研 究 速 報
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A Note on Finite Element Synthesis of Structures (Part 3)
——Shape Modification for Stress Reduction——
有限要素法による構造シンセシスに関するノート (第3報)
——応力低減のための形状変更——

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1. Introduction

The finite element method has enabled us to analyse numerically but in detail the structural response corresponding to the given structural shape, mechanical and geometrical boundary conditions and meterial constants. Several methods have been proposed so far to modify the structural shape on the basis of the inverse variational principle and optimization technique[1),2),3),4)]. These methods necessitate the iteration of structural modification until the objective design is accomplished. The number of iterations is desired as small as possible when the finite element method is employed which requires sizable CPU time for an implementation. The uniqueness of the objective design is not assured in inverse problems, and the objective design is to be searched in the vast domain of many candidate design. It turns out that the methods proposed hitherto are likely to need a large number of iterations.

This paper presents the numerical examples of the shape modification for stress reduction by means of the structural synthesis technique based on the notion that the objective design is searched in the vicinity of the baseline design[5)]. The design variables chosen are the nodal coordinates of the finite elements along the contour. The objective response is set so as to reduce the elastic stress in the structure of initial shape.

2. Nodal coordinates and Stress Sensitivity

The shape of structures is described by the nodal coordinates of the elements along the contour in the context of the finite element method. The sensitivities of the stress with respect to the nodal coordinates are to be evaluated at first in the case of the proposed technique in order to modify the shape for the purpose of stress reduction. In this section, how to evaluate the stress sensitivities is formulated in regard to the triangular, three-noded, constant strain element.

2.1 Sensitivity of stiffness matrix

The stiffness matrix is calculated by Eq. (1), where t denotes the element thickness, S the area of the triangular element, $[B]$ the strain-nodal displacement matrix and $[D]$ the stress-strain matrix. $[D]$ and t are not affected by the change of the nodal coordinates.

$$[k]=t\ S\ [B]^T\ [D][B] \tag{1}$$

Suppose that the coordinates of the nodes i, j and k of an element are changed through the design variables α_n assigned as in Eq. (2). The upper bar indicates the values concerning the baseline design hereafter.

$$x_i=\bar{x}_i\ (1+\alpha_1), \cdots\cdots, \ y_k=\bar{y}_k\ (1+\alpha_6) \tag{2}$$

The expression of $[B]$ is given by Eq. (3),

$$[B]=\frac{1}{A}\begin{pmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{pmatrix} \tag{3}$$

where

$$b_1=y_j-y_k, \cdots\cdots, \ c_3=x_j-x_i \tag{4}$$

and

$$A=x_ib_1+x_jb_2+x_kb_3=y_ic_1+y_jc_2+y_kc_3 \tag{5}$$

The variation of $[B]$ with respect to that of the nodal coordinates of Eq. (2) can be approximated by the Taylor series expansion truncated at the first-order as given below.

$$[B]=[\bar{B}]+\sum_{n=1}^{6}[B_n]\alpha_n \tag{6}$$

In the above, $[\bar{B}]$ means the matrix defined by the quantities at the baseline design, that is, Eqs. (3)

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through (5) with the upper bar. The sensitivities $[B_n]$ can be evaluated by differentiating $[B]$ with respect to α_n, for instance, $[B_1]$ is obtained in the following form.

$$[B_1]=\frac{-\bar{b}_1\bar{x}_1}{\bar{A}^2}\begin{Bmatrix} \bar{b}_1 & 0 & \bar{b}_2 & 0 & \bar{b}_3 & 0 \\ 0 & \bar{c}_1 & 0 & \bar{c}_2-\bar{A}/\bar{b}_1 & 0 & \bar{c}_3-\bar{A}/\bar{b}_1 \\ \bar{c}_1 & \bar{b}_1 & \bar{c}_2-\bar{A}/\bar{b}_1 & \bar{b}_2 & \bar{c}_3-\bar{A}/\bar{b}_1 & \bar{b}_3 \end{Bmatrix} \tag{7}$$

The area S is also influenced by the change of the nodal coordinates and can be approximated in the form of Eq. (8), where S is $A/2$,

$$S=\bar{S}+\sum_{n=1}^{6} S_n\alpha_n \tag{8}$$

with the sensitivities of

$$S_1=\bar{b}_1\bar{x}_i/2$$

and so forth. Based on the above results, the variation of $[k]$ can be summarized in the following form of the first-order approximation.

$$\begin{aligned}[k] &= t[\bar{S}[\bar{B}]^T[D][\bar{B}]+\sum_{n=1}^{6}\{\bar{S}([\bar{B}]^T[D][B_n] \\ &\quad +[B_n]^T[D][\bar{B}])+S_n[\bar{B}]^T[D][\bar{B}]\}] \\ &= [\bar{k}]+\sum_{n=1}^{6}[k_n]\alpha_n \end{aligned} \tag{9}$$

2.2 Stress sensitivity

The variation of the nodal displacements caused by that of the stiffness matrix can be evaluated by means of the perturbation technique[6] and is expressed by Eq. (10), where $\{u\}$ indicates the nodal displacements in regard to an element.

$$\{u\}=\{\bar{u}\}+\sum_{n=1}^{6}\{u_n\}\alpha_n \tag{10}$$

Then the stress components in an element is calculated by Eq. (11).

$$\begin{aligned}\{\sigma\} &= [D][B]\{u\} \\ &= [D][\bar{B}]\{\bar{u}\}+\sum_{n=1}^{6}[D]([\bar{B}]\{u_n\} \\ &\quad +[B_n]\{\bar{u}\})\alpha_n=\{\bar{\sigma}\}+\sum_{n=1}^{6}\{\sigma_n\}\alpha_n \end{aligned} \tag{11}$$

This means that any stress index under consideration can be expressed by the first-order approximation as follows,

$$z_j=\bar{z}_j+\sum_{n=1}^{N} z_{jn}\alpha_n \tag{12}$$

where N indicates the total number of the design variables taken. The above result is employed to determine the design variables based on the functional composed of the squared sum of α_n and the equality constraint conditions representing the attainment of the objective response by Eq. (12). The design variables are determined together with the Lagrange multipliers μ_j as the solution of Eq. (13)[5].

$$\left[\begin{array}{ccc|ccc} 2 & & 0 & -z_{11} & \cdots & -z_{j1} \\ & \ddots & & \vdots & & \vdots \\ 0 & & 2 & -z_{1N} & \cdots & -z_{jN} \\ \hline & & & & & \\ & & & & 0 & \\ SYM. & & & & & \end{array}\right]\left\{\begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_N \\ \hline \mu_1 \\ \vdots \\ \mu_J \end{array}\right\}=\left\{\begin{array}{c} 0 \\ \hline \bar{z}_1-z_1{}^* \\ \vdots \\ \bar{z}_J-z_J{}^* \end{array}\right\} \tag{13}$$

It is necessary to iterate the above solution procedure by renewing the baseline design and evaluating the sensitivities for the renewed baseline design in order to overcome the deficiency of the first-order approximation included.

3. Numerical Examples

3.1 Shape modification of tapered plate

Suppose that a tapered plate is subjected to the tension as showin in Fig. 1. The plate is divided into twenty elements, and the stress in the upper elements, the edge of which is located on the contour, is set to be equal to 2 MPa. The y-coordinates of ten nodes on the contour are chosen as the design variables. Figure 1 illustrates that the uniform stress state is accomplished by only three iterations even for the large change of the plate width. It is found that

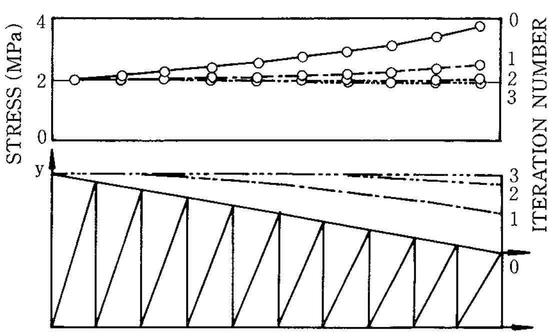

Fig. 1 Shape modification of tapered plate for stress uniformization

the criterion of the uniform stress should be imposed to the whole length of the plate to realize the objective design of the uniform stress state.

3. 2 Hole shape of plate under bi-axial tension

Figure 2 shows a quarter model of a square plate perforated at the center and subject to uniform bi-axial tension. The vertical tension is two times as large as the horizontal tension. The hole is circle-shaped initially. The distribution of the equivalent stress along the hole edge is shown in Fig. 3. It is aimed at to reduce the distribution of the equivalent stress indicated as INITIAL to the value indicated as FINAL in the figure. In doing so, the objective stresses in all the eight elements, whose edge is located on the contour, are used to compose the constraint conditions, while the x- and/or y-coordinates of all the nine nodes on the contour are taken as the design variables. The coordinates of the inner nodes are changed by such a simple rule that the ratio of their change to the change of the nearest contour node is to be in proportion to the distance from the right upper corner of the plate.

The iteration history of the axis lengths of the hole is shown in Fig. 4. In this example, it is found that there takes place the remarkable distortion of the elements along the hole edge as shown in Fig. 2 (d) when the FINAL objectives are imposed by a single step. The distortion, probably caused by the simple rule for the change of the inner nodes, can be avoided by setting the objectives by two steps, the first one of which is indicated by INTERMEDIATE in Fig. 3, to suppress the large deviation of the objective response from the baseline response appearing on the right-hand side of Eq. (13). Th numerals in Fig. 4 indicate the largest deviation of the stress value from the objective in the unit of percentage. Figure 4 shows that the stress is converged sufficiently by six iterations, and that at least twelve iterations are required for the convergence of the long and short axes. This implies that the convergence of the shape is slower than that of the stress taken as the objective.

3. 3 Reduction of fillet bending stress of gear

The shape of involute spur gear teeth should be

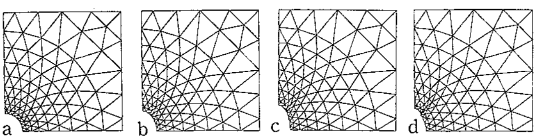

Fig. 2 Change of circular hole to elliptic hole in plate subjected to bi-axial tension

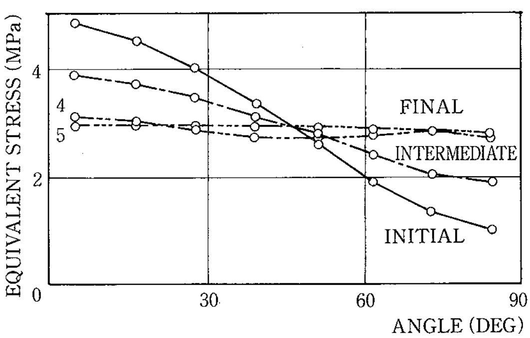

Fig. 3 Distribution of equivalent stress along hole edge

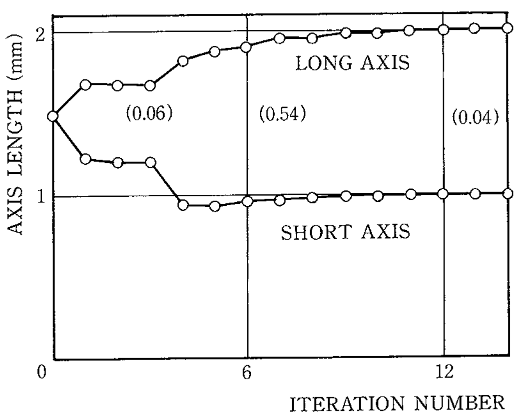

Fig. 4 Iteration history of long and short axes of hole

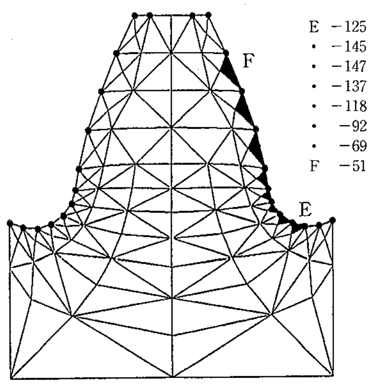

Fig. 5 Objective stress values and finite element division of involute spur gear

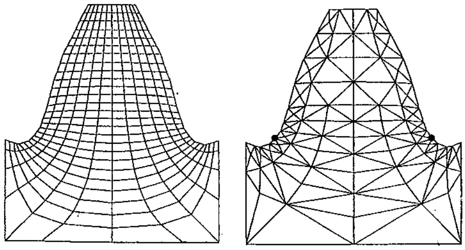

Fig. 6 Net generation by boundary fitting for inner nodes

symmetric with respect to their center line in spite of the loading is asymmetric[7]. It is aimed at in this example to reduce the fillet bending stress of a steel gear to the values indicated in Fig. 5 in the unit of MPa. The width of the domain subjected to the element division is 116mm, and the loading to the left-side face is 2kN. The objective values are set approximately equal to 80% of the initial values. The solid circles in Fig. 5 indicate the nodes, the coordinates of which are taken as the design variables. The shape symmetry of the tooth is imposed as the additional constraint conditions for the coordinate change. The location of the inner nodes are renewed in the course of the iteration based on the net generated by the boundary fitting technique as shown in Fig. 6 to avoid the excessive element distortion. The stress reduction aimed at is obtained by seven iterations, and the largest coordinate change, 3.49mm in the horizontal direction, occurs at the point indicated by the solid circle in Fig. 6. Figure 7 illustrates the distribution of the equivalent stress before and after the shape modification.

4. Concluding remarks

The validity of the structural synthesis based on the notion of the minimum change of design is exhibited through the numerical examples of the stress reduction in problems of elastic plane stress state. Rather large change from the baseline stress to the objective stress or considerable change of shape can be resulted by less than ten iterations even though the first-order approximation is used. It should be noted for the completion of the involute gear design that other constraint conditions should be taken into account such that the contour runs on the involute curve.

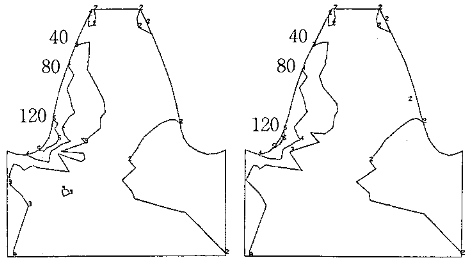

Fig. 7 Stress distribution before and after shape modification

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References

1) Oda, J. and Yamazaki, K., Determination of an Optimum Strength Shape of an Axisymmetric Body by the Finite Element (in Japanese), Trans. JSME., Ser. I, Vol. 43, No. 368 (1977), pp. 1182-1190.

2) Tsuta, N. and Yamaji, S., Shape Optimization Method of High Stressed Part by Using the Finite Element (in Japanese), Trans. JSME., Ser. I, Vol. 44, No. 386 (1978), pp. 3333-3342.

3) Murakawa, E., Myojin, K., Tokumasu, S. and Aso, T., Optimum Design Technique Using Boundary Element Method (in Japanese), Trans. JSME., Ser. A, Vol. 49, No. 439 (1983), pp. 396-401.

4) Tada, Y., Seguchi, Y. and Konishi, H., Shape Determination Problem of Structures Considering Variation of Loads (in Japanese), Trans, JSME., Ser. A, Vol. 52, No. 478 (1986), pp. 1608-1614.

5) Nakagiri, S. and Suzuki, K., A Note on Finite Element Synthesis of Structures (Part 2)-Indeterminate Shift Synthesis of Vibration Eigenvalues and Eigenvectors, Seisan-Kenkyu, Vol. 40, No. 4 (1988), pp. 209-212.

6) Nakagiri, S. and Hisada, T., Stochastic Finite Element Method, An Introduction (in Japanese), Baifu-kan, 1985.

7) Andrews, J.D. and Hearn, E.J, An Automatic Design Optimization Procedure to Minimize Fillet Bending Stresses in Involute Spur Gear, Int. J. num. Meth. Eng., Vol. 24, No. 5 (1987), pp. 975-986.