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On the Constitutive Equation of Discontinuous Crack Model

非連続き裂モデルにおける構成方程式

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1. INTRODUCTION

Crack problems are usually dealt with based on continuum model in the same way as in other strength problems of structures. But generally the necessary condition to be a continuum is not satisfied around an actual crack tip. From this point of view, previously, a discontinuous crack model which expresses the discountinuities around a crack tip as the first-order approximation and can be regarded as the most generalized model of Dugdale's was proposed and its usefulness was shown through the evaluation of crack parameters [1].

The purpose of this study is to consider a general method to give the constitutive equation of the plane taking the discontinuity into consideration in the discontinuous crack model. A method to give the constitutive equation in case of elasto−plastic body was shown in the previous paper, however, it partly contains the theoretically inconsistent part although it possesses the usefulness that the matrix concerned is given in a symmetrical form. Furthermore, the discontinuous plane may become useful not only for the former discontinuous crack model but also for other problems. From these reasons, here we consider again, also taking not only the elasto-plastic problem but also the creep problem into account, the way to give the constitutive equation of the discontinuous plane in which a generality and a theoretical consistency are kept.

2. DISCONTINUOUS CRACK MODEL AND CRACK PARAMETERS

In the discontinuous crack model, a cracked body is considered to be composed of two parts, i. e. the plane ahead of the initial crack front containing the crack and the other, before deformation (the time $\tau=0$) as shown in Fig.1 (a) . After deformation ($\tau=t$) , the former is stretched as shown in Figs. 1 (b) and (c) (Fig.1(b) shows the state in which the plane has been stretched and, furtheremore, the crack has extended to present crack front) , and the latter behaves as an ordinary continuum[1]. So, for the region except for the plane, we give an ordinary constitutive equation that is expressed by a relation between stress components and strain components. Therefore, when we define $\lfloor \sigma_n \rfloor$ and $\lfloor \delta_n \rfloor$, referring to Fig.1 (c) , by

$$\lfloor \sigma_n \rfloor = \lfloor \sigma_{22}\ \sigma_{23}\ \sigma_{12} \rfloor \tag{1}$$

$$\lfloor \delta_n \rfloor = \lfloor \delta_{22}\ \delta_{23}\ \delta_{21} \rfloor \tag{2}$$

and we give a constitutive equation for the plane as a relation between $\lfloor \sigma_n \rfloor$ and $\lfloor \delta_n \rfloor$, this crack

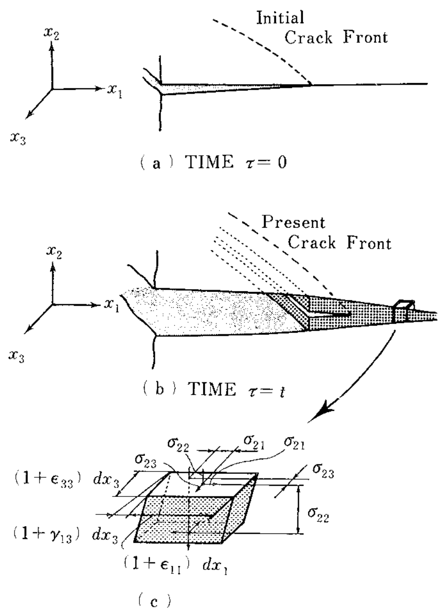

Fig.1 Discontinuous crack model

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model becomes one that enables us to analyze crack problems according to actual history of loading process.

In this model, when the strain energy plane density W_{plane} [2)3)] is given by

$$W_{plane}=\int dW_{plane},\quad dW_{plane}=\lfloor \sigma_n \rfloor \{d\delta_n\} \tag{3}$$

the crack energy density[2)3)] at $(x_1,x_2,x_3)=(X_1,0,X_3)$ is defined by

$$\mathcal{E}(t,X_1,X_3)=\int dW_{plane}(t,X_1,X_3) \tag{4}$$

and it distributes in x_1-x_3 plane. Moreover, the crack tip opening displacement CTOD at $(x_1,x_2,x_3)=(X_1^f,0,X_3^f)$ on the present crack front is given, even after the onset of crack growth, by

$$CTOD(t)=\delta_{22}(t,X_1^f,X_3^f) \tag{5}$$

where $\delta_{22}(t,X_1,X_3)$ distributes in x_1-x_3 plane. So we can immediately estimate these parameters when we give an appropriate constitutive relation for the discontinuous plane.

3. CONSTITUTIVE EQUATION OF DISCONTINUOUS PLANE

It is desired that the constitutive equation for the discontinuous plane given by a relation between $\lfloor \sigma_n \rfloor$ and $\lfloor \delta_n \rfloor$ can be estimated based on the constitutive equation for the region considered to be a continuum without any contradiction. So we think as follows.

Suppose that a point shown in Fig.2 (a) is stretched under the application of stress $\lfloor \sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \sigma_{23}\ \sigma_{31}\ \sigma_{12} \rfloor$ as shown in Fig.2 (b) and the relative displacement δ_{ij} between the opposite planes is generated as the result (in the figure, two dimensional case is shown for simplicity). Here, δ_{ij} is the relative displacement in the direction of x_i-axis between the opposite planes perpendicular to x_j-axis. Moreover, in Fig.2 (b), we show the state in which a point at the initial state is stretched into a region with finite dimensions under the stress of finite value, though here we think that the applied force is zero. In order to relate the constitutive equation given by a relation between stress and δ_{ij} with the ordinary one for the continuum, it is natural to assume the characteristic length h_{ij}, define a strain-like quantity by

$$\lfloor \tilde{\varepsilon}_{11}\ \tilde{\varepsilon}_{22}\ \tilde{\varepsilon}_{33}\ \tilde{\gamma}_{23}\ \tilde{\gamma}_{31}\ \tilde{\gamma}_{12} \rfloor = \lfloor \delta_{11}/h_{11}\ \delta_{22}/h_{22}\ \delta_{33}/h_{33}\ \delta_{23}/h_{23}+\delta_{32}/h_{32}\ \delta_{31}/h_{31}+\delta_{13}/h_{13}\ \delta_{12}/h_{12}+\delta_{21}/h_{21} \rfloor \tag{6}$$

and give the constitutive equation between this strain-like quantity and the stress by the same relation as that between the strain and the stress for the continuum. Then, we can adjust parameter h_{ij} so as to realize the actual relation between δ_{ij} and the stress.

When we think as the above-mentioned, the discontinuous plane in Fig.1 can be considered as the limiting case when $\delta_{11},\delta_{33},\delta_{32},\delta_{31},\delta_{13},\delta_{12}\to 0$. Here, even at the state shown in Fig.1 (b), the displacement $u_i(X_1,0,X_3)$ can be defined in the discontinuous plane, so the in-plane strain $\varepsilon_{11}=\partial u_1/\partial X_1$, $\varepsilon_{33}=\partial u_3/\partial X_3$ and $\gamma_{31}=\partial u_3/\partial X_1+\partial u_1/\partial X_3$, $\partial u_2/\partial X_3$ and $\partial u_2/\partial X_1$ are still existing. Accordingly, in the limitting case when δ_{11}, δ_{33}, δ_{32}, δ_{31}, δ_{13}, $\delta_{12}\to 0$, it is reasonable to consider that the relations as

$$\frac{\delta_{11}}{h_{11}}=\tilde{\varepsilon}_{11}=\varepsilon_{11}=\frac{\partial u_1}{\partial X_1},\quad \frac{\delta_{33}}{h_{33}}=\tilde{\varepsilon}_{33}=\varepsilon_{33}=\frac{\partial u_3}{\partial X_3},$$
$$\frac{\delta_{31}}{h_{31}}+\frac{\delta_{13}}{h_{13}}=\tilde{\gamma}_{31}=\gamma_{31}=\frac{\partial u_3}{\partial X_1}+\frac{\partial u_1}{\partial X_3},\quad \frac{\delta_{32}}{h_{32}}=\frac{\partial u_2}{\partial X_3},$$
$$\frac{\delta_{12}}{h_{12}}=\frac{\partial u_2}{\partial X_1} \tag{7}$$

hold and, therefore, h_{11}, h_{33}, h_{32}, h_{31}, h_{13}, h_{12} go zero when δ_{11}, δ_{33}, δ_{32}, δ_{31}, δ_{13}, $\delta_{12}\to 0$. So, the strain-like quantity in Eq. (6) can be shown as follows:

$$\lfloor \varepsilon_{11}\ \tilde{\varepsilon}_{22}\ \varepsilon_{33}\ \tilde{\gamma}_{23}\ \gamma_{31}\ \tilde{\gamma}_{12} \rfloor = \lfloor \partial u_1/\partial X_1\ \delta_{22}/h_{22}\ \partial u_3/\partial X_3\ \delta_{23}/h_{23}+\partial u_2/\partial X_3\ \partial u_3/\partial X_1+\partial u_1/\partial X_3\ \partial u_2/\partial X_1+\delta_{21}/h_{21} \rfloor \tag{8}$$

Hence, the constitutive equation for the discontinuous plane is given by using the above strain-like quantity instead of the strain in the constitutive equation for the continuum.

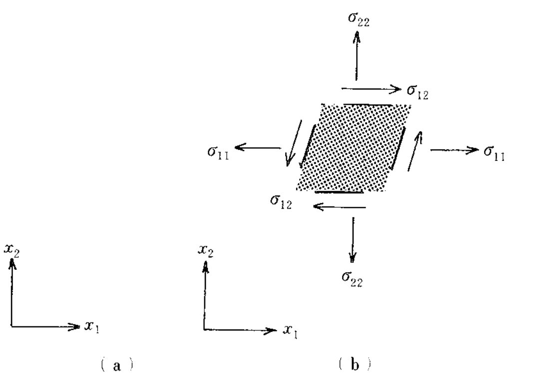

Fig.2 A point stretched under stress

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We give the constitutive relations for linear elastic, elasto−plastic and creep cases concretely in the following. Hereafter, we use the following expressions:

$$\lfloor \sigma_t \rfloor = \lfloor \sigma_{11}\ \sigma_{33}\ \sigma_{31} \rfloor \qquad (9)$$

$$\lfloor \varepsilon_t \rfloor = \lfloor \varepsilon_{11}\ \varepsilon_{33}\ \varepsilon_{31} \rfloor \qquad (10)$$

$$\lfloor \varepsilon_n \rfloor = \lfloor \varepsilon_{22}\ \gamma_{23}\ \gamma_{12} \rfloor \qquad (11)$$

$$\lfloor \tilde{\varepsilon}_n \rfloor = \lfloor \tilde{\varepsilon}_{22}\ \tilde{\gamma}_{23}\ \tilde{\gamma}_{12} \rfloor = \lfloor \delta_n \rfloor [h]^{-1} + \lfloor \partial u_n/\partial X \rfloor = \lfloor \delta'_n \rfloor [h]^{-1} \qquad (12)$$

$$[\mathrm{H}] = \left[\begin{array}{c|c} \mathrm{I} & 0 \\ \hline 0 & h \end{array}\right] \qquad (13)$$

where

$$[h] = \begin{bmatrix} h_{22} & 0 & 0 \\ 0 & h_{23} & 0 \\ 0 & 0 & h_{21} \end{bmatrix}, \qquad [\mathrm{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\lfloor \partial u_n/\partial X \rfloor = \lfloor 0\ \partial u_2/\partial X_3\ \partial u_2/\partial X_1 \rfloor$$

$$\lfloor \delta'_n \rfloor = \lfloor \delta_n \rfloor + \lfloor \partial u_n/\partial X \rfloor\ [h] \qquad (14)$$

In case of linear elastic body When the constitutive equation for the continuum is given, by using an elastic matrix $[D^e]$ as

$$\begin{Bmatrix} \sigma_t \\ \sigma_n \end{Bmatrix} = \left[\begin{array}{c|c} D^e_{tt} & D^e_{tn} \\ \hline D^e_{nt} & D^e_{nn} \end{array}\right] \begin{Bmatrix} \varepsilon^e_t \\ \varepsilon^e_n \end{Bmatrix} \qquad (15)$$

that for the discontinuous plane is given by

$$\begin{Bmatrix} \sigma_t \\ \sigma_n \end{Bmatrix} = \left[\begin{array}{c|c} D^e_{tt} & D^e_{tn} \\ \hline D^e_{nt} & D^e_{nn} \end{array}\right] \begin{Bmatrix} \varepsilon^e_t \\ \tilde{\varepsilon}^e_n \end{Bmatrix} \qquad (16)$$

or, using the relations of Eqs. (12) and (14), by

$$\{\sigma_t\} = [D^e_{tt}]\{\varepsilon^e_t\} + [D^e_{tn}][h^e]^{-1}\{\delta^e_n\} + [D^e_{tn}]\left\{\frac{\partial u^e_n}{\partial X}\right\}$$

$$\{\sigma_n\} = [D^e_{nt}]\{\varepsilon^e_t\} + [D^e_{nn}][h^e]^{-1}\{\delta^e_n\} + [D^e_{nn}]\left\{\frac{\partial u^e_n}{\partial X}\right\} \qquad (17)$$

where the superscript e signifies that the quantity is that for elastic behavior.

In case of elasto−plastic body In the same way, when the constitutive equation for the continuum is given, by using an elasto−plastic matrix $[D^p]$, as

$$\begin{Bmatrix} d\sigma_t \\ d\sigma_n \end{Bmatrix} = \left[\begin{array}{c|c} D^p_{tt} & D^p_{tn} \\ \hline D^p_{nt} & D^p_{nn} \end{array}\right] \begin{Bmatrix} d\varepsilon^t_t \\ d\varepsilon^t_n \end{Bmatrix} \qquad (18)$$

that for the discontinuous plane is given by

$$\begin{Bmatrix} d\sigma_t \\ d\sigma_n \end{Bmatrix} = \left[\begin{array}{c|c} D^p_{tt} & D^p_{tn} \\ \hline D^p_{nt} & D^p_{nn} \end{array}\right] \begin{Bmatrix} d\varepsilon^t_t \\ d\tilde{\varepsilon}^t_n \end{Bmatrix} \qquad (19)$$

or

$$\{d\sigma_t\} = [D^p_{tt}]\{d\varepsilon^t_t\} + [D^p_{tn}][h^t]^{-1}\{d\delta^t_n\} + [D^p_{tn}]\left\{\frac{\partial du^t_n}{\partial X}\right\}$$

$$\{d\sigma_n\} = [D^p_{nt}]\{d\varepsilon^t_t\} + [D^p_{nn}][h^t]^{-1}\{d\delta^t_n\} + [D^p_{nn}]\left\{\frac{\partial du^t_n}{\partial X}\right\} \qquad (20)$$

where the superscript t signifies that the quantity is that for the total behavior that consists of the elastic and plastic contributions.

When $[h^t]$ for the total behavior is given, the constitutive equation is immediately obtained from the elasto−plastic matrix $[D^p]$ as is shown above. On the other hand, when $[h^e]$ for elastic and $[h^p]$ for plastic are given independently, the constitutive equation is obtained as follows (the case of $[h^e]=[h^p]$ is included in the above case).

Assume that the yield condition is given, in the same way as in the continuum, by

$$f(\sigma_{ij}) = \bar{\sigma} = c \qquad (21)$$

and the plastic strain−like quantity increment $\lfloor d\tilde{\varepsilon}^p_s \rfloor = \lfloor d\varepsilon^p_t d\tilde{\varepsilon}^p_n \rfloor$ is given, by using $f(\sigma_{ij})$ in the above as the plastic potential, by

$$\{d\tilde{\varepsilon}^p_s\} = [H^p]^{-1}\begin{Bmatrix} d\varepsilon^p_t \\ d\delta'^p_n \end{Bmatrix} = \bar{c}\left\{\frac{\partial f}{\partial \sigma}\right\} df \qquad (22)$$

where $f(\sigma_{ij})$, $\bar{\sigma}$, c and $\bar{c}$ is the yield function, the equivalent stress, the parameter of hardening and the proportional constant respectively. Next, using $\lfloor \sigma \rfloor = \lfloor \sigma_t\ \sigma_n \rfloor$ and representing the equivalent plastic strain−like quantity increment by $\overline{d\tilde{\varepsilon}^p_s}$, we define the plastic work−like quantity increment by

$$d\tilde{W}^p = \lfloor \sigma \rfloor \{d\tilde{\varepsilon}^p_s\} = \bar{\sigma}\,\overline{d\tilde{\varepsilon}^p_s} \qquad (23)$$

and consider that the relation

$$\lfloor \sigma \rfloor \{\partial f/\partial \sigma\} = f = \bar{\sigma} \qquad (24)$$

holds when von Mises yield condition is used. Then, the relation

$$\bar{c}\,df = \overline{d\tilde{\varepsilon}^p_s} \qquad (25)$$

is obtained from Eq. (22) and, further, if c in Eq. (21) is assumed to be a function of $\tilde{W}^p$, H' exists uniquely so that the relation

$$\overline{d\tilde{\varepsilon}^p_s} = d\bar{\sigma}/H' \qquad (26)$$

holds. Therefore, considering the relation

$$d\bar{\sigma} = df = \lfloor \partial f/\partial \sigma \rfloor \{d\sigma\} \qquad (27)$$

the relation

$$\bar{c}\,df = \frac{\lfloor \partial f/\partial \sigma \rfloor \{d\sigma\}}{H'} \qquad (28)$$

is obtained from Eqs. (25) and (26). By the way, when $\lfloor d\tilde{\varepsilon}^e_s \rfloor = \lfloor d\varepsilon^e_t d\tilde{\varepsilon}^e_n \rfloor$ and $\lfloor d\tilde{\varepsilon}^t_s \rfloor = \lfloor d\varepsilon^t_t\ d\tilde{\varepsilon}^t_n \rfloor$, $\{d\sigma\}$ can be given by

$$\{d\sigma\} = [D^e]\{d\tilde{\varepsilon}^e_s\} = [D^e][H^e]^{-1}\begin{Bmatrix} d\varepsilon^t_t - d\varepsilon^p_t \\ d\delta'^t_n - d\delta'^p_n \end{Bmatrix} \qquad (29)$$

Therefore, using Eq. (21), we obtain

$$\{d\sigma\}=[D^e][H^e]^{-1}\left\{\frac{d\varepsilon_t^t}{d\delta_n'^t}\right\}$$
$$-\bar{c}[D^e][H^e]^{-1}[H^p]\{\partial f/\partial\sigma\}df \qquad (30)$$

Substituting this into Eq. (28), solving with respect to $\bar{c}df$ and substituting this $\bar{c}df$ into Eq. (30), we obtain the final formula:

$$\{d\sigma\}=\left([D^e][H^e]^{-1} - \frac{[D^e][H^e]^{-1}[H^p]\left\{\frac{\partial f}{\partial\sigma}\right\}\left\lfloor\frac{\partial f}{\partial\sigma}\right\rfloor[D^e][H^e]^{-1}}{H'+\left\lfloor\frac{\partial f}{\partial\sigma}\right\rfloor[D^e][H^e]^{-1}[H^p]\left\{\frac{\partial f}{\partial\sigma}\right\}}\right)\left\{\begin{matrix}d\varepsilon_t^t\\ d\delta_n'^t\end{matrix}\right\}$$
$$=\left[\begin{array}{c|c}\bar{D}_{tt}^p & \bar{D}_{tn}^p\\ \bar{D}_{nt}^p & \bar{D}_{nn}^p\end{array}\right]\left\{\begin{matrix}d\varepsilon_t^t\\ d\delta_n'^t\end{matrix}\right\} \qquad (31)$$

Using Eq. (13), Eq. (31) can be expressed as

$$\{d\sigma_t\}=[\bar{D}_{tt}^p]\{d\varepsilon_t^t\}+[\bar{D}_{tn}^p]\{d\delta_n^t\}+[\bar{D}_{tn}^p][h^t]\left\{\frac{\partial du_n^t}{\partial X}\right\}$$
$$\{d\sigma_n\}=[\bar{D}_{nt}^p]\{d\varepsilon_t^t\}+[\bar{D}_{nn}^p]\{d\delta_n^t\}$$
$$+[\bar{D}_{nn}^p][h^t]\left\{\frac{\partial du_n^t}{\partial X}\right\} \qquad (32)$$

Here, $[h^t]$ in Eq. (32) can be dealt with as follows. As the strain-like quantity increment can be expressed as

$$\{d\tilde{\varepsilon}_n^t\}=\{d\tilde{\varepsilon}_n^e\}+\{d\tilde{\varepsilon}_n^p\} \qquad (33)$$

the relation

$$[h^t]^{-1}\{d\delta_n'^t\}=[h^e]^{-1}\{d\delta_n'^e\}+[h^p]^{-1}\{d\delta_n'^p\} \qquad (34)$$

holds from Eq. (12). $[h^t]$ changes momentarily with the increase of load, and we may use $[h^t]$ at the time τ that is determined by substituting really generated $\{d\delta'_n\}$ in the period from $\tau-d\tau$ to τ into Eq. (34) as $[h^t]$ for the variation between τ and $\tau+d\tau$.

Here, we consider the difference between Eq. (32) and Eq. (20). Eq. (20) is the relation when $[h^t]$ is given (for example, it can be given as the function of equivalent plastic strain-like quantity), and, on the other hand, Eq. (32) is for the case that $[h^e]$ and $[h^p]$ are given independently, therefore, $[h^t]$ is determined by Eq. (34) for each loading process. For this reason, the meanings of $[h^t]$ in the both equations are different. But, in case of $[h^e]=[h^p]$, $[h^t]$ is also equal to these and Eq. (20) and Eq. (32) completely coincide with each other.

In case of creep The constitutive equation for the continuum is usually given by

$$\left\{\begin{matrix}\dot{\varepsilon}_t^c\\ \dot{\varepsilon}_n^c\end{matrix}\right\}=\dot{\bar{\varepsilon}}^c\left\{\begin{matrix}\partial f/\partial\sigma_t\\ \partial f/\partial\sigma_n\end{matrix}\right\} \qquad (35)$$

where the superscript c signifies that the quantity is that for creep behavior, and $\lfloor\dot{\varepsilon}_t^c/\dot{\varepsilon}_n^c\rfloor$ is the creep strain rate, $f(\sigma_{ij})$ is the creep potential of the type of von Mises and $\dot{\bar{\varepsilon}}^c$ is the equivalent creep strain rate that can be given as a function of equivalent stress $\bar{\sigma}(=f)$ and time τ as

$$\dot{\bar{\varepsilon}}^c=\dot{\varepsilon}_c(\bar{\sigma},\tau) \qquad (36)$$

Therefore, for the discontinuous plane, assuming that the equivalent strain-like quantity rate $\lfloor\dot{\tilde{\varepsilon}}_s^c\rfloor=\lfloor\dot{\varepsilon}_t^c\ \dot{\tilde{\varepsilon}}_n^c\rfloor$ can be given by

$$\dot{\bar{\varepsilon}}_s^c=\dot{\varepsilon}_c(\bar{\sigma},\tau) \qquad (37)$$

the constitutive equation is given as follows:

$$\left\{\begin{matrix}\dot{\varepsilon}_t^c\\ \dot{\tilde{\varepsilon}}_n^c\end{matrix}\right\}=\dot{\bar{\varepsilon}}_s^c\left\{\begin{matrix}\partial f/\partial\sigma_t\\ \partial f/\partial\sigma_n\end{matrix}\right\} \qquad (38)$$

Therefore, using Eqs. (12) and (14), we obtain

$$\left\{\begin{matrix}\dot{\varepsilon}_t^c\\ \dot{\delta}_n'^c\end{matrix}\right\}=\dot{\bar{\varepsilon}}_s^c[H^c]\left\{\begin{matrix}\partial f/\partial\sigma_t\\ \partial f/\partial\sigma_n\end{matrix}\right\} \qquad (39)$$

4. CONCLUSION

A general constitutive equation for the plane considering the discontinuity in the discontinuous crack model was discussed and it was shown that it can be given based on the constitutive equation of the continuum with sufficient rationality. The discontinuous crack model can be formulated by finite element method and, therefore, it is expected to be a crack model to analyze various types of crack problems and to estimate the crack parameters. Moreover, we should like to point out that the discontinuous plane of which the constitutive equation was given in this paper may be successfully applied to other problems besides crack problems. (Manuscript received, May 9,1986)

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