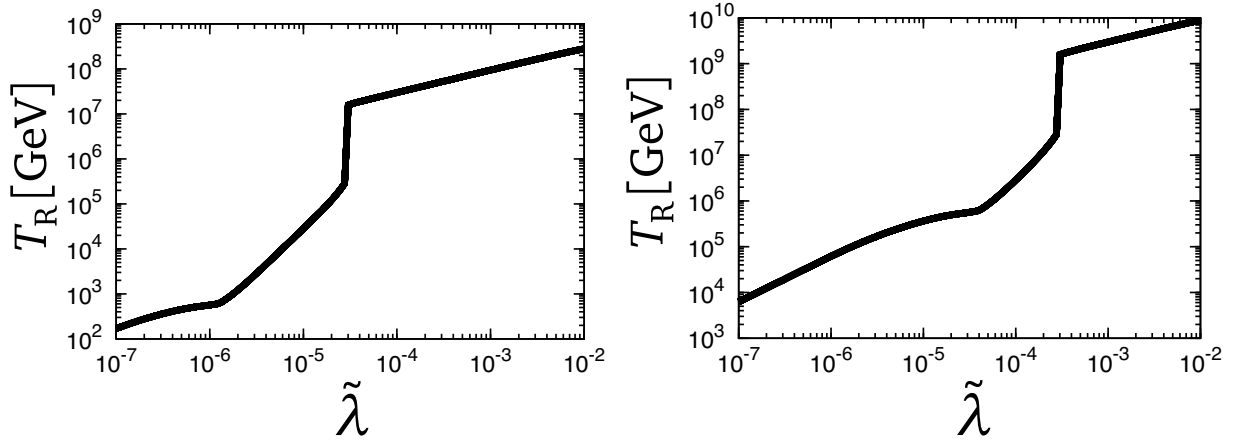


Figure 5.2: The reheating temperature T_R as a function of $\tilde{\lambda}$ is shown. **Left:** $m_\phi = 1$ TeV and **Right:** $m_\phi = 10^3$ TeV.

constant during the regime where the dominant dissipation rate is given by $\Gamma_\phi^{\text{eff}} \sim \tilde{\lambda} T^2 / (\alpha \tilde{\phi})$. Finally, the reheating is completed at $\Gamma_\phi^{\text{eff}} \sim H$. In the top panel, the reheating takes place via $\Gamma_\phi^{\text{eff}} \sim \tilde{\lambda} T^2 / (\alpha \tilde{\phi})$ at $H \sim 2 \times 10^{-2}$ GeV, and the reheating temperature is $T_R \sim 10^8$ GeV.

The middle and bottom panels are computed with $(m_\phi, \tilde{\lambda}, \phi_i) = (1 \text{ TeV}, 10^{-5}, 10^{18} \text{ GeV})$ and $(m_\phi, \tilde{\lambda}, \phi_i) = (1 \text{ TeV}, 10^{-7}, 10^{18} \text{ GeV})$ respectively. The subsequent evolution is the same as the top panel case in the both middle and bottom panels. First, the thermal plasma is produced via the preheating, and the condition for non-perturbative production soon saturates. Then, the plateau region follows $H \sim 5 \times 10^{-1} - 10^{-4}$ GeV (middle) and $H \sim 5 \times 10^{-1} - 10^{-3}$ GeV (bottom). After that, since $\tilde{\phi}$ decreases due to the cosmic expansion, the dominant dissipation rate becomes $\Gamma_\phi^{\text{eff}} \sim \tilde{\lambda}^2 \alpha T$. In the middle panel, the reheating takes place via $\Gamma_\phi^{\text{eff}} \sim \tilde{\lambda}^2 \alpha T$ at $H \sim 10^{-8}$ GeV, and the reheating temperature is $T_R \sim 10^5$ GeV. On the other hand, in the bottom panel, the reheating occurs via $\Gamma_\phi^{\text{eff}} \sim \tilde{\lambda}^2 m_\phi$ at $H \sim 10^{-13}$ GeV, and its temperature is given by $T_R \sim 3 \times 10^2$ GeV.^{♦5}

Analytically, the reheating temperature can be roughly estimated as follows in the four cases: reheating via (0) $\Gamma_\phi^{\text{eff}} \sim \Gamma_\phi^{\text{NP}}$, (i) $\Gamma_\phi^{\text{eff}} \sim \tilde{\lambda} T^2 / (\alpha \tilde{\phi})$, (ii) $\Gamma_\phi^{\text{eff}} \sim \tilde{\lambda}^2 \alpha T$ and (iii) $\Gamma_\phi^{\text{eff}} \sim$

^{♦5} Usually, the reheating temperature T_R is defined as the temperature at which the radiation dominated Universe begins and it roughly corresponds to the epoch $H \sim \Gamma_\phi$ as (2.25). In the present situation with thermal dissipation effect, this definition is ambiguous because of the peculiar behavior of Γ_ϕ^{eff} . As seen in the middle panel of Fig. 5.1, Γ_ϕ^{eff} can once become equal to H but the relation $\rho_{\text{rad}} \sim \rho_\phi$ may hold thereafter without exponential decay of the inflaton for a while. This is because the dissipation rate decreases faster than the Hubble parameter during the regime: $\Gamma_\phi^{\text{eff}} \propto \tilde{\phi}^2$. Therefore, the reheating temperature T_R here is defined as the temperature at which the inflaton energy density begins to decrease exponentially. One should note that, although the parameter T_R is a convenient quantity which describes a global picture of the early Universe, actual thermal history before the reheating would be significantly different from a conventional one.

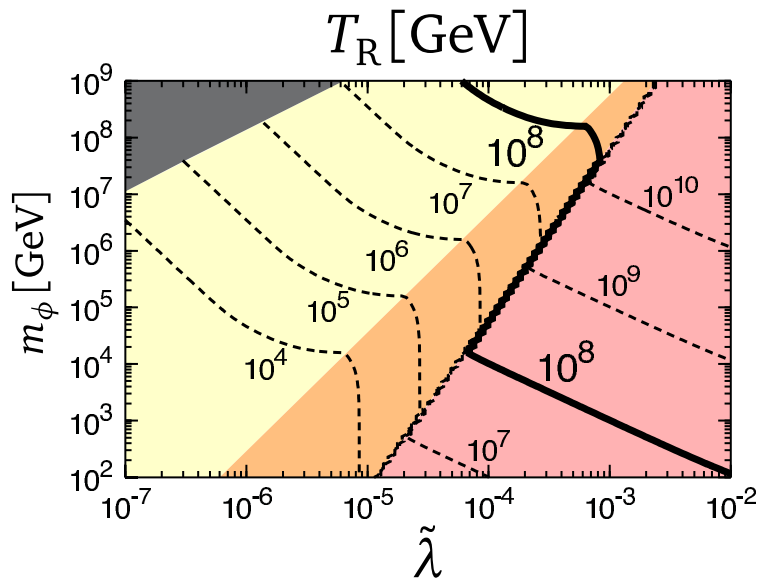


Figure 5.3: Contour plot of reheating temperature T_R as a function of $\tilde{\lambda}$ and m_ϕ for $\phi_i = 10^{18}$ GeV. Inside the Black shaded region, the condition $g^2 \tilde{\lambda} \tilde{\phi} > m_\phi$ is violated, and this region depends on the initial amplitude ϕ_i . In the Yellow shaded region, the reheating is completed by the perturbative decay via the Yukawa interaction, and in the Orange and Red shaded region, the reheating is dominated by the thermal dissipation.

$\tilde{\lambda}^2 m_\phi$ [36]:

$$T_R \sim \begin{cases} c_0^{1/2} \left(\frac{90}{\pi^2 g_*} \right)^{1/4} \left(\frac{1}{2\pi^4 |\epsilon g|} \right)^{1/2} \sqrt{\tilde{\lambda}^2 m_\phi M_{\text{pl}}} & \cdots (0) \\ c_1^{1/2} \left(\frac{30\sqrt{3}}{\pi^2 g_* \alpha} \right)^{1/2} \sqrt{\tilde{\lambda} m_\phi M_{\text{pl}}} & \cdots (i) \\ c_2 \left(\frac{90}{\pi^2 g_*} \right)^{1/2} \alpha \tilde{\lambda}^2 M_{\text{pl}} & \cdots (ii) \\ c_2^{1/2} \left(\frac{90}{\pi^2 g_*} \right)^{1/4} \sqrt{\tilde{\lambda}^2 m_\phi M_{\text{pl}}} & \cdots (iii) \end{cases} \quad (5.25)$$

Note that the resultant reheating temperature contains the uncertainty c_1 in Eq. (5.14) (See also footnote ♦3). Importantly, the coupling $\tilde{\lambda}$ dependence differs among (i) – (iii) and the initial amplitude ϕ_i dependence is absent even in the case (i). These behavior can be seen in Figs. 5.2 and 5.3. In Fig. 5.2, reheating temperature is plotted as a function of $\tilde{\lambda}$ for $m_\phi = 1$ TeV (left) and $m_\phi = 10^3$ TeV (right) with $\phi_i = 10^{18}$ GeV. It is seen that in the small $\tilde{\lambda}$ limit, the reheating temperature is determined by the standard perturbative decay scenario [case (iii)]. As T_R increases and approaches to m_ϕ for larger $\tilde{\lambda}$, it begins to saturate due to the effect of thermal blocking. For larger $\tilde{\lambda}$, however, thermal dissipation comes in and again T_R increases [case (ii) and (i)]. This figure does not depend on ϕ_i for $\phi_i \gtrsim 10^{15}$ GeV. In Fig. 5.3, contours of reheating temperature as a function of $\tilde{\lambda}$ and m_ϕ are shown.

As mentioned in footnote ♦5, the reheating temperature T_R here is defined as the temperature at which the inflaton energy density begins to decrease exponentially. The sharp discontinuity between two regimes [(i) and (ii)] seen in Fig. 5.2 is related to the definition of reheating temperature T_R . The reheating cannot be completed during the regime where the effective dissipation rate is given by $\Gamma_\phi^{\text{eff}} \propto \tilde{\phi}^2$, with the definition of reheating that we employed. This is clearly seen in the middle panel of Fig. 5.1 ($\tilde{\lambda} = 10^{-5}$); Γ_ϕ^{eff} crosses H two times, but at the first crossing the reheating is not completed and the reheating temperature

is roughly determined at the second crossing. For larger $\tilde{\lambda}$, the situation becomes close to the top panel of Fig. 5.1 ($\tilde{\lambda} = 10^{-3}$), where reheating is completed soon after the first crossing. Thus the reheating temperature jumps somewhere around $10^{-5} < \tilde{\lambda} < 10^{-3}$ (for $m_\phi = 1$ TeV) if it is plotted as a function of $\tilde{\lambda}$. This is the reason for the behavior in Fig. 5.2.

Here we note that, for the purpose of estimating the reheating temperature alone, it is not necessary to impose the condition that the light degrees of freedom should be always kept in thermal equilibrium for every ϕ 's oscillation. The only requirement is that the typical interaction time scale of plasma should be much faster than the Hubble parameter and the dissipation rate of ϕ before the reheating is completed.

5.2 Curvaton

In this section, we study the effects of the background plasma on the curvaton mechanism, *i.e.* ϕ represents the curvaton in this section, and the reheating is assumed to take place in a conventional way (*e.g.* Planck-suppressed decay) for simplicity. A characteristic feature of the curvaton model is that it can produce the large non-Gaussianity which is parameterized by f_{NL} . As briefly explained in Sec. 2.2, in a simple scenario where the curvaton oscillates with a quadratic potential and convert its energy into radiation via a vacuum decay rate, f_{NL} is given by [Eq. (2.47)]

$$\frac{6}{5}f_{\text{NL}} = \frac{3}{2r} - 2 - r; \quad r \equiv \frac{3\tilde{R}}{3\tilde{R} + 4}; \quad \tilde{R} \equiv \left. \frac{\rho_\phi}{\rho_{\text{rad}}} \right|_{\text{dec}}, \quad (5.26)$$

and also the power spectrum is

$$\mathcal{P}_\zeta = \left(\frac{H_*}{2\pi\phi_i} \frac{2r}{3} \right)^2, \quad (5.27)$$

with H_* being the Hubble parameter at the horizon exit of the CMB scale. Here and hereafter we assume that the curvaton is the main source of the curvature perturbation for simplicity. If the curvaton energy density at its decay is smaller than that of radiation, then f_{NL} is enhanced by a factor $1/\tilde{R}$ and it might contradict with the Planck result [65]. So, one needs to know the energy fraction of the curvaton at its decay to examine the curvaton scenario.

However, since the curvaton should convert its primordial fluctuations acquired during the inflation, it inevitably couples to light particles. Hence, as one might guess from Eqs. (5.2) and (5.3), one has to take into account effects of interactions with light particles because they change when the curvaton oscillates, how the curvaton evolves after the onset of oscillation and when it decays. Moreover, in general, the oscillation time depends on the initial value of the curvaton and the effective dissipation rate also depends on both the amplitude of the curvaton and the background temperature. Therefore, the above formulas are expected to be modified. Thermal effects on the dynamics of the curvaton were partly studied in the literature [155, 156]. In Ref. [46], we have first taken into account all these effects.

In the following, we first derive rather general formulas of curvature perturbations for completeness, and see that the typical behavior of f_{NL} is still roughly given by $1/\tilde{R}$ except for some special cases. Then, we numerically compute $1/\tilde{R}$ for two cases; $\lambda \gg y \sim 0$ and $\lambda \sim y$, and show that if the curvaton cannot survive from interactions with the thermal plasma, it becomes difficult to explain today's amount of power spectrum and also tends to produce too large non-Gaussianity. We put an upper bounds on the couplings λ and y . This section closely follows Ref. [46].

Before going into details, we summarize basic evolution equations in this case:^{♦6}

$$\frac{d}{dt} \ln \tilde{\phi} = -\frac{1}{n_1} \left[(3 - n_2 n_3) H + \Gamma_\phi^{\text{eff}} \right], \quad (5.28)$$

$$\dot{\rho}_I = -[3H + \Gamma_I] \rho_I, \quad (5.29)$$

$$\dot{\rho}_{\text{rad}} = -4H \rho_{\text{rad}} + \Gamma_I \rho_I + \frac{n}{n_1} \Gamma_\phi^{\text{eff}} \rho_\phi, \quad (5.30)$$

$$3M_{\text{pl}}^2 H^2 = \rho_\phi + \rho_I + \rho_{\text{rad}}, \quad (5.31)$$

where

$$(n, n_1, n_2) = \begin{cases} \left(n, \frac{n+2}{2}, 0 \right) & \text{for vacuum potential with } V \propto |\phi|^n, \\ (2, 2, 1) & \text{for thermal mass,} \\ (1, 1, 2) & \text{for thermal log,} \end{cases} \quad (5.32)$$

and

$$n_3 = \begin{cases} 1 & \text{for radiation dominant era,} \\ \frac{3}{8} & \text{for inflaton dominant era.} \end{cases} \quad (5.33)$$

Here ρ_I and Γ_I represent the energy density and decay rate of inflaton. And we assume that the inflaton has a quadratic potential and reheats the Universe via a vacuum decay rate (*e.g.* Planck-suppressed decay). As we will see later, the CW-potential correction has important effects on curvature perturbations and hence we keep this term in this section.

5.2.1 Curvature Perturbation

As explained in Sec. 2.2, according to the δN -formalism [66–69], the primordial curvature perturbation ζ is obtained from

$$\zeta(\mathbf{x}) = N(\mathbf{x}) - \bar{N}, \quad (5.34)$$

where $N(\mathbf{x})$ denotes the e-folding number from the initial spatially flat slice to the final uniform density slice, and \bar{N} is that of the background one. Thus, the curvature perturbation is expanded as

$$\zeta = N_{\phi_i} \delta\phi_i + \frac{1}{2} N_{\phi_i \phi_i} \delta\phi_i \delta\phi_i + \dots, \quad (5.35)$$

where $\delta\phi_i$ denotes fluctuations around ϕ_i . This relation implies that the power spectrum and the non-linearity parameter can be expressed as

$$\mathcal{P}_\zeta = \left(\frac{H_*}{2\pi} \right)^2 N_{\phi_i}^2; \quad \frac{6}{5} f_{\text{NL}} = \frac{N_{\phi_i \phi_i}}{N_{\phi_i}^2}. \quad (5.36)$$

^{♦6} Again note that there is subtlety on the definition of energy density of ϕ , ρ_ϕ and energy conversion from ϕ to radiation, $\frac{n}{n_1} \Gamma_\phi^{\text{eff}}$, in the case of oscillation with thermal potential. However, in this case, the energy density of ϕ is at most that of one degree of freedom in thermal bath $\sim T^4$. Hence, it is merely a small change of g_* and can be neglected practically within an accuracy of our estimation.

Hence, we need to know the ϕ_i -dependence of N .

However, as mentioned at the beginning of this section, the ϕ_i -dependence on the e-folding number N from the spatially flat surface to the uniform density surface is complicated in our setup [Eq. (4.2)]. The large scale curvature perturbation is produced via four steps in our case: (i) Fluctuations of ϕ_i acquired during the inflation yield those of ϕ 's energy density, (ii) The beginning of ϕ 's oscillation may depend on ϕ_i [157–159], (iii) The time when ϕ 's equation of state changes may depend on the amplitude of ϕ [46, 160] and (iv) The effective dissipation rate may also depend on the amplitude of ϕ (*self-modulated reheating*; a variant of the modulated reheating [161]). As can be seen from the basic evolution equations (5.28)–(5.33), all these effects could contribute to the final curvature perturbation in general.

We take the final uniform density slice well after ϕ -decay. For our purpose, we can take the initial spatially flat surface freely as long as the curvaton energy density is negligible. We take this surface so that ϕ 's equation of state does not change after that. We refer to this surface as the uniform density surface of others (except for ϕ), and denote with the superscript (uds-o). In addition, we assume that the inflaton decays into radiation after the change of ϕ 's equation of state but before the decay of ϕ for simplicity, and thus take the uds-o surface after the reheating hereafter. One can express ρ_ϕ on the uds-o surface by using the initial energy density $\rho_\phi^{(\text{ini})}$ as

$$\rho_\phi^{(\text{uds-o})}(\mathbf{x}) = \rho_\phi^{(\text{ini})}(\mathbf{x}) \left(\frac{H_{\text{os}}(\mathbf{x})}{H_w(\mathbf{x})} \right)^{-\frac{2(1+w_\phi^{(\text{osc})})}{1+w_{\text{tot}}}} \left(\frac{H_w(\mathbf{x})}{H^{(\text{uds-o})}} \right)^{-\frac{2(1+w_\phi^{(\text{dec})})}{1+w_{\text{tot}}}}, \quad (5.37)$$

where $w_\phi^{(\text{osc/dec})}$ is the equation of state of ϕ just after the oscillation/before the dissipation respectively, w_{tot} is the equation of state of the Universe, H_{os} is the Hubble parameter at the beginning of oscillation, and H_w is the Hubble parameter when the equation of state changes; $\phi(\mathbf{x}) = \phi_w$. Here notice that since the energy density of ϕ at the change of e.o.s., $\rho_\phi^{(w)}$, does not depend on $\phi_i(\mathbf{x})$, the ϕ_i -dependence only enters via $H_w(\mathbf{x})$. We define ζ_ϕ as

$$\zeta_\phi(\mathbf{x}) \equiv N_\phi(\mathbf{x}) - \bar{N}_\phi, \quad (5.38)$$

where $N_\phi(\mathbf{x})$ is the e-folding number from the spatially flat surface to the uniform ρ_ϕ surface. This quantity is conserved once the equation of motion is fixed. Recalling that $\rho_\phi \propto a^{-3(1+w_\phi^{(\text{dec})})}$, one can express ζ_ϕ as

$$\frac{\rho_\phi^{(\text{uds-o})}(\mathbf{x})}{\bar{\rho}_\phi^{(\text{uds-o})}} = e^{3(1+w_\phi^{(\text{dec})})(N_\phi(\mathbf{x}) - \bar{N}_\phi)} \leftrightarrow \zeta_\phi = \frac{1}{3(1+w_\phi^{(\text{dec})})} \ln \frac{\rho_\phi^{(\text{uds-o})}(\mathbf{x})}{\bar{\rho}_\phi^{(\text{uds-o})}}. \quad (5.39)$$

The relation between $\rho_\phi^{(\text{uds-o})}/\bar{\rho}_\phi$ and the primordial fluctuation can be read from Eq. (5.37):

$$\frac{\rho_\phi^{(\text{uds-o})}(\mathbf{x})}{\bar{\rho}_\phi^{(\text{uds-o})}} = \left(\frac{H_w(\mathbf{x})}{\bar{H}_w} \right)^{-b_w} = \left(\frac{H_{\text{os}}(\mathbf{x})}{\bar{H}_{\text{os}}} \right)^{-b_w} \left(\frac{\rho_\phi^{(\text{ini})}(\mathbf{x})}{\bar{\rho}_\phi^{(\text{ini})}} \right)^{b_w/a_w}, \quad (5.40)$$

where $a_w \equiv 2(1+w_\phi^{(\text{osc})})/(1+w_{\text{tot}})$ and $b_w \equiv 2(1+w_\phi^{(\text{dec})})/(1+w_{\text{tot}})$. In the second equality, we have used

$$1 = \frac{\rho_\phi^{(\text{ini})}(\mathbf{x})}{\bar{\rho}_\phi^{(\text{ini})}} \left(\frac{H_{\text{os}}(\mathbf{x})}{\bar{H}_{\text{os}}} \frac{\bar{H}_w}{H_w(\mathbf{x})} \right)^{-a_w}. \quad (5.41)$$

Therefore one can express ζ_ϕ as

$$\zeta_\phi = \frac{b_w}{3(1+w_\phi^{(\text{dec})})} \ln \left(\frac{\bar{H}_w}{H_w(\mathbf{x})} \right) = \frac{b_w}{3(1+w_\phi^{(\text{dec})})} \left[\frac{1}{a_w} \ln \frac{\rho_\phi^{(\text{ini})}(\mathbf{x})}{\bar{\rho}_\phi^{(\text{ini})}} - \ln \frac{H_{\text{os}}(\mathbf{x})}{\bar{H}_{\text{os}}} \right] \quad (5.42)$$

$$\equiv \frac{k}{3(1+w_\phi^{(\text{dec})})} \ln \left(1 + \frac{\delta\phi_i(\mathbf{x})}{\bar{\phi}_i} \right), \quad (5.43)$$

where k represents a model-dependent order one constant; for instance in a conventional case $k = 2$. In addition, if the initial value of ϕ is larger, then it takes more time to satisfy $\phi(\mathbf{x}) = \phi_w$. Therefore we expect that the parameter k is positive. After the equation of motion is fixed, ζ_ϕ conserves as already mentioned, so let us move to another contribution.

Then, we take the ϕ -decay surface $H^{(\text{dec})}(\mathbf{x}) = \Gamma_\phi^{(\text{dec})}(\mathbf{x})$:

$$\rho_{\text{rad}}^{(\text{dec})}(\mathbf{x}) + \rho_\phi^{(\text{dec})}(\mathbf{x}) = 3M_{\text{pl}}^2 H^{(\text{dec})2}(\mathbf{x}) = \bar{\rho}_{\text{tot}}^{(\text{dec})} \left(\frac{\Gamma_\phi^{(\text{dec})}(\mathbf{x})}{\bar{\Gamma}_\phi^{(\text{dec})}} \right)^2. \quad (5.44)$$

Here note that Γ_ϕ represents the effective dissipation rate, but we omit the eff superscript for notational simplicity, and also note that the superscript (dec) stands for just before the decay. We define $\delta N_1(\mathbf{x}) \equiv N_1(\mathbf{x}) - \bar{N}_1$ where N_1 is the e-folding number from the uniform ρ_ϕ surface to the ϕ -decay surface. Then, one finds

$$\frac{\rho_\phi^{(\text{dec})}(\mathbf{x})}{\bar{\rho}_\phi^{(\text{dec})}} = e^{-3(1+w_\phi^{(\text{dec})})\delta N_1(\mathbf{x})}; \quad \frac{\rho_{\text{rad}}^{(\text{dec})}(\mathbf{x})}{\bar{\rho}_{\text{rad}}^{(\text{dec})}} = e^{-4(\zeta_\phi(\mathbf{x}) + \delta N_1(\mathbf{x}))}. \quad (5.45)$$

We consider a rather general form of the dissipation rate in the following:

$$\Gamma_\phi(\tilde{\phi}; T) \propto \phi^l T^m \propto a^{-ql} \rho_{\text{rad}}^{\frac{m}{4}}, \quad (5.46)$$

which can be expressed as

$$\frac{\Gamma_\phi^{(\text{dec})}(\mathbf{x})}{\bar{\Gamma}_\phi^{(\text{dec})}} = e^{-ql\delta N_1(\mathbf{x}) - m(\zeta_\phi(\mathbf{x}) + \delta N_1(\mathbf{x}))} = e^{-p\delta N_1(\mathbf{x}) - m\zeta_\phi(\mathbf{x})}, \quad (5.47)$$

with $p \equiv ql + m$. Finally, inserting Eqs. (5.45) and (5.47) into (5.44), we obtain the following relation:

$$R_\phi e^{-3(w_\phi^{(\text{dec})} + 1)\delta N_1(\mathbf{x})} + R_{\text{rad}} e^{-4(\delta N_1(\mathbf{x}) + \zeta_\phi(\mathbf{x}))} - e^{-2(p\delta N_1(\mathbf{x}) + m\zeta_\phi(\mathbf{x}))} = 0, \quad (5.48)$$

where

$$R_\phi \equiv \frac{\tilde{\rho}_\phi^{(\text{dec})}}{\tilde{\rho}_{\text{tot}}^{(\text{dec})}}; \quad R_{\text{rad}} \equiv \frac{\tilde{\rho}_{\text{rad}}^{(\text{dec})}}{\tilde{\rho}_{\text{tot}}^{(\text{dec})}}. \quad (5.49)$$

Eq. (5.48) relates the δN_1 with the fluctuations $\delta\phi_i/\bar{\phi}_i$ through ζ_ϕ . For later usage, it is convenient to express δN_1 in terms of ζ_ϕ order by order:

$$N_1^{(1)} = \frac{2(2R_{\text{rad}} - m)}{2p - 4R_{\text{rad}} - 3(1 + w_\phi^{(\text{dec})})R_\phi} \zeta_\phi, \quad (5.50)$$

$$N_1^{(2)} = \frac{-2R_r R_\phi}{[2p - 3(1 + w_\phi^{(\text{dec})})R_\phi - 4R_r]^3} \left[4(p - m) - 3(1 + w_\phi^{(\text{dec})})(2 - m) \right]^2 \zeta_\phi^2, \quad (5.51)$$

$$\dots \quad (5.52)$$

Finally, let us connect the decay slice to the uniform density slice well after the ϕ -decay. We define $\delta N_2(\mathbf{x}) \equiv N_2(\mathbf{x}) - \bar{N}_2$ where $N_2(\mathbf{x})$ is the e-folding number from the ϕ -decay surface to the uniform density surface. The energy density of radiation can be expressed as

$$\frac{\bar{\rho}_{\text{rad}}^{(\text{dec},+)}}{\rho_{\text{rad}}^{(\text{dec},+)}(\mathbf{x})} = e^{-4\delta N_2(\mathbf{x})}, \quad \rho_{\text{rad}}^{(\text{dec},+)}(\mathbf{x}) = 3H^{(\text{dec},+)}(\mathbf{x})M_{\text{pl}}^2 = \bar{\rho}_{\text{rad}}^{(\text{dec},+)} \left(\frac{\Gamma_{\phi}^{(\text{dec},+)}(\mathbf{x})}{\bar{\Gamma}_{\phi}^{(\text{dec},+)}} \right)^2. \quad (5.53)$$

Here (dec,+) superscript denotes the ϕ -decay surface right after the complete decay of ϕ . Thus, we obtain the following relation:

$$e^{4\delta N_2(\mathbf{x})} = \left(\frac{\Gamma_{\phi}^{(\text{dec},+)}(\mathbf{x})}{\bar{\Gamma}_{\phi}^{(\text{dec},+)}} \right)^2 = e^{-2p\delta N_1(\mathbf{x}) - 2m\zeta_{\phi}(\mathbf{x})} \leftrightarrow \delta N_2(\mathbf{x}) = -\frac{1}{2}(p\delta N_1(\mathbf{x}) + \zeta_{\phi}(\mathbf{x})). \quad (5.54)$$

See also Ref. [162]; the validity of sudden decay approximation that we employed here is discussed.

Now we can compute the curvature perturbation ζ well after the decay of ϕ . The resulting curvature perturbation is given by their sum:

$$\zeta = \zeta_{\phi} + \delta N_1 + \delta N_2. \quad (5.55)$$

At the leading order, it is given by

$$\zeta^{(1)} = \frac{R_{\phi}}{3(1 + w_{\phi}^{(\text{dec})})R_{\phi} + 4R_r} \left[3(1 + w_{\phi}^{(\text{dec})})\zeta_{\phi} + (3w_{\phi}^{(\text{dec})} - 1)\frac{\delta\Gamma_{\phi}^{(\text{dec})}}{2\bar{\Gamma}_{\phi}^{(\text{dec})}} \right]. \quad (5.56)$$

One can see that the curvature perturbation is proportional to R_{ϕ} and the effect of self-modulated reheating vanishes for $w_{\phi}^{(\text{dec})} = 1/3$, as expected. By substituting (5.47), we obtain

$$\zeta^{(1)} = \frac{R_{\phi} [4(p - m) - 3(1 + w_{\phi}^{(\text{dec})})(2 - m)]}{2[2p - 3(1 + w_{\phi}^{(\text{dec})})R_{\phi} - 4R_r]} \zeta_{\phi}, \quad (5.57)$$

at the leading order, and

$$\zeta^{(2)} = \frac{(p - 2)R_r R_{\phi}}{[2p - 3(1 + w_{\phi}^{(\text{dec})})R_{\phi} - 4R_r]^3} [4(p - m) - 3(1 + w_{\phi}^{(\text{dec})})(2 - m)]^2 \zeta_{\phi}^2, \quad (5.58)$$

at the second order in ζ_{ϕ} .

Then, let us evaluate power spectrum \mathcal{P}_{ζ} and the non-linearity parameter f_{NL} in terms of $p, m, w_{\phi}^{(\text{dec})}, k$ and r which is defined as

$$r \equiv \frac{R_{\phi} [3(1 + w_{\phi}^{(\text{dec})})(2 - m) - 4(p - m)]}{2[(3(1 + w_{\phi}^{(\text{dec})}) - 2p)R_{\phi} + (4 - 2p)R_r]}. \quad (5.59)$$

As one can see, it coincides with $3\rho_{\phi}/(3\rho_{\phi} + 4\rho_r)$ in the standard curvaton scenario where parameters are given by $(p, m, w_{\phi}^{(\text{dec})}) = (0, 0, 0)$. The power spectrum is obtained as

$$\mathcal{P}_{\zeta} = \left(\frac{H_*}{2\pi\phi_i} \frac{kr}{3(1 + w_{\phi}^{(\text{dec})})} \right)^2. \quad (5.60)$$

The non-linearity parameter f_{NL} has the following form:

$$f_{\text{NL}} = A^{(0)} + A^{(1)}r + A^{(-1)}\frac{1}{r}, \quad (5.61)$$

with

$$A^{(0)} = -\frac{5}{3} \times \frac{6(1 + w_{\phi}^{(\text{dec})}) - 4 - 2p}{2 - p}, \quad (5.62)$$

$$A^{(1)} = -\frac{5}{6} \times \frac{4(4 - 3(1 + w_{\phi}^{(\text{dec})}))(3(1 + w_{\phi}^{(\text{dec})}) - 2p)}{(2 - p)(3(1 + w_{\phi}^{(\text{dec})}))(2 - m) - 4(p - m)}, \quad (5.63)$$

$$A^{(-1)} = \frac{5}{4} \times \left[\frac{2(3(1 + w_{\phi}^{(\text{dec})}))(2 - m) - 4(p - m)}{3(2 - p)} - \frac{2(1 + w_{\phi}^{(\text{dec})})}{k} \right]. \quad (5.64)$$

In most cases including the cases which we deal with, $A^{(-1)}$ is a factor of order unity, hence we have $f_{\text{NL}} \sim 1/r$ for $r \ll 1$ as in the ordinary curvaton model.

However, there exists an exceptional situation in which the enhancement of non-linearity parameter does not exist, $A^{(-1)} = 0$, keeping $\zeta \sim 5 \times 10^{-5}$. Thus the constraint on a curvaton scenario is drastically relaxed [160]. This scenario is quite simple. First, the curvaton starts to oscillate with the quartic potential, such as the CW potential, before the completion of reheating. Then, the curvaton potential becomes dominated by the quadratic one. At this stage, we assume that the curvaton energy density is still negligible. Finally, after the completion of reheating, the curvaton decays into radiation with a constant decay rate. This case corresponds to the following parameters; $(p, m, w_{\phi}^{(\text{dec})}, k) = (0, 0, 0, 1)$. Obviously, the factor vanishes, $A^{(-1)} = 0$, in this case.

5.2.2 Numerical Results

As shown in the previous subsection, the non-linearity parameter still follows the typical behavior $f_{\text{NL}} \propto 1/r$ for $r \ll 1$ except for the special case [160]. In this subsection, we discuss the viability of the curvaton scenario which falls into Eq. (4.2). In particular, we estimate the energy fraction of the curvaton at its dissipation:

$$\tilde{R} \equiv \left. \frac{\rho_{\phi}}{\rho_{\text{rad}}} \right|_{\text{dec}}, \quad (5.65)$$

which characterizes the properties of ϕ as a curvaton.

Here we assume that the curvaton is a dominant contribution to the observed density perturbation. Hence, the energy fraction should be relatively large, $\tilde{R} \gtrsim 0.1$, to avoid the constraints on f_{NL} . In addition, the observed power spectrum should be explained by the curvaton, which indicates

$$(5 \times 10^{-5})^2 \simeq \mathcal{P}_{\zeta} \sim \left(\frac{H_*}{2\pi\phi_i} r(\tilde{R}) \right)^2 \leftrightarrow \mathcal{O}(10^{-4}) \times \phi_i \frac{1}{r(\tilde{R})} \sim H_* \lesssim 10^{14} \text{ GeV}. \quad (5.66)$$

Note that unless non-trivial cancellation occurs, the typical behavior of r is $r(\tilde{R}) \sim \tilde{R}/(\mathcal{O}(1) + \tilde{R})$. Here the upper bound on the inflation scale H_* comes from the condition that the amplitude of tensor mode should not be too large. The typical behavior of this constraint is the following;

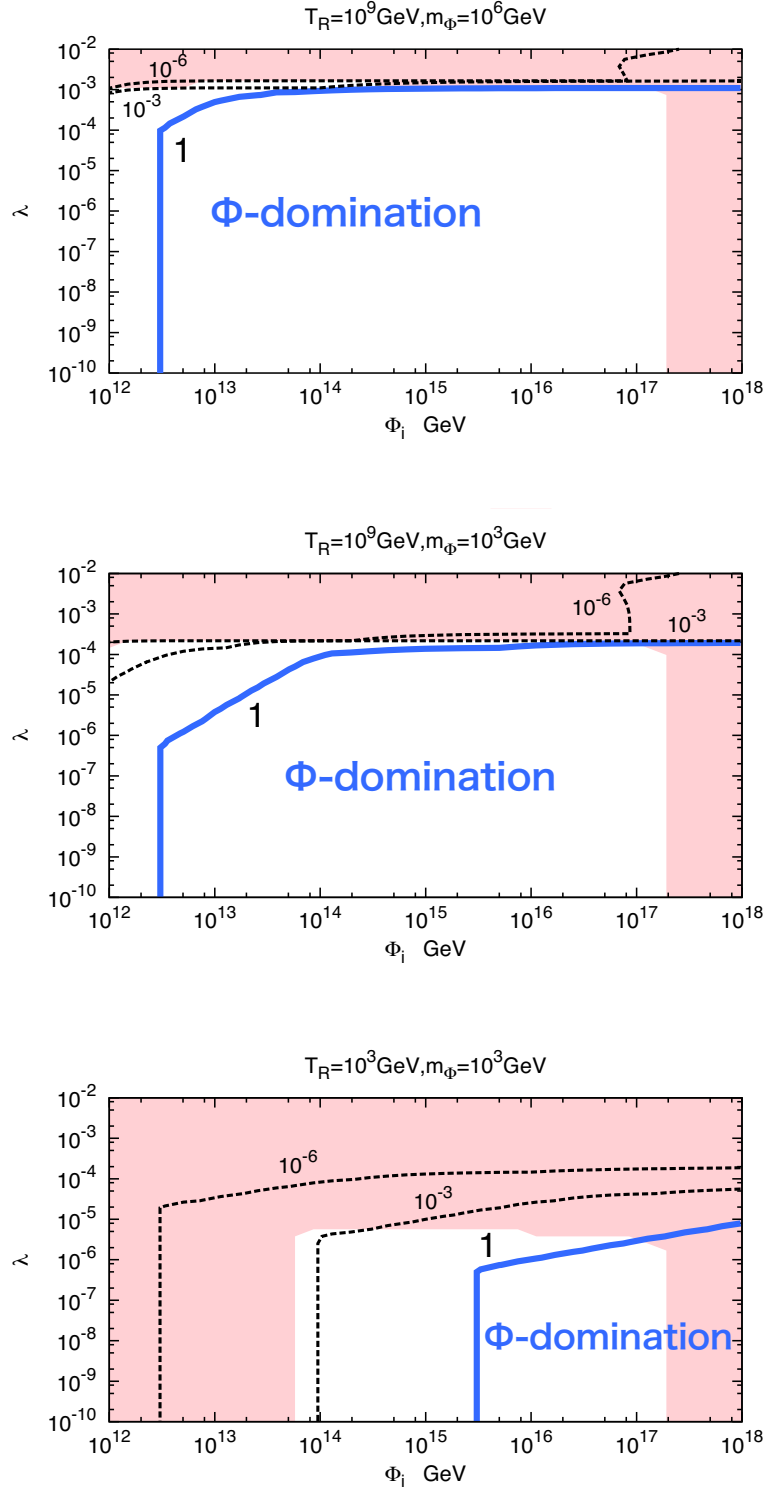


Figure 5.4: Contours of \tilde{R} on (ϕ_i, λ) plane. We take $(m_\phi, T_R) = (10^3 \text{ TeV}, 10^9 \text{ GeV})$ [**Top**], $(m_\phi, T_R) = (1 \text{ TeV}, 10^9 \text{ GeV})$ [**Middle**], $(m_\phi, T_R) = (1 \text{ TeV}, 10^3 \text{ GeV})$ [**Bottom**]. In the **Pink** shaded region, the condition (5.66) is violated.

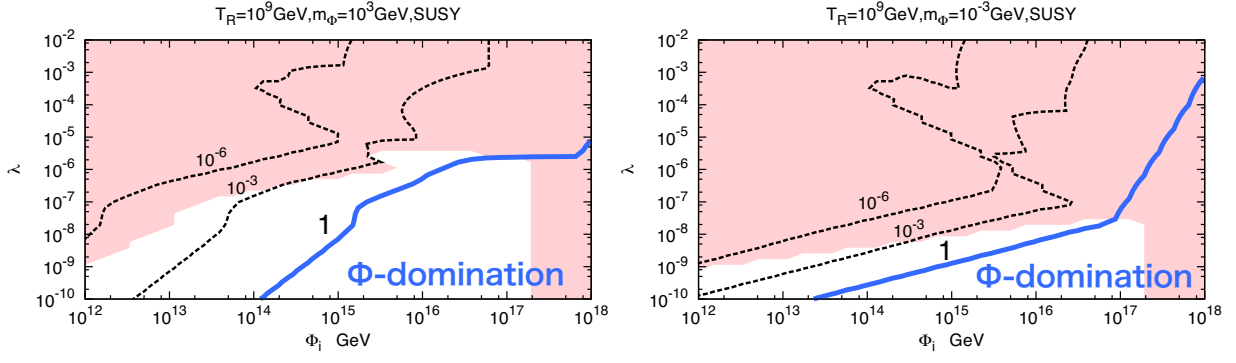


Figure 5.5: Contours of \tilde{R} on (ϕ_i, λ) plane in SUSY case $\lambda = y$. We take $(m_\phi, T_R) = (1 \text{ TeV}, 10^9 \text{ GeV})$ (left) and $(10^{-6} \text{ TeV}, 10^9 \text{ GeV})$ (right). In the **Pink** shaded region, the condition (5.66) is violated.

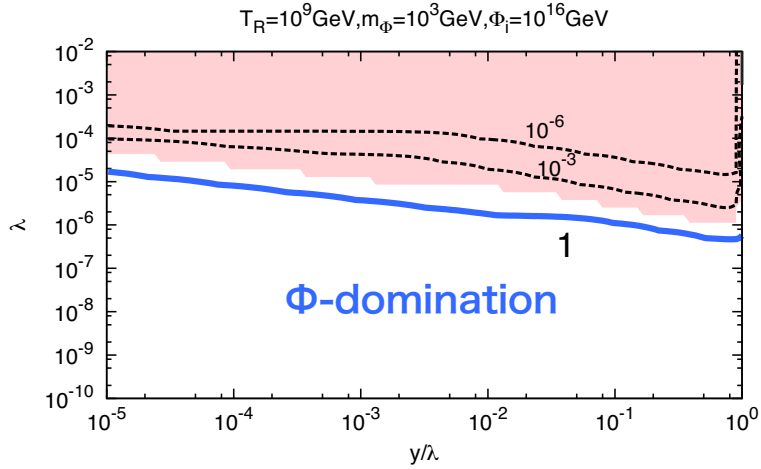


Figure 5.6: Contours of \tilde{R} on $(\lambda, y/\lambda)$ plane with $\phi_i = 10^{16} \text{ GeV}$. We have taken $(m_\phi, T_R) = (1 \text{ TeV}, 10^9 \text{ GeV})$.

$\phi_i \lesssim \mathcal{O}(10^{18}) \text{ GeV}$ for $\tilde{R} > 1$ and $\phi_i \lesssim \tilde{R} \times \mathcal{O}(10^{18})$ for $\tilde{R} < 1$. On the other hand, H_* must be larger than H_{os} , which is the Hubble parameter at the onset of ϕ oscillation, otherwise ϕ starts to oscillate during or before the inflation. Thus, we impose a following condition:

$$H_{\text{os}} \lesssim \mathcal{O}(10^{-4}) \times \phi_i \frac{1}{r(\tilde{R})}. \quad (5.67)$$

We have checked these conditions (5.66) and (5.67) in the numerical study.

First, we consider the suppressed Yukawa case, $\lambda \gg y \sim 0$. In such a case, the scalar condensation ϕ cannot completely disappear if the coupling λ is smaller than the critical value λ_c because Z_2 -symmetry forbids the perturbative decay of ϕ . Therefore, we assume non-zero $\Gamma_\phi^{\text{higher}}$ for ϕ to obtain a small but nonzero perturbative decay rate (very small y is equivalent). In order to see the typical situation, we assume the following form $\Gamma_\phi^{\text{higher}} \sim m_\phi^3/M_*^2$ with a decay temperature $T_{\text{dec}} \sim \mathcal{O}(1) \text{ MeV}$, which indicates

$$\Gamma_\phi^{\text{higher}} = \sqrt{\frac{g_* \pi^2 T_{\text{dec}}^4}{90 M_{\text{pl}}^2}}. \quad (5.68)$$

Here M_* denotes a cutoff scale. This ensures that ϕ condensation decays before BBN. Thus, for small λ in which ϕ condensation cannot be dissipated away, our calculation gives the upper bound on \tilde{R} . We take $\alpha = 0.05$ in our numerical computation. This set up is close to the minimal Higgs curvaton model [163]. The remaining parameters are the reheating temperature of the universe T_R , the tree-level mass m_ϕ and the coupling constant λ .

Fig. 5.4 shows a contour plot of \tilde{R} on (ϕ_i, λ) -plane. We take $(m_\phi, T_R) = (10^3 \text{ TeV}, 10^9 \text{ GeV})$ [top], $(m_\phi, T_R) = (1 \text{ TeV}, 10^9 \text{ GeV})$ [middle], and $(m_\phi, T_R) = (1 \text{ TeV}, 10^3 \text{ GeV})$ [bottom]. Inside the pink shaded region, the condition (5.66) is violated. One can see that $\tilde{R} \sim 1$ can be realized just below the line of $\lambda \sim \lambda_c$. This fact is easily understood because for $\lambda > \lambda_c$, the condensation ϕ is dissipated and \tilde{R} becomes suppressed, while for $\lambda \ll \lambda_c$, the condensation survives until it decays via the higher dimensional term. Therefore, the line of $\tilde{R} \sim 1$ exists just below $\lambda = \lambda_c$. The difference between top and middle panels mainly comes from the position of the line $\lambda = \lambda_c$. For smaller T_R , ϕ oscillates in the inflaton dominant era for a long time and \tilde{R} tends to be smaller.

Now we consider more general case, *i.e.* $y \neq 0$. If we set $\lambda \sim y$, the quartic potential of ϕ coming from the CW correction is suppressed. Fig. 5.5 shows contours of \tilde{R} in the case of vanishing CW potential. We take $(m_\phi, T_R) = (1 \text{ TeV}, 10^9 \text{ GeV})$ [left] and $(10^{-6} \text{ TeV}, 10^9 \text{ GeV})$ [right]. In the pink shaded region, the condition (5.66) is violated. Since the Yukawa coupling induces the earlier dissipation/decay compared with the $y = 0$ case, the energy fraction \tilde{R} tends to be smaller. Hence the constraints become severer. The contours of \tilde{R} in the figure are relatively curved in the upper side and tend to have large values compared with the previous case $y = 0$. This is because thermal potential is more likely to affect the dynamics of the condensation of ϕ in this case and because the absence of four point self interaction delays the beginning of oscillation, respectively. As in the previous case, above the critical coupling y_c , the curvaton dissipates its energy thermally.

Fig. 5.6 indicates contour of \tilde{R} for a general set of (λ, y) with $\phi_i = 10^{16} \text{ GeV}$. We take $(m_\phi, T_R) = (1 \text{ TeV}, 10^9 \text{ GeV})$. Similar to the case of $y = 0$, the line of $\tilde{R} = 1$ lies a bit below $\lambda = \lambda_c$. This figure indicates that the effect of CW potential vanishes at the vicinity of $y \sim \lambda$.

Chapter 6

Conclusions and Discussion

In this thesis, we have investigated the dynamics of scalar condensation in the early Universe; with special emphasis on the interplay between the scalar condensate and particle-like excitations around it. For this purpose, employing the CTP formalism to follow the dynamics of correlators, we have obtained coarse-grained equations of motion under the *separation of time scale* assumptions that are often satisfied in the cosmological setup. In other words, in a single framework, we have shown that, owing to interactions of the scalar condensate with abundant particles, the following effects should be taken into account appropriately;

- corrections to the effective potential from abundant particles,
- corrections to the dissipation rate from abundant particles,
- non-perturbative particle production if the scalar condensate acts as a non-adiabatic background.

Importantly, these effects drastically change the dynamics of scalar condensates;

- its beginning time of oscillation which is determined by $H^2 \simeq 2\partial V_{\text{eff}}(\phi; T)/\partial \phi^2$ is altered [See Fig. 4.3 and Eqs. (4.76)–(4.80)],
- its equation of state is significantly modified [Eqs. (4.194)–(4.197)],
- its decay time is also changed $\Gamma_{\phi}^{\text{eff}}(\tilde{\phi}; T) \simeq H$ [Eqs. (4.201)–(4.211)].

Then, we have revisited several important roles of scalar condensates in the early Universe in order to clarify their impacts on cosmology. Comments are given in turn for each roles of scalar condensation.

6.1 Reheating after Inflation

We have studied issues of reheating and thermalization after inflation. In particular, we have shown that if the inflaton is relatively light and its coupling to light fields is not suppressed (unlike the Planck-suppressed operator), then the standard picture of reheating where the perturbative decay of the inflaton dominates the process of reheating does not hold. We have seen that in such a case the following two effects play crucial roles; the production of quite high temperature plasma at the preheating stage and the subsequent dissipation due to interactions with the plasma. Roughly speaking, they become important for $m_{\phi} \lesssim \alpha^2 \tilde{\lambda}^2 M_{\text{pl}}$ [$\leftrightarrow m_{\phi}/(10^{13} \text{ GeV}) \lesssim (10\tilde{\lambda})^2$]. As a result, the reheating temperature, which characterizes

the transition time from inflaton oscillation dominant era to radiation dominant era, can be dramatically changed [See also Eq. (5.25), Fig. 5.2 and Fig. 5.3]:

$$T_R \sim \begin{cases} c_0^{1/2} \left(\frac{90}{\pi^2 g_*} \right)^{1/4} \left(\frac{1}{2\pi^4 |\epsilon g|} \right)^{1/2} \sqrt{\tilde{\lambda}^2 m_\phi M_{\text{pl}}} & \cdots (0) \\ c_1^{1/2} \left(\frac{30\sqrt{3}}{\pi^2 g_* \alpha} \right)^{1/2} \sqrt{\tilde{\lambda} m_\phi M_{\text{pl}}} & \cdots (i) \\ c_2 \left(\frac{90}{\pi^2 g_*} \right)^{1/2} \alpha \tilde{\lambda}^2 M_{\text{pl}} & \cdots (ii) \\ c_2^{1/2} \left(\frac{90}{\pi^2 g_*} \right)^{1/4} \sqrt{\tilde{\lambda}^2 m_\phi M_{\text{pl}}} & \cdots (iii) \end{cases} \quad (6.1)$$

for the reheating via (0) $\Gamma_\phi^{\text{eff}} \sim \Gamma_\phi^{\text{NP}}$, (i) $\Gamma_\phi^{\text{eff}} \sim \tilde{\lambda} T^2 / (\alpha \tilde{\phi})$, (ii) $\Gamma_\phi^{\text{eff}} \sim \tilde{\lambda}^2 \alpha T$, and (iii) $\Gamma_\phi^{\text{eff}} \sim \tilde{\lambda}^2 m_\phi$.

We comment on in what kinds of models these effects might play important roles. Basically, a class of inflaton models with a small inflaton mass and with a renormalizable interaction to radiation is relevant. Below, we list several attractive models of inflaton within this class and explicitly write down the dominant interactions which connect the inflaton to radiation. One may roughly estimate the reheating temperature for these models by using the above formulas by plugging the typical coupling (between the inflaton and light particles) and the inflaton mass. However, it should be noted that the reheating temperature can depend on how to flatten the potential of inflaton because the effective mass, m_ϕ , in the inflaton oscillation era can vary during the course of dynamics. Hence, one has to study separately in each cases for further investigation.

- The singlet DM inflaton model [151–153] exactly shares essential features of our phenomenological setup of Eq. (4.2) with $y = 0$; where the singlet Z_2 -odd scalar field plays the roles of inflaton and DM simultaneously. The interaction between the singlet and light particles is caused by

$$\mathcal{L}_{\text{int}} = -\frac{1}{2} \lambda^2 \phi^2 |H|^2, \quad (6.2)$$

where ϕ is a singlet Z_2 -odd scalar field and H is the Higgs field, which has the large top Yukawa coupling and thus can decay immediately at the preheating stage. As already explained, in this case, the inflaton condensate completely dissipates its energy at some time and the inflaton particles participate in the thermal plasma. Later, the number density of inflaton particles freezes out and they can be the present DM. To account for the present DM abundance, its mass is typically weak scale and the coupling should not be suppressed; there are two regimes: $\lambda^2 \sim \mathcal{O}(10^{-3})$ for $m_\phi \sim m_h/2$ and $\lambda^2 \sim \mathcal{O}(1)$ for $m_\phi \sim \mathcal{O}(1)$ TeV, with m_h being the SM Higgs mass (See [164] for instance).

- Another interesting example is the Higgs inflation; where the SM Higgs field acts as the inflaton [133, 165]. The relevant interaction terms for reheating are given by

$$\mathcal{L}_{\text{int}} = -\frac{y_t}{\sqrt{2}} \phi \bar{t} t + \frac{g_2^2}{4} \phi^2 W_\mu^+ W^{\mu-} + \frac{g_Y^2 + g_2^2}{8} \phi^2 Z_\mu Z^\mu, \quad (6.3)$$

where y_t is the top-Yukawa coupling, $g_{Y/2}$ represents gauge coupling of $U(1)_Y/SU(2)_W$, and ϕ, t, W, Z denote the SM Higgs, top quark, W-boson and Z-boson respectively. Notice

that these couplings should be evaluated at the energy scale of reheating. Though one should treat the background plasma carefully since the electro-weak symmetry $U(1)_Y \times SU(2)_W$ is broken during the inflaton oscillation regime, still the final process of reheating cannot be understood without taking the dissipation effect into account, as studied in this paper.

- In the context of leptogenesis, the right-handed sneutrino inflation [13, 14] is an attractive candidate and also within a class of our setup. The superpotential is given by

$$W = \frac{1}{2} M_i N_i N_i + y_{i\alpha}^N N_i L_\alpha H_u + W_{\text{MSSM}}, \quad (6.4)$$

where N_i ($i = 1, 2, 3$), L_α ($\alpha = e, \mu, \tau$) and H_u represent the right-handed neutrinos, lepton doublets and up-type Higgs superfields respectively. W_{MSSM} indicates the MSSM superpotential. Here we take the basis where the mass matrix of right-handed neutrinos, M_i , becomes diagonal and real. To account for the tiny neutrino masses, the typical value of the Yukawa coupling is $y^N \sim \mathcal{O}(0.1)$ for $M \sim \mathcal{O}(10^{13})$ GeV. This superpotential encodes the following interaction terms which may play important roles in the dynamics of reheating:^{♦1}

$$\mathcal{L}_{\text{int}} = -y_{i\alpha}^N \tilde{N}_i L_\alpha \tilde{H}_u - |y_{i\alpha}^N \tilde{N}_i H_u|^2 - |y_{i\alpha}^N \tilde{N}_i \tilde{L}_\alpha|^2 - (y_{i\alpha}^N M_i \tilde{N}_i^* \tilde{L}_\alpha H_u + \text{h.c.}). \quad (6.5)$$

Roughly speaking, if the sneutrino condensate thermally dissipates its energy completely, then the “thermal” leptogenesis takes place since it participates in the thermal plasma; if this is not the case, “non-thermal” leptogenesis takes place where the lepton-asymmetry is completely determined by the non-equilibrium decay of right-handed sneutrino condensate.^{♦2}

- The MSSM inflation [166] is also an interesting possibility where the MSSM flat direction plays the role of inflaton. The relevant interaction terms during the dynamics of reheating depend on the choice of the flat direction (*e.g.* LLE , UDD). Since its expectation value (partly) breaks the SM gauge group, it couples with sfermions via D-terms, with gauge bosons via their masses, and with fermions via Yukawa couplings. Notice that these interactions are controlled by the gauge couplings of SM gauge group (See Ref. [167]). Its dynamics after inflation is partly studied in Ref. [167], but the final reheating might be caused by interactions with the thermal plasma. Similar results might hold for the alchemical inflation scenario [168], in which the inflaton turns into the MSSM flat direction after the inflation and then the flat direction oscillates around the origin with a mass of soft SUSY breaking scale.
- Though we have stuck to the simple phenomenological model [Eq. (4.2)], our methods can be also applied to the case where the inflaton oscillates around a VEV; that is, it interacts with the thermal plasma via a higher-dimensional operator. Hence, it will also be useful for reheating after a class of thermal inflation model [169]. For a larger VEV ($\sim M_{\text{pl}}$), these effects become smaller, but it is non-trivial for an intermediate scale (*c.f.* PQ phase transition [45, 119]).

As we have seen, the reheating temperature and the reheating dynamics before the complete dissipation of inflaton can significantly differ from that of conventional one due to the

^{♦1} Here we omit the cross terms including the top-Yukawa coupling.

^{♦2} Also, the dynamics of sneutrino condensate other than the inflaton (*e.g.* curvaton) is interesting [15–20].

effects of interactions with radiation. Obviously, since the value of reheating temperature differs from the conventional one, the abundance that depends on the reheating temperature can be drastically modified. In addition, since the evolution of preexisting radiation before the completion of reheating is also modified, the abundance of heavy particles which are produced thermally at $T > T_R$ is altered. Therefore, thermal particle production at that stage, such as gravitino, right-handed neutrino and heavy DM, can differ from that of conventional arguments.

6.2 Curvaton

We have studied the dynamics of curvaton model which falls into Eq. (4.2) in detail, and shown that its dynamics can be drastically modified due to interactions with the thermal plasma; in particular, its energy fraction of the curvaton at its complete decay/dissipation, \tilde{R} , can be significantly changed, which is an important parameter to determine the curvature perturbation generated by the curvaton. In addition, we have also shown that the curvature perturbation has the following additional contributions compared with the simple case with a quadratic potential and constant decay rate:

- The beginning of oscillation may depend on its initial field value ϕ_i ,
- The equation of state can change and its transition time may depend on the amplitude of ϕ
- The decay/dissipation time may also depend on the amplitude of ϕ .

We have derived the formulas for the power spectrum and the non-linearity parameter taking account of all these effects listed above [Eqs. (5.60)–(5.64)].

Then, taking all these effects into account, we have examined the viability of the curvaton scenario where the observed curvature perturbation is dominantly produced by the curvaton. From these equations, we can see that the power spectrum is roughly given by $\mathcal{P}_\zeta \propto [H_* r(\tilde{R})/\phi_i]^2$ with $r(\tilde{R}) \sim \tilde{R}/(\mathcal{O}(1) + \tilde{R})$ unless non-trivial cancellations occur. The non-discovery of primordial tensor perturbations constrains the parameter space as $\phi_i/r(\tilde{R}) \lesssim \mathcal{O}(10^{17})$ GeV, whose typical behavior is $\phi_i \lesssim \mathcal{O}(10^{18})$ GeV for $\tilde{R} > 1$ and $\phi_i \lesssim \tilde{R} \times \mathcal{O}(10^{18})$ for $\tilde{R} < 1$. Also, the non-linearity parameter roughly behaves as $f_{\text{NL}} \propto 1/\tilde{R}$ for $\tilde{R} < 1$ unless non-trivial cancellations occur, which implies the following lower bound on the energy fraction; $\tilde{R} \gtrsim \mathcal{O}(0.1)$. Typically, the interaction with the thermal plasma makes the beginning of oscillation and the complete dissipation earlier. And thus it tends to decrease the value of \tilde{R} and narrow down the allowed parameter region. To demonstrate it explicitly, we have computed the energy fraction \tilde{R} as a function of $\tilde{\lambda}$ and ϕ_i for two typical cases; $\lambda \gg y \sim 0$ and $\lambda \sim y > 0$. In particular, we have found the upper-bound on the coupling between the curvaton and light fields; $y \lesssim y_c$ and $\lambda \lesssim \lambda_c$ with $y_c = [m_\phi/(\alpha M_{\text{pl}})]^{1/2}$ and $\lambda_c = [m_\phi/M_{\text{pl}}]^{1/4}$. Otherwise, the energy fraction becomes too small and it contradicts with the constraint on the amplitude of tensor perturbation.

6.3 Discussion on Other Mechanisms

Though we have focused on the above roles of scalar condensation in the early Universe in this thesis, our analysis has relevance to other mechanisms. For instance, in the case of Affleck-Dine baryogenesis [10], it was already shown in Refs. [120–122] that efficiencies of baryogenesis is

modified by thermal corrections to the effective potential. This is because the baryon/lepton number density is fixed at the beginning time of oscillation, but it is changed drastically by them [See also Fig. 4.3 and Eqs. (4.76)–(4.80)]. After the onset of oscillation, the Affleck-Dine condensate may fragment into localized objects so-called Q-ball if the potential is shallower than the quadratic, and the subsequent evolution of the Universe is dramatically altered if it is formed [170–180]. As a next step, it is interesting to study how the dissipative effects caused by interactions with the background plasma can affect the formation of Q-ball.

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Appendix A

Notation and Conventions

Notation and conventions used throughout this thesis are summarized.

A.1 Units

We adopt natural units where the speed of light c , the reduced Planck constant \hbar and the Boltzmann constant k_B are taken to be unity. We do not use the Planck units (the special natural units) where the Newton constant G is also taken to be unity. Instead, we explicitly write down the reduced Planck mass $M_{\text{pl}} \simeq 2.4 \times 10^{18} \text{ GeV}$.

A.2 Metric

We use the $(+, -, -, -)$ convention:

$$(\eta_{\mu\nu}) = \text{diag}(1, -1, -1, -1). \quad (\text{A.1})$$

A.3 Propagators

Let us consider a system which is described by a density matrix $\hat{\rho}$ in the Heisenberg picture.^{♦1} Two basic propagators are defined as

$$G_{>}(x, y) \equiv \langle \hat{\phi}(x) \hat{\phi}(y) \rangle_{\text{con}}, \quad (\text{A.3})$$

$$G_{<}(x, y) \equiv (-)^{|\phi|} \langle \hat{\phi}(y) \hat{\phi}(x) \rangle_{\text{con}}, \quad (\text{A.4})$$

where the subscript “con” denotes the connected part and $\langle \bullet \rangle \equiv \text{Tr}[\bullet]$ and $|\phi| = 0, 1$ for bosonic/fermionic field respectively. For brevity we drop “con” in the following. Here possible internal degree of freedom is implicit: if there are several species ($i = 1, 2, \dots, n$), x should be replaced with a set $\{x, i\}$. Note that all the propagators which we introduce in the following can be expressed in terms of these two propagators.

^{♦1} In the Schrödinger picture a density matrix obeys the von-Neumann Liouville Eq.:

$$i\partial_t \hat{\rho}(t) = [\hat{H}, \hat{\rho}(t)], \quad (\text{A.2})$$

and hence it can be understood as an initial condition $\hat{\rho}(t_{\text{ini}}) = \hat{\rho}$.

As explained in Sec. 3.1, a path integral on the closed time path (CTP) contour \mathcal{C} is useful in studying the evolution of expectation values of some operators. Hence it is convenient to define a propagator on the CTP contour, so-called Schwinger-Keldysh propagator:

$$G(x, y) \equiv \langle T_{\mathcal{C}} \hat{\phi}(x) \hat{\phi}(y) \rangle = \theta_{\mathcal{C}}(x^0, y^0) G_{>}(x, y) + \theta_{\mathcal{C}}(y^0, x^0) G_{<}(x, y), \quad (\text{A.5})$$

where $\theta_{\mathcal{C}}$ denotes a step function defined on the contour \mathcal{C} . It is instructive to clarify the relation of the Schwinger-Keldysh propagator with other common ones:

$$G(x, y) = \begin{cases} G_{\text{Fyn}}(x, y) & \text{for } x^0 \in \mathcal{C}_+, y^0 \in \mathcal{C}_+ \\ G_{>}(x, y) & \text{for } x^0 \in \mathcal{C}_-, y^0 \in \mathcal{C}_+ \\ G_{<}(x, y) & \text{for } x^0 \in \mathcal{C}_+, y^0 \in \mathcal{C}_- \\ G_{\text{Dys}}(x, y) & \text{for } x^0 \in \mathcal{C}_-, y^0 \in \mathcal{C}_- \end{cases}, \quad (\text{A.6})$$

where other common propagators, which are frequently used in the in-out formalism, are given by

$$\text{Feynmann Propagator : } G_{\text{Fyn}}(x, y) \equiv \theta(x^0 - y^0) G_{>}(x, y) + \theta(y^0 - x^0) G_{<}(x, y), \quad (\text{A.7})$$

$$\text{Dyson Propagator : } G_{\text{Dys}}(x, y) \equiv \theta(y^0 - x^0) G_{>}(x, y) + \theta(x^0 - y^0) G_{<}(x, y). \quad (\text{A.8})$$

In some cases^{♦2} it makes their physical meanings clearer to express the above propagators in terms of following two propagators:

$$\text{Hadamard Propagator : } G_{\text{H}}(x, y) \equiv \langle [\hat{\phi}(x), \hat{\phi}(y)]_+ \rangle = G_{>}(x, y) + G_{<}(x, y), \quad (\text{A.9})$$

$$\text{Jordan Propagator : } G_{\text{J}}(x, y) \equiv \langle [\hat{\phi}(x), \hat{\phi}(y)]_- \rangle = G_{>}(x, y) - G_{<}(x, y), \quad (\text{A.10})$$

with $[\bullet, \bullet]_{\pm}$ being a commutator/anti-commutator, which are defined as

$$[A, B]_{\pm} \equiv AB \pm (-)^{|A||B|} BA. \quad (\text{A.11})$$

These two propagators are also known as the “statistical/spectral function” and encode the number/spectrum of quasi-particle excitation respectively. The Schwinger-Keldysh propagator can be expressed as

$$G(x, y) = \frac{1}{2} [G_{\text{H}}(x, y) + \text{sgn}_{\mathcal{C}}(x^0, y^0) G_{\text{J}}(x, y)], \quad (\text{A.12})$$

where $\text{sgn}_{\mathcal{C}}$ denotes a sign function defined on the contour \mathcal{C} .^{♦3} We sometimes use the retarded and advanced propagators defined as follows:

$$G_{\text{ret}}(x, y) = i\theta(x_0 - y_0) G_{\text{J}}(x, y), \quad (\text{A.14})$$

$$G_{\text{adv}}(x, y) = -i\theta(y_0 - x_0) G_{\text{J}}(x, y). \quad (\text{A.15})$$

^{♦2} e.g., if the quasi-particle picture is valid

^{♦3} In some literature, the following convention of statistical/spectral function is used

$$F(x, y) \equiv \frac{1}{2} G_{\text{H}}(x, y); \quad \rho(x, y) \equiv i G_{\text{J}}(x, y). \quad (\text{A.13})$$

Appendix B

Basic Ingredients of Thermal Field Theory

Here we summarize basic ingredients of thermal field theory that are frequently used in this thesis. See [86, 181] for details. If a system is close to thermal equilibrium, at least lower order correlators can be well approximated with ones computed in terms of the Gibbs-state:

$$\hat{\rho}_{\text{eq}} = \frac{e^{-\beta(\hat{H} - \sum_i \mu_i \hat{Q}_i)}}{Z}; \quad Z = \text{Tr} \left[e^{-\beta(\hat{H} - \sum_i \mu_i \hat{Q}_i)} \right], \quad (\text{B.1})$$

where Q_i represents possible conserved charges and β is the inverse temperature $1/T$. Since it obviously commutes with the Hamiltonian and $e^{-\beta\hat{H}}$ can be interpreted as a complex time evolution operator, Green functions have interesting properties as we see in the following. From cosmological point of view, the chemical potential for Baryon and Lepton asymmetries is much smaller than the background temperature, and hence we can use the simple canonical ensemble in most cases. Thus, in the following, we summarize basic relations for the canonical ensemble neglecting μ .

B.1 Kubo-Martin-Schwinger Relation

As mentioned above, since the canonical ensemble,

$$\hat{\rho}_{\text{eq}} = \frac{e^{-\beta\hat{H}}}{Z}; \quad Z = \text{Tr} \left[e^{-\beta\hat{H}} \right], \quad (\text{B.2})$$

commutes with the Hamiltonian \hat{H} and the translation operator \hat{P} , all the Green functions depend only on the difference of space time:

$$G_{\bullet}(x, y) = G_{\bullet}(x - y). \quad (\text{B.3})$$

Therefore it is convenient to perform the Fourier transform

$$G_{\bullet}(x) \xrightarrow{\text{Fourier tr.}} G_{\bullet}(P). \quad (\text{B.4})$$

Also the canonical ensemble can be understood as a complex time evolution operator, and hence there is a relation between two basic Green functions:

$$G_{>}(x^0, \mathbf{x}) = (-)^{|\phi|} G_{<}(x^0 + i\beta, \mathbf{x}) \xrightarrow{\text{Fourier tr.}} G_{>}(P) = (-)^{|\phi|} e^{\beta p^0} G_{<}(P), \quad (\text{B.5})$$

which is the so-called ‘‘Kubo-Martin-Schwinger’’ (KMS) relation [83, 84]. As it connects two independent propagators, all the propagators can be expressed by a single propagator. We

choose the Jordan propagator (spectral function) as this role and call its Fourier transform the spectral density, which is denoted as

$$\rho(P) \equiv G_J(P). \quad (\text{B.6})$$

Then the other propagators can be obtained as

$$G_>(P) = (1 \pm f_{\text{B/F}}(p_0)) \rho(P), \quad (\text{B.7})$$

$$G_<(P) = \pm f_{\text{B/F}}(p_0) \rho(P), \quad (\text{B.8})$$

$$G_{\text{Fyn}}(P) = \int \frac{dk_0}{2\pi} \frac{\rho(k_0, \mathbf{p})}{ik_0 - ip_0 + \epsilon} \pm f_{\text{B/F}}(p_0) \rho(P), \quad (\text{B.9})$$

$$G_{\text{Dys}}(P) = \int \frac{dk_0}{2\pi} \frac{\rho(k_0, \mathbf{p})}{ip_0 - ik_0 + \epsilon} \pm f_{\text{B/F}}(p_0) \rho(P), \quad (\text{B.10})$$

$$G_{\text{H}}(P) = (1 \pm 2f_{\text{B/F}}(p_0)) \rho(P), \quad (\text{B.11})$$

for $|\phi| = 0, 1$ respectively. Here the Bose-Einstein/Fermi-Dirac distribution is denoted as $f_{\text{B/F}}(p_0)$.

B.2 Real and Imaginary Time Formalism

As explained in the previous chapter, the generating functional for sources J_1, J_2, \dots, J_n is given by

$$\begin{aligned} Z[\{J\}; \rho] &\equiv \text{Tr} \left[\hat{\rho} T_{\mathcal{C}} \exp \left(i \sum_m \frac{1}{m!} \int_{\mathcal{C}} d^4x_1 \cdots d^4x_m J_m(x_1, \dots, x_m) \hat{\phi}(x_1) \cdots \hat{\phi}(x_m) \right) \right] \quad (\text{B.12}) \\ &= \int d\varphi_+ d\varphi_- \langle \varphi_+ | \hat{\rho} | \varphi_- \rangle \int_{\varphi_+}^{\varphi_-} \mathcal{D}\varphi \\ &\quad \exp \left(i \int_{\mathcal{C}} d^4x \mathcal{L}(x) + i \sum_m \frac{1}{m!} \int_{\mathcal{C}} d^4x_1 \cdots d^4x_m J_m(x_1, \dots, x_m) \varphi(x_1) \cdots \varphi(x_m) \right) \end{aligned} \quad (\text{B.13})$$

For a thermal equilibrium system, one can proceed further because the canonical ensemble can be interpreted as the complex time evolution operator:

$$\langle \varphi_+ | \hat{\rho}_{\text{eq}} | \varphi_- \rangle = \int_{\varphi_-}^{\varphi_+} \mathcal{D}\phi \exp \left(i \int_{\mathcal{C}_\beta} d^4x \mathcal{L}(x) \right), \quad (\text{B.14})$$

where \mathcal{C}_β denotes the Matsubara contour. Therefore it is convenient to extend the integral domain of sources as

$$Z_\beta[\{J\}] \equiv \text{Tr} \left[\hat{\rho}_{\text{eq}} T_{\mathcal{C} + \mathcal{C}_\beta} \exp \left(i \sum_m \frac{1}{m!} \int_{\mathcal{C} + \mathcal{C}_\beta} d^4x_1 \cdots d^4x_m J_m(x_1, \dots, x_m) \hat{\phi}(x_1) \cdots \hat{\phi}(x_m) \right) \right] \quad (\text{B.15})$$

$$= \int_{[\text{anti-}] \text{periodic}} \mathcal{D}\varphi \exp \left(i \int_{\mathcal{C} + \mathcal{C}_\beta} d^4x \mathcal{L}(x) + i \sum_m \frac{1}{m!} \int_{\mathcal{C} + \mathcal{C}_\beta} d^4x_1 \cdots d^4x_m J_m(x_1, \dots, x_m) \varphi(x_1) \cdots \varphi(x_m) \right) \quad (\text{B.16})$$

where the path integral is performed under the condition $\varphi(0_+, \mathbf{x}) = (-)^{|\phi|} \varphi(-i\beta, \mathbf{x})$ for $|\phi| = [1], 0$.

There are well known two conventions to construct the thermal field theory.^{♦1} If all the source terms lie on the contour \mathcal{C} , then all the Green functions have real time arguments, which results in so-called “real time formalism (RTF)” [75–77]. On the other hand, if all the sources lie solely on the contour \mathcal{C}_β (which implies $t_f = 0$), then all the Green functions have imaginary time arguments, which is so-called “imaginary time formalism (ITF)” [85].

The Schwinger-Keldysh propagator generated from this functional becomes

$$G(x, y) = \langle T_{\mathcal{C}+\mathcal{C}_\beta} \hat{\phi}(x) \hat{\phi}(y) \rangle. \quad (\text{B.17})$$

To see the relation between two formulation, the following expression of Shwinger-Keldysh propagator is useful

$$G(x, y) = \int \frac{d^4 P}{(2\pi)^4} e^{-iP \cdot (x-y)} \left[\theta_{\mathcal{C}+\mathcal{C}_\beta}(x_0, y_0) \pm f_{\text{B/F}}(p_0) \right] \rho(P), \quad (\text{B.18})$$

for $|\phi| = 0, 1$. Here we use the KMS relation. On the one hand, if one chooses both x_0, y_0 on the contour \mathcal{C} , then the Schwinger-Keldysh propagator encodes the following ones as can be seen from Eq. (A.6)

$$G^{(\text{RTF})}(x) = \begin{cases} \int \frac{d^4 P}{(2\pi)^4} e^{-iP \cdot x} \left[\int \frac{dk_0}{2\pi} \frac{\rho(k_0, \mathbf{p})}{ik_0 - i\epsilon} \pm f_{\text{B/F}}(p_0) \rho(P) \right] & \text{for } x^0 \in \mathcal{C}_+, y^0 \in \mathcal{C}_+ \\ \int \frac{d^4 P}{(2\pi)^4} e^{-iP \cdot x} [1 \pm f_{\text{B/F}}(p_0)] \rho(P) & \text{for } x^0 \in \mathcal{C}_-, y^0 \in \mathcal{C}_+ \\ \int \frac{d^4 P}{(2\pi)^4} e^{-iP \cdot x} \pm f_{\text{B/F}}(p_0) \rho(P) & \text{for } x^0 \in \mathcal{C}_+, y^0 \in \mathcal{C}_- \\ \int \frac{d^4 P}{(2\pi)^4} e^{-iP \cdot x} \left[\int \frac{dk_0}{2\pi} \frac{\rho(k_0, \mathbf{p})}{ip_0 - ik_0 + \epsilon} \pm f_{\text{B/F}}(p_0) \rho(P) \right] & \text{for } x^0 \in \mathcal{C}_-, y^0 \in \mathcal{C}_- \end{cases}, \quad (\text{B.19})$$

for $|\phi| = 0, 1$. On the other hand, for $x_0, y_0 \in \mathcal{C}_\beta$, one finds the following periodicity/anti-periodicity condition of the Schwinger-Keldysh propagator for ITF:

$$G^{(\text{ITF})}(x_0, \mathbf{x}) = \int \frac{d^4 P}{(2\pi)^4} e^{-iP \cdot x} [1 \pm f_{\text{B/F}}(p_0)] \rho(P) \quad (\text{B.20})$$

$$= \int \frac{d^4 P}{(2\pi)^4} e^{-ip_0(x_0 + i\beta) + i\mathbf{p} \cdot \mathbf{x}} f_{\text{B/F}}(p_0) \rho(P) \quad (\text{B.21})$$

$$= \pm G^{(\text{ITF})}(x_0 + i\beta, \mathbf{x}), \quad (\text{B.22})$$

for $|\phi| = 0, 1$. Therefore it can be expanded as a Fourier series:

$$G^{(\text{ITF})}(x) = i \frac{1}{\beta} \sum_n e^{-i\omega_n x_0} \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p} \cdot \mathbf{x}} G^{(\text{ITF})}(\omega_n, \mathbf{p}); \quad (\text{B.23})$$

$$\omega_n = \frac{\pi}{\beta} \begin{cases} 2n & \text{for } |\phi| = 0 \\ 2n + 1 & \text{for } |\phi| = 1 \end{cases}. \quad (\text{B.24})$$

Obviously Schwinger-Keldysh propagators in both RTF and ITF can be constructed if one knows the spectral density ρ . Let us consider the other way around: how to obtain the spectral density ρ from the Schwinger-Keldysh propagator. In RTF it is rather trivial: since the

^{♦1} Of course, it is possible to deform the contour so there exists many other conventions in general. One of these is the Umezawa formalism, which is also known as the “*Thermo-Field-Dynamics*” [182]

Schwinger-Keldysh propagator for $x_0 \in \mathcal{C}_\pm, y_0 \in \mathcal{C}_\mp$ results in $G_{>/<}$, one can easily obtain the spectral density by their difference $G_{>} - G_{<}$. In ITF, the inverse Fourier transform implies

$$G^{(\text{ITF})}(\omega_n, \mathbf{p}) = \int_0^{-i\beta} d^4x e^{i\omega_n x_0 - i\mathbf{p} \cdot \mathbf{x}} \int \frac{d^4K}{(2\pi)^4} e^{-iK \cdot x} \rho(K) [1 \pm f_{\text{B/F}}(k_0)] \quad (\text{B.25})$$

$$= i \int \frac{dk_0}{2\pi} \frac{\rho(k_0, \mathbf{p})}{\omega_n - k_0}. \quad (\text{B.26})$$

From the definition of the retarded and advanced propagators given in Eqs. (A.14) and (A.15), we have following relations

$$G_{\text{ret}}(P) = - \int \frac{dk_0}{2\pi} \frac{\rho(k_0, \mathbf{p})}{p_0 - k_0 + i\epsilon}, \quad (\text{B.27})$$

$$G_{\text{adv}}(P) = - \int \frac{dk_0}{2\pi} \frac{\rho(k_0, \mathbf{p})}{p_0 - k_0 - i\epsilon}. \quad (\text{B.28})$$

Thus one finds

$$G_{\text{ret}}(P) = iG^{(\text{ITF})}(p_0 + i\epsilon, \mathbf{p}), \quad (\text{B.29})$$

$$G_{\text{adv}}(P) = iG^{(\text{ITF})}(p_0 - i\epsilon, \mathbf{p}). \quad (\text{B.30})$$

Then, recalling $\rho(P) = -i[G_{\text{ret}}(P) - G_{\text{adv}}(P)]$, we can construct the spectral density from the ITF propagator:

$$\rho(P) = G^{(\text{ITF})}(p_0 + i\epsilon, \mathbf{p}) - G^{(\text{ITF})}(p_0 - i\epsilon, \mathbf{p}). \quad (\text{B.31})$$

B.3 Breit-Wigner Approximation

For instance let us consider a scalar field with a mass m_ϕ . In the free field limit, its spectral density takes the following form:

$$\rho(P) = (2\pi) \text{sgn}(p_0) \delta(P^2 - m_\phi^2). \quad (\text{B.32})$$

For a weakly interacting system, the spectral density may be approximated with the Breit-Wigner form:

$$\rho(P) \simeq Z_p \frac{2p_0 \Gamma_p}{[p_0^2 - \Omega_p^2]^2 + [p_0 \Gamma_p]^2} + \rho^{(\text{cont})}(P), \quad (\text{B.33})$$

where Ω_p and Γ_p represent a dispersion relation and a relaxation rate of quasi-particle excitation respectively, which can be computed perturbatively if one specifies interactions of the theory. Z_p is the wave function renormalization which comes from the shift of the pole position due to the thermal correction. Typically, the dispersion relation has the following form; $\Omega_p = \sqrt{\mathbf{p}^2 + m_{\phi, \text{th}}^2}$ with $m_{\phi, \text{th}} \sim gT$. Here g is a typical coupling constant. The first part describes a quasi-particle pole and the latter, $\rho^{(\text{cont})}$, gives the continuum spectrum.

In the fermion case, there are complications associated with the chiral symmetry. In the free field limit, its spectral density is nothing but

$$\rho(P) = (m_\psi + \not{P}) (2\pi) \text{sgn}(p_0) \delta(P^2 - m_\psi^2). \quad (\text{B.34})$$

If the Dirac mass term is smaller than the typical size of thermal corrections to the dispersion relation, $m_\psi \ll gT$, then the chiral symmetry is approximately conserved. In this case, the spectral density typically has the following form:

$$\rho(P) = \sum_{s=\pm} \frac{Z_p^s}{2} \left[\frac{\Gamma_p^s}{[p_0 - \Omega_p^s]^2 + \Gamma_p^{s2}/4} (\gamma_0 - \hat{\mathbf{q}} \cdot \boldsymbol{\gamma}) + \frac{\Gamma_p^s}{[p_0 + \Omega_p^s]^2 + \Gamma_p^{s2}/4} (\gamma_0 + \hat{\mathbf{q}} \cdot \boldsymbol{\gamma}) \right] + \rho^{(\text{cont})}(P), \quad (\text{B.35})$$

where $s = +$ correspond to the ordinary particle-like excitation, and $s = -$ represents the collective excitation in the plasma, which is so-called “plasmino” [117]. Again, the dispersion relations have the threshold energy due to the thermal correction; $\Omega_0^s = m_{\psi, \text{th}} \sim gT$. For a large momentum $p \gg gT$, the plasmino contribution tends to be exponentially suppressed, which is imprinted in the wave function renormalization Z_p^- [86].

On the other hand, if the Dirac mass term is larger than the thermal correction, $m_\psi \gg gT$, then the spectral density can be approximated with

$$\rho(P) \simeq (m_\psi + \not{p}) \frac{2p_0 \Gamma_p}{[p_0^2 - \omega_p^2]^2 + [p_0 \Gamma_p]^2} + \rho^{(\text{cont})}(P). \quad (\text{B.36})$$

Appendix C

Standard Cosmology

Here we briefly summarize basic ingredients of the Standard Cosmology which we use frequently throughout this thesis. See for instance [50].

C.1 Friedmann-Lemaître-Robertson-Walker Universe

C.1.1 Metric

In the Standard Cosmology, the Universe is assumed to be homogeneous and isotropic on large scales (aside from small perturbations), which leads to the so-called Friedmann-Lemaître-Robertson-Walker (FLRW) metric:^{♦1}

$$ds^2 = dt^2 - a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (\text{C.1})$$

where $a(t)$ is the scale factor which characterizes a relative size of space-like surface, t denotes the cosmic time and k is the curvature. The geodesic equation implies that a free-fall particle without a peculiar motion ($x^i = \text{const.}$) keeps fixed on this coordinate (t, r, θ, ϕ) , and hence it is known as the comoving coordinate.

C.1.2 Einstein Equations

Assuming the general covariance, one finds the minimal action for general relativity, which is so-called the Einstein-Hilbert action. Differentiating with respect to the metric $g_{\mu\nu}$, we obtain the celebrated Einstein equation:

$$M_{\text{pl}}^2 G_{\mu\nu} = T_{\mu\nu}, \quad (\text{C.2})$$

where the Einstein tensor is defined as $G_{\mu\nu} \equiv R_{\mu\nu} - Rg_{\mu\nu}/2$ and $T_{\mu\nu}$ is the energy-momentum tensor. As one can see, the energy-momentum tensor should have the *perfect fluid* form^{♦2}, $T_{\mu\nu}(t) = \text{diag}(\rho(t), -p(t), -p(t), -p(t))$, since the space-time is homogeneous and isotropic in the FLRW universe. In the FLRW universe, the Einstein equations lead to the following two equations:

$$H^2 \equiv \left(\frac{\dot{a}}{a} \right)^2 = \frac{\rho}{3M_{\text{pl}}^2} - \frac{k}{a^2}, \quad (\text{C.3})$$

^{♦1} ϕ should not be confused with the scalar field ϕ .

^{♦2} Aside from small perturbations.

and

$$\dot{H}^2 + H^2 = \frac{\ddot{a}}{a} = -\frac{\rho + 3p}{6M_{\text{pl}}^2}. \quad (\text{C.4})$$

They are also known as the *Friedmann equations*. From Friedmann equations, one can show

$$\frac{d}{dt}(\rho a^3) = -p \frac{d}{dt}(a^3), \quad (\text{C.5})$$

which indicates the entropy conservation as we will see in the next section. To solve these equations, one should specify the equation of state, which is often parametrized as

$$p \equiv w\rho, \quad (\text{C.6})$$

where w is a parameter depending on what dominates the Universe. Using Eq. (C.5), one finds

$$\rho \propto a^{-3(1+w)}. \quad (\text{C.7})$$

Typically, there are three types of equation of state:

$$\text{Matter: } w = 0; \quad \rho_{\text{mat}} \propto a^{-3}, \quad (\text{C.8})$$

$$\text{Radiation: } w = \frac{1}{3}; \quad \rho_{\text{rad}} \propto a^{-4}, \quad (\text{C.9})$$

$$\text{Vacuum: } w = -1; \quad \rho_{\text{vac}} = \text{const.} \quad (\text{C.10})$$

It is convenient to define the density parameters for each species:

$$\Omega_{\bullet} \equiv \frac{\rho_{\bullet}}{\rho_{\text{crit}}}; \quad \rho_{\text{crit}} \equiv 3M_{\text{pl}}^2 H^2. \quad (\text{C.11})$$

Obviously, Eq. (C.3) can be expressed as

$$1 - \Omega_k = \Omega_{\text{mat}} + \Omega_{\text{rad}} + \Omega_{\text{vac}}; \quad \Omega_k \equiv -\frac{k}{a^2 H^2}. \quad (\text{C.12})$$

The present values of these cosmological parameters are summarized as follows. The Universe is flat; $\Omega_k = 0.042^{+0.043}_{-0.048}$ (95% C.L.). The present Hubble parameter is given by $H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1}$ with $h = 0.673 \pm 0.012$ (68% C.L.). The density parameters for baryon, cold dark matter and dark energy are $\Omega_b h^2 = 0.02207 \pm 0.00027$ (68% C.L.), $\Omega_c h^2 = 0.1198 \pm 0.0026$ (68% C.L.) and $\Omega_{\Lambda} = 0.685^{+0.017}_{-0.016}$ (68% C.L.) respectively. See [47] for details.

Since the Universe is flat, we take $k = 0$ in the following. Then, the scale factor obeys the scaling solution for each era:

$$\text{Matter Dominant (MD): } a \propto t^{2/3}; \quad H = \frac{2}{3t}, \quad (\text{C.13})$$

$$\text{Radiation Dominant (RD): } a \propto t^{1/2}; \quad H = \frac{1}{2t}, \quad (\text{C.14})$$

$$\text{Vacuum Dominant (VD): } a \propto e^{Ht}; \quad H = \text{const.} \quad (\text{C.15})$$

C.1.3 Particles on FLRW Universe

Since we know the dynamics of the Universe, now we can consider how a particle on this space-time propagates. For later convenience, let us define the notion how far a massless test particle can propagate. The massless particle obeys the null-geodesic equation, $ds^2 = 0$. Due to the homogeneity and isotropy of the Universe, a comoving distance which the test particle can propagate from t_i to t can be expressed as

$$\int_{t_i}^t \frac{dt}{a} = \eta - \eta_i, \quad (\text{C.16})$$

where we define the conformal time, $d\eta = dt/a$. Then, we define the *particle horizon*, which indicates the maximum comoving distance a particle can propagate from the beginning of the Universe to now:

$$\eta_p(t) = \int_0^t \frac{dt}{a} = \int_{a_i}^a \frac{d \ln a}{aH}, \quad (\text{C.17})$$

where $(aH)^{-1}$ is the comoving Hubble radius. Its physical distance is given by $d_p(t) = a(t)\eta_p(t)$. The particles horizon can be expressed as

$$\eta_p(a) \propto \begin{cases} a^{1/2} & \text{MD} \\ a & \text{RD} \\ a_i^{-1} & \text{VD} \end{cases}, \quad (\text{C.18})$$

where we have assumed $a \gg a_i$. Also, one can define the *event horizon*, which is the maximum comoving distance a particle can propagate from now to the future:

$$\eta_e(t) = \int_t^\infty \frac{dt}{a} = \int_a^\infty \frac{d \ln a}{aH}. \quad (\text{C.19})$$

Its physical distance is given by $d_e(t) = a(t)\eta_e(t)$. Obviously, the event horizon diverges for RD and MD but it approaches to a^{-1} for VD, which indicates that particles comes into the event horizon for RD/MD but, for VD, particles goes outside the event horizon due to the rapid expansion.

C.1.4 Horizon and Flatness problems

Here we briefly explain the horizon and flatness problem which lead us to the idea of inflation. Eq. (C.18) indicates that the comoving Hubble radius grows monotonically. If our Universe experiences only RD or MD, then observed Universe in the comoving horizon should be far outside at the early epoch, and thus there seems to be no a priori reasons to expect the homogeneity of observed CMB. This is the *horizon problem*. Also, the density parameter of the curvature grows with time because it is proportional to the comoving Hubble radius squared in RD or MD. Nevertheless, observed Ω_k is almost zero, which implies that its initial value should be extremely suppressed or exactly vanish. This is the *flatness problem*. As one can see, these fine-tuning issues can be ameliorated by assuming that our Universe experiences VD era at its early epoch, which is so-called *inflation*, since the comoving Hubble radius decreases at this era.

C.2 Thermodynamics in the Early Universe

In this section, we summarize basics of thermodynamics in the early Universe; In particular, that of particle-like excitations (on-shell pole) in thermal equilibrium neglecting finite density corrections that appears with the coupling. The number density, energy density and pressure can be expressed as

$$n_i = g_i \int_p f_{B/F}(\omega_{i,p}); \quad \rho_i = g_i \int_p \omega_{i,p} f_{B/F}(\omega_{i,p}); \quad p_i = g_i \int_p \frac{p^2}{3\omega_{i,p}} f_{B/F}(\omega_{i,p}), \quad (C.20)$$

where g_i represents the degree of freedom for each species i and $\omega_{i,p}$ represents the dispersion relation of i . Let us see their typical behavior in two limits; Non-relativistic $m_i \gg T$ and Ultra-relativistic $m_i \ll T$:

$$\text{Non-relativistic:} \quad \begin{cases} n_i & \simeq g_i \left(\frac{m_i}{2\pi} \right)^{3/2} e^{-\beta(m_i - \mu_i)}, \\ \rho_i & \simeq m_i n_i + \frac{3}{2} n_i T, \\ p_i & = n_i T, \end{cases} \quad (C.21)$$

and

$$\text{Ultra-relativistic:} \quad \begin{cases} n_i & \simeq g_i \left(\frac{\zeta(3)}{\pi^2} \right) T^3 \begin{cases} 1 & \text{for Boson,} \\ 3/4 & \text{for Fermion,} \end{cases} \\ \rho_i & \simeq g_i \left(\frac{\pi^2}{30} \right) T^4 \begin{cases} 1 & \text{for Boson,} \\ 7/8 & \text{for Fermion,} \end{cases} \\ p_i & = \frac{\rho_i}{3}. \end{cases} \quad (C.22)$$

$\zeta(3)$ is the Riemann-zeta function and its value is $\zeta(3) \simeq 1.2020 \dots$. As one can see, Non-relativistic particles behave as *matter* and ultra-relativistic particles do as *radiation*. The total energy density of radiation can be written as follows:

$$\rho_{\text{rad}} = \frac{\pi^2 g_*(T)}{30} T^4; \quad g_*(T) \equiv \sum_{b: \text{boson}} g_b \left(\frac{T_b}{T} \right)^4 + \frac{7}{8} \sum_{b: \text{fermion}} g_b \left(\frac{T_b}{T} \right)^4, \quad (C.23)$$

where the summation is taken over relativistic particles. If there exists relativistic species with different temperature T_i , then the factor T_i^4/T^4 should be multiplied.

Let us recall the thermodynamic relation:

$$G(T, p, N_i) = \sum_i N_i \mu_i(T, p); \quad G(T, p, N_i) \equiv [U - TS + pV](T, p, N_i), \quad (C.24)$$

where U , S and N represent energy $U = \rho V$, entropy $S = sV$ and number $N_i = n_i V$ respectively. In the first equality, we have used the fact that the Gibbs free energy G is a linear function in an extensive variable N_i , and that $\partial G / \partial N_i = \mu_i$. Thus, we obtain the following expression for the entropy density:

$$s = \frac{\rho + p - \sum_i \mu_i n_i}{T}. \quad (C.25)$$

In our Universe, the chemical potential is much less than the temperature in most cases, and hence we neglect it hereafter. Then, together with Eq. (C.5), we can see that the cosmic expansion conserves the entropy. Importantly, the entropy is dominated by relativistic particles and it can be expressed as

$$s = \frac{4\rho_{\text{rad}}}{3T} = \frac{2\pi^2 g_{s,*}(T)}{45} T^3; \quad g_{s,*}(T) \equiv \sum_{b:\text{boson}} g_b \left(\frac{T_b}{T}\right)^3 + \frac{7}{8} \sum_{b:\text{fermion}} g_b \left(\frac{T_f}{T}\right)^3, \quad (\text{C.26})$$

where again the summation is taken over relativistic particles.

C.3 Scalar Field on FLRW

In this section, we derive the equation of motion for a scalar field in the FLRW background with vanishing curvature $k = 0$.^{♦3} The action of a scalar field which minimally couples to gravity is given by

$$S_\phi = \int a^3 dt d^3x \left[\frac{1}{2} (\partial_t \phi)^2 - \frac{1}{2a^2(t)} (\nabla \phi)^2 - V(\phi) \right]. \quad (\text{C.27})$$

Here we insert the FLRW metric. For later convenience, let us rescale the scalar field $\varphi = a\phi$ and use the conformal time η . Then, in terms of the *conformal variables* this action can be expressed as

$$S_\varphi = \int d\eta d^3x \left[\frac{1}{2} (\partial_\eta \varphi)^2 - \frac{1}{2} (\nabla \varphi)^2 - \frac{a''}{2a} \varphi^2 - a^4 V(\varphi/a) \right], \quad (\text{C.28})$$

where $(\dots)'$ represents a derivative with respect to the conformal time η .

Differentiating the action with respect to φ , we can obtain the equation of motion of a scalar field on FLRW background:

$$0 = \varphi'' - \nabla^2 \varphi - \frac{a''}{a} \varphi + a^3 V'(\varphi/a) = \varphi'' - \nabla^2 \varphi - \frac{a''}{a} \varphi + a^2 m_\phi^2 \varphi + \frac{\lambda_\phi^2}{6} \varphi^3. \quad (\text{C.29})$$

In the second equality, we have assumed that the potential is given by $V = m_\phi^2 \phi^2/2 + \lambda_\phi^2 \phi^4/4!$. In terms of the physical variables, this equation of motion can be expressed as

$$0 = \ddot{\phi} - \nabla^2 \phi + 3H\dot{\phi} + V'(\phi) = \ddot{\phi} - \nabla^2 \phi + 3H\dot{\phi} + m_\phi^2 \phi + \frac{\lambda_\phi^2}{6} \phi^3. \quad (\text{C.30})$$

Obviously, if the oscillation period is much faster than the expansion of the Universe H^{-1} , then this equation of motion can be well approximated with that of Minkowski-background.

For the sake of completeness, let us see how to quantize a free scalar field on the FLRW. Performing the Fourier transform, the *conformal* scalar field can be expressed as

$$\hat{\phi}(\eta, \mathbf{x}) = \int_{\mathbf{k}} [f_k(\eta) \hat{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + f_k^*(\eta) \hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}}]; \quad 0 = f_k''(\eta) + \left(\mathbf{k}^2 + a^2 m_\phi^2 - \frac{a''}{a} \right) f_k(\eta), \quad (\text{C.31})$$

with $[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}')$, which indicates the following normalization of Wronskian; $i = (f_k, f_k^*) \equiv f_k \partial_\eta f_k^* - f_k^* \partial_\eta f_k$. As is usual the case with quantum fields in an evolving background,

^{♦3} We only consider the scalar sector and neglect interactions with radiation. Their effects are discussed in the main part of this thesis.

a positive frequency mode and negative frequency mode can mix (See also Sec. 4.3). First, as done in Sec. 4.3, let us consider the adiabatic expansion of the Universe; that is,

$$1 \gg \frac{\omega'_k(\eta)}{\omega_k^2(\eta)}; \quad \omega_k(\eta) \equiv \sqrt{k^2 + a^2(\eta)m_\phi^2 - \frac{a''(\eta)}{a(\eta)}}. \quad (\text{C.32})$$

For instance, this condition is satisfied for $m_\phi \gg \mathcal{H}/a = H$ with $\mathcal{H} = a'/a$. Then, we can construct the WKB solution,

$$f_k(\eta) \simeq \frac{e^{-i \int^\eta d\eta' \omega_k(\eta')}}{\sqrt{2\omega_k(\eta)}}, \quad (\text{C.33})$$

and the adiabatic vacuum; $\hat{a}_k|0\rangle = 0$. Here we take the boundary condition of f_k so that there are no particles in the adiabatic vacuum initially. Its physical interpretation is obvious, that is, if the de Broglie wave length is much shorter than the Hubble scale, then the expansion of the Universe can be neglected for such particles.

For comparison, it is instructive to consider super-horizon modes in a de Sitter space. Let us consider a light scalar field $m_\phi \ll H$ to break the above condition. In a de Sitter background, the wave function satisfies the following equation:

$$0 = f_k''(\eta) + \left(k^2 - \frac{2}{\eta^2}\right)f_k(\eta), \quad (\text{C.34})$$

where we take the conformal time as $\eta = -1/(aH)$ so that $\eta = 0$ corresponds to the infinite future. Fortunately, this equation can be solved analytically:

$$f_k(\eta) = \alpha \frac{e^{-ik\eta}}{\sqrt{2k}} \left(1 - \frac{i}{k\eta}\right) + \beta \frac{e^{ik\eta}}{\sqrt{2k}} \left(1 + \frac{i}{k\eta}\right). \quad (\text{C.35})$$

Then, we impose that there are no particles for modes well inside the horizon, which implies $f_k \rightarrow e^{-ik\eta}/\sqrt{2k}$ for $k \gg aH = 1/|\eta|$, that is, $\alpha = 1$ and $\beta = 0$:

$$f_k^{(\text{BD})}(\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}} \left(1 - \frac{i}{k\eta}\right). \quad (\text{C.36})$$

This boundary condition corresponds to the Bunch-Davies Vacuum. The equal time two-point function on the Bunch-Davies Vacuum is given by

$$\langle \hat{\phi}_k(t) \hat{\phi}_{k'}(t) \rangle_{\text{BD}} = \frac{1}{a^2(\eta)} \langle \hat{\phi}_k(\eta) \hat{\phi}_{k'}(\eta) \rangle_{\text{BD}} = (2\pi^3) \delta(\mathbf{k} + \mathbf{k}') \frac{1}{a^2} \frac{1}{2k} \left(1 + \frac{H^2 a^2}{k^2}\right). \quad (\text{C.37})$$

In terms of the physical variables, this equation can be rewritten as

$$\langle \hat{\phi}_p(t) \hat{\phi}_{p'}(t) \rangle_{\text{BD}} = a^4 \langle \hat{\phi}_k(t) \hat{\phi}_{k'}(t) \rangle_{\text{BD}} = (2\pi)^3 \delta(\mathbf{p} + \mathbf{p}') \frac{1}{2p} \left(1 + \frac{H^2}{p^2}\right), \quad (\text{C.38})$$

with the physical momentum being $\mathbf{p} = \mathbf{k}/a$. The first term is nothing but the ordinary vacuum contribution in Minkowski-space. As one can see, there is another contribution which becomes important for super-horizon modes; $p \ll H$. Moving back to the conformal expression, one can see that the two-point function approaches a constant value for super-horizon modes:

$$\langle \hat{\phi}_k(t) \hat{\phi}_{k'}(t) \rangle_{\text{BD}} \xrightarrow{H \gg k/a} (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') \frac{H^2}{2k^3}. \quad (\text{C.39})$$

This expression remains in the physical variables. Note that the obtained result is applicable to gravitons since its (second order) action is the same as that of a massless free scalar field up to a normalization factor.

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