A Solution Of Chandrasekhar's Integral Equation For Radiative Transfer In Plane-Parallel Atmospheres With Thin Optical Thickness

(薄い光学的厚さを有する平行平面大気の放射過程の

チャンドラセカール積分方程式による一解法)

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1 Introduction

In the area of satellite remote sensing, we observe the radiance at the top of atmosphere (TOA) by instruments aboard satellites. We then retrieve the surface reflectance and the optical thickness from the observed radiance (inversion problem). The observing equation based on the single scattering approximation is given as

$$\rho(\tau, r : \vec{i}_1, \vec{i}_0) = r + \frac{P(\vec{i}_1, \vec{i}_0)}{4|\cos\theta_0|\cos\theta_1}\tau, \tag{1}$$

where ρ is the observed generalized reflectance, $\vec{i}_1$ ($\vec{i}_0$) is the direction of the observation (the incident sun light), P is the scattering phase function, θ_1 (θ_0) is the zenith angle of $\vec{i}_1$ ($\vec{i}_0$), and τ is the optical thickness of the atmosphere. We retrieve the two unknown variables r and τ from ρ.

The Coastal Zone Color Scanner (CZCS) aboard Nimbus-7 launched in 1978 was the first satellite instrument to retrieve chlorophyll concentration in the ocean.[1] As a high accuracy atmospheric correction is needed to retrieve chlorophyll concentration, many methods to solve the inversion problem of radiative transfer have been developed. The single scattering approximation was first introduced to solve this problem.

In the mid-1990s, new algorithms were required for new satellite instruments, such as Ocean Color and Temperature Scanner (OCTS) aboard ADEOS and Sea-WiFS aboard Sea Star.[2] [3] The single scattering approximation does not yield the radiance at the TOA to meet the required accuracy, so the forward calculation and look-up table method are employed. The radiance at the TOA is evaluated by the forward calculation and look-up table and is very accurate but implicit in term of the optical thickness.

In this thesis, we seek the solution of radiance at the TOA in the form of a polynomial in τ.

Radiative transfer in a plane parallel atmosphere has been a major scientific and mathematical subject for many years.[4] [5] Fig. 1 illustrates the geometry of the radiative transfer process. The solar irradiance F_0 enters the layer of vertical optical thickness τ. The radiance scattered by the layer, $I(0, \vec{i}_1)$, emerges from the top, while the the radiance transmitted by the layer, $I(0, \vec{i}_4)$, emerges from the bottom. We assume a perfectly absorbing

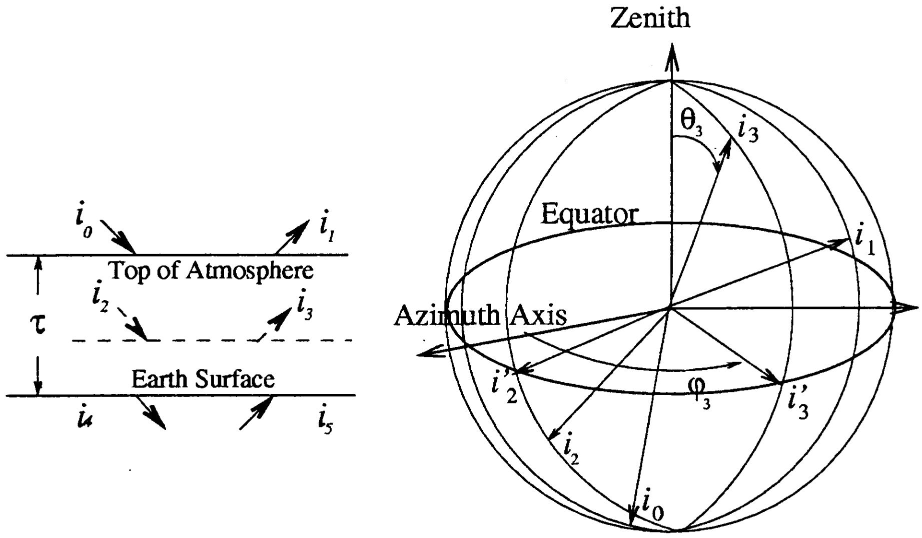

Figure 1: Geometry of Radiative Transfer

surface at the bottom. The direction of the radiance $\vec{i}_n$ with a suffix of an even (odd) number is the lower (upper) bound. We employ a polar coordinate system in which the zenith angle θ is measured from the zenith and the azimuth angle φ is measured between the projection of the direction onto the equator and the fixed direction on the equator. We denote the cosine of the zenith angle of the direction $\vec{i}_n$ as μ_n and the azimuth angle of the direction $\vec{i}_n$ as φ_n.

We introduce the scattering function $S(\tau, i_1, i)$ and the transmission function $T(\tau, i_1, i)$. The scattered radiance in the direction i_1 from the top of the layer $I(0, \vec{i}_1)$ is given by

$$I(0, \vec{i}_1) = \frac{1}{4\pi\mu_1} \int_U S(\tau, \vec{i}_1, \vec{i}) I(0, \vec{i}) d\Omega, \tag{2}$$

where $I(0, \vec{i})$ is the incident radiance from the direction $\vec{i}$, Ω is the solid angle subtended by the incident radiance and the integral domain is the upper unit hemisphere. The transmitted radiance in the direction i_4 from the bottom of the layer $I(0, \vec{i}_4)$ is given by

$$I(\tau, \vec{i}_4) = \frac{1}{4\pi|\mu_4|} \int_U T(\tau, \vec{i}_4, \vec{i}) I(0, \vec{i}) d\Omega + \exp(-\frac{\tau}{\mu_4}) I(0, \vec{i}_4), \tag{3}$$

where the second term is the attenuated direct solar radiance.

The input solar radiance $I(0,\vec{i})$ is expressed as

$$I(0,i) = F_0\delta(\vec{i} - \vec{i}_0) \tag{4}$$

where $\delta(\vec{i})$ is Dirac's delta function. We assume no incident radiance from the bottom. Substituting the above equation into the scattering and transmission functions, we obtain

$$I(0,\vec{i}_1) = \frac{F_0}{4\pi\mu_1}S(\tau,\vec{i}_1,\vec{i}_0) \tag{5}$$

and

$$I(0,\vec{i}_4) = \frac{F_0}{4\pi|\mu_4|}T(\tau,\vec{i}_4,\vec{i}_0). \tag{6}$$

If τ is considered sufficiently small, we have the first approximation of $S_1(\tau,\vec{i}_1,\vec{i}_0)$ and $T_1(\tau,\vec{i}_4,\vec{i}_0)$,

$$S_1(\tau,\vec{i}_1,\vec{i}_0) = P(i_1,i_0)\tau, \tag{7}$$

and

$$T_1(\tau,\vec{i}_4,\vec{i}_0) = P(i_4,i_0)\tau. \tag{8}$$

The radiative transfer process for the layer of atmosphere is governed by the integro-differential equation given as

$$\mu\frac{dI(\tau,\vec{i})}{d\tau} = -I(\tau,\vec{i}) + \int_{4\pi} P(\vec{i},\vec{i'})I(\tau,\vec{i'})\frac{d\Omega'}{4\pi} + \frac{F_0}{4\pi}\exp(-\frac{\tau}{\mu_s})P(\vec{i},\vec{i}_s) \tag{9}$$

where $I(\tau,\vec{i})$ is the radiance at the optical thickness τ and in the direction $\vec{i}$, $\vec{i}_s$ is the direction of the sun, μ is the cosine of the direction $\vec{i}$, F_0 is the solar irradiance, μ_s is the cosine of the zenith angle of the sun light, and $d\Omega'$ is an infinitesimal solid angle around the direction $\vec{i'}$.

Based on this integro-differential equation, there have been various methods to calculate scattered radiance at the top of the atmosphere. Hansen[6] and Goody[7] summarized advantages and disadvantages of these methods. The primary thrust for those methods has been to solve the forward problem, that is, to evaluate the radiance at the TOA in terms of an optical thickness and a surface reflectance. Since τ is a variable in the integro-differential equation, it is impossible to obtain the radiance in the explicit form in τ.

Chandrasekhar's work[4] in 1960 derived the simultaneous integral equation for radiative transfer by the principles of invariance. Since τ is not a variable but is a parameter in this integral equation, we can obtain the solution of radiance at the TOA in the form of a polynomial in τ by iteratively integrations with regard to the zenith angle. The integral equation is given in the form of non-linear, two unknown functions X and Y.

$$X(\mu) = 1 + \int_0^1 [X(\mu)X(\mu') - Y(\mu)Y(\mu')]\frac{\mu d\mu'}{2(\mu + \mu')} \tag{10}$$

$$Y(\mu) = \exp\left(-\frac{\tau}{\mu}\right) + \int_0^1 [Y(\mu)X(\mu') - X(\mu)Y(\mu')]\frac{\mu d\mu'}{2(\mu - \mu')} \tag{11}$$

Using X and Y, $S(\tau, \mu, \mu_0)$ and $T(\tau, \mu_4^-, \mu_0^-)$ are given below.

$$\left(\frac{1}{\mu_0} + \frac{1}{\mu}\right)S(\tau, \mu, \mu_0) = [X(\mu)X(\mu_0) - Y(\mu)Y(\mu_0)] \tag{12}$$

$$\left(\frac{1}{\mu} - \frac{1}{\mu_0}\right)T(\tau, \mu, \mu_0) = [Y(\mu)X(\mu_0) - X(\mu)Y(\mu_0)] \tag{13}$$

The solution of integral equation for X and Y, however, is not unique. Chandrasekhar gave an expression of a family of solution[4].

Busbridge derived the same X and Y functions from the Milne integral equation not from the principle of the invariance[5]. The Milne integral equation is an integral equation with two variables, μ and τ but is linear different from Chandrasekhar's integral equation.

$$J(x, \mu_0) = \exp\left(-\frac{x}{\mu_0}\right) + \int_0^1 \int_0^\tau \exp\left(-\frac{|y - x|}{\mu}\right)J(y, \mu_0)dy\frac{d\mu}{2\mu} \tag{14}$$

where $J(x, \mu_0)$ is the unknown function to solve. Using $J(x, \mu_0)$, $S(\tau, \mu_1, \mu_0)$ and $T(\tau, \mu_1, \mu_0)$ are expressed as,

$$S(\tau, \mu_1, \mu_0) = \int_0^\tau \exp\left(-\frac{y}{\mu_1}\right)J(y, \mu_0)dy \tag{15}$$

$$T(\tau, \mu_1, \mu_0) = \int_0^\tau \exp\left(-\frac{\tau - y}{\mu_1}\right)J(y, \mu_0)dy \tag{16}$$

Busbridge showed that the iterative integration of the Milne equation gives the unique solution[5]. Busbridge also derived a linearized integral equation for X and Y^5.

$$T(\mu)X(\mu) = 1 - \mu \int_0^1 \frac{X(u)du}{2(\mu - u)} - \mu \exp\left(\frac{-\tau}{\mu}\right) \int_0^1 \frac{Y(u)du}{2(\mu + u)} \quad (17)$$

$$T(\mu)Y(\mu) = \exp\left(-\frac{\tau}{\mu}\right)[1 - \mu \int_0^1 \frac{X(\mu)du}{2(\mu + \mu)}] - \mu \int_0^1 \frac{Y(u)du}{2(\mu - u)} \quad (18)$$

where $T(\mu)$ is given as

$$T(\mu) = 1 - \int_0^1 \frac{\mu^2 du}{\mu^2 - u^2}. \quad (19)$$

Based on this linearized equation Mullikin [8] gave the the family of the solution of X and Y functions. Mullikin [9] also showed that the iterative integration of X and Y with initial function $X(\mu) = 1$ and $Y(\mu) = e^{(-\tau/\mu)}$ converges for atmospheres with thin optical thicknesses and high albedos and that this solution is unique. Chandrasekhar [4] integrated the second iteration with the same first iteration for X and Y and made a complemental correction. He concluded that the second approximation agreed well with the exact calculation up to $\tau = 0.5$.

We employ the similar iterative integration as Chandrasekhar obtained the second approximation for X and Y but begin with integral equation of S and T. In section five we show the agreement of the our iterations U and V with the iteration for X and Y. And we also show that our iterated solution is only possible solution in the family of the solutions given by Mullikin.

We discuss the iterative integration of Chandrasekhar's integral equation to solve the inversion problem of the radiative transfer in section two.

In section three, we evaluate the second and third approximation for an isotropic atmosphere. We often encounter integrations of the exponential-like functions shown below

$$U_2(\tau, \vec{i_0}) = \int_0^1 \left(\frac{1}{\mu_3} + \frac{1}{|\mu_0|}\right)^{-1}\left(1 - \exp\left(-\frac{\tau}{\mu_3} - \frac{\tau}{|\mu_0|}\right)\right)\frac{d\mu_3}{2\mu_3}. \quad (20)$$

Expanding "exp" in the above equation into the power series in $(-\tau)$ and integrating indefinitely, we obtain

$$\begin{aligned} U_2(\tau, \vec{i_0}) &= \int_0^1 [-(-\tau) - \left(\frac{1}{\mu_3} + \frac{1}{|\mu_0|}\right)\frac{(-\tau)^2}{2!} - \cdots]\frac{d\mu_3}{2\mu_3} \\ &= \frac{1}{2}[-\log\mu_3(-\tau) - \left(-\frac{1}{\mu_3} + \frac{\log\mu_3}{|\mu_0|}\right)\frac{(-\tau)^2}{2!} - \frac{1}{\mu_3} - \cdots]. \quad (21) \end{aligned}$$

Substituting $\mu_3 = 1$ at the upper integral limit, we can obtain the upper integral value. However, substituting $\mu_3 = 0$ at the lower integral limit, each coefficient of the powers in τ is either ∞ or $-\infty$. We must sum up all the powers on the interval $(\epsilon, 1)$ and then make ϵ approach 0. In other words, we cannot change the order of lim and $\sum$ at the lower integral limit. The problem of the radiative transfer is thus deduced to evaluating convergences of exponential-like functions in the form of a series expansion at the lower integral limit. We evaluated several series expansions, which are necessary to obtain the second and third approximation for the isotropic atmosphere. We evaluate the third approximations numerically for isotropic atmospheres. The results of the first, second, and third approximations are then compared.

In section four, we evaluate the second approximation of the scattering function for anisotropic atmospheres. In addition to the integration in equation (20), we must evaluate moment integrations shown as shown below

$$U_2^n(\tau, i_3) = \int_0^1 (\frac{1}{\mu} + \frac{1}{\mu_3})^{-1}(1 - \exp(-\frac{\tau}{\mu} - \frac{\tau}{\mu_3}))\frac{\mu^n d\mu}{2\mu}. \tag{22}$$

We show that we can evaluate the higher powers in τ of the power series expansion of $U_2^n(\tau, i_3)$ separately at the lower integral limit. Based on this consideration, we show that the coefficients of the quadratic polynomial of the second approximation of the scattering function can be expressed as the surface integration of products of the scattering phase functions on the unit hemisphere. We evaluate the second approximations numerically for the Rayleigh scattering and the maritime aerosol atmosphere, and compare the result with the single scattering approximation and the exact solution.

In section five we summarize the properties of the solutions obtained by the iterative integration of Chandrasekhar's integral equation.

We conclude the thesis in section six.

2 Radiation Transfer Process

2.1 Chandrasekhar's Integral Equation

The radiative transfer process is governed by Chandrasekhar's integral equation[3] in which the two unknown functions, $S(\tau,\vec{i}_1,\vec{i}_0)$ and $T(\tau,\vec{i}_1,\vec{i}_0)$, are described

$$
\begin{aligned}
S(\tau,\vec{i}_1,\vec{i}_0) &= (\frac{1}{\mu_1}+\frac{1}{|\mu_0|})^{-1}\times[\ \{1-\exp(\frac{(-\tau)}{\mu_1}+\frac{(-\tau)}{|\mu_0|})\}P(\vec{i}_1,\vec{i}_0) \\
&+ \int_U P(\vec{i}_1,\vec{i}_3)S(\tau,\vec{i}_3,\vec{i}_0)\frac{d\Omega_3}{4\pi\mu_3} - \exp(-\frac{\tau}{|\mu_0|})\int_U T(\tau,\vec{i}_1,\vec{i}_3)P(\vec{i}_3,\vec{i}_0)\frac{d\Omega_3}{4\pi\mu_3} \\
&+ \int_L S(\tau,\vec{i}_1,\vec{i}_2)P(\vec{i}_2,\vec{i}_0)\frac{d\Omega_2}{4\pi|\mu_2|} - \exp(-\frac{\tau}{\mu_1})\int_L P(\vec{i}_1,\vec{i}_2)T(\tau,\vec{i}_2,\vec{i}_0)\frac{d\Omega_2}{4\pi|\mu_2|} \\
&+ \int_U\int_L S(\tau,\vec{i}_1,\vec{i}_2)P(\vec{i}_2,\vec{i}_3)S(\tau,\vec{i}_3,\vec{i}_0)\frac{d\Omega_2}{4\pi|\mu_2|}\frac{d\Omega_3}{4\pi\mu_3} \\
&- \int_U\int_L T(\tau,\vec{i}_1,\vec{i}_3)P(\vec{i}_3,\vec{i}_2)T(\tau,\vec{i}_2,\vec{i}_0)\frac{d\Omega_2}{4\pi|\mu_2|}\frac{d\Omega_3}{4\pi\mu_3}\ ],
\end{aligned}
\tag{23}
$$

and

$$
\begin{aligned}
T(\tau,\vec{i}_4,\vec{i}_0) &= (\frac{1}{|\mu_4|}-\frac{1}{|\mu_0|})^{-1}[\ \{\exp(-\frac{\tau}{|\mu_0|})-\exp(-\frac{\tau}{|\mu_4|})\}P(\vec{i}_4,\vec{i}_0) \\
&+ \int_L P(\vec{i}_4,\vec{i}_2)T(\tau,\vec{i}_2,\vec{i}_0)\frac{d\Omega_2}{4\pi|\mu_2|} - \exp(-\frac{\tau}{|\mu_4|})\int_U P(\vec{i}_4,\vec{i}_3)S(\tau,\vec{i}_3,\vec{i}_0)\frac{d\Omega_3}{4\pi\mu_3} \\
&+ \exp(-\frac{\tau}{|\mu_0|})\int_U S(\tau,\vec{i}_4,\vec{i}_3)P(\vec{i}_3,\vec{i}_0)\frac{d\Omega_3}{4\pi\mu_3} - \int_L T(\tau,\vec{i}_4,\vec{i}_2)P(\vec{i}_2,\vec{i}_0)\frac{d\Omega_2}{4\pi|\mu_2|} \\
&+ \int_U\int_L S(\tau,\vec{i}_4,\vec{i}_3)P(\vec{i}_3,\vec{i}_2)T(\tau,\vec{i}_2,\vec{i}_0)\frac{d\Omega_2}{4\pi|\mu_2|}\frac{d\Omega_3}{4\pi\mu_3} \\
&- \int_U\int_U T(\tau,\vec{i}_4,\vec{i}_2)P(\vec{i}_2,\vec{i}_3)S(\tau,\vec{i}_3,\vec{i}_0)\frac{d\Omega_2}{4\pi|\mu_2|}\frac{d\Omega_3}{4\pi\mu_3}\ ].
\end{aligned}
\tag{24}
$$

The cosine of the zenith angle in the lower bound directions, which has an even index, is negative. For the convenience of the following evaluation, we replace μ_n by a new parameter, μ_n^- which is equal to $-\mu_n$ for $n = even$. Using the new variables, the integration on the lower unit hemisphere is changed to the integration on the upper unite hemisphere, shown below

$$
\int_L \frac{f(\mu_2,\cdot\cdot)d\Omega_2}{4\pi|\mu_2|} = \int_0^{2\pi} d\varphi_2 \int_0^{-1} \frac{f(\mu_2,\cdot,\cdot)(-d\mu_2)}{4\pi(-\mu_2)}
$$

$$= \int_0^{2\pi} d\varphi_2 \int_0^1 \frac{f(-\mu_2^-, \cdot, \cdot) d\mu_2^-}{4\pi\mu_2^-} = \int_U f(-\mu_2^-, \cdot, \cdot) \frac{d\Omega_2^-}{4\pi\mu_2^-}. \qquad (25)$$

Since we adopt the new variable μ_2^- and the integral domain is changed to the upper hemisphere, we omit the U for the integral domain unless the domain has a special meaning. The cosine of the angle between the upper and lower bound directions is thus expressed as negative as shown below

$$\begin{aligned} \cos(\theta_{e,o}) &= \mu_e\mu_o + (1-\mu_e^2)^{1/2}(1-\mu_o^2)^{1/2}\cos(\varphi_e - \varphi_o) \\ &= -\mu_e^-\mu_o + (1-(\mu_e^-)^2)^{1/2}(1-\mu_o^2)^{1/2}\cos(\varphi_e - \varphi_o). \end{aligned} \qquad (26)$$

The phase function is assumed to be a function of the angle θ_{mn} between the directions m and n and is normalized in the whole solid angle 4π. We express the phase function $P(i_m, i_n)$ as

$$P(i_m, i_n) = P(\cos\theta_{mn}) = \sum_{j=0}^{\infty} w_j P_j(\cos\theta_{mn}), \qquad (27)$$

where $P_j(x)$ is a Legendre function of the first kind.

2.2 Iteration Scheme

The simultaneous integral equations of radiative transfer can be solved by successive iteration. The first approximations $S_1(\tau, \vec{i}_1, \vec{i}_0)$ and $T_1(\tau, \vec{i}_1, \vec{i}_0)$ are the first terms on the right-hand side in equations (23) and (24), and are expressed below

$$S_1(\tau, \vec{i}_1, \vec{i}_0) = (\frac{1}{\mu_1} + \frac{1}{\mu_0^-})^{-1}[1 - \exp(\frac{(-\tau)}{\mu_1} + \frac{(-\tau)}{\mu_0^-})]P(\vec{i}_1, \vec{i}_0), \qquad (28)$$

$$T_1(\tau, \vec{i}_4, \vec{i}_0) = (\frac{1}{\mu_4^-} - \frac{1}{\mu_0^-})^{-1}(\exp(-\frac{\tau}{\mu_0^-}) - \exp(-\frac{\tau}{\mu_4^-}))P(\vec{i}_4, \vec{i}_0). \qquad (29)$$

Substituting the first approximations into the integrals in the original equations, the second approximation $S_2(\tau, \vec{i}_1, \vec{i}_0)$ is obtained

$$S_2(\tau, \vec{i}_1, \vec{i}_0) = S_1(\tau, \vec{i}_1, \vec{i}_0) + (\frac{1}{\mu_1} + \frac{1}{\mu_0^-})^{-1}$$

$$[\quad \int P(\vec{i}_1, \vec{i}_3) S_1(\tau, \vec{i}_3, \vec{i}_0) \frac{d\Omega_3}{4\pi\mu_3} - \exp(-\frac{\tau}{\mu_0^-}) \int P(\vec{i}_3, \vec{i}_0) T_1(\tau, \vec{i}_1, \vec{i}_3) \frac{d\Omega_3}{4\pi\mu_3}$$

$$
\begin{aligned}
&+ \int P(\vec{i}_2,\vec{i}_0)S_1(\tau,\vec{i}_1,\vec{i}_2)\frac{d\Omega_2^-}{4\pi\mu_2^-} - \exp(-\frac{\tau}{\mu_1})\int P(\vec{i}_1,\vec{i}_2)T_1(\tau,\vec{i}_2,\vec{i}_0)\frac{d\Omega_2}{4\pi\mu_2^-} \\
&+ \int\int S_1(\tau,\vec{i}_1,\vec{i}_2)P(\vec{i}_2,\vec{i}_3)S_1(\tau,\vec{i}_3,\vec{i}_0)\frac{d\Omega_2^-}{4\pi\mu_2^-}\frac{d\Omega_3}{4\pi\mu_3} \\
&- \int\int T_1(\tau,\vec{i}_1,\vec{i}_3)P(\vec{i}_3,\vec{i}_2)T_1(\tau,\vec{i}_2,\vec{i}_0)\frac{d\Omega_2^-}{4\pi\mu_2^-}\frac{d\Omega_3}{4\pi\mu_3}].
\end{aligned}
\tag{30}
$$

We also obtain the second approximation of $T_1(\tau,\vec{i}_0,\vec{i}_4)$ in the same manner. We designate the integral part in the above equation as the second iteration $\Delta S_2(\tau,\vec{i}_1,\vec{i}_0)$. The second approximation is thus expressed as below,

$$
S_2(\tau,\vec{i}_1,\vec{i}_0) = S_1(\tau,\vec{i}_1,\vec{i}_0) + \Delta S_2(\tau,\vec{i}_1,\vec{i}_0). \tag{31}
$$

Iterating this procedure, we obtain the nth iteration $\Delta S_n(\tau,\vec{i}_1,\vec{i}_0)$, given below

$$
\begin{aligned}
&\Delta S_n(\tau,\vec{i}_1,\vec{i}_0) = (\frac{1}{\mu_1}+\frac{1}{\mu_0^-})^{-1}\times \\
&[\ \int P(\vec{i}_1,\vec{i}_3)\Delta S_{n-1}(\tau,\vec{i}_3,\vec{i}_0)\frac{d\Omega_3}{4\pi\mu_3} - \exp(-\frac{\tau}{\mu_0^-})\int P(\vec{i}_3,\vec{i}_0)\Delta T_{n-1}(\tau,\vec{i}_1,\vec{i}_3)\frac{d\Omega_3}{4\pi\mu_3} \\
&+ \int P(\vec{i}_2,\vec{i}_0)\Delta S_{n-1}(\tau,\vec{i}_1,\vec{i}_2)\frac{d\Omega_2^-}{4\pi\mu_2^-} - \exp(-\frac{\tau}{\mu_1})\int P(\vec{i}_1,\vec{i}_2)\Delta T_{n-1}(\tau,\vec{i}_2,\vec{i}_0)\frac{d\Omega_2}{4\pi\mu_2^-} \\
&+ \int\int \Delta S_{n-1}(\tau,\vec{i}_1,\vec{i}_2)P(\vec{i}_2,\vec{i}_3)\Delta S_{n-1}(\tau,\vec{i}_3,\vec{i}_0)\frac{d\Omega_2^-}{4\pi\mu_2^-}\frac{d\Omega_3}{4\pi\mu_3} \\
&- \int\int \Delta T_{n-1}(\tau,\vec{i}_1,\vec{i}_3)P(\vec{i}_3,\vec{i}_2)\Delta T_{n-1}(\tau,\vec{i}_2,\vec{i}_0)\frac{d\Omega_2^-}{4\pi\mu_2^-}\frac{d\Omega_3}{4\pi\mu_3}\ ].
\end{aligned}
\tag{32}
$$

The nth approximation of $S_n(\tau,\vec{i}_1,\vec{i}_0)$ is evaluated as

$$
S_n(\tau,\vec{i}_1,\vec{i}_0) = \sum_{j=1}^{n}\Delta S_j(\tau,\vec{i}_1,\vec{i}_0) \tag{33}
$$

We can obtain the nth iteration $\Delta T_n(\tau,\vec{i}_4,\vec{i}_0)$ and then the nth approximation $T_n(\tau,\vec{i}_4,\vec{i}_0)$ in the same manner.

Since the first approximations, $S_1(\tau,\vec{i}_1,\vec{i}_0)$ and $T_1(\tau,\vec{i}_4,\vec{i}_0)$, have factors $P(\vec{i}_n,\vec{i}_m)$, the first approximation accounts for the single scattering. As the number of multiplications of $P(\vec{i}_n,\vec{i}_m)$ in $\Delta S_n(\tau,\vec{i}_1,\vec{i}_0)$ and $\Delta T_n(\tau,\vec{i}_4,\vec{i}_0)$ is equal to or greater than n, they account for the nth and higher order scattering.

3 Integration of the Isotropic Atmosphere

3.1 Iteration Scheme

The phase function for the isotropic atmosphere $P(\vec{i}_n, \vec{i}_m)$ equals 1. The first iterations are expressed below

$$S_1(\tau, \vec{i}_1, \vec{i}_0) = (\frac{1}{\mu_1} + \frac{1}{\mu_0^-})^{-1}[1 - \exp(-\tau)(\frac{1}{\mu_1} + \frac{1}{\mu_0^-})] \tag{34}$$

$$T_1(\tau, \vec{i}_4, \vec{i}_0) = (\frac{1}{\mu_4^-} - \frac{1}{\mu_0^-})^{-1}[\exp(-\frac{\tau}{\mu_0^-}) - \exp(-\frac{\tau}{\mu_4^-})]. \tag{35}$$

As the first approximations do not explicitly include the azimuth angle difference between $\vec{i}_1$ and $\vec{i}_0$, we denote, hereafter, the arguments of S_1 and T_1 as μ_0^- and μ_1. Exchanging the order of the arguments of the cosines does not change $S_1(\tau, \mu_1, \mu_0^-)$ and $T_1(\tau, \mu_4^-, \mu_0^-)$.

$$S_1(\tau, \mu_1, \mu_0^-) = S_1(\tau, \mu_0^-, \mu_1) \tag{36}$$

$$T_1(\tau, \mu_4^-, \mu_0^-) = T_1(\tau, \mu_0^-, \mu_4^-). \tag{37}$$

Integrating with respect to the azimuth directions, we obtain the recurrence relation of $\Delta S_n(\tau, \mu_1, \mu_0^-)$ and $\Delta T_n(\tau, \mu_4^-, \mu_0^-)$ for $n \geq 2$, given below,

$$\begin{aligned}
\Delta S_n(\tau, \mu_1, \mu_0^-) &= (\frac{1}{\mu_1} + \frac{1}{\mu_0^-})^{-1} \times \\
[& \int_0^1 \Delta S_{n-1}(\tau, \mu_3, \mu_0^-)\frac{d\mu_3}{2\mu_3} - \exp(-\frac{\tau}{\mu_0^-})\int_0^1 \Delta T_{n-1}(\tau, \mu_1, \mu_3)\frac{d\mu_3}{2\mu_3} \\
+ & \int_0^1 \Delta S_{n-1}(\tau, \mu_1, \mu_2^-)\frac{d\mu_2^-}{2\mu_2^-} - \exp(-\frac{\tau}{\mu_1})\int_0^1 \Delta T_{n-1}(\tau, \mu_2^-, \mu_0^-)\frac{d\mu_2^-}{2\mu_2^-} \\
+ & \int_0^1 \Delta S_{n-1}(\tau, \mu_1, \mu_2^-)\frac{d\mu_2^-}{2\mu_2^-}\int_0^1 \Delta S_{n-1}(\tau, \mu_3, \mu_0^-)\frac{d\mu_3}{2\mu_3} \\
- & \int_0^1 \Delta T_{n-1}(\tau, \mu_2^-, \mu_0^-)\frac{d\mu_2^-}{2\mu_2^-}\int_0^1 \Delta T_{n-1}(\tau, \mu_1, \mu_3)\frac{d\mu_3}{2\mu_3}],
\end{aligned} \tag{38}$$

and

$$\begin{aligned}
\Delta T_n(\tau, \mu_4^-, \mu_0^-) &= (\frac{1}{\mu_4^-} - \frac{1}{\mu_0^-})^{-1} \times \\
[& \int_0^1 \Delta T_{n-1}(\tau, \mu_2^-, \mu_0^-)\frac{d\mu_2^-}{2\mu_2^-} - \exp(-\frac{\tau}{\mu_4^-})\int_0^1 \Delta S_{n-1}(\tau, \mu_3, \mu_0^-)\frac{d\mu_3}{2\mu_3}
\end{aligned}$$

$$
\begin{aligned}
- & \int_0^1 \Delta T_{n-1}(\tau,\mu_4^-,\mu_2^-)\frac{d\mu_2^-}{2\mu_2^-} + \exp(-\frac{\tau}{\mu_0^-})\int_0^1 \Delta S_{n-1}(\tau,\mu_4^-,\mu_3)\frac{d\mu_3}{2\mu_3} \\
+ & \int_0^1 \Delta T_{n-1}(\tau,\mu_2^-,\mu_0^-)\frac{d\mu_2^-}{2\mu_2^-}\int_0^1 \Delta S_{n-1}(\tau,\mu_4^-,\mu_3)\frac{d\mu_3}{2\mu_3} \\
- & \int_0^1 \Delta T_{n-1}(\tau,\mu_4^-,\mu_2^-)\frac{d\mu_2^-}{2\mu_2^-}\int_0^1 \Delta S_{n-1}(\tau,\mu_3,\mu_0^-)\frac{d\mu_3}{2\mu_3}\ ],
\end{aligned} \tag{39}
$$

where $\Delta S_1(\tau,\mu_0^-,\mu_3) = S_1(\tau,\mu_0^-,\mu_3)$ and $\Delta T_1(\tau,\mu_0^-,\mu_4^-) = T_1(\tau,\mu_0^-,\mu_4^-)$.

We introduce new functions $U_2(\tau,\mu^*)$ and $V_2(\tau,\mu^*)$ as below

$$
U_2(\tau,\mu^*) = \int_0^1 (\frac{1}{\mu}+\frac{1}{\mu^*})^{-1}[1-\exp(-\tau)(\frac{1}{\mu}+\frac{1}{\mu^*})]\frac{d\mu}{2\mu} \tag{40}
$$

$$
V_2(\tau,\mu^*) = \int_0^1 (\frac{1}{\mu}-\frac{1}{\mu^*})^{-1}[[\exp(-\frac{\tau}{\mu^*})-\exp(-\frac{\tau}{\mu})]\frac{d\mu}{2\mu}. \tag{41}
$$

It is noted that $U_2(\tau,\mu^*)$ and $V_2(\tau,\mu^*)$ are not independent, and one is calculated from the other without integration.

$$
U_2(\tau,\mu^*) = \exp(-\frac{\tau}{\mu^*})V_2(\tau,-\mu^*) \tag{42}
$$

$$
V_2(\tau,\mu^*) = \exp(-\frac{\tau}{\mu^*})U_2(\tau,-\mu^*). \tag{43}
$$

Using $U_2(\tau,\mu^*)$ and $V_2(\tau,\mu^*)$, we obtain $S_2(\tau,\mu^*)$ and $T_2(\tau,\mu^*)$,

$$
\begin{aligned}
& \Delta S_2(\tau,\mu_1,\mu_0^-) = (\frac{1}{\mu_1}+\frac{1}{\mu_0^-})^{-1}\times \\
[\quad & U_2(\tau,\mu_0^-)+U_2(\tau,\mu_1)-\exp(-\frac{\tau}{\mu_1})V_2(\tau,\mu_0^-)-\exp(-\frac{\tau}{\mu_0^-})V_2(\tau,\mu_1) \\
+ \quad & U_2(\tau,\mu_0^-)U_2(\tau,\mu_1)-V_2(\tau,\mu_0^-)V_2(\tau,\mu_1)\ ],
\end{aligned} \tag{44}
$$

and

$$
\begin{aligned}
& \Delta T_2(\tau,\mu_4^-,\mu_0^-) = (\frac{1}{\mu_4^-}-\frac{1}{\mu_0^-})^{-1}\times \\
[\quad & V_2(\tau,\mu_0^-)-V_2(\tau,\mu_4^-)-\exp(-\frac{\tau}{\mu_4^-})U_2(\tau,\mu_0^-)+\exp(-\frac{\tau}{\mu_0^-})U_2(\tau,\mu_4^-) \\
+ \quad & V_2(\tau,\mu_0^-)U_2(\tau,\mu_4^-)-V_2(\tau,\mu_4^-)U_2(\tau,\mu_0^-)].
\end{aligned} \tag{45}
$$

Exchanging the order of the arguments in equations (44) and (45) does not change $\Delta S_2(\tau,\vec{i}_1,\vec{i}_0)$ and $\Delta T_2(\tau,\vec{i}_4,\vec{i}_0)$.

$$
\Delta S_2(\tau,\mu_1,\mu_0^-) = \Delta S_2(\tau,\mu_0^-,\mu_1) \tag{46}
$$

$$
\Delta T_2(\tau,\mu_4^-,\mu_0^-) = \Delta T_2(\tau,\mu_0^-,\mu_4^-). \tag{47}
$$

Repeating this procedure for $n = 3, 4, \cdots$, we obtain the recurrence relations for $U_n(\tau, \mu^*)$ and $V_n(\tau, \mu^*)$ that are defined as the first integration in equations (38) and (39).

$$U_n(\tau, \mu^*) = \int_0^1 \Delta S_{n-1}(\tau, \mu, \mu^*) \frac{d\mu}{2\mu} \tag{48}$$

$$V_n(\tau, \mu^*) = \int_0^1 \Delta T_{n-1}(\tau, \mu, \mu^*) \frac{d\mu}{2\mu} \tag{49}$$

$U_n(\tau, \mu^*)$ and $V_n(\tau, \mu^*)$ satisfy the recurrence relation given below,

$$\begin{aligned} U_{n+1}(\tau, \mu^*) = \int_0^1 & [U_n(\tau, \mu^*) + U_n(\tau, \mu) \\ & - \exp(-\frac{\tau}{\mu}) V_n(\tau, \mu^*) - \exp(-\frac{\tau}{\mu^*}) V_n(\tau, \mu) \\ & + U_n(\tau, \mu^*) U_n(\tau, \mu) - V_n(\tau, \mu^*) V_n(\tau, \mu)] (\frac{1}{\mu} + \frac{1}{\mu^*})^{-1} \frac{d\mu}{2\mu}, \end{aligned} \tag{50}$$

and

$$\begin{aligned} V_{n+1}(\tau, \mu^*) = \int_0^1 & [V_n(\tau, \mu^*) - V_n(\tau, \mu) \\ & - \exp(-\frac{\tau}{\mu}) U_n(\tau, \mu^*) + \exp(-\frac{\tau}{\mu^*}) U_n(\tau, \mu) \\ & + V_n(\tau, \mu^*) U_n(\tau, \mu) - V_n(\tau, \mu) U_n(\tau, \mu^*)] (\frac{1}{\mu} - \frac{1}{\mu^*})^{-1} \frac{d\mu}{2\mu}. \end{aligned} \tag{51}$$

We obtain $\Delta S_n(\tau, \vec{i}_1, \vec{i}_0)$ and $\Delta T_n(\tau, \vec{i}_4, \vec{i}_0)$ in the same manner as in the equations (44) and (45).

$$\begin{aligned} \Delta S_n(\tau, \mu_1, \mu_0^-) &= (\frac{1}{\mu_1} + \frac{1}{\mu_0^-})^{-1} \times \\ [\quad & U_n(\tau, \mu_0^-) + U_n(\tau, \mu_1) - \exp(-\frac{\tau}{\mu_1}) V_n(\tau, \mu_0^-) - \exp(-\frac{\tau}{\mu_0^-}) V_n(\tau, \mu_1) \\ + \quad & U_n(\tau, \mu_0^-) U_n(\tau, \mu_1) - V_n(\tau, \mu_0^-) V_n(\tau, \mu_1) \ ], \end{aligned} \tag{52}$$

and

$$\begin{aligned} \Delta T_n(\tau, \mu_4^-, \mu_0^-) &= (\frac{1}{\mu_4^-} - \frac{1}{\mu_0^-})^{-1} \times \\ [\quad & V_n(\tau, \mu_0^-) - V_n(\tau, \mu_4^-) - \exp(-\frac{\tau}{\mu_4^-}) U_n(\tau, \mu_0^-) + \exp(-\frac{\tau}{\mu_0^-}) U_n(\tau, \mu_4^-) \\ + \quad & V_n(\tau, \mu_0^-) U_n(\tau, \mu_4^-) - V_n(\tau, \mu_4^-) U_n(\tau, \mu_0^-)]. \end{aligned} \tag{53}$$

Changing the order of arguments of the cosines in equations (52) and (53) does not change $\Delta S_n(\tau,\mu_1,\mu_0^-)$ and $\Delta T_n(\tau,\mu_4^-,\mu_0^-)$,

$$\Delta S_n(\tau,\mu_1,\mu_0^-) = \Delta S_n(\tau,\mu_0^-,\mu_1) \tag{54}$$

$$\Delta T_n(\tau,\mu_4^-,\mu_0^-) = \Delta T_n(\tau,\mu_0^-,\mu_4^-). \tag{55}$$

From equations (42), (43), (50), and (51),we obtain an relation between U_n and V_n.

$$\exp(-\tau/\mu^*)U_n(\tau,-\mu^*) = V_n(\tau,\mu^*) \tag{56}$$

Once we integrate $U_n(\tau,\mu^*)$, we can obtain $V_n(\tau,\mu^*)$ by a simple multiplication by $\exp(-\tau/\mu^*)$, and vice versa.

Changing the variable and the parameter from μ to $p = 1/\mu$ and from μ^* to $p^* = 1/\mu^*$, we rewrite the recurrence relation as

$$\begin{aligned} U_n(\tau,p^*) &= \int_1^\infty [\ U_{n-1}(\tau,p^*) - \exp(-\tau p)V_{n-1}(\tau,p^*) \\ &+ \ U_{n-1}(\tau,p) - \exp(-\tau p^*)V_{n-1}(\tau,p) \\ &+ \ U_{n-1}(\tau,p^*)U_{n-1}(\tau,p) - V_{n-1}(\tau,p^*)V_{n-1}(\tau,p)\ ]\frac{dp}{2(p+p^*)p}, \end{aligned} \tag{57}$$

with the initial integration,

$$U_2(\tau,p^*) = \int_1^\infty \frac{1-\exp(-\tau(p+p^*))}{(p+p^*)}\frac{dp}{2p}. \tag{58}$$

Since the denominators of the integrand in the above two equations are $o(p^2)$, the integrations converge as p goes to ∞, if the numerators are bounded. It is shown later that all the terms in the numerator are bounded.

3.2 Integration of $U_n(\tau,\mu^*)$ and $V_n(\tau,\mu^*)$

3.2.1 Integration of $U_2(\tau,\mu^*)$ and $V_2(\tau,\mu^*)$

We now evaluate $V_2(\tau,p^*)$.

$$V_2(\tau,p^*) = \int_1^\infty \frac{\exp(-\tau p^*) - \exp(-\tau p)}{(p-p^*)}\frac{dp}{2p} \tag{59}$$

The integrand has an apparent singularity at $p = p^*$, but not a true singularity. Expanding the exponential function into the power series, the apparent singularity vanishes as shown below

$$V_2(\tau, p^*) = \sum_{n=1}^{\infty} \frac{(-\tau)^{n-1}}{n!} \int_1^{\infty} \sum_{r=0}^{n-1} \frac{p^r p^{*(n-1-r)}}{p} \frac{dp}{2}. \tag{60}$$

The function $\exp(-\tau p)$ can be expanded in a power series around 0, and the expanded series converges uniformly except for $-\tau p = \pm\infty$. We can thus interchange the order of summation and integration. Expanding the integrand into a power series in p, we obtain the following (refer to derivation 1).

$$\begin{aligned} V_2(\tau, p^*) &= \int_1^{\infty} [\sum_{n=0}^{\infty} \frac{(-\tau p^*)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-\tau p)^n}{n!}] \frac{dp}{2(p-p^*)p} \\ &= \frac{(-1)}{2p*} [\sum_{n=1}^{\infty} (\sum_{r=1}^{\infty} \frac{(-\tau p)^r}{r(n+r)!})(-\tau p^*)^n + \log p \sum_{n=1}^{\infty} \frac{(-\tau p^*)^n}{n!}]_1^{\infty} \end{aligned} \tag{61}$$

Substituting $p = 1$ into the equation above, we easily obtain the lower limit integral value. For the upper limit at $p = \infty$, we need the following further derivation (refer to Derivation 2 and section 3.3).

$$\begin{aligned} \lim_{p\to\infty} [(\sum_{r=1}^{\infty} \frac{(-\tau p)^r n!}{r(n+r)!}) + \log p] &= \lim_{p\to\infty} [\sum_{r=1}^{\infty} \frac{(-\tau p)^r}{r r!} + \log p] + \sum_{q=1}^{n} \frac{1}{q} \\ &= -\gamma - \log\tau + \sum_{q=1}^{n} \frac{1}{q} = -C + \sum_{q=1}^{n} \frac{1}{q} \end{aligned} \tag{62}$$

where $\gamma = 0.577216$ is Euler's constant and $C = \log\tau + \gamma$. We thus obtain $V_2(\tau, p^*)$ as

$$\begin{aligned} V_2(\tau, p^*) &= \frac{(-1)}{2p*} \sum_{n=1}^{\infty} [-C + \sum_{m=1}^{n} \frac{1}{m} - \sum_{m=1}^{\infty} \frac{(-\tau)^m n!}{m(n+m)!})] \frac{(-\tau p^*)^n}{n!} \\ &= \frac{\tau}{2} \sum_{n=0}^{\infty} [-C + \sum_{m=1}^{n+1} \frac{1}{m} - \sum_{m=1}^{\infty} \frac{(-\tau)^m (n+1)!}{m(n+1+m)!})] \frac{(-\tau p^*)^n}{(n+1)!}. \end{aligned} \tag{63}$$

Substituting the above equations into $U_2(\tau, p^*) = \exp(-\tau p^*) V_2(\tau, -p^*)$, we obtain the function $U_2(\tau, p^*)$ (refer to Derivation 3).

$$U_2(\tau, p^*) = \frac{\tau}{2} \sum_{n=0}^{\infty} \{-C - \sum_{m=1}^{\infty} \frac{(-\tau)^m}{m m!} + \sum_{m=0}^{\infty} \frac{(-\tau)^m}{(n+1+m)m!}\} \frac{(-\tau p^*)^n}{(n+1)!} \tag{64}$$

The truncated forms of $U_2(\tau,p^*)$ and $V_2(\tau,p^*)$ in the power series in $(-\tau)$ are given below

$$\begin{aligned} U_2(\tau,p^*) &= \frac{1}{2}[\,(C-1)(-\tau)+\{1+(C-\frac{1}{2})p^*\}\frac{(-\tau)^2}{2!} \\ &+ \{\frac{1}{2}+2p^*+(C-\frac{1}{3})p^{*2}\}\frac{(-\tau)^3}{3!}\,] \end{aligned} \tag{65}$$

and

$$\begin{aligned} V_2(\tau,p^*) &= \frac{1}{2}[(C-1)(-\tau)+\{1-(-C+\frac{3}{2})p^*\}\frac{(-\tau)^2}{2!} \\ &+ \{\frac{1}{2}+p^*-(-C+\frac{11}{6})p^{*2}\}\frac{(-\tau)^3}{3!}]. \end{aligned} \tag{66}$$

It is noted that $U_2(\tau,p^*)$ and $V_2(\tau,p^*)$ are polynomials in $(-\tau)$ with no 0th power of $(-\tau)$. The lowest power of their expanded series is not simply τ but $(-\log\tau)\tau$. Near $\tau\sim 0$, $U_2(\tau,p^*)$ and $V_2(\tau,p^*)$ are given as

$$U_2(\tau,p^*)\sim V_2(\tau,p^*)\sim\frac{-(\log\tau)\tau}{2}. \tag{67}$$

When τ approaches ∞, $U_2(\tau,p*)$ converges to a fixed value.

$$\lim_{\tau\to\infty}\int_1^\infty\frac{1-\exp(-\tau(p+p^*))}{2(p+p^*)p}dp=\int_1^\infty\frac{dp}{2(p+p^*)p} \tag{68}$$

Thus we have

$$U_2(\infty,p^*)=\frac{\log(1+p^*)}{2p^*}. \tag{69}$$

When τ approaches ∞, $V_2(\infty,p^*)$ converges to 0, due to the equation (59),

$$V_2(\infty,p^*)=0. \tag{70}$$

3.2.2 Integration of $U_3(\tau,p^*)$ and $V_3(\tau,p^*)$

The third iteration, $V_3(\tau,p^*)$, is evaluated by integrating second iterations $U_2(\tau,p^*)$ and $V_2(\tau,p^*)$ given below.

$$
\begin{aligned}
&V_3(\tau, p^*) \\
= &\int_1^\infty [\ V_2(\tau,p^*) - \exp(-\tau p)U_2(\tau,p^*) + \exp(-\tau p^*)U_2(\tau,p) - V_2(\tau,p) \\
&+U_2(\tau,p)V_2(\tau,p^*) - V_2(\tau,p)U_2(\tau,p^*)\ ]\frac{dp}{2p(p-p^*)} \\
= &\int_1^\infty \frac{V_2(\tau,p^*)-V_2(\tau,p)}{p-p^*}\frac{dp}{2p} - \exp(-\tau p^*)\int_1^\infty \frac{U_2(\tau,p^*)-U_2(\tau,p)}{p-p^*}\frac{dp}{2p} \\
+ &\ U_2(\tau,p^*)V_2(\tau,p^*) + U_2(\tau,p^*)\int_1^\infty \frac{V_2(\tau,p^*)-V_2(\tau,p)}{p-p^*}\frac{dp}{2p} \\
- &\ V_2(\tau,p^*)\int_1^\infty \frac{U_2(\tau,p^*)-U_2(\tau,p)}{p-p^*}\frac{dp}{2p} \qquad (71)
\end{aligned}
$$

We need two integrations,

$$
U_3^-(\tau,p^*) = \int_1^\infty \frac{U_2(\tau,p^*)-U_2(\tau,p)}{p-p^*}\frac{dp}{2p} \qquad (72)
$$

and

$$
V_3^-(\tau,p^*) = \int_1^\infty \frac{V_2(\tau,p^*)-V_2(\tau,p)}{p-p^*}\frac{dp}{2p}. \qquad (73)
$$

Using $U_3^-(\tau,p^*)$ and $V_3^-(\tau,p^*)$, we obtain $V_3(\tau,p^*)$ expressed below.

$$
\begin{aligned}
&V_3(\tau,p^*) = U_2(p^*)V_2(p^*) + V_3^-(\tau,p^*) - \exp(-\tau p^*)U_3^-(\tau,p^*) \\
+ &\ U_2(p^*)V_3^-(\tau,p^*) - V_2(p^*)U_3^-(\tau,p^*) \qquad (74)
\end{aligned}
$$

As $U_2(\tau,p^*)$ and $V_2(\tau,p^*)$ are expressed as summations of exponential-like functions in the equations (64) and (63), we need to evaluate each exponential-like function to obtain both $V_3^-(\tau,p^*)$ and $U_3^-(\tau,p^*)$. Designating the coefficient of each exponential-like functions as a_n, the indefinite integration is performed as below (refer to derivation 4).

$$
\begin{aligned}
&\int [\frac{\tau}{2}\sum_{r=0}^\infty a_{r+1}\frac{(-\tau p^*)^r}{(r+1)!} - \frac{\tau}{2}\sum_{r=0}^\infty a_{r+1}\frac{(-\tau p)^r}{(r+1)!}]\frac{dp}{2(p-p^*)p} \\
= &\ \frac{-\tau}{4p*}\sum_{m=1}^\infty [a_{m+1}(\log p) + \sum_{q=1}^\infty \frac{a_{m+1+q}(m+1)!(-\tau p)^q}{q(m+1+q)!})]\frac{(-\tau p^*)^m}{(m+1)!} \\
= &\ \frac{(-\tau)^2}{4}\sum_{m=0}^\infty [a_{m+2}(\log p) + \sum_{q=1}^\infty \frac{a_{m+2+q}(m+2)!(-\tau p)^q}{q(m+2+q)!})]\frac{(-\tau p^*)^m}{(m+2)!} \qquad (75)
\end{aligned}
$$

Inserting $p = 1$ in the above equation, we obtain the coefficient c_m at the lower integral limit given below.

$$c_m = \sum_{q=1}^{\infty} \frac{a_{m+q} m! (-\tau)^q}{q(m+q)!} \tag{76}$$

For the upper limit, the converged value b_m as p approaches to ∞ is evaluated as

$$b_m = \lim_{p \to \infty} [\sum_{q=1}^{\infty} \frac{a_{m+q} m! (-\tau p)^q}{q(m+q)!} + a_m \log p]. \tag{77}$$

The integration in equation(75) is then expressed as a new exponential-like function given below,

$$= \frac{(-\tau)^2}{4} \sum_{m=0}^{\infty} [b_{m+2} - c_{m+2}] \frac{(-\tau p^*)^m}{(m+2)!}. \tag{78}$$

Evaluation of b_m for several exponential functions is discussed in subsection 3.3.

$U_2(\tau, p^*)$ is composed of the three exponential-like functions expressed in equation (64). Substituting b_m and c_m for those exponential-like functions that constitute $U_2(\tau, p^*)$, we obtain $U_3^-(\tau, p^*)$ as below.

$$\begin{aligned} U_3^-(\tau, p^*) &= \frac{(-\tau)^2}{4}(-C - \sum_{m=1}^{\infty} \frac{(-\tau)^m}{m m!}) \sum_{n=0}^{\infty}(-C + \sum_{r=1}^{n+2} \frac{1}{r}) \frac{(-\tau p^*)^n}{(n+2)!} \\ &+ \frac{(-\tau)^2}{4} \sum_{n=0}^{\infty} [\sum_{m=0}^{\infty} [-C + \sum_{r=1}^{n+2} \frac{1}{r} + \frac{1}{n+2+m}] \frac{(-\tau)^m}{(n+2+m)m!}] \frac{(-\tau p^*)^n}{(n+2)!} \\ &- \frac{(-\tau)^2}{4} \sum_{n=0}^{\infty}(-C - \sum_{m=1}^{\infty} \frac{(-\tau)^m}{m m!})[\sum_{q=1}^{\infty} \frac{(n+2)!(-\tau)^q}{q(n+2+q)!})] \frac{(-\tau p^*)^n}{(n+2)!} \\ &- \frac{(-\tau)^2}{4} \sum_{n=0}^{\infty} [\sum_{q=1}^{\infty} \{ \sum_{m=0}^{\infty} \frac{(-\tau)^m}{(n+2+q+m)m!} \} \frac{(n+2)!(-\tau)^q}{q(n+2+q)!}] \frac{(-\tau p^*)^n}{(n+2)!} \end{aligned} \tag{79}$$

In the same manner, we obtain $V_3^-(\tau, p^*)$ as below.

$$\begin{aligned} V_3^-(\tau, p^*) &= \frac{(-\tau)^2}{4}[-C \sum_{n=0}^{\infty}(-C + \sum_{r=1}^{n+2} \frac{1}{r})] \frac{(-\tau p^*)^n}{(n+2)!} \\ &+ \frac{(-\tau)^2}{4} \sum_{n=0}^{\infty} [(\sum_{r=1}^{n+2} \frac{1}{r})(-C + \sum_{q=1}^{n+2} \frac{1}{q}) + \sum_{r=1}^{n+2} \frac{1}{r^2} - \frac{\pi^2}{6}] \frac{(-\tau p^*)^n}{(n+2)!} \end{aligned} \tag{80}$$

$$
\begin{aligned}
- & \frac{(-\tau)^2}{4}\sum_{n=0}^{\infty}[\sum_{m=1}^{\infty}(-C+\sum_{q=1}^{n+2}\frac{1}{q})\frac{(-\tau)^m(n+2)!}{m(n+2+m)!}]\frac{(-\tau p^*)^n}{(n+2)!} \\
- & \frac{(-\tau)^2}{4}\sum_{n=0}^{\infty}\sum_{q=1}^{\infty}(-C+\sum_{r=1}^{n+2+q}\frac{1}{r})\frac{(n+2)!(-\tau)^q}{q(n+2+q)!}\}\frac{(-\tau p^*)^n}{(n+2)!} \\
+ & \frac{(-\tau)^2}{4}\sum_{n=0}^{\infty}[\sum_{q=1}^{\infty}\{\sum_{m=1}^{\infty}\frac{(n+2+q)!(-\tau)^m}{m(n+2+q+m)!}\}\frac{(n+2)!(-\tau)^q}{q(n+2+q)!})]\frac{(-\tau p^*)^n}{(n+2)!}
\end{aligned}
$$

Substituting $U_3^-(\tau,p^*)$ and $V_3^-(\tau,p^*)$ into equation (74), we obtain $V_3(\tau,p^*)$, truncated to the third power in τ.

$$
\begin{aligned}
V_3(\tau,p^*) &= \frac{1}{4}[2(-C)^2+5(-C)+\frac{9}{2}-\frac{\pi^2}{6}]\frac{(-\tau)^2}{2!} \\
&+\frac{1}{4}[\{-\frac{3}{2}(-C)^2+(1\frac{3}{4}+\frac{\pi^2}{4})(-C)+1\frac{11}{12}+\frac{\pi^2}{4}\} \\
&+\{3(-C)^2+7\frac{1}{2}(-C)+7-\frac{\pi^2}{6}\}p^*]\frac{(-\tau)^3}{3!}
\end{aligned}
\tag{81}
$$

$U_3(\tau,p^*)$ is obtained by $\exp(-\tau p^*)V_3(\tau,-p^*)$.

$$
\begin{aligned}
U_3(\tau,p^*) &= \frac{1}{4}[2(-C)^2+5(-C)+\frac{9}{2}-\frac{\pi^2}{6}]\frac{(-\tau)^2}{2!} \\
&+ \frac{1}{4}[\{-\frac{3}{2}(-C)^2+(1\frac{3}{4}+\frac{\pi^2}{4})(-C)+1\frac{11}{12}+\frac{\pi^2}{4}\} \\
&+ \frac{1}{4}[3(-C)^2+7\frac{1}{2}(-C)+\frac{13}{2}-\frac{\pi^2}{3}]p^*\frac{(-\tau)^3}{3!}
\end{aligned}
\tag{82}
$$

It is noted that the lowest power of $U_3(\tau,p^*)$ and $V_3(\tau,p^*)$ in $(-\tau)$ is the second order,or the first powers in $(-\tau)$ vanish. The lowest power of their expanded series is not simply τ^2 but $(-\tau\log\tau)^2$.

Near $\tau\sim 0$, the following relation holds

$$
U_3(\tau,p^*)\sim V_3(\tau,p^*)\sim\frac{1}{4}(-\log\tau)^2\tau^2. \tag{83}
$$

We can then choose the ratio of $U_3(\tau,p^*)/U_2(\tau,p^*)$, as less than 1 ,

$$
\frac{U_3(\tau,p^*)}{U_2(\tau,p^*)}<1. \tag{84}
$$

The similar relation for V holds

$$\frac{V_3(\tau, p^*)}{V_2(\tau, p^*)} < 1. \tag{85}$$

At $\tau = \infty$, $V_3(\tau, p^*)$ has the following form.

$$V_3(\infty, p^*) = \int_1^\infty [\ V_2(\infty, p^*) - V_2(\infty, p) - \exp(-\infty \cdot p) U_2(\infty, p^*) \tag{86}$$

$$+ \quad \exp(-\infty \cdot p^*) U_2(\infty, p) + U_2(\infty, p) V_2(\infty, p^*) - V_2(\infty, p) U_2(\infty, p^*)\ ] \frac{dp}{2p(p - p^*)}.$$

Since $V_2(\infty, p^*) = 0$, and $U_2(\infty, p^*)$ is bounded, we conclude,

$$\lim_{\tau \to \infty} V_3(\tau, p^*) = V_3(\infty, p^*) = 0. \tag{87}$$

Substituting $U_2(\infty, p^*)$ into the equation (57), we have $U_3(\infty, p^*)$

$$\begin{aligned} U_3(\infty, p^*) &= \int_1^\infty \frac{U_2(\infty, p^*) dp}{2(p + p^*)p} + \int_1^\infty \frac{U_2(\infty, p) dp}{2(p + p^*)p} \\ &+ \int_1^\infty \frac{U_2(\infty, p^*) U_2(\infty, p) dp}{2(p + p^*)p} \\ &= U_2(\infty, p^*)^2 + (1 + U_2(\infty, p^*)) \int_1^\infty \frac{U_2(\infty, p) dp}{2(p + p^*)p} \end{aligned} \tag{88}$$

The last integration is evaluated as below (refer to Derivation (5)).

$$\begin{aligned} &\int_1^\infty \frac{U_2(\infty, p) dp}{2(p + p^*)p} = \int_1^\infty \frac{\log(p + 1) dp}{4(p + p^*)p^2} \\ &= (1/4p^*)[2 \log 2 - \frac{1}{p^*}\{\frac{\pi^2}{12} - \frac{(\log 2)^2}{2} + (\log 2) \log(p^* + 1) - \sum_{m=1}^\infty \frac{1}{m^2}(\frac{1 - p^*}{2})^m\}] \\ &\qquad (1 \le p^* \le 3) \\ &= (1/4p^*)[2 \log 2 - \frac{1}{p^*}[\frac{(\log 2)^2}{2} + \frac{\pi^2}{4} - \log 2 \log \frac{p^* - 1}{p^* + 1} + \sum_{m=1}^\infty \frac{1}{m^2}(\frac{-2}{p^* - 1})^m] \\ &\qquad (p^* \ge 3) \end{aligned} \tag{89}$$

Due to the equation (69), $U_2(\infty, p^*) \le U_2(\infty, 1)$, we also obtain

$$U_3(\infty, p^*) \le U_3(\infty, 1). \tag{90}$$

$U_3(\infty, 1)$ and $U_2(\infty, 1)$ satisfy the following relation.

$$\begin{aligned} U_3(\infty, 1) &< U_2(\infty, 1)\{U_2(\infty, 1) + U_2(\infty, 1)(1 + U_2(\infty, 1))\} \\ &< U_2(\infty, 1). \end{aligned} \tag{91}$$

3.2.3 Higher Iterations

Repeating the same procedure that derives the third approximation, we obtain the higher approximation,

$$V_{n+1}(\tau,p^*) = U_2(\tau,p^*)V_n(\tau,p^*) + V^-_{n+1}(\tau,p^*) - \exp(-\tau p^*)U^-_{n+1}(\tau,p^*)$$
$$+ \quad U_n(p^*)V^-_{n+1}(\tau,p^*) - V_n(p^*)U^-_{n+1}(\tau,p^*), \tag{92}$$

where the two integrations are evaluated as

$$U^-_{n+1}(\tau,p^*) = \int_1^\infty \frac{U_n(\tau,p^*) - U_n(\tau,p)}{2p(p-p^*)}dp \tag{93}$$

$$V^-_{n+1}(\tau,p^*) = \int_1^\infty \frac{V_n(\tau,p^*) - V_n(\tau,p)}{2p(p-p^*)}dp. \tag{94}$$

$U_n(\tau,p^*)$ and $V_n(\tau,p^*)$ are expressed as summations of exponential-like functions. We designate $U_n(\tau,p^*)$ and $V_n(\tau,p^*)$ as $W_n(\tau,p^*)$ and their coefficients as a_n given below,

$$W_n(\tau,p^*) = \frac{(\tau)^{n-1}}{2^{n-1}}\sum_{m=0}^{\infty} a_{m+n-1}(\tau)\frac{(-\tau p^*)^m}{(m+n-1)!}. \tag{95}$$

Substituting $W_n(\tau,p^*)$ into the integration of $U^-_{n+1}(\tau,p^*)$ or $V^-_{n+1}(\tau,p^*)$, we obtain $W_{n+1}(\tau,p^*)$ as

$$W^-_{n+1}(\tau,p^*) = \int_1^\infty \frac{W_n(\tau,p^*) - W_n(\tau,p)}{2p(p-p^*)}dp,$$
$$= \frac{(\tau)^n}{2^n}\sum_{m=0}^{\infty}[b_{m+n} - c_{m+n}]\frac{(-\tau p^*)^m}{(m+n)!} \tag{96}$$

where b_{m+n} and c_{m+n} are given below (Derivation 4).

$$b_{m+n} = \lim_{p\to\infty}[a_{m+n}\log p + \sum_{q=1}^{\infty}\frac{a_{m+q+n}(m+n)!(-\tau p)^q}{q(m+q+n)!}] \tag{97}$$

$$c_{m+n} = \sum_{q=1}^{\infty}\frac{a_{m+q+n}(m+n)!(-\tau)^q}{q(m+q+n)!} \tag{98}$$

For the upper limit, we evaluate following,

$$b_m = \lim_{p\to\infty}[a_m\log p + \sum_{q=1}^{\infty}\frac{a_{m+q}(m)!(-\tau p)^q}{q(m+q)!}]. \tag{99}$$

As we point out in the section 1, the problem of radiative transfer is deduced to evaluate b_m in the above equation. The convergence of the series b_m is affirmed because the denominator of the integrand in equation (96) is $o(p^2)$ and the numerator is bounded.

We begin the iterative integration with $V_2(\tau, p^*)$ in equation (61), and all the coefficients of the power series of the integrand are 1. The resulting integration of $V_2(\tau, p^*)$ is given in equation (63) and consists of three functions. The coefficients of those functions in the power series expansions are given below,

$$-C, \qquad \sum_{m=1}^{n+1} \frac{1}{m}, \qquad and \qquad \sum_{m=1}^{\infty} \frac{(-\tau)^m (n+1)!}{m(n+1+m)!}. \tag{100}$$

By evaluating $\exp(-\tau p^*)V_2(\tau, -p^*)$, we obtain $U_2(\tau, -p^*)$ given by equation (64).

It is noted that the lowest power of $U_2(\tau, p^*)$ and $V_2(\tau, p^*)$ in $(-\tau)$ is

$$\frac{-\tau \log \tau}{2}. \tag{101}$$

For higher iterative integration, we integrate equations (93) and (94). The resulting coefficients of those functions that constitute $V_{n+1}^-(\tau, p^*)$ and $U_{n+1}^-(\tau, p^*)$ are given below,

$$(-C)^n, \quad (-C)^{n-1} \sum_{m=1}^{n+1} \frac{1}{m}, \quad (-C)^{n-1} \sum_{m=1}^{\infty} \frac{(-\tau)^m (n+1)!}{m(n+1+m)!}, \quad \cdots \quad . \tag{102}$$

Substituting $V_{n+1}^-(\tau, p^*)$ and $U_{n+1}^-(\tau, p^*)$ into equation (92), we obtain $V_{n+1}(\tau, p^*)$. By the relation $U_{n+1}(\tau, p^*) = \exp(-\tau p*)V_{n+1}(-\tau, p^*)$, we obtain $U_{n+1}(\tau, p^*)$.

It is noted that and the lowest powers in $(-\tau)$ of $V_{n+1}(\tau, p^*)$ and $U_{n+1}(\tau, p^*)$ are $U_2(\tau, p^*)V_n(\tau, p^*)$. Thus near $\tau \sim 0$, the following relation holds.

$$U_{n+1}(\tau, p^*) \sim V_{n+1}(\tau, p^*) \sim \frac{(-\log \tau)}{2} \tau U_n(\tau, p^*) \tag{103}$$

We can then choose a ratio of $U_{n+1}(\tau, p^*)/U_n(\tau, p^*)$ as less than 1,

$$\frac{U_{n+1}(\tau, p^*)}{U_n(\tau, p^*)} < 1, \tag{104}$$

$$\frac{V_{n+1}(\tau, p^*)}{V_n(\tau, p^*)} < 1. \tag{105}$$

At $\tau = \infty$, $V_3(\tau, p^*)$ has the following form.

$$V_3(\tau, p^*) = \int_1^\infty [\ V_2(\tau, p^*) - V_2(\tau, p) - \exp(-\tau p)U_2(\tau, p^*) \tag{106}$$

$$+ \ \exp(-\tau p^*)U_2(\tau, p) + U_2(\tau, p)V_2(\tau, p^*) - V_2(\tau, p)U_2(\tau, p^*)\ ]\frac{dp}{2p(p - p^*)}$$

$V_2(\tau, p^*) = 0$, and $U_2(\tau, p^*)$ is bounded, we conclude

$$\lim_{\tau \to \infty} V_3(\tau, p^*) = V_3(\infty, p^*) = 0. \tag{107}$$

Substituting $U_n(\infty, p^*)$ into equation (57), we have $U_{n+1}(\infty, p^*)$

$$U_{n+1}(\infty, p^*) = \int_1^\infty \frac{U_n(\infty, p^*)dp}{2(p + p^*)p} + \int_1^\infty \frac{U_n(\infty, p)dp}{2(p + p^*)p}$$

$$+ \int_1^\infty \frac{U_n(\infty, p^*)U_n(\infty, p)dp}{2(p + p^*)p}$$

$$= \ U_2(\infty, p^*)U_n(\infty, p^*) + (1 + U_n(\infty, p^*)) \int_1^\infty \frac{U_n(\infty, p)dp}{2(p + p^*)p} \tag{108}$$

Because of the relation in equation (90), the following recurrent relation holds.

$$U_n(\infty, p^*) \le U_n(\infty, 1) \tag{109}$$

$U_3(\infty, 1)$ is evaluated as below.

$$\begin{aligned} U_{n+1}(\infty, 1) &< U_n(\infty, 1)\{U_2(\infty, 1) + U_2(\infty, 1)(1 + U_n(\infty, 1))\} \\ &< U_n(\infty, 1)\{U_2(\infty, 1) + U_2(\infty, 1)(1 + U_2(\infty, 1))\} \\ &< U_n(\infty, 1) \end{aligned} \tag{110}$$

3.3 Convergence of Integrated Functions at Infinity

The converged values of integrated functions, which are necessary for the second and third iterations, are discussed in this section. The detailed derivations are given in the references.[10] [11] [12]

3.3.1 Converged Values (1)

We evaluate the converged value for the series given below.

$$\lim_{p \to \infty} [\sum_{m=1}^{\infty} \frac{(-\tau p)^m}{m m!} + \log p\ ] \tag{111}$$

We begin to evaluate $1/1 + 1/2 + \cdots + 1/n$ as

$$\begin{aligned}\sum_{r=1}^{n} \frac{1}{r} &= \int_0^1 (1 + x + x^2 + + \cdots + x^{n-1}) dx = \int_0^1 \frac{1 - x^n}{1 - x} dx \\ &= \int_0^1 \frac{1 - (1 - y)^n}{y} dy = \int_0^n \frac{1 - (1 - \frac{p}{n})^n}{p} dp, \end{aligned} \tag{112}$$

where the variables are changed ($y = 1 - x$, $p = ny$). Subtracting $\int_1^n \frac{dp}{p}$ from both sides in the equation above, we obtain

$$\sum_{r=1}^{n} \frac{1}{r} - \int_1^n \frac{dp}{p} = \int_0^n \frac{1 - (1 - \frac{p}{n})^n}{p} dp - \log n. \tag{113}$$

As $n \to \infty$, the term $(1 - \frac{p}{n})^n$ approaches $\exp(-p)$ and the left hand side approaches γ,

$$\gamma = \lim_{n \to \infty} [\int_0^n \frac{1 - \exp(-p)}{p} dp - \log n]. \tag{114}$$

where $\gamma = 0.577216$ is Euler's constant. Dividing the integral interval $(0, \tau p)$ and $(\tau p, n)$, we obtain

$$\begin{aligned}\gamma &= \lim_{n \to \infty} [\int_0^{\tau p} \frac{1 - \exp(-p)}{p} dp + \int_{\tau p}^{n} \frac{1 - \exp(-p)}{p} dp - \log n] \\ &= -\sum_{m=1}^{\infty} \frac{(-\tau p)^m}{m m!} - \log \tau - \log p - \lim_{n \to \infty} \int_{\tau p}^{n} \frac{\exp(-p)}{p} dp. \end{aligned} \tag{115}$$

Thus we obtain the converged value

$$\lim_{p \to \infty} (\sum_{m=1}^{\infty} \frac{(-\tau p)^m}{m m!} + \log p) = -\log(\tau) - \gamma = -C, \tag{116}$$

where C is designated as $\log \tau + \gamma$.

3.3.2 Converged Values (2)

We evaluate the converged value for the series given below

$$\lim_{p \to \infty} \sum_{q=1}^{\infty} \frac{(-p)^q}{(q + r) q!} \tag{117}$$

For $r = 1$, we derive the series as below.

$$\sum_{q=1}^{\infty} \frac{(-p)^q}{(q+1)!} = \frac{1}{(-p)} \sum_{q=2}^{\infty} \frac{(-p)^q}{q!} = \frac{1}{(-p)} \{\exp(-p) - \sum_{q=0}^{1} \frac{(-p)^q}{q!}\} \quad (118)$$

As $p \to \infty$, we obtain the converged value for $r = 1$.

$$\lim_{p \to \infty} \sum_{q=1}^{\infty} \frac{(-p)^q}{(q+1)!} = -1 \quad (119)$$

The recurrence relation for $r \neq 1$ is given below (refer to Derivation 6).

$$\begin{aligned} & \sum_{q=1}^{\infty} \frac{(-p)^q}{(q+r)q!} \\ = & -\frac{r-1}{(-p)} \sum_{q=1}^{\infty} \frac{(-p)^q}{(q+r-1)q!} + \frac{r-1}{r} + \sum_{q=1}^{\infty} \frac{(-p)^q}{(q+1)!} \end{aligned} \quad (120)$$

As $p \to \infty$, we obtain the converged value.

$$\lim_{p \to \infty} \sum_{q=1}^{\infty} \frac{(-p)^q}{(q+r)q!} = -\frac{1}{r} \quad (121)$$

Next, we evaluate the converged value for the series, given below.

$$g_r^n(p) = \sum_{m=1}^{\infty} \frac{n!(-p)^m}{(r+n+m)(n+m)!} \quad (122)$$

We derive it as below.

$$\begin{aligned} g_r^n &= \lim_{p \to \infty} \frac{n!}{(-p)^n} [\sum_{q=1}^{\infty} \frac{(-p)^q}{(r+q)(q)!} - \sum_{q=1}^{n} \frac{(-p)^q}{(r+q)(q)!}] \\ &= -\frac{1}{n+r} + \lim_{p \to \infty} \frac{n!}{(-p)^n} [\sum_{q=1}^{\infty} \frac{(-p)^q}{(r+q)q!}] \end{aligned} \quad (123)$$

As $p \to \infty$, we obtain the converged value.

$$g_r^n = -\frac{1}{n+r} \quad (124)$$

3.3.3 Converged Values (3)

We evaluate the convergence of b_m in equation (76) for different a_n.

First we evaluate the converged value for $a_n = 1$.

$$b_m = \lim_{p\to\infty}[\sum_{q=1}^{\infty}\frac{m!(-\tau p)^q}{q(m+q)!} + \log p] \tag{125}$$

Due to the Derivation 2, we obtain the following.

$$b_m = -C + \sum_{r=1}^{m}\frac{1}{r} \tag{126}$$

Second we evaluate the converged value for $a_n = \sum_{l=1}^{n}\frac{1}{l}$.

$$b_m = \lim_{p\to\infty}[\sum_{q=1}^{\infty}(\sum_{l=1}^{m+q}\frac{1}{l})\frac{m!(-\tau p)^q}{q(m+q)!} + (\sum_{l=1}^{m}\frac{1}{l})\log p] \tag{127}$$

Due to the Deviation 7, we obtain the following.

$$b_m = (\sum_{r=1}^{m}\frac{1}{r})(-C + \sum_{q=1}^{m}\frac{1}{q}) + \sum_{r=1}^{m}\frac{1}{r^2} - \frac{\pi^2}{6} \tag{128}$$

Third we evaluate the converged value for $a_n = \sum_{l=1}^{\infty}\frac{(-\tau)^l n!}{l(n+l)!}$. The derivation is given below.

$$\begin{aligned} b_m &= \lim_{p\to\infty}[\sum_{q=1}^{\infty}\sum_{r=1}^{\infty}\frac{(-\tau)^r(m+q)!}{r(m+q+r)!}\frac{n!(-\tau p)^q}{q(m+q)!} + \sum_{r=1}^{\infty}\frac{(-\tau)^r m!}{r(m+r)!}\log p] \\ &= \lim_{p\to\infty}[\sum_{r=1}^{\infty}[\sum_{q=1}^{\infty}\frac{(m+r)!(-\tau p)^q}{q(m+r+q)!} + \log p]\frac{(-\tau)^r m!}{r(m+r)!} \\ &= \sum_{l=1}^{\infty}e_{m+l}\frac{(-\tau)^l m!}{l(m+l)!} = \sum_{l=1}^{\infty}(-C + \sum_{q=1}^{m+l}\frac{1}{q})\frac{(-\tau)^l m!}{l(m+l)!} \end{aligned} \tag{129}$$

And finally we evaluate the converged value for $a_n = \sum_{l=0}^{\infty}\frac{(-\tau)^l}{(n+l)l!}$.

$$b_m = \lim_{p\to\infty}[\sum_{q=1}^{\infty}\sum_{r=1}^{\infty}\frac{(-\tau)^r}{(m+q+r)r!}\frac{n!(-\tau p)^q}{q(m+q)!} + \sum_{r=0}^{\infty}\frac{(-\tau)^r}{(m+r)r!}\log p] \tag{130}$$

The derivation is given in Derivation (8),

$$b_m = \sum_{l=0}^{\infty}[e_m + \frac{1}{m+l}]\frac{(-\tau)^l}{(m+l)l!} = \sum_{l=0}^{\infty}[-C + \sum_{q=1}^{m}\frac{1}{q} + \frac{1}{m+l}]\frac{(-\tau)^l}{(m+l)l!} \tag{131}$$

3.4 Evaluation of $S_n(\tau, \mu_1, \mu_0^-)$

Based on the iterations of $U_n(\tau, \mu^*)$ and $V_n(\tau, \mu^*)$, we evaluate $S_n(\tau, \mu_1, \mu_0^-)$.

3.4.1 $\tau \simeq 0$

The nth approximation of the scattering function $S(\tau, \mu_1, \mu_0^-)$ is expressed as

$$\begin{aligned} S_n(\tau, \mu_1, \mu_0^-) &= \sum_{j=1}^{n} \Delta S_j(\tau, \mu_1, \mu_0^-) = S_1(\tau, \mu_1, \mu_0^-) + (\frac{1}{\mu_1} + \frac{1}{\mu_0^-})^{-1} \sum_{j=2}^{n} \\ &[U_j(\tau, \mu_0^-) + U_j(\tau, \mu_1) - \exp(-\frac{\tau}{\mu_1})V_j(\tau, \mu_0^-) - \exp(-\frac{\tau}{\mu_0^-})V_j(\tau, \mu_1) \\ &+U_j(\tau, \mu_0^-)U_j(\tau, \mu_1) - V_j(\tau, \mu_0^-)V_j(\tau, \mu_1)\ ]. \end{aligned} \tag{132}$$

Near $\tau \simeq 0$, $U_n(\tau, p^*)$ and $V_n(\tau, p^*)$ hold the relations in equation (104) and (105). Substituting equation (104) into the first term in the bracket of the summation in the above equation, we obtain

$$\sum_{j=2}^{n} U_j(\tau, \mu_0^-) < U_2(\tau, \mu_0^-) \sum_{j=2}^{n} k^{j-2}. \tag{133}$$

All the other terms in the bracket of the summation are also expressed in the same manner as the above equation. Since we can choose $k < 1$, the series $S_n(\tau, \mu_1, \mu_0^-)$ converges absolutely.

$$\lim_{n\to\infty} S_n(\tau, \mu_1, \mu_0^-) = S(\tau, \mu_1, \mu_0^-) \tag{134}$$

We also conclude the convergence of series $T_n(\tau, \mu_1, \mu_0^-)$ in the same manner.

$$\lim_{n\to\infty} T_n(\tau, \mu_1, \mu_0^-) = T(\tau, \mu_1, \mu_1) \tag{135}$$

3.4.2 $\tau = \infty$

The nth approximation of the scattering function $S(\infty, \mu_1, \mu_1)$ is expressed as

$$S_n(\infty, \mu_1, \mu_0^-) = \sum_{j=1}^{n} \Delta S_j(\infty, \mu_1, \mu_0^-) = S_1(\infty, \mu_1, \mu_0^-) \tag{136}$$
$$+(\frac{1}{\mu_1} + \frac{1}{\mu_0^-})^{-1} \sum_{j=2}^{n} [U_j(\infty, \mu_0^-) + U_j(\infty, \mu_1) + U_j(\infty, \mu_0^-) U_j(\infty, \mu_1)].$$

At $\tau = \infty$, $U_n(\infty, p^*)$ holds the relations in the equation (110). Substituting the relations in the above equation, we obtain

$$\sum_{j=2}^{n} U_j(\infty, \mu_0^-) < U_2(\infty, \mu_0^-) \sum_{j=2}^{n} k^{j-2}. \tag{137}$$

All the terms in $S_n(\infty, \mu_1, \mu_0^-)$ are also expressed in the same manner as the above equation. Since we can choose $k = 0.813 < 1$, the series $S_n(\infty, \mu_1, \mu_0^-)$ converges absolutely.

$$\lim_{n \to \infty} S_n(\infty, \mu_1, \mu_0^-) = S(\infty, \mu_1, \mu_0^-) \tag{138}$$

Since $V_2(\infty, \mu_0^-)$ is 0, $T_n(\infty, \mu_1, \mu_0^-)$ is 0 due to equation (53),

$$\lim_{n \to \infty} T_n(\infty, \mu_1, \mu_0^-) = T(\infty, \mu_1, \mu_1) = 0 \tag{139}$$

3.4.3 Truncated Polynomials of Scattering and Transmitted Functions

Substituting $V_2(\tau, p^*)$ (66) and $V_3(\tau, p^*)$ (81) into equation (71), we obtain the third approximation $S_3(\tau, \mu_1, \mu_0^-)$ of the scattering function in the polynomial up to the third power in τ nearer $\tau \simeq 0$,

$$\begin{aligned}
& S_3(\tau, \mu_1, \mu_0^-) = S_1(\tau, \mu_1, \mu_0^-) + \Delta S_2(\tau, \mu_1, \mu_0^-) + \Delta S_3(\tau, \mu_1, \mu_0^-) \\
= \ & \tau - \frac{1}{2}(\frac{1}{\mu_1} + \frac{1}{\mu_0^-})\tau^2 + \frac{1}{6}(\frac{1}{\mu_1} + \frac{1}{\mu_0^-})^2 \tau^3 \\
& + \frac{1}{2}(-\log \tau)\tau^2 + (\frac{3}{4} - \frac{\gamma}{2})\tau^2 \qquad (140) \\
& + (\frac{1}{8} - \frac{1}{4}(\frac{1}{\mu_1} + \frac{1}{\mu_0^-}))(-\log \tau)\tau^3 + [\frac{-\gamma}{8} + \frac{7}{24} - (-\frac{\gamma}{4} + \frac{3}{4})(\frac{1}{\mu_1} + \frac{1}{\mu_0^-})]\tau^3 \\
& + \frac{1}{4}(-\log \tau)^2 \tau^3 + (-\frac{\gamma}{2} + \frac{5}{8})(-\log \tau)\tau^3 + (\frac{\gamma^2}{4} - \frac{5}{8}\gamma + \frac{7}{12} - \frac{\pi^2}{72})\tau^3
\end{aligned}$$

In the above equation, the second line is the first iteration, the third and fourth lines are the second iteration, and the fifth line is the third iteration.

In the same manner as for $S_3(\tau, \mu_1, \mu_0^-)$, we obtain $T_3(\tau, \mu_4^-, \mu_0^-)$ of the scattering function.

$$
\begin{aligned}
T_3(\tau, \mu_4^-, \mu_0^-) &= T_1(\tau, \mu_4^-, \mu_0^-) + \Delta T_2(\tau, \mu_4^-, \mu_0^-) + \Delta T_3(\tau, \mu_4^-, \mu_0^-) \\
&= \tau - \frac{1}{2}(\frac{1}{\mu_4^-} + \frac{1}{\mu_0^-})\tau^2 + \frac{1}{6}(\frac{1}{\mu_4^{-2}} + \frac{1}{\mu_4^-\mu_0^-} + \frac{1}{\mu_0^{-2}})\tau^3 \\
&+\frac{1}{2}(-\log\tau)\tau^2 + (\frac{3}{4} - \frac{\gamma}{2})\tau^2 \qquad (141) \\
&+[\frac{1}{8} - \frac{1}{4}(\frac{1}{\mu_1} + \frac{1}{\mu_0^-})](-\log\tau)\tau^3 + [-\frac{\gamma}{8} + \frac{7}{24} - (-\frac{\gamma}{4} + \frac{3}{8})(\frac{1}{\mu_1} + \frac{1}{\mu_0^-})]\tau^3 \\
&+\frac{1}{4}(-\log\tau)^2\tau^3 + (-\frac{\gamma}{2} + \frac{5}{8})(-\log\tau)\tau^3 + (\frac{\gamma^2}{4} - \frac{5}{8}\gamma + \frac{7}{12} - \frac{\pi^2}{72})\tau^3
\end{aligned}
$$

We rewrite the above equations in cubic polynomials in τ and insert albedo ω where $P(i_1, i_0)$ might occupy.

$$
\begin{aligned}
&S_3(\tau, \mu_1, \mu_0^-) = \omega_0\tau \qquad (142) \\
&+s_{210}\omega_0^2(-\log\tau)\tau^2 + \{s_{200}\omega_0^2 + s_{201}\omega_0(\frac{1}{\mu_1} + \frac{1}{\mu_0^-})\}\tau^2 \\
&+s_{320}\omega_0^3(-\log\tau)^2\tau^3 + \{s_{310}\omega_0^3 + s_{311}\omega_0^2(\frac{1}{\mu_1} + \frac{1}{\mu_0^-})\}(-\log\tau)\tau^3 \\
&+\{s_{3002}\omega_0^2 + s_{3003}\omega_0^3 + s_{301}\omega_0^2(\frac{1}{\mu_1} + \frac{1}{\mu_0^-}) + s_{302}\omega_0(\frac{1}{\mu_1} + \frac{1}{\mu_0^-})^2\}\tau^3
\end{aligned}
$$

$$
\begin{aligned}
&T_3(\tau, \mu_4^-, \mu_0^-) = \omega_0\tau \qquad (143) \\
&+t_{210}\omega_0^2(-\log\tau)\tau^2 + t_{200}\omega_0^2\tau^2 + t_{201}\omega_0(\frac{1}{\mu_4^-} + \frac{1}{\mu_0^-})\tau^2 \\
&+t_{320}\omega_0^3(-\log\tau)^2\tau^3 + \{t_{310}\omega_0^3\tau^3 + t_{311}\omega_0^2(\frac{1}{\mu_4^-} + \frac{1}{\mu_0^-})\}(-\log\tau)\tau^3 \\
&+\{t_{3002}\omega_0^2 + t_{3003}\omega_0^3 + t_{301}\omega_0^2(\frac{1}{\mu_1} + \frac{1}{\mu_0^-}) \\
&+t_{302}\omega_0(\frac{1}{\mu_4^{-2}} + \frac{1}{\mu_4^-}\frac{1}{\mu_0^-} + \frac{1}{\mu_0^{-2}})\}\tau^3
\end{aligned}
$$

where

$s_{210} = 0.5 \qquad s_{200} = 0.461 \qquad s_{201} = -0.5$

$s_{320} = 0.25 \qquad s_{310} = 0.461 \qquad s_{311} = -0.25$

$s_{3002} = 0.167 \qquad s_{3003} = 0.221 \qquad s_{301} = -0.231 \qquad s_{302} = 0.167$

$t_{lmno} = s_{lmno}.$

Substituting equations (69) and (89) into equation (52), we obtain the third approximation $S_3(\infty, \mu_1, \mu_0^-)$ of the scattering function in the polynomial at $\tau = \infty$.

$$S_3(\tau, \mu_1, \mu_0^-) = S_1(\infty, \mu_1, \mu_0^-) + \Delta S_2(\infty, \mu_1, \mu_0^-) + \Delta S_3(\infty, \mu_1, \mu_0^-)$$
$$= \frac{\mu_1 \mu_0^-}{\mu_1 + \mu_0^-}[1 + (\{\frac{\mu_1}{2}\log(1 + \frac{1}{\mu_1}) + 1\}\{\frac{\mu_0^-}{2}\log(1 + \frac{1}{\mu_0^-}) + 1\} - 1)$$
$$+ \frac{\mu_1 \mu_0^-}{\mu_1 + \mu_0^-}[\{U_3(\infty, \mu_1) + 1\}\{U_3(\infty, \mu_0^-) + 1\} - 1] \tag{144}$$

3.5 Numerical Calculation of Scattering

We calculate the first, second, and third approximations as linear, quadratic, and cubic polynomials. Fig.2 illustrates the scattering function for $\omega = 1$ and $\omega = 0.7$ and the pairs of the incident solar and observing angles are $(0°, 0°)$, $(30°, 30°)$ and $(45°, 45°)$.

We recognize the characteristics of the approximation of the scattering function as
(1) the third approximation is nearer to the first approximation than to the second approximation in all cases,
(2) the effect of the albedo ω is recognized but not is so significant,
(3) the effect of the incident and observing angles is also recognized.

It is noted that the truncated $S_2(\tau, \mu_i, \mu_j)$ and $T_2(\tau, \mu_i, \mu_j)$ up to the second order in τ are identical. And the truncated $S_3(\tau, \mu_i, \mu_j)$ and $T_3(\tau, \mu_i, \mu_j)$ to the third order in τ are slightly different. The last terms of the truncated polynomials of $S_3(\tau, \mu_i, \mu_j)$ is given as

$$s_{302}\omega_0(\frac{1}{\mu_1} + \frac{1}{\mu_0^-})^2\}\tau^3,$$

while that of $T_3(\tau, \mu_i, \mu_j)$ is given as

$$s_{302}\omega_0(\frac{1}{\mu_4^{-2}} + \frac{1}{\mu_4^-}\frac{1}{\mu_0^-} + \frac{1}{\mu_0^{-2}})\}\tau^3.$$

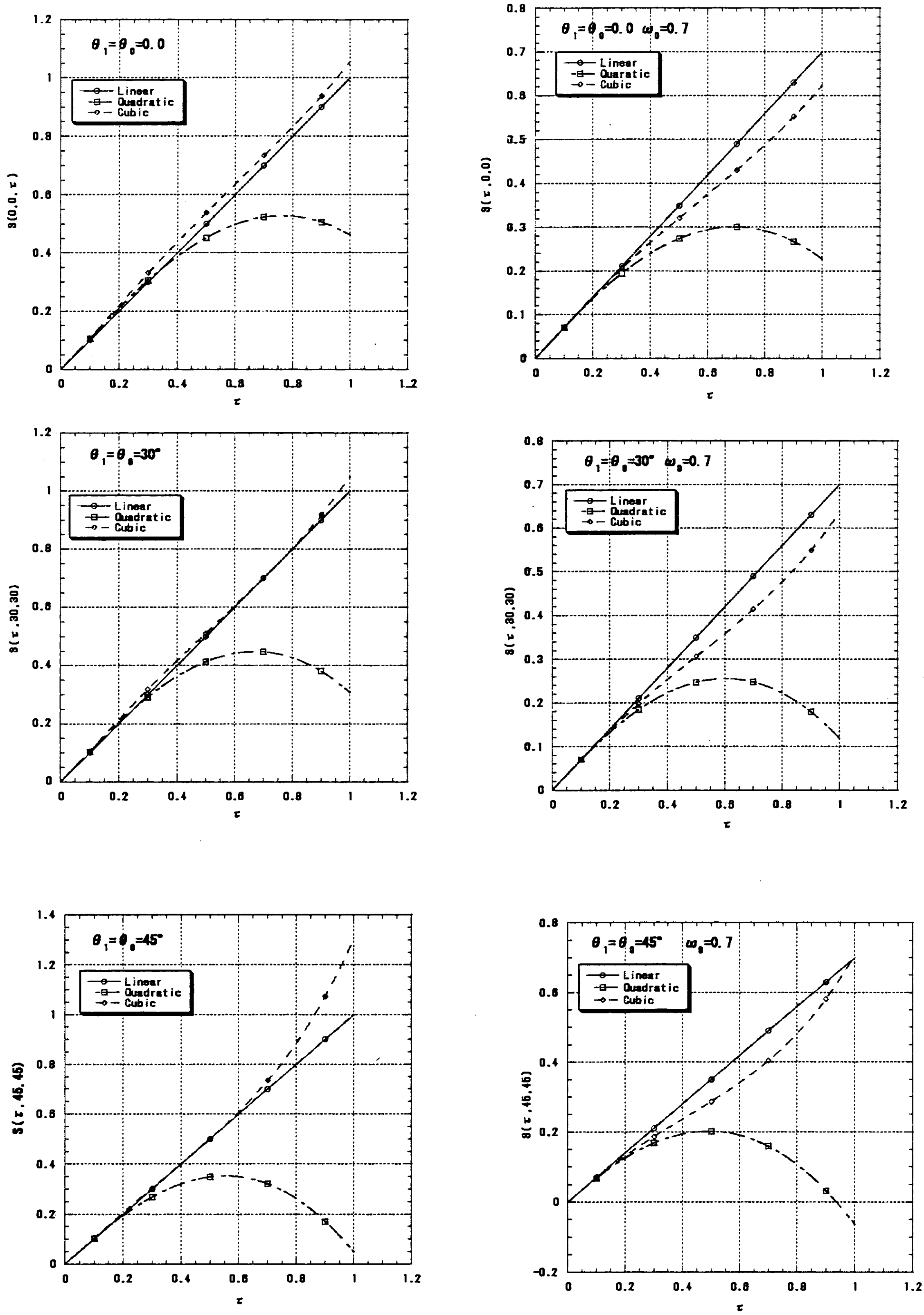

Figure 2: Scattering for Isotropic Atmosphere

4 Integration of the Anisotropic Atmosphere

The second iterations, $\Delta S_2(\tau, \vec{i}_1, \vec{i}_0)$ for the anisotropic atmosphere is expressed from equation (23) as

$$
\begin{aligned}
&\Delta S_2(\tau, \vec{i}_1, \vec{i}_0) = (\frac{1}{\mu_1} + \frac{1}{\mu_0^-})^{-1} \\
&[\ \int P(\vec{i}_1, \vec{i}_3)P(\vec{i}_3, \vec{i}_0)U_1(\mu_0^-)\frac{d\Omega_3}{4\pi\mu_3} - \int P(\vec{i}_1, \vec{i}_3)P(\vec{i}_3, \vec{i}_0)U_1(-\mu_1)\frac{d\Omega_3}{4\pi\mu_3} \\
&+ \int P(\vec{i}_1, \vec{i}_2)P(\vec{i}_2, \vec{i}_0)U_1(\mu_1)\frac{d\Omega_2^-}{4\pi\mu_2^-} - \int P(\vec{i}_1, \vec{i}_2)P(\vec{i}_2, \vec{i}_0)U_1(-\mu_0^-)\frac{d\Omega_2}{4\pi\mu_2^-} \\
&- \ \{\exp(-\frac{\tau}{\mu_0^-} - \frac{\tau}{\mu_1}) - 1\} \times \\
&\{\ \int P(\vec{i}_1, \vec{i}_3)P(\vec{i}_3, \vec{i}_0)U_1(-\mu_1)\frac{d\Omega_3}{4\pi\mu_3} + \int P(\vec{i}_1, \vec{i}_2)P(\vec{i}_2, \vec{i}_0)U_1(-\mu_0^-)\frac{d\Omega_2^-}{4\pi\mu_2^-}\} \\
&+ \ \int\int P(\vec{i}_1, \vec{i}_2)P(\vec{i}_2, \vec{i}_3)P(\vec{i}_3, \vec{i}_0)U_1(\mu_1, \mu_2^-)U_1(\mu_3, \mu_0^-)\frac{d\Omega_2^-}{4\pi\mu_2^-}\frac{d\Omega_3}{4\pi\mu_3} \\
&- \ \int\int P(\vec{i}_1, \vec{i}_3)P(\vec{i}_3, \vec{i}_2)P(\vec{i}_2, \vec{i}_0)U_1(-\mu_0^-, \mu_2^-)U_1(-\mu_1, \mu_3))\frac{d\Omega_2^-}{4\pi\mu_2^-}\frac{d\Omega_3}{4\pi\mu_3}] \\
&- \ \{\exp(-\frac{\tau}{\mu_0^-} - \frac{\tau}{\mu_1}) - 1\} \\
&\times \ \int\int P(\vec{i}_1, \vec{i}_3)P(\vec{i}_3, \vec{i}_2)P(\vec{i}_2, \vec{i}_0)U_1(-\mu_0^-, \mu_2^-)U_1(-\mu_1, \mu_3))\frac{d\Omega_2^-}{4\pi\mu_2^-}\frac{d\Omega_3}{4\pi\mu_3}]
\end{aligned}
\tag{145}
$$

where $U_1(\tau, \mu_0, \mu_1)$ is given as

$$
U_1(\tau, \mu_1, \mu_0^-) = (\frac{1}{\mu_1} + \frac{1}{\mu_0^-})^{-1}[1 - \exp(\frac{(-\tau)}{\mu_1} + \frac{(-\tau)}{\mu_0^-})]. \tag{146}
$$

In equation (145) the arguments in the functions $U_1(\tau, \vec{i}_1, \vec{i}_0)$ are omitted except for the significant ones.

Using the addition theorem of the Legendre function, we decompose the phase function into the the associate Legendre functions, $P_l^m(\mu_3)$, as

$$
P(\vec{i}_1, \vec{i}_3) = \{\sum_{m=0}^{\infty}(2 - \delta_{0,m})(\sum_{l=m}^{\infty} w_l^m P_l^m(\mu_1)P_l^m(\mu_3))\cos m(\varphi_3 - \varphi_1)\} \tag{147}
$$

$$
P(\vec{i}_3, \vec{i}_0) = \{\sum_{m=0}^{\infty}(2 - \delta_{0,m})(\sum_{k=m}^{\infty} w_k^m P_k^m(\mu_3)P_k^m(-\mu_0^-))\cos m(\varphi_3 - \varphi_0)\}, \tag{148}
$$

where $\delta_{0,m}$ and w_l^m are given as

$$\delta_{0,m} = 1 \quad (m = 0), \qquad \delta_{0,m} = 0 \quad (m \neq 0), \tag{149}$$

$$w_l^m = w_l \frac{(l-m)!}{(l+m)!}. \tag{150}$$

The minus sign in equation (148) is due to the opposite direction of the vectors $\vec{i}_3$ and $\vec{i}_0$, or a pair of directions: one is the upper bound and the other is the lower bound (refer to equation(26)).

Due to the orthogonality of the functions $P_l^m(\mu_3)\cos m(\varphi_3 - \varphi_1)$ for different m, we obtain the integration

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} P(\vec{i}_1, \vec{i}_3)P(\vec{i}_3, \vec{i}_0)d\varphi_3 \\
= &\sum_{m=0}^{\infty}(2-\delta_{0,m})(\sum_{l,k=m}^{\infty}(-1)^k w_l^m w_k^m P_l^m(\mu_1)P_k^m(\mu_0^-)P_l^m(\mu_3)P_k^m(\mu_3)) \\
\times &\cos m(\varphi_1 - \varphi_0).
\end{aligned} \tag{151}$$

The minus sign $(-1)^k$ is due to the odd/even characteristic of the associated Legendre function $P_k^m(\mu_0^-)$.

In the same manner, we obtain the integration of the triple product of phase functions as

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi} P(\vec{i}_1, \vec{i}_2)P(\vec{i}_2, \vec{i}_3)P(\vec{i}_3, \vec{i}_0)\frac{d\varphi_2}{2\pi}\frac{d\varphi_3}{2\pi} \\
= &\int_0^{2\pi}\int_0^{2\pi}[\sum_{m=0}^{\infty}(2-\delta_{0,m})\{\sum_{l=m}^{\infty} w_l^m P_l^m(\mu_1)P_l^m(-\mu_2^-)\}\cos m(\varphi_2 - \varphi_1)] \\
\times &[\sum_{m=0}^{\infty}(2-\delta_{0,m})\{\sum_{l=m}^{\infty} w_l^m P_l^m(-\mu_2^-)P_l^m(\mu_3)\}\cos m(\varphi_3 - \varphi_2)]\frac{d\varphi_2}{2\pi} \\
\times &[\sum_{m=0}^{\infty}(2-\delta_{0,m})\{\sum_{l=m}^{\infty} w_l^m P_l^m(\mu_3)P_l^m(-\mu_0^-)\}\cos m(\varphi_0 - \varphi_3)]\frac{d\varphi_3}{2\pi} \\
= &\sum_{m=0}^{\infty}(2-\delta_{0,m})\cos m(\varphi_0 - \varphi_1)[\sum_{a,b,c=m}^{\infty}(-1)^{a+b+c}\{ \\
&w_a^m w_b^m w_c^m P_a^m(\mu_1)P_a^m(\mu_2)P_b^m(\mu_2)P_b^m(\mu_3)P_c^m(\mu_3)P_c^m(\mu_0)\}].
\end{aligned} \tag{152}$$

We define an integration, $Q_{l,k}^m(\tau, \mu_0^-)$, as

$$Q_{l,k}^m(\tau, \mu_0^-) = \int_0^1 P_l^m(\mu_3)P_k^m(\mu_3)U_1(\tau, \mu_3, \mu_0^-)\frac{d\mu_3}{2\mu_3}. \tag{153}$$

Using $Q^m_{l,k}(\tau, \mu_0^-)$, we obtain ΔS_2.

$$\Delta S_2(\tau, \vec{i}_1, \vec{i}_0) = (\frac{1}{\mu_1} + \frac{1}{\mu_0^-})^{-1} \sum_{m=0}^{\infty} (2 - \delta_{0,m}) \cos m(\varphi_1 - \varphi_0) \times$$

$$[\quad \sum_{l,k=m}^{\infty} w_l^m w_k^m \{P_l^m(\mu_1) P_k^m(\mu_0^-)(-1)^k \{Q^m_{l,k}(\tau, \mu_0^-) - (-1)^{l+k} Q^m_{l,k}(\tau, -\mu_0^-)\}$$

$$+ \sum_{l,k=m}^{\infty} w_l^m w_k^m \{P_l^m(\mu_1) P_k^m(\mu_0^-)(-1)^l \{Q^m_{l,k}(\tau, \mu_1) - (-1)^{l+k} Q^m_{l,k}(\tau, -\mu_1)\}$$

$$-\{\exp(-\frac{\tau}{\mu_1} - \frac{\tau}{\mu_0^-}) - 1\}$$

$$\times \quad \sum_{l,k=m}^{\infty} w_l^m w_k^m \{P_l^m(\mu_1) P_k^m(\mu_0^-)(-1)^l \{Q^m_{l,k}(-\mu_0^-) + (-1)^{l+k} Q^m_{l,k}(-\mu_1)\}$$

$$+ \quad \frac{1}{2} \sum_{a,b,c=m}^{\infty} (-1)^{a+b+c} w_a^m w_b^m w_c^m P_a^m(\mu_1) P_c^m(\mu_0^-) Q^m_{a,b}(\tau, \mu_1) Q^m_{b,c}(\tau, \mu_0^-)$$

$$- \quad \frac{1}{2} \sum_{a,b,c=m}^{\infty} (-1)^b w_a^m w_b^m w_c^m P_a^m(\mu_1) P_c^m(\mu_0^-) Q^m_{a,b}(\tau, -\mu_1) Q^m_{b,c}(\tau, -\mu_0^-)]$$

$$- \quad \{\exp(-\frac{\tau}{\mu_0^-} - \frac{\tau}{\mu_1}) - 1\}$$

$$\times \quad \frac{1}{2} \sum_{a,b,c=m}^{\infty} (-1)^b w_a^m w_b^m w_c^m P_a^m(\mu_1) P_c^m(\mu_0^-) Q^m_{a,b}(\tau, -\mu_1) Q^m_{b,c}(\tau, -\mu_0^-)] \quad (154)$$

In the above equation, the sign of each term is determined by the choice of l, k, a, b, and c. This determination originates from the even/odd characteristic of the associated Legendre functions.

4.1 Zenith Angle Integration

The second iteration $\Delta S_2(\tau, i_1, i_0)$ is expressed as $Q^m_{l,k}(\tau, \mu)$ in equation (154). The integrand of $Q^m_{l,k}(\tau, \mu_0^-)$ has a factor $P_l^m(\mu_3) P_k^m(\mu_3)$ and it is reduced to the polynomial of the degree $l + k$ in τ given below.

$$P_l^m(\mu_3) P_k^m(\mu_3) = \sum_{n=0}^{l+k} a_n^{mlk} \mu_3^n \quad (155)$$

We integrate the moment integration, $U_2^n(\tau, \mu_0^-)$, as below.

$$U_2^n(\tau, \mu_0^-) = \int_0^1 \mu_3^n U_1(\tau, \mu_3, \mu_0^-) \frac{d\mu_3}{2\mu_3} \quad (156)$$

$U_2^n(\tau,\mu_0^-)$ is evaluated by partial integration and its recurrence relation is given in Derivation 9. The coefficients of the series expansion of $U_2^n(\tau,\mu_0^-)$ in $(-\tau)$ are also polynomials of $1/\mu_0^-$. The truncated form of $U_2^n(\tau,\mu_0^-)$ for $n \geq 1$ is expressed as

$$U_2^n(\tau,\mu_0^-) = u_{1,0}^n(-\tau) + u_{2,0}^n\frac{(-\tau)^2}{2!} + u_{2,1}^n\frac{1}{\mu_0^-}\frac{(-\tau)^2}{2!} + \cdots, \quad (157)$$

where $u_{l,k}^n$ is the coefficient of nth power and is given in Derivation 9. It is noted that the coefficient of the first power $(-\tau)$ does not include μ_0^-.

Substituting equations (157) and (155) into equation (153), we obtain $Q_{l,k}^m(\tau,\mu)$ as a power series expansion in $(-\tau)$,

$$Q_{l,k}^m(\tau,\mu_0) = q_{1,0}^{mlk}(-\tau) + q_{2,0}^{mlk}\frac{(-\tau)^2}{2!} + q_{2,1}^{mlk}\frac{(-\tau)^2}{2!}\frac{1}{\mu_0^-} + \cdots. \quad (158)$$

where a_n^{mlk} is the coefficient of the nth degree power in μ_3, given below.

$$q_{d,e}^{mlk} = \sum_{n=0}^{l+k} a_n^{mlk} u_{d,e}^n \quad (159)$$

It is noted that the coefficient of the first power $(-\tau)$ in $Q_{l,k}^m(\tau,\mu_0)$ does not include μ_0^-.

Substituting equation (158) into the first term in the bracket of equation (154), we obtain following.

$$\begin{aligned} Q_{l,k}^m(\tau,\mu_0^-) \quad - \quad & (-1)^{l+k}Q_{l,k}^m(\tau,-\mu_0^-) = q_{2,1}^{mlk}\frac{(-\tau)^2}{2!}\frac{2}{\mu_0^-} + \cdots \quad (l+k = even) \\ = \quad & 2q_{1,0}^{mlk}(-\tau) + 2q_{2,0}^{mlk}\frac{(-\tau)^2}{2!} + \cdots \qquad (l+k = odd) \quad (160) \end{aligned}$$

For the case $(l+k = odd)$, adding the second term in equation (154) to the first term, we obtain

$$\begin{aligned} & P_l^m(\mu_1)P_k^m(\mu_0)(-1)^k\{Q_{l,k}^m(\mu_0) - (-1)^{l+k}Q_{l,k}^m(-\mu_0)\}, \\ + \quad & P_l^m(\mu_1)P_k^m(\mu_0)(-1)^l\{Q_{l,k}^m(\mu_1) - (-1)^{l+k}Q_{l,k}^m(-\mu_1)\} \\ = \quad & 0\frac{(-\tau)^2}{2!} + \cdots. \qquad (161) \end{aligned}$$

From the first and second terms in equation (154), the first power in $(-\tau)$ vanishes and the second power emerges only for the case $(l+k = even)$.

From the third terms in the bracket of equation (154), we only need the first power in $(-\tau)$ because it is multiplied by another first power in $(-\tau)$. Substituting the equation (158) into this term, we obtain

$$
\begin{aligned}
Q^m_{l,k}(-\mu_0) + (-1)^{l+k} Q^m_{l,k}(-\mu_1) &= 2q^{mlk}_{1,0}(-\tau) + \cdots, \quad (l+k = even) \\
&= 0(-\tau) + \cdots \quad (l+k = odd). \qquad (162)
\end{aligned}
$$

From the fourth and fifth term, substituting the equation (154) we obtain

$$
\begin{aligned}
& \frac{1}{2} \sum_{a,b,c=m}^{\infty} (-1)^{a+b+c} w^m_a w^m_b w^m_c P^m_a(\mu_1) P^m_c(\mu_0) Q^m_{a,b}(\mu_1) Q^m_{b,c}(\mu_0) \\
- & \frac{1}{2} \sum_{a,b,c=m}^{\infty} (-1)^{b} w^m_a w^m_b w^m_c P^m_a(\mu_1) P^m_c(\mu_0) Q^m_{a,b}(-\mu_1) Q^m_{b,c}(-\mu_0) \\
= & \ 0(-\tau)^2 + \cdots \quad (a+c = even) \\
= & \ [\sum_{a,b,c=m}^{\infty} (-1)^{b+1} w^m_a w^m_b w^m_c P^m_a(\mu_1) P^m_c(\mu_0) q^{mab}_{1,0} q^{mbc}_{1,0}](-\tau)^2 + \cdots \\
& \qquad (a+c = odd). \qquad (163)
\end{aligned}
$$

No $(-\tau)^2$ term emerges from the sixth term in the bracket of equation (154).

Thus we obtain the truncated second iterations given below,

$$
\begin{aligned}
& \Delta S_2(\tau, \vec{i}_1, \vec{i}_0) \\
= & \sum_{m=0}^{\infty} (2 - \delta_{0,m}) [\sum_{(l,k)^1}^{\infty} (-1)^k w^m_l w^m_k (q^{mlk}_{2,1} - 2q^{mlk}_{1,0}) P^m_l(\mu_1) P^m_k(\mu_0) \\
+ & \ 2(\frac{1}{\mu_1} + \frac{1}{\mu_0})^{-1} \sum_{(a,b,c)^2}^{\infty} (-1)^{b+1} w^m_a w^m_b w^m_c q^{mab}_{1,0} q^{mbc}_{1,0} P^m_a(\mu_1) P^m_c(\mu_0)] \\
\times & \ \cos m(\varphi_1 - \varphi_0) \frac{(-\tau)^2}{2!}, \qquad (164)
\end{aligned}
$$

where $(l,k)^1$ denotes $l, k \geq m$ and $l+k$ is even, and $(a,b,c)^2$ denotes $a, b, c \geq m$ and $a+c$ is odd.

It is noted that the truncated second iteration does not have a first power in $(-\tau)$. It is also noted that the fourth and fifth terms vanish for $w_a = 0$ or $w_c = 0$ for all the possible combinations of $a + c = odd$ such as in Rayleigh scattering.

4.2 Separated Integration

We consider changing the order of lim and $\sum$ in the integration of ΔS_2. We begin with the integration of $U_2^n(\tau, i_7)$.

$$U_2^n(\tau, i_7) = \int_0^1 (1 - \exp(-\frac{\tau}{\mu} - \frac{\tau}{\mu_7}))\frac{\mu_7 \mu^n d\mu}{2(\mu_7 + \mu)}$$

$$= -\lim_{\varepsilon \to 0} \int_\varepsilon^1 \sum_{m=1}^{\infty} \frac{(-\tau)^m}{m!}(\frac{1}{\mu_7} + \frac{1}{\mu})^{m-1}\mu^n \frac{d\mu}{2\mu} \qquad (165)$$

On the integral interval $(\varepsilon, 1)$, we interchange the order of $\int_\varepsilon^1$ and $\sum$ and divide the summation into two parts: one is $m \leq n$ and the other is $m > n$

$$U_2^n(\tau, i_7) = -\sum_{m=1}^{n} [\int_0^1 (\frac{\mu}{\mu_7} + 1)^{m-1} \mu^{n-m} d\mu] \frac{(-\tau)^m}{2m!}$$

$$- \quad (-\tau)^n \lim_{\varepsilon \to 0} \sum_{m=n+1}^{\infty} [\int_\varepsilon^1 (\frac{1}{\mu_7} + \frac{1}{\mu})^{m-1} (\frac{\mu}{\mu_7} + 1)^n \frac{d\mu}{\mu}] \frac{(-\tau)^m}{2(m+n)!}. \qquad (166)$$

The first part of the summation ($m \leq n$) consists of ordinary integrations because the minimum power in the μ in the integrand is 0 and there is no singularity at $\mu = 0$. We can separately integrate the powers of $U_2^n(\tau, i_7)$ in $(-\tau)$. We call this part the Separated Integration Part (SIP). The order of the power in τ of $U_2^n(\tau, i_7)$ that is integrated separately is equal to or less than n.

The second part ($m > n$) involves negative powers in μ or $\log p$, and has singularities at $\mu = 0$. Therefore, we cannot interchange the order of lim and $\sum$ and so we cannot integrate the terms separately. We call the second part the Non-Separated Integration Part (NIP). The order of the power in τ of $U_2^n(\tau, i_7)$ that cannot be integrated separately is greater than n. NIP converges as μ approaches 0, because it is equal to the converged $U_2^n(\tau, i_6)$ minus the bounded SIP.

We apply the above observation to equation (154).

4.2.1 First and Second Terms

Due to equation (164), the first power in τ vanishes from the first and the second terms in equation (154) and we can select only the case ($l + k = even$)

of $P_l^m(\mu_3)P_k^m(\mu_3)$. The polynomial of $P_l^m(\mu_3)P_k^m(\mu_3)$ in μ_3 are given below (refer to derivation 9).

$$\begin{aligned} P_l^m(\mu_3)P_k^m(\mu_3) &= a_0 + a_2\mu_3^2 + \cdots, \qquad (m+l = even), \\ &= a_2\mu_3^2 + a_4\mu_3^4 + \cdots (m+l = odd). \end{aligned} \tag{167}$$

We must calculate the moment integrations: $U_2^0(\mu_3)$, $U_2^2(\mu_3)$, $U_2^4(\mu_3)$, $\cdots$. In $U_2^0(\mu_3)$ the second power in τ is include in a NIP, and in the other moment integrations the second powers in τ are included in SIPs. Taking account of $a_0 = P_l^m(0)P_k^m(0)$, the integration $Q_{l,k}^m(\tau, \mu_0^-)$ is then divided into SIP and NIP as below.

$$\begin{aligned} Q_{l,k}^m(\tau, \mu_0^-) &= \int_0^1 P_l^m(\mu_3)P_k^m(\mu_3)U_1(\tau, \mu_3, \mu_0^-)\frac{d\mu_3}{2\mu_3} \\ &= \lim_{\varepsilon \to 0} P_l^m(0)P_k^m(0)\int_\varepsilon^1 U_1(\tau, \mu_3, \mu_0^-)\frac{d\mu_3}{2\mu_3} \\ &+ \int_0^1 (P_l^m(\mu_3)P_k^m(\mu_3) - P_l^m(0)P_k^m(0))U_1(\tau, \mu_3, \mu_0^-)\frac{d\mu_3}{2\mu_3} \end{aligned} \tag{168}$$

Substituting the second power of $(-\tau)$ in the polynomial $U_2^0(p_0)$ into NIP and truncating SIP to the second power of $(-\tau)$, we obtain

$$\begin{aligned} Q_{l,k}^m(\tau, \mu_0^-) &= \frac{1}{2}\{1 + (C - \frac{1}{2})\frac{1}{\mu_0^-}\}P_l^m(0)P_k^m(0) \\ &- \int_0^1 (P_l^m(\mu_3)P_k^m(\mu_3) - P_l^m(0)P_k^m(0))(\frac{1}{\mu_3} + \frac{1}{\mu_0^-})\frac{d\mu_3}{2\mu_3}]\frac{(-\tau)^2}{2!}. \end{aligned} \tag{169}$$

4.2.2 Third Term

We only need the first order power in τ from the third term integration in equation (154). Due to equation (164), we can select only the case $(l + k = even)$ of $P_l^m(\mu_3)P_k^m(\mu_3)$. We can select the first power in the same manner as in the first and second terms. We then obtain the integration $Q_{l,k}^m(\tau, -\mu_0^-)$.

$$\begin{aligned} Q_{l,k}^m(\tau, -\mu_0^-) &= [\frac{1}{2}(C - 1)P_l^m(0)P_k^m(0) \\ &- \int_0^1 (P_l^m(\mu_3)P_k^m(\mu_3) - P_l^m(0)P_k^m(0))\frac{d\mu_3}{2\mu_3}](-\tau) \end{aligned} \tag{170}$$

4.2.3 Fourth and Fifth Terms

Due to equation (164), the first power in τ vanishes from the fourth and fifth terms in equation (154) and we can select only the case $(a+c=odd)$ of $P_l^m(\mu_3)P_k^m(\mu_3)$. The polynomial of $P_l^m(\mu_3)P_k^m(\mu_3)$ in μ_3 are given below (refer to derivation 9).

The lowest powers of μ_2^- and μ_3 for the product of $P_a^m(\mu_2^-)$ are given below (refer to the Derivation 9),

$$\begin{aligned} & P_a^m(\mu_2^-)P_b^m(\mu_2^-)P_b^m(\mu_3)P_c^m(\mu_3) \\ &= c_{12}(\mu_2^-)(\mu_3)^2 && (m+b=odd, a+b=even) \\ &= c_{21}(\mu_2^-)^2(\mu_3) && (m+b=odd, a+b=odd) \\ &= c_{01}(\mu_3) && (m+b=even, a+b=even) \\ &= c_{01}(\mu_2^-) && (m+b=even, a+b=odd) \end{aligned}$$

We integrate the fourth and fifth term below.

$$\int_0^1\int_0^1 P_a^m(\mu_2^-)P_b^m(\mu_2^-)P_b^m(\mu_3)P_c^m(\mu_3)U_1(\mu_1,\mu_2^-)\frac{d\mu_2^-}{2\mu_2^-}U_1(\mu_3,\mu_0^-)\frac{d\mu_3}{2\mu_3} \quad (171)$$

For $m+b=odd$ the integration is the product of two SIPs: one for μ_2^- and the other for μ_3. For $m+b=even$ the integration is a product of one SIP and NIP. The integration is then expressed for the following,

$$\begin{aligned} &= [\int_0^1\int_0^1 P_a^m(\mu_2^-)P_b^m(\mu_2^-)P_b^m(\mu_3)P_c^m(\mu_3)\frac{d\mu_2^-}{2\mu_2^-}\frac{d\mu_3}{2\mu_3}](-\tau)^2 \\ &+ \cdot\cdot \qquad (m+b=odd), \\ &= [-\frac{1}{2}(C-1)P_a^m(0)P_b^m(0)\int_0^1 P_b^m(\mu_3)P_c^m(\mu_3)\frac{d\mu_3}{2\mu_3} \\ &+ \int_0^1\int_0^1\{P_a^m(\mu_2^-)P_b^m(\mu_2^-)-P_a^m(0)P_b^m(0)\}P_b^m(\mu_3)P_c^m(\mu_3)\frac{d\mu_2^-}{2\mu_2^-}\frac{d\mu_3}{2\mu_3}](-\tau)^2 \\ &+\cdot\cdot \qquad (m+b=even, a+b=even), \\ &= [-\frac{1}{2}(C-1)P_b^m(0)P_c^m(0)\int_0^1 P_a^m(\mu_2^-)P_b^m(\mu_2^-)\frac{d\mu_2^-}{2\mu_2^-} \\ &+ \int_0^1\int_0^1 P_a^m(\mu_2^-)P_b^m(\mu_2^-)\{P_b^m(\mu_3)P_c^m(\mu_3)-P_b^m(0)P_c^m(0)\}\frac{d\mu_2^-}{2\mu_2^-}\frac{d\mu_3}{2\mu_3}](-\tau)^2 \\ &+\cdot\cdot \qquad (m+b=even, a+b=odd) \qquad (172) \end{aligned}$$

Sine the lowest power in τ is 3 in the last term in equation (154), we do not need to integrate this last term.

4.3 Surface Integration

We can bring back the idea of division of integration to the surface integration on the unit hemisphere expressed in equation (145). This division is possible because the integrations in equation (145) are expressed as the summation of the integrations of the products of the associated Legendre functions shown in equation (153) and further the products become polynomials of definite order in μ_3 (155). It is noted that from the last integration in equation (154) no second power in $(-\tau)^2$ emerges.

4.3.1 First four integrations

The first integration in equation (145) is divided into SIP and NIP and expressed as

$$\begin{aligned}
&\int P(\vec{i}_1,\vec{i}_3)P(\vec{i}_3,\vec{i}_0)U_1(\tau,\mu_0^-,\mu_3)\frac{d\Omega_3}{4\pi\mu_3} \\
= &\int\{P(\vec{i}_1,\vec{i}_3)P(\vec{i}_3,\vec{i}_0) - P(\vec{i}_1,\vec{i}_3')P(\vec{i}_3',\vec{i}_0)\}U_1(\tau,\mu_0^-,\mu_3)\frac{d\Omega_3}{4\pi\mu_3} \\
+ &\lim_{\mu_3\to 0}\int\{P(\vec{i}_1,\vec{i}_3')P(\vec{i}_3',\vec{i}_0)\}U_1(\tau,\mu_0^-,\mu_3)\frac{d\Omega_3}{4\pi\mu_3},
\end{aligned} \tag{173}$$

where $\vec{i}_3'$ denotes the projected direction of $\vec{i}_3$ onto the equator of the unit sphere (refer to Figure 1). At the equator of the unit sphere, μ_3' is 0. Substituting the second power term of the polynomial $U_1(\tau,\mu_1,\mu_0^-)$ into SIP and the second term of the polynomial $U_2(\tau,\mu_0^-)$ into NIP in the above equation, we obtain the truncated polynomial of the second degree in τ,

$$\begin{aligned}
&\int P(\vec{i}_1,\vec{i}_3)P(\vec{i}_3,\vec{i}_0)U_1(\tau,\mu_0^-,\mu_3)\frac{d\Omega_3}{4\pi\mu_3} \\
= &-[\int\{P(\vec{i}_1,\vec{i}_3)P(\vec{i}_3,\vec{i}_0) - P(\vec{i}_1,\vec{i}_3')P(\vec{i}_3',\vec{i}_0)\}(\frac{1}{\mu_0^-}+\frac{1}{\mu_3})\frac{d\Omega_3}{4\pi\mu_3}]\frac{(-\tau)^2}{2!} \\
+ &\{1+(C-\frac{1}{2})\frac{1}{\mu_0^-}\}[\int_0^{2\pi} P(\vec{i}_1,\vec{i}_3')P(\vec{i}_3',\vec{i}_0)\frac{d\varphi_3}{4\pi}]\frac{(-\tau)^2}{2!}.
\end{aligned} \tag{174}$$

Substituting the above equations into the first and second integrations in the bracket in equation (145), we obtain

$$\int P(\vec{i}_1,\vec{i}_3)P(\vec{i}_3,\vec{i}_0)U_1(\mu_0^-)\frac{d\Omega_3}{4\pi\mu_3} - \int P(\vec{i}_1,\vec{i}_3)P(\vec{i}_3,\vec{i}_0)U_1(-\mu_1)\frac{d\Omega_3}{4\pi\mu_3}$$

$$
\begin{aligned}
&= -(\frac{1}{\mu_0^-}+\frac{1}{\mu_1})[\int\{P(\vec{i}_1,\vec{i}_3)P(\vec{i}_3,\vec{i}_0)-P(\vec{i}_1,\vec{i}_3')P(\vec{i}_3',\vec{i}_0)\}\frac{d\Omega_3}{4\pi\mu_3}]\frac{(-\tau)^2}{2!} \\
&+ (C-\frac{1}{2})(\frac{1}{\mu_1}+\frac{1}{\mu_0^-})[\int_0^{2\pi}P(\vec{i}_1,\vec{i}_3')P(\vec{i}_3',\vec{i}_0)\frac{d\varphi_3}{4\pi}]\frac{(-\tau)^2}{2!}. \qquad (175)
\end{aligned}
$$

We evaluate the third and fourth integrations in equation (145) in the same manner.

4.3.2 Fifth and sixth integrations

The fifth integration is divided into SIP and NIP as below.

$$
\begin{aligned}
&-\{\exp(-\frac{\tau}{\mu_0^-}-\frac{\tau}{\mu_1})-1\}\int P(\vec{i}_1,\vec{i}_3)P(\vec{i}_3,\vec{i}_0)U_1(-\mu_1)\frac{d\Omega_3}{4\pi\mu_3} \\
&= -2(\frac{1}{\mu_1}+\frac{1}{\mu_0^-})[(C-1)\int_0^{2\pi}P(\vec{i}_1,\vec{i}_3')P(\vec{i}_3',\vec{i}_0)\frac{d\varphi_3}{4\pi} \\
&- \int\{P(\vec{i}_1,\vec{i}_3)P(\vec{i}_3,\vec{i}_0)-P(\vec{i}_1,\vec{i}_3')P(\vec{i}_3',\vec{i}_0)\}\frac{d\Omega_3}{4\pi\mu_3}]\frac{(-\tau)^2}{2!}+\cdots \qquad (176)
\end{aligned}
$$

The equation above holds for the two cases described in (170). We can evaluate the sixth integration in equation (145) in the same manner.

4.3.3 Seventh and eighth integrations

The seventh integration is divided into SIP and NIP as below.

$$
\begin{aligned}
&\int\int P(\vec{i}_1,\vec{i}_2)P(\vec{i}_2,\vec{i}_3)P(\vec{i}_3,\vec{i}_0)U_1(\mu_1,\mu_2^-)U_1(\mu_3,\mu_0^-)\frac{d\Omega_2^-}{4\pi\mu_2^-}\frac{d\Omega_3}{4\pi\mu_3} \\
&= \int\int[P(\vec{i}_1,\vec{i}_2)P(\vec{i}_2,\vec{i}_3)P(\vec{i}_3,\vec{i}_0)-P(\vec{i}_1,\vec{i}_2')P(\vec{i}_2',\vec{i}_3)P(\vec{i}_3,\vec{i}_0) \\
&\quad -P(\vec{i}_1,\vec{i}_2)P(\vec{i}_2,\vec{i}_3')P(\vec{i}_3',\vec{i}_0)]U_1(\mu_1,\mu_2^-)U_1(\mu_3,\mu_0^-)\frac{d\Omega_2^-}{4\pi\mu_2^-}\frac{d\Omega_3}{4\pi\mu_3} \\
&+ \int\int P(\vec{i}_1,\vec{i}_2')P(\vec{i}_2',\vec{i}_3)P(\vec{i}_3,\vec{i}_0)U_1(\mu_1,\mu_2^-)U_1(\mu_3,\mu_0^-) \\
&+ \int\int P(\vec{i}_1,\vec{i}_2)P(\vec{i}_2,\vec{i}_3')P(\vec{i}_3',\vec{i}_0)U_1(\mu_1,\mu_2^-)U_1((\mu_3,\mu_0^-)]\frac{d\Omega_2^-}{4\pi\mu_2^-}\frac{d\Omega_3}{4\pi\mu_3} \\
&= [\int\int\{P(\vec{i}_1,\vec{i}_2)P(\vec{i}_2,\vec{i}_3)P(\vec{i}_3,\vec{i}_0)-P(\vec{i}_1,\vec{i}_2')P(\vec{i}_2',\vec{i}_3)P(\vec{i}_3,\vec{i}_0) \\
&\quad -P(\vec{i}_1,\vec{i}_2)P(\vec{i}_2,\vec{i}_3')P(\vec{i}_3',\vec{i}_0)\}\frac{d\Omega_2^-}{4\pi\mu_2^-}\frac{d\Omega_3}{4\pi\mu_3}
\end{aligned}
$$

$$- \quad (C-1)\int\int P(\vec{i}_1,\vec{i}'_2)P(\vec{i}'_2,\vec{i}_3)P(\vec{i}_3,\vec{i}_0)\frac{d\varphi_2}{4\pi}\frac{d\Omega_3}{4\pi\mu_3}$$

$$- \quad (C-1)\int\int P(\vec{i}_1,\vec{i}_2)P(\vec{i}_2,\vec{i}'_3)P(\vec{i}'_3,\vec{i}_0)\frac{d\Omega_2^-}{4\pi\mu_2^-}\frac{d\varphi_3}{4\pi}](-\tau)^2 \qquad (177)$$

The equation above holds for all the cases described in (172). We can evaluate the eighth integration in equation (145) in the same manner.

4.4 Result of the Second Approximation

Expanding the first approximated scattering function into a power series expansion in τ and truncating up to the second degree, we obtain

$$S_1(\tau,\vec{i}_1,\vec{i}_0) = [\tau - (\frac{1}{\mu_1}+\frac{1}{|\mu_0|})\frac{\tau^2}{2!}]P(\vec{i}_1,\vec{i}_0). \qquad (178)$$

The second iteration is expressed by the integration of the products of the phase functions as below.

$$\Delta S_2(\tau,\vec{i}_1,\vec{i}_0) = [(3-2C)I_e(\vec{i}_1,\vec{i}_0) + I_u(\vec{i}_1,\vec{i}_0) + I_l(\vec{i}_1,\vec{i}_0)$$
$$+ \quad 2(\frac{1}{\mu_1}+\frac{1}{\mu_0^-})^{-1}\{(C-1)(I_{uu}(\vec{i}_1,\vec{i}_0)+I_{ll}(\vec{i}_1,\vec{i}_0)) + I_{ul}^2(\vec{i}_0,\vec{i}_1)\}]\frac{(-\tau)^2}{2!} \qquad (179)$$

The coefficients are evaluated on the unit hemisphere (we return the variable μ_2^- to μ_2).

$$I_u(\vec{i}_1,\vec{i}_0) = \int_U\{P(\vec{i}_1,\vec{i}_3)P(\vec{i}_3,\vec{i}_0) - P(\vec{i}_1,\vec{i}'_3)P(\vec{i}'_3,\vec{i}_0)\}\frac{d\Omega_3}{4\pi\mu_3} \qquad (180)$$

$$I_l(\vec{i}_1,\vec{i}_0) = \int_L\{P(\vec{i}_1,\vec{i}_2)P(\vec{i}_2,\vec{i}_0) - P(\vec{i}_1,\vec{i}'_2)P(\vec{i}'_2,\vec{i}_0)\}\frac{d\Omega_2}{4\pi|\mu_2|} \qquad (181)$$

$$I_e(\vec{i}_1,\vec{i}_0) = \int_0^{2\pi} P(\vec{i}_1,\vec{i}'_3)P(\vec{i}'_3,\vec{i}_0)\frac{d\varphi_3}{4\pi} \qquad (182)$$

and

$$I_{uu}(\vec{i}_1,\vec{i}_0) = \int_U\int_0^{2\pi}\{P(\vec{i}_1,\vec{i}'_2)P(\vec{i}'_2,\vec{i}_3)P(\vec{i}_3,\vec{i}_0) - P(\vec{i}_0,\vec{i}'_2)P(\vec{i}'_2,\vec{i}_3)P(\vec{i}_3,\vec{i}_1)\}\frac{d\varphi_2 d\Omega_3}{16\pi^2\mu_3} \qquad (183)$$

$$I_{ll}(\vec{i}_1,\vec{i}_0) = \int_L\int_0^{2\pi}\{P(\vec{i}_1,\vec{i}_2)P(\vec{i}_2,\vec{i}_3')P(\vec{i}_3',\vec{i}_0) - P(\vec{i}_0,\vec{i}_2)P(\vec{i}_2,\vec{i}_3')P(\vec{i}_3',\vec{i}_1)\}\frac{d\varphi_3 d\Omega_2}{16\pi^2|\mu_2|} \quad (184)$$

$$\begin{aligned} I_{ul}(\vec{i}_1,\vec{i}_0) &= \int_U\int_L[\{P(\vec{i}_1,\vec{i}_2)P(\vec{i}_2,\vec{i}_3)P(\vec{i}_3,\vec{i}_0) - P(\vec{i}_0,\vec{i}_2)P(\vec{i}_2,\vec{i}_3)P(\vec{i}_3,\vec{i}_1)\} \\ &- \{P(\vec{i}_1,\vec{i}_2')P(\vec{i}_2',\vec{i}_3)P(\vec{i}_3,\vec{i}_0) - P(\vec{i}_0,\vec{i}_2')P(\vec{i}_2',\vec{i}_3)P(\vec{i}_3,\vec{i}_1)\} \\ &- \{P(\vec{i}_1,\vec{i}_2)P(\vec{i}_2,\vec{i}_3')P(\vec{i}_3',\vec{i}_0) - P(\vec{i}_0,\vec{i}_2)P(\vec{i}_2,\vec{i}_3')P(\vec{i}_3',\vec{i}_1)\}]\frac{d\Omega_2 d\Omega_3}{16\pi^2|\mu_2|\mu_3} \end{aligned} \quad (185)$$

Adding the first approximation to the second iteration, we obtain the second approximation.

$$\begin{aligned} S_2(\tau,\vec{i}_1,\vec{i}_0) &= P(\vec{i}_1,\vec{i}_0)\tau - (\frac{1}{\mu_1}+\frac{1}{|\mu_0|})P(\vec{i}_1,\vec{i}_0)\frac{(-\tau)^2}{2!} \\ &+ \{-2I_e(\vec{i}_1,\vec{i}_0) + 2(\frac{1}{\mu_1}+\frac{1}{|\mu_0|})^{-1}(I_{uu}(\vec{i}_1,\vec{i}_0) + I_{ll}(\vec{i}_1,\vec{i}_0))\}\log\tau\frac{(-\tau)^2}{2!} \\ &+ [I_u(\vec{i}_1,\vec{i}_0) + I_l(\vec{i}_1,\vec{i}_0) + (3-2\gamma)I_e(\vec{i}_1,\vec{i}_0) + \\ &\quad 2(\frac{1}{\mu_1}+\frac{1}{|\mu_0|})^{-1}\{(\gamma-1)(I_{uu}(\vec{i}_1,\vec{i}_0) + I_{ll}(\vec{i}_1,\vec{i}_0)) + I_{ul}(\vec{i}_1,\vec{i}_0)\}]\frac{(-\tau)^2}{2!}. \end{aligned} \quad (186)$$

4.5 Numerical Calculation of Scattering

To evaluate the accuracy of the second approximation discussed in the previous subsections, we carry out calculations for the Rayleigh and aerosol scattering atmospheres for various cases and compare the result with the exact solutions. To appreciate the improvement of accuracy from single scattering, we calculate the the single scattering reflectance by the equation (1) as,

$$\rho(\tau,\vec{i}_1,\vec{i}_0) = \frac{\tau\omega_0 P(\vec{i}_1,\vec{i}_0)}{4\mu_1|\mu_0|}, \quad (187)$$

where ω_0 is the single scattering albedo. The albedo for the Rayleigh scattering is 1 and its phase function is given as,

$$P(\vec{i}_1,\vec{i}_0) = \frac{3(1+\cos{(\vec{i}_1\wedge\vec{i}_0)}^2)}{4}. \quad (188)$$

The maritime aerosol with relative humidity of 80.0% at a wavelength of 443 nm is used and its phase function is given in Figure 3. The Maritime aerosol model has a single scattering albedo of 0.9929.

The successive-order-of-scattering (SOS) [13] method is used in solving the radiative transfer equation for the Rayleigh-scattering atmosphere and the aerosol-scattering atmosphere. The SOS code for the ocean-atmosphere system was developed for the atmospheric correction algorithm for the ocean color sensors. [14] [15] This code computes the upward radiance at the top and the downward radiance at the base of the medium for the ocean-atmosphere system. The code is capable of yielding radiances that are accurate to nearly 0.1%. [14]

4.5.1 Rayleigh Scattering Atmosphere

Figure 3 quantitatively evaluates accuracy using the second approximation for a Rayleigh scattering atmosphere for the various solar-sensor geometries and for the eight wavelengths from 412nm to 865nm. The Rayleigh optical thicknesses corresponding to the eight wavelengths are 0.3185(412nm), 0.2361(443nm), 0.1560(490nm), 0.1324(510nm), 0.0938(560nm), 0.0436(670nm), 0.0255(765nm) and 0.0155(865nm). Figures 3(a) to (d) show the error (%) in the computed upward reflectance as a function of the solar zenith angle and for a sensor viewing angle of 45°. The left (right) part of each plot in Figure 3 corresponds to a relative azimuth angle of 0° (180°) (principal scattering plane).

Results in Figure 3 show the significant improvement in accuracy with using the second approximation compared with the single scattering formula. Using the new formula, the errors are all within 1% for the red and near-infrared wavelengths, while errors are within nearly 10% for the blue bands for zenith angles less than 60°. It is important to note that, with the second approximation, the error in a given wavelength (optical thickness) is almost independent of the viewing geometry.

4.5.2 Maritime Aerosol Atmosphere

The double scattering approximation is the second approximation without triple scattering. Computations for various coefficients are much more involved due predominantly to the aerosol forward scattering characteristics. Figure 4 shows examples of error (%) for a sensor viewing angle of 45° as a function of the solar zenith angle for the principal scattering plane. Figures 4(a) to (d) are the results for the aerosol optical thicknesses of 0.05(a), 0.1(b), 0.2(c), and 0.3(d).

The results in Figure 4 show significant improvement in accuracy using the double scattering approximation compared with the single scattering formula particularly for the part of the results with $\Delta\phi = 0°$. It is interesting to note that the scattering angles for the $\Delta\phi = 0°$ part (the left part of the plot) vary from 55° to 135°. (all forward scattering), while the scattering angles for $\Delta\phi = 180°$ (the right part of the plot) change from 135° to 180° (all backward scattering). We are usually more interested in the results with the large scattering angles because measurements with the large scattering angles are generally accessible for satellite remote sensing. With the double scattering approximation and scattering angles $> 100°$, the reflectance error is usually within 1% for an aerosol optical thickness of 0.05, while the error is within 4% for an optical thickness of 0.1. As expected, the error increases for the turbid atmosphere. The error is proportional to the slant path of the optical thickness, i.e., $\tau_a/cos\theta$. The aerosol reflectance errors are within about 7% (12%) for an optical thickness of 0.2 (0.3).

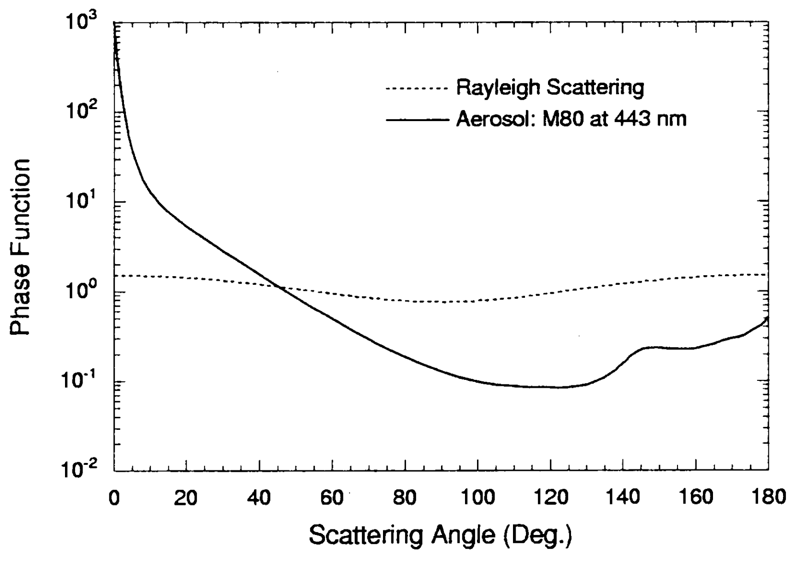

Figure 3: Phase Functions

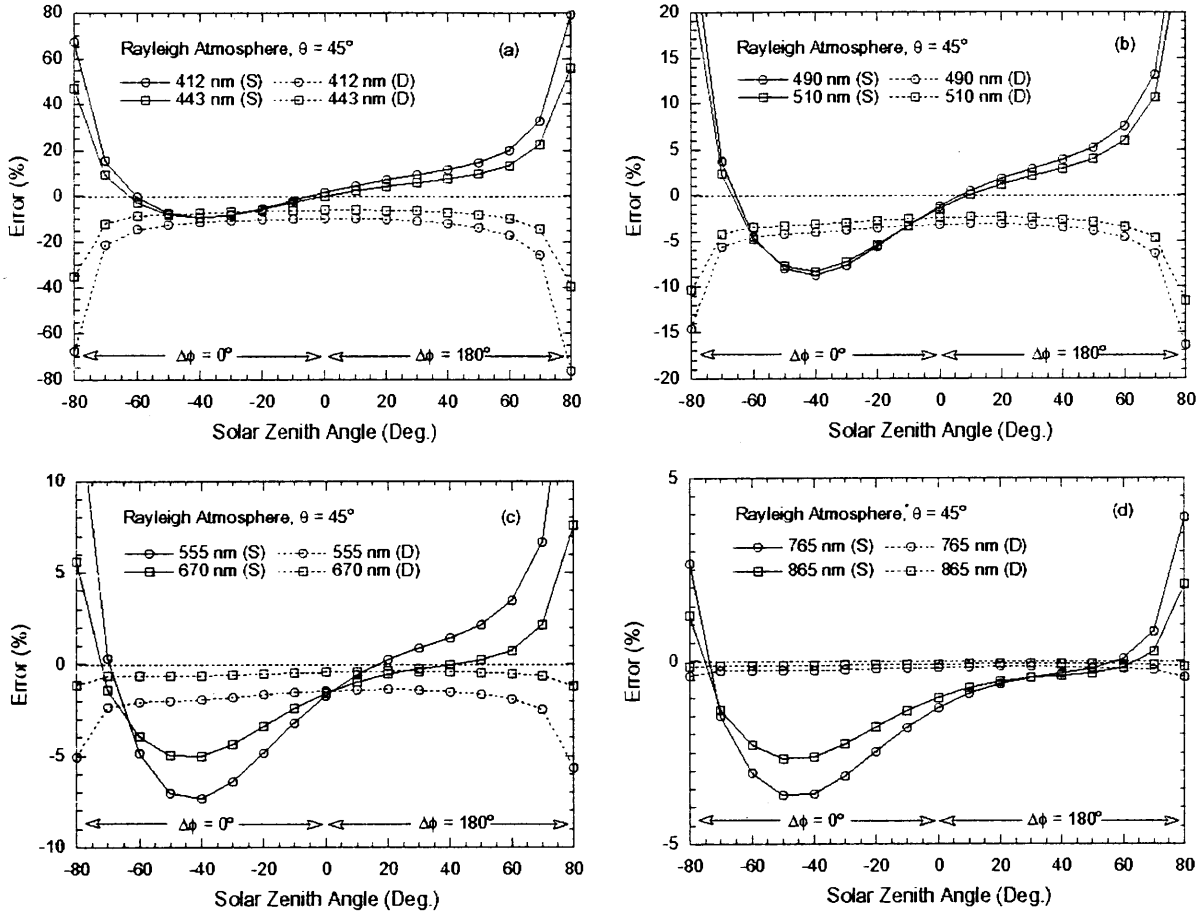

Figure 4: Rayleigh Scattering

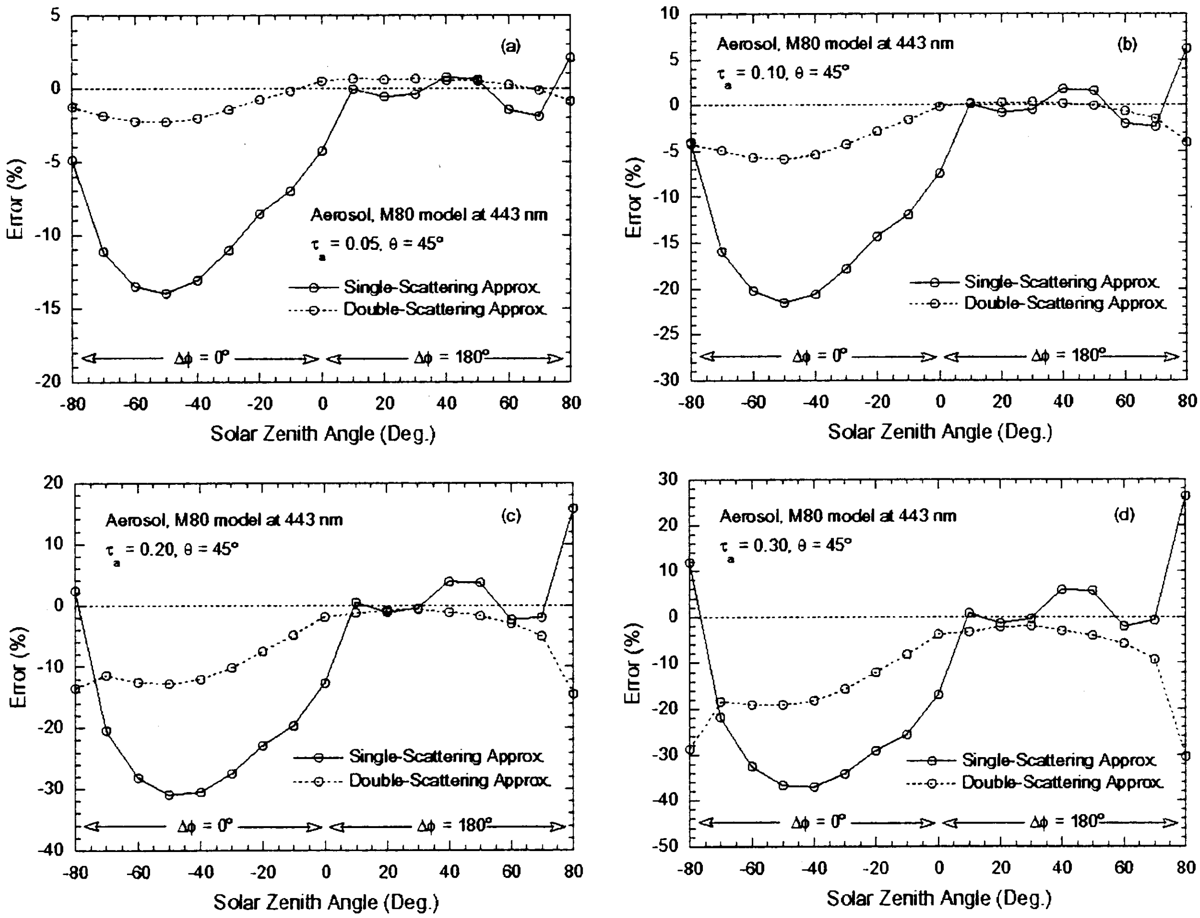

Figure 5: Maritime Aerosol Scattering

5 Properties of the Solution

5.1 Uniqueness of the solution

The functions $U_n(\tau,\mu)$ and $V_n(\tau,\mu)$ are identical to the nth iteration of X and Y respectively because of equation (48) and (49) for U_n and V_n and corresponding equations for X and Y.

$$\begin{aligned} X(\tau,\mu) &= 1 + U_2(\tau,\mu) + U_3(\tau,\mu) + \cdot + \cdot \\ Y(\tau,\mu) &= \exp(-\tau/\mu) + V_2(\tau,\mu) + V_3(\tau,\mu) + \cdot + \cdot \end{aligned} \tag{189}$$

$S(\tau,\mu_1,\mu_0^-)$ and $T(\tau,\mu_4^-,\mu_0^-)$ are given below.

$$\begin{aligned} (\frac{1}{\mu_1} + \frac{1}{\mu_0})S(\tau,\mu_1,\mu_0^-) &= [X(\mu_1)X(\mu_0^-) - Y(\mu_1)Y(\mu_0^-)] \\ (\frac{1}{\mu_4^-} - \frac{1}{\mu_0})T(\tau,\mu_4^-,\mu_0^-) &= [Y(\mu_0^-)X(\mu_4^-) - X(\mu_0^-)Y(\mu_4^-)] \end{aligned} \tag{190}$$

Due to Mullikin, the family of the solution X and Y for $\omega = 1$ are given as

$$X(\tau,\mu) = X_0(\mu) + (bg+a)X_0(\mu)\mu + bX_0(\mu)\mu^2 + aY_0(\mu)\mu + bY_0(\mu)\mu^2 \tag{191}$$

$$Y(\tau,\mu) = Y_0(\mu) - (bg-a)Y_0(\mu)\mu + bY_0(\mu)\mu^2 - aX_0(\mu)\mu + bX_0(\mu)\mu^2 \tag{192}$$

where $X_0(\tau,\mu)$ and $Y_0(\tau,\mu)$ are the solution of the iterative integration with 1 ($\exp(-\tau/\mu)$) as the initial function for X (Y), a and b are arbitrary constants, and g is the given as

$$g = (\frac{1}{2}\int_0^1 X_0(\tau,\mu)\mu d\mu + \frac{1}{2}\int_0^1 Y_0(\tau,\mu)\mu d\mu)(\frac{1}{2}\int_0^1 Y_0(\tau,\mu)d\mu)^{-1}. \tag{193}$$

Substituting $X(\tau,\mu)$ and $Y(\tau,\mu)$ into $S(\tau,\mu_1,\mu_0^-)$ and taking account of the following,

$$\lim_{\tau\to 0} S(\tau,\mu_1,\mu_0^-) = \tau + o(\tau), \tag{194}$$

we conclude that

$$a = b = 0 \tag{195}$$

The condition in equation(194) comes from the single scattering for the thin layer in the plane-parallel atmosphere in equation (34).

The inclusion of $\log\tau$ in $S(\tau, \mu_1, \mu_0^-)$ in the equation (140) implies the possibility of the non-uniqueness of $S(\tau, \mu_1, \mu_0^-)$. From the theory of function, $\log z$ is a many valued function, given as

$$\log z = \log|z| + n2\pi i \tag{196}$$

where n is an integer and z is a real number. On a different branch, n is different. In most cases, we select the branch where n is 0. On branches where n is not 0, $\log z$ explicitly has an imaginary component. However a function $f(\log z)$ that has powers in $\log z$ may become a real valued function on branches where n is not 0. The function $S(\tau, \mu_1, \mu_0^-)$ is such a function as it has powers in $\log\tau$.

5.2 Polynomials

The lowest power in τ of the expanded series of the second iterations $\Delta S_2(\tau, i_1, i_0)$ and $\Delta T_2(\tau, i_1, i_4)$ for the isotropic atmosphere is 2 and that of the third iterations are 3. In general the lowest power in τ of the expanded series of the iterations is raised by 1 as the iteration step is raised. For an anisotropic atmosphere, we only show that the lowest power in τ of the second iterations $\Delta S_2(\tau, i_1, i_0)$ is 2. We conjecture that the lowest power in τ of the iterations is raised by 1 as the iteration step is raised. Therefore, we need only up to the third iteration, if we want the cubic polynomial approximation for $S_(\tau, i_1, i_0)$ and $T(\tau, i_1, i_4)$.

The second approximations $S_2(\tau, i_1, i_0)$ and $T_2(\tau, i_1, i_4)$ are not pure quadratic polynomials but quadratic polynomials with terms $\tau^2\log(\tau)$. The log term comes from the singularity at the lower integral limit. It is possible that, by expanding $\log\tau$ around $\tau \neq 0.0$, we can obtain the quadratic polynomials of the scattering function $S(\tau, \vec{i_1}, \vec{i_0})$. Rather, we leave it as log to maintain the validity of the expression in the range $\tau \geq 0.0$. The third approximations $S_3(\tau, i_1, i_0)$ and $T_3(\tau, i_1, i_4)$ are similarly 'quasi' cubic polynomials with $\tau^3(\log\tau)^2$ and $\tau^3\log\tau$. The term $\tau^3(\log\tau)^2$ is more significant than τ^3 but less significant than τ^2, as expressed below.

$$\lim_{\tau\to 0}\frac{\tau^3(\log\tau)^2}{\tau^3} = \infty \tag{197}$$

$$\lim_{\tau\to 0}\frac{\tau^3(\log\tau)^2}{\tau^2} = \lim_{\tau\to 0}\frac{(\log\tau)^2}{1/\tau} = \lim_{\tau\to 0}\frac{2(\log\tau)}{-1/\tau} = \lim_{\tau\to 0}\tau = 0 \qquad (198)$$

Therefore the terms $\tau^3\log\tau$ and $\tau^3(\log\tau)^2$ do not affect the second approximation truncated up to the second power in τ. The order of significance in the third approximation, as τ approaches to 0, is given below.

$$\tau,\quad \tau^2\log\tau,\quad \tau^2,\quad \tau^3(\log\tau)^2,\quad \tau^3\log\tau,\quad \tau^3 \qquad (199)$$

Near $\tau \sim 0$, the dominant term in the polynomials of the expanded series of $\Delta S_2(\tau,i_1,i_0)$ and $\Delta T_2(\tau,i_1,i_4)$ is $(1/2)(-\log\tau)\tau^2$ and that of $\Delta S_3(\tau,i_1,i_0)$ and $\Delta T_3(\tau,i_1,i_4)$ is $(1/4)(-\log\tau)^2\tau^3$. In general, the dominant term in the polynomials of the expanded series of $\Delta S_n(\tau,i_1,i_0)$ and $\Delta T_n(\tau,i_1,i_4)$ is $(1/2^{n-1})(-\log\tau)^{n-1}\tau^n$. The ratio $S_n(\tau,i_1,i_0)/S_{n+1}(\tau,i_1,i_0) = (1/2)(-\log\tau)\tau$ is far less than 1. Therefore $S_1(\tau,i_1,i_0)+S_2(\tau,i_1,i_0)+\cdot+\cdot+\cdot$ converges to $S(\tau,i_1,i_0)$ near $\tau\sim 0$. Furthermore the series also converges to $S(\infty,i_1,i_0)$ at $\tau=\infty$.

5.3 Surface Integration for Anisotropic Atmosphere

In the polynomial of the second approximation for the anisotropic atmosphere, the coefficient of the powers in τ are characterized by the number of multiplications of the scattering phase function: one phase function corresponds to single scattering, two multiplications corresponds to double scattering and three multiplications corresponds to triple scattering. The coefficients that include $P(\vec{i}_1,\vec{i}_0)$ are single scattering, the surface integrals I_e,I_l,I_u are double scattering and the surface integrals I_{uu}, I_{ll}, I_{ul} are triple scattering. For the atmosphere with a scattering phase function that does not have the odd order Legendre functions in its series expansion in the cosine of the angle, such as the Rayleigh scattering, there are no triple scattering terms in the second approximation.

In the second approximation for the anisotropic atmosphere we evaluate the coefficients of the quadratic polynomial by integrating products of the phase functions on the surface of the unit hemisphere. We do not need to decompose the phase functions into Legendre functions and trigonometric functions. This greatly simplifies calculations.

At $\tau=\infty$, we can integrate the iterations easily without the difficulties that exist in the integration when τ is finite.

6 Conclusion

We can integrate Chandrasekhar's integral equation based on two important mathematical break-throughs: evaluating the converged value of several exponential-like functions at ∞; and evaluating the coefficients of the polynomial by integrating products of phase functions on the surface of the unit hemisphere.

We obtain the approximate scattering and transmission functions, as the cubic polynomials in the optical thickness for the isotropic atmosphere.

We obtain the approximated scattering function as a quadratic polynomial in the optical thickness for the anisotropic atmosphere. The coefficients of the polynomial are surface integrated values of the phase functions with respect to the unit hemisphere.

The numerical calculation of approximation for the isotropic atmosphere shows that the cubic approximation is nearer to the linear approximation than to the second approximation for $\tau < 0.5$.

The numerical calculation by the quadratic approximation for the anisotropic atmosphere yields generalized reflectively more accurate than by the single scattering approximation. In particular the improvement is remarkable when the observing direction is close to the the solar input direction.

The computation time for all the numerical evaluation is significantly short.

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Derivation 1

$$V_2(\tau,p^*) = \exp(-\tau p^*)U_2(\tau,-p^*) = \int_1^\infty \frac{\exp(-\tau p^*)-\exp(-\tau p)}{2(p-p^*)p}dp$$
$$= \int_1^\infty \sum_{n=1}^{\infty} \frac{(-\tau p^*)^n - (-\tau p)^n}{n!}\frac{dp}{2(p-p^*)p}$$
$$= \sum_{n=1}^{\infty} \frac{(-\tau)^n}{2n!}\int_1^\infty \frac{(p^*)^n - p^n dp}{(p-p^*)p}$$
$$= \frac{(-1)}{2p*}[\sum_{n=1}^{\infty}\frac{(-\tau)^n}{n!}\int_1^\infty \frac{(p^n-p^{*n})dp}{(p-p^*)} - \sum_{n=1}^{\infty}\frac{(-\tau)^n}{n!}\int_1^\infty \frac{(p^n-p^{*n})dp}{p}$$
$$= \frac{(-1)}{2p*}[\sum_{n=1}^{\infty}\frac{(-\tau)^n}{n!}\int_1^\infty \sum_{r=0}^{n-1} p^r p^{*(n-1-r)}dp - \sum_{n=1}^{\infty}\frac{(-\tau)^n}{n!}(\frac{p^n}{n} - p^{*n}\log p)]$$
$$= \frac{(-1)}{2p*}[\sum_{n=1}^{\infty}\frac{(-\tau)^n}{n!}\sum_{r=0}^{n-1}\frac{p^{r+1}p^{*(n-1-r)}}{r+1} - \sum_{n=1}^{\infty}\frac{(-\tau)^n}{n!}(\frac{p^n}{n} - p^{*n}\log p)]_1^\infty$$

$$= \frac{(-1)}{2p*}[\sum_{n=1}^{\infty} \frac{(-\tau)^n}{n!} \sum_{r=1}^{n} \frac{p^r p^{*(n-r)}}{r} - \sum_{n=1}^{\infty} \frac{(-\tau p)^n}{nn!} + (\sum_{n=1}^{\infty} \frac{(-\tau p^*)^n}{n!}) \log p)]_1^{\infty}$$

$$= \frac{(-1)}{2p*}[\sum_{m=0}^{\infty} \sum_{r=1}^{\infty} \frac{(-\tau p)^r (-\tau p^*)^m}{r(m+r)!} - \sum_{n=1}^{\infty} \frac{(-\tau p)^n}{nn!} + (\sum_{n=1}^{\infty} \frac{(-\tau p^*)^n}{n!}) \log p)]_1^{\infty}$$

$$= \frac{(-1)}{2p*}[\sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \frac{(-\tau p)^r (-\tau p^*)^m}{r(m+r)!} + (\sum_{n=1}^{\infty} \frac{(-\tau p^*)^n}{n!}) \log p)]_1^{\infty}$$

$$= \frac{(-1)}{2p*}[\sum_{m=1}^{\infty} (\sum_{r=1}^{\infty} \frac{(-\tau p)^r}{r(m+r)!})(-\tau p^*)^m + \log p \sum_{n=1}^{\infty} \frac{(-\tau p^*)^n}{n!}]_1^{\infty}$$

$$= \frac{(-1)}{2p*}[\sum_{m=1}^{\infty} [-C + \sum_{q=1}^{m} \frac{1}{q} - \sum_{r=1}^{\infty} \frac{(-\tau)^r m!}{r(m+r)!})\ ] \frac{(-\tau p^*)^m}{m!}$$

Derivation 2

$$\begin{aligned}
e_n &= \lim_{p\to\infty} [\sum_{q=1}^{\infty} \frac{n!(-\tau p)^q}{q(n+q)!} + \log p] \\
&= \lim_{p\to\infty} [\sum_{q=1}^{\infty} \frac{(n-1)!(-\tau p)^q}{q(n-1+q)!} - \sum_{q=1}^{\infty} \frac{(n-1)!(-\tau p)^q}{(n+q)(n-1+q)!} + \log p] \\
&= \lim_{p\to\infty} [\sum_{q=1}^{\infty} \frac{(n-1)!(-\tau p)^q}{q(n-1+q)!} - \sum_{q=1}^{\infty} \frac{(n-1)!(-\tau p)^q}{(n+q)!} + \log p] \\
&= \lim_{p\to\infty} [\log p + \sum_{q=1}^{\infty} \frac{(n-1)!(-\tau p)^q}{q(n-1+q)!} - \sum_{q=1}^{\infty} \frac{(n-1)!(-\tau p)^q}{(n+q)!}] \\
&= \lim_{p\to\infty} [\log p + \sum_{q=1}^{\infty} \frac{(n-1)!(-\tau p)^q}{q(n-1+q)!} - \frac{(n-1)!}{(-\tau p)^n} \sum_{q=1}^{\infty} \frac{(-\tau p)^{q+n}}{(n+q)!}] \\
&= \lim_{p\to\infty} [\log p + \sum_{q=1}^{\infty} \frac{(n-1)!(-\tau p)^q}{q(n-1+q)!} - \frac{(n-1)!}{(-\tau p)^n} (e^{-\tau p} - \sum_{q=0}^{n} \frac{(-\tau p)^q}{(q)!})] \\
&= \lim_{p\to\infty} [\log p + \sum_{q=1}^{\infty} \frac{(n-1)!(-\tau p)^q}{q(n-1+q)!}] + \frac{1}{n} \\
&= \lim_{p\to\infty} [\log p + \sum_{q=1}^{\infty} \frac{(n-2)!(-\tau p)^q}{q(n-2+q)!}] + \frac{1}{n} + \frac{1}{n-1} \\
&= \lim_{p\to\infty} [\sum_{q=1}^{\infty} \frac{(-\tau p)^q}{qq!} + \log p] + \sum_{r=1}^{n} \frac{1}{r}
\end{aligned}$$

Derivation 3

$$\begin{aligned}
\exp(\pm\tau p^*)V_2(\pm p^*) &= (\sum_{q=0}^{\infty}\frac{(\pm\tau p^*)^q}{q!})(\frac{\tau}{2}\sum_{m=0}^{\infty}a_{m+1}(\tau)\frac{(\mp\tau p^*)^m}{(m+1)!}) \\
&= (\sum_{q=0}^{\infty}\frac{(\pm\tau p^*)^q}{q!})(\frac{\mp 1}{2p^*}\sum_{m=1}^{\infty}a_m(\tau)\frac{(\mp\tau p^*)^m}{m!}) \\
&= \frac{\mp 1}{2p^*}(\sum_{q=0}^{\infty}\frac{(\pm\tau p^*)^q}{q!})(\sum_{m=1}^{\infty}a_m(\tau)\frac{(\mp\tau p^*)^m}{m!}) \\
&= \frac{\mp 1}{2p^*}\sum_{m=1}^{\infty}\{\sum_{q=0}^{\infty}\frac{(-1)^m a_m(\tau)(\pm\tau p^*)^{m+q}}{m!q!}\} \\
&= \frac{\mp 1}{2p^*}\sum_{n=1}^{\infty}\{\sum_{m=1}^{n}\frac{(-1)^m a_m(\tau)(\pm\tau p^*)^n}{m!(n-m)!}\} \\
&= \frac{\pm 1}{2p^*}\sum_{n=1}^{\infty}\{\sum_{m=1}^{n}\frac{(-1)^{m+1}a_m(\tau)n!}{m!(n-m)!}\}\frac{(\pm\tau p^*)^n}{n!} \\
&= \frac{\pm 1}{2p^*}\sum_{n=0}^{\infty}a_n^{'}(\tau)\frac{(\pm\tau p^*)^n}{n!} = \frac{\tau}{2}\sum_{n=0}^{\infty}a_{n+1}^{'}(\tau)\frac{(\pm\tau p^*)^n}{(n+1)!}
\end{aligned}$$

where $a_n^{'}$ is expressed as below A,

$$a_n'(\tau) = \sum_{m=1}^{n}a_m(\tau)\frac{(-1)^{m+1}n!}{m!(n-m)!}.$$

For $a_n = 1$,

$$a_n'(\tau) = \sum_{m=1}^{n}\frac{(-1)^{m+1}n!}{m!(n-m)!} = -(\sum_{m=0}^{n}\frac{(-1)^m n!}{m!(n-m)!} - n!/n!) = 1$$

For $a_n = \sum_{m=1}^{n}\frac{1}{m}$,

$$\begin{aligned}
a_n'(\tau) &= \sum_{m=1}^{n}(\sum_{q=1}^{m}\frac{1}{q})\frac{(-1)^{m+1}n!}{m!(n-m)!} = \sum_{q=1}^{n}\frac{1}{q}(\sum_{m=q}^{n}\frac{(-1)^{m+1}n!}{m!(n-m)!}) \\
&= \sum_{q=1}^{n}\frac{1}{q}(\sum_{m=0}^{n}\frac{(-1)^{m+1}n!}{m!(n-m)!} - \sum_{m=0}^{q-1}\frac{(-1)^{m+1}n!}{m!(n-m)!}) \\
&= \sum_{q=1}^{n}\frac{1}{q}(\sum_{m=0}^{q-1}\frac{(-1)^m n!}{m!(n-m)!}) = \sum_{q=1}^{n}\frac{1}{q}\frac{(-1)^{q-1}n!}{(q-1)!(n-q+1)!} \\
&= \sum_{q=0}^{n-1}\frac{(-1)^q n!}{(q+1)q!(n-q)!} = \frac{1}{n}
\end{aligned}$$

For $a_n = \sum_{m=1}^{\infty} \frac{(-\tau)^m n!}{m(n+m)!}$,

$$
\begin{aligned}
a_n'(\tau) &= \sum_{m=1}^{n}(\sum_{q=1}^{\infty}\frac{(-\tau)^q m!}{q(m+q)!})\frac{(-1)^{m+1}n!}{m!(n-m)!} = \sum_{q=1}^{\infty}\frac{(-\tau)^q n!}{q}(\sum_{m=1}^{n}\frac{(-1)^{m+1}}{(m+q)!(n-m)!}) \\
&= \sum_{q=1}^{\infty}\frac{(-\tau)^q n!}{q}(\sum_{m=1}^{n}\frac{(-1)^{m+1}}{(m+q)!(n+q-(m+q))!}) \\
&= \sum_{q=1}^{\infty}\frac{(-\tau)^q n!}{q}(\sum_{r=q+1}^{n+q}\frac{(-1)^{r-q+1}}{r!(n+q-r)!}) \\
&= \sum_{q=1}^{\infty}\frac{(-\tau)^q n!}{q(n+q)!}(\sum_{r=0}^{n+q}\frac{(-1)^{r-q+1}(n+q)!}{r!(n+q-r)!} - \sum_{r=0}^{q}\frac{(-1)^{r-q+1}(n+q)!}{r!(n+q-r)!}) \\
&= \sum_{q=1}^{\infty}\frac{(-\tau)^q n!}{q(n+q)!}((-1)^q\sum_{r=0}^{q}\frac{(-1)^r(n+q)!}{r!(n+q-r)!}) \\
&= \sum_{q=1}^{\infty}\frac{(-\tau)^q n!}{q(n+q)!}((-1)^q(-1)^q\frac{(n+q-1)!}{q!(n+q-1-q)!}) \\
&= \sum_{q=1}^{\infty}\frac{(-\tau)^q n}{q(n+q)}(\frac{1}{q!}) = \sum_{q=1}^{\infty}\frac{(-\tau)^q}{q}(\frac{1}{q!}) - \sum_{q=1}^{\infty}\frac{(-\tau)^q}{n+q}(\frac{1}{q!}) \\
&= \sum_{q=1}^{\infty}\frac{(-\tau)^q}{qq!} - \sum_{q=1}^{\infty}\frac{(-\tau)^q}{(n+q)q!}
\end{aligned}
$$

Derivation 4

$$
\begin{aligned}
&\int_1^{\infty}[\frac{\tau}{2}\sum_{r=0}^{\infty}a_{r+1}\frac{(-\tau p^*)^r}{(r+1)!} - \frac{\tau}{2}\sum_{r=0}^{\infty}a_{r+1}\frac{(-\tau p)^r}{(r+1)!}]\frac{dp}{2(p-p^*)p} \\
=\ & \frac{\tau}{4}\sum_{r=1}^{\infty}\frac{a_{r+1}}{(r+1)!}\int_1^{\infty}\frac{(-\tau p^*)^r-(-\tau p)^r}{(p-p^*)p}dp \\
=\ & \frac{-\tau}{4p*}\sum_{r=1}^{\infty}\frac{a_{r+1}(-\tau)^r}{(r+1)!}[\int_1^{\infty}\frac{p^{*r}-p^r}{p}dp - \int_1^{\infty}\frac{p^{*r}-p^r}{p-p^*}dp] \\
=\ & \frac{-\tau}{4p*}\sum_{r=1}^{\infty}\frac{a_{r+1}(-\tau)^r}{(r+1)!}[\ (\log p)p^{*r} - \frac{p^r}{r} + \int_1^{\infty}\sum_{q=0}^{r-1}p^{*(r-1-q)}p^q dp] \\
=\ & \frac{-\tau}{4p*}\sum_{r=1}^{\infty}\frac{a_{r+1}(-\tau)^r}{(r+1)!}[\ (\log p)p^{*r} + \sum_{q=0}^{r-2}\frac{p^{*(r-1-q)}p^{q+1}}{q+1})] \\
=\ & \frac{-\tau}{4p*}[(\sum_{r=1}^{\infty}\frac{a_{r+1}(-\tau p^*)^r}{(r+1)!})\log p + \sum_{r=2}^{\infty}\frac{a_{r+1}(-\tau)^r}{(r+1)!}\sum_{m=1}^{r-1}\frac{p^{*m}p^{r-m}}{r-m})]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-\tau}{4p*}[\sum_{r=1}^{\infty}(\frac{a_{r+1}(-\tau p^*)^r}{(r+1)!})(\log p) + \sum_{m=1}^{\infty}\sum_{q=1}^{\infty}\frac{a_{m+1+q}}{(m+1+q)!}\frac{(-\tau p^*)^m(-\tau p)^q}{q})] \\
&= \frac{-\tau}{4p*}\sum_{m=1}^{\infty}[a_{m+1}(\log p) + \sum_{q=1}^{\infty}\frac{a_{m+1+q}(m+1)!(-\tau p)^q}{q(m+1+q)!})]\frac{(-\tau p^*)^m}{(m+1)!} \\
&= \frac{(-\tau)^2}{4}\sum_{m=0}^{\infty}[a_{m+2}(\log p) + \sum_{q=1}^{\infty}\frac{a_{m+2+q}(m+2)!(-\tau p)^q}{q(m+2+q)!})]\frac{(-\tau p^*)^m}{(m+2)!}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{(\tau)^{n-1}}{2^{n-1}}\int_1^{\infty}\sum_{m=1}^{\infty}a_{m+n-1}(\tau)[\frac{(-\tau p^*)^m}{(m+n-1)!} - \frac{(-\tau p)^m}{(m+n-1)!}]\frac{dp}{2(p-p^*)p} \\
&= \frac{(\tau)^{n-1}}{2^n}\sum_{m=1}^{\infty}a_{m+n-1}(\tau)\int_1^{\infty}[\frac{(-\tau p^*)^m}{(m+n-1)!} - \frac{(-\tau p)^m}{(m+n-1)!}]\frac{dp}{(p-p^*)p} \\
&= \frac{(\tau)^{n-1}}{2^n}\sum_{m=1}^{\infty}\frac{a_{m+n-1}(-\tau)^m}{(m+n-1)!}\int_1^{\infty}\frac{p^{*m}-p^m}{(p-p^*)p}dp \\
&= \frac{(\tau)^{n-1}}{2^n p^*}\sum_{m=1}^{\infty}\frac{a_{m+n-1}(-\tau)^m}{(m+n-1)!}[\int_1^{\infty}\frac{p^{*m}-p^m}{p-p^*}dp - \int_1^{\infty}\frac{p^{*m}-p^m}{p}dp] \\
&= \frac{-(\tau)^{n-1}}{2^n p^*}\sum_{m=1}^{\infty}\frac{a_{m+n-1}(-\tau)^m}{(m+n-1)!}[\int_1^{\infty}\frac{p^m-p^{*m}}{p-p^*}dp - \int_1^{\infty}\frac{p^m-p^{*m}}{p}dp] \\
&= \frac{-(\tau)^{n-1}}{2^n p^*}\sum_{m=1}^{\infty}\frac{a_{m+n-1}(-\tau)^m}{(m+n-1)!}[\,(\log p)p^{*m} - \frac{p^m}{m} + \int_1^{\infty}\sum_{q=0}^{m-1}p^{*(m-1-q)}p^q dp] \\
&= \frac{-(\tau)^{n-1}}{2^n p^*}\sum_{m=1}^{\infty}\frac{a_{m+n-1}(-\tau)^m}{(m+n-1)!}[\,(\log p)p^{*m} + \sum_{q=0}^{m-2}\frac{p^{*(m-1-q)}p^{q+1}}{q+1}] \\
&= \frac{-(\tau)^{n-1}}{2^n p^*}[\sum_{m=1}^{\infty}\frac{a_{m+n-1}(-\tau p^*)^m}{(m+n-1)!}\log p + \sum_{m=2}^{\infty}\frac{a_{m+n-1}(-\tau)^m}{(m+n-1)!}\sum_{q=0}^{m-2}\frac{p^{*(m-1-q)}p^{q+1}}{q+1}] \\
&= \frac{-(\tau)^{n-1}}{2^n p^*}[\cdot\cdot + \sum_{m=2}^{\infty}\sum_{q=0}^{m-2}\frac{a_{m+n-1}}{(m+n-1)!}\frac{(-\tau p)^{*(m-1-q)}(-\tau p)^{q+1}}{q+1}] \\
&= \frac{-(\tau)^{n-1}}{2^n p^*}[\cdot\cdot + \sum_{m=2}^{\infty}\sum_{q=1}^{m-1}\frac{a_{m+n-1}}{(m+n-1)!}\frac{(-\tau p)^{*(m-q)}(-\tau p)^q}{q}] \\
&= \frac{-(\tau)^{n-1}}{2^n p^*}[\cdot\cdot + \sum_{r=1}^{\infty}\sum_{q=1}^{\infty}\frac{a_{r+q+n-1}}{(r+q+n-1)!}\frac{(-\tau p)^{*r}(-\tau p)^q}{q}] \\
&= \frac{-(\tau)^{n-1}}{2^n p^*}\sum_{m=1}^{\infty}[a_{m+n-1}\log p + \sum_{q=1}^{\infty}\frac{a_{m+q+n-1}(m+n-1)!(-\tau p)^q}{q(m+q+n-1)!}]\frac{(-\tau p^*)^m}{(m+n-1)!}
\end{aligned}
$$

$$= \frac{(\tau)^n}{2^n}\sum_{m=0}^{\infty}[a_{m+n}\log p + \sum_{q=1}^{\infty}\frac{a_{m+q+n}(m+n)!(-\tau p)^q}{q(m+q+n)!}]\frac{(-\tau p^*)^m}{(m+n)!}$$

Derivation 5

$$\int_1^\infty \frac{U_2(\infty,p)dp}{2(p+p^*)p} = \int_1^\infty \frac{\log(p+1)dp}{4(p+p^*)p^2}$$

$$= (1/4p^*)[\int_1^\infty \frac{\log(p+1)dp}{p^2} - \frac{1}{p^*}(\int_1^\infty \frac{\log(p+1)dp}{p} - \int_1^\infty \frac{\log(p+1)dp}{(p+p^*)})]$$

$$= (1/4p^*)[2\log 2 - \frac{1}{p^*}(\int_2^\infty \frac{\log q}{q-1}dq - \int_2^\infty \frac{\log q}{q+q^*}dq)\ ]$$

For $1 < p^* < 3 \qquad 0 < q^* = p^* - 1 < 2$

$$= (1/4p^*)[2\log 2 - \frac{1}{p^*}(\int_2^\infty \frac{\log q}{q(1-1/q)}dq - \int_2^\infty \frac{\log q}{q(1+q^*/q)}dq)\ ]$$

$$= \cdot - \frac{1}{p^*}[(\int_2^\infty \frac{\log q}{q}(1+\frac{1}{q}+\frac{1}{q^2}+\cdot)dq - \int_2^\infty \frac{\log q}{q}(1-\frac{q^*}{q}+\frac{q^{*2}}{q^2}-\cdot)dq\ ]$$

$$= \cdot - \frac{1}{p^*}[\int_2^\infty \sum_{m=2}^{\infty}\frac{\log q}{q^m}dq - \int_2^\infty \sum_{m=2}^{\infty}(-q^*)^{m-1}\frac{\log q}{q^m}dq)]$$

$$= \cdot - \frac{1}{p^*}[\sum_{m=2}^{\infty}\{-\frac{(\log q)}{(m-1)q^{m-1}} - \frac{1}{(m-1)^2q^{m-1}}\}_2^\infty - \sum_{m=2}^{\infty}(-q^*)^{m-1}\{\cdot\cdot\}]$$

$$= \cdot - \frac{1}{p^*}[\sum_{m=1}^{\infty}\{-\frac{(\log q)}{mq^m} - \frac{1}{m^2q^m}\}_2^\infty - \sum_{m=1}^{\infty}(-q^*)^m\{\cdot\cdot\}]$$

$$= \cdot - \frac{1}{p^*}[\sum_{m=1}^{\infty}\{\frac{(\log 2)}{m2^m} + \frac{1}{m^22^m}\} - \sum_{m=1}^{\infty}\{\frac{\log 2}{m}(\frac{-q^*}{2})^m + \frac{1}{m^2}(\frac{-q^*}{2})^m\}]$$

$$= \cdot - \frac{1}{p^*}[\{\ (\log 2)(\log(\frac{1}{1-1/2}) + \frac{\pi^2}{12} - \frac{(\log 2)^2}{2}\}$$

$$-\{(\log 2)(\log(\frac{1}{1+q^*/2})) + \sum_{m=1}^{\infty}\frac{1}{m^2}(\frac{-q^*}{2})^m\}]$$

$$= \cdot - \frac{1}{p^*}[\{\ (\log 2)^2 + \frac{\pi^2}{12} - \frac{(\log 2)^2}{2}\} - \{(\log 2)(\log 2 - \log(2+q^*)) + \sum_{m=1}^{\infty}\frac{1}{m^2}(\frac{-q^*}{2})^m\}]$$

$$= \cdot - \frac{1}{p^*}[(\log 2)\log(2+q^*) + \frac{\pi^2}{12} - \frac{(\log 2)^2}{2} - \sum_{m=1}^{\infty}\frac{1}{m^2}(\frac{-q^*}{2})^m\}]$$

$$= (1/4p^*)[2\log 2 - \frac{1}{p^*}[+\frac{\pi^2}{12} - \frac{(\log 2)^2}{2} + (\log 2)\log(p^*+1) - \sum_{m=1}^{\infty}\frac{1}{m^2}(\frac{1-p^*}{2})^m\}]$$

For $p^* \geq 3$ $(q^* \geq 2)$

$$= \quad (1/4p^*)[2\log 2 - \frac{1}{p^*}(\int_1^\infty \frac{\log(p+1)dp}{p} - \int_1^\infty \frac{\log(p+1)dp}{(p+p^*)})\ ]$$

$$= \quad (1/4p^*)[2\log 2 - \frac{1}{p^*}(\int_2^\infty \frac{\log q}{q-1}dq - \int_2^\infty \frac{\log q}{q+q^*}dq)\ ]$$

$$= \quad \cdot - \frac{1}{p^*}(\int_2^\infty \frac{\log q}{q}(1+\frac{1}{q}+\cdot\cdot)dq$$

$$-\int_2^{q^*} \frac{\log q}{q^*}(1-\frac{q}{q^*}+\frac{q^2}{q^{*2}}\cdot)dq - \int_{q^*}^\infty \frac{\log q}{q}(1-\frac{q^*}{q}+\frac{q^{*2}}{q^2}\cdot)dq\ ]$$

$$= \quad \cdot - \frac{1}{p^*}[\{\ (\log 2)^2 + \frac{\pi^2}{12} - \frac{(\log 2)^2}{2}\} - \sum_{m=1}^\infty (-q^*)^m\{\frac{(\log q^*)}{mq^{*m}} + \frac{1}{m^2 q^{*m}}\}]$$

$$-\int_2^{q^*} \sum_{m=0}^\infty \frac{(-1)^m}{(q^*)^{(m+1)}} q^m \log q dq)]$$

$$= \quad \cdot - \frac{1}{p^*}[\{\ (\log 2)^2 + \frac{\pi^2}{12} - \frac{(\log 2)^2}{2}\} - \sum_{m=1}^\infty \{\log q^* \frac{(-1)^m}{m} + \frac{(-1)^m}{m^2}\}]$$

$$-\sum_{m=0}^\infty \{\ -\frac{(\log q)(-q)^{m+1}}{(m+1)q^{*(m+1)}} + \frac{(-q)^{m+1}}{(m+1)^2 q^{*(m+1)}}\}_2^{q^*})]$$

$$= \quad \cdot - \frac{1}{p^*}[\{\ (\log 2)^2\} + \frac{\pi^2}{12} - \frac{(\log 2)^2}{2} - \log q^* \log(\frac{1}{1-(-1)}) + \frac{\pi^2}{12}]$$

$$+\sum_{m=1}^\infty \{\ \frac{(\log q^*)(-1)^m}{m} - \frac{(-1)^m}{(m)^2} - \frac{(\log 2)(-2)^m}{(m)q^{*(m)}} + \frac{(-2)^m}{(m)^2 q^{*(m)}}\}]$$

$$= \quad \cdot - \frac{1}{p^*}[(\log 2)^2 + (\frac{\pi^2}{12} - \frac{(\log 2)^2}{2}) + \frac{\pi^2}{12}$$

$$+\frac{\pi^2}{12} - (\log 2)\log\frac{q^*}{q^*+2} + \sum_{m=1}^\infty \frac{1}{m^2}(\frac{-2}{q^*})^m]$$

$$= \quad (1/4p^*)[2\log 2 - \frac{1}{p^*}[\frac{(\log 2)^2}{2} + \frac{\pi^2}{4} - \log 2 \log\frac{p^*-1}{p^*+1} + \sum_{m=1}^\infty \frac{1}{m^2}(\frac{-2}{p^*-1})^m]$$

Derivation 6

$$\sum_{q=1}^\infty \frac{(-p)^q}{(q+r)q!} = \sum_{q=1}^\infty \frac{(-p)^q}{(q+r)q!} - \sum_{q=1}^\infty \frac{(-p)^q}{(q+1)!} + \sum_{q=1}^\infty \frac{(-p)^q}{(q+1)!}$$

$$= -\sum_{q=1}^\infty \frac{(r-1)(-p)^q}{(q+r)(q+1)q!} + \sum_{q=1}^\infty \frac{(-p)^q}{(q+1)!}$$

$$= -\frac{r-1}{(-p)}\sum_{q=2}^{\infty}\frac{(-p)^q}{(q+r-1)q!}+\sum_{q=1}^{\infty}\frac{(-p)^q}{(q+1)!}$$

$$= -\frac{r-1}{(-p)}(\sum_{q=1}^{\infty}\frac{(-p)^q}{(q+r-1)q!}-\frac{(-p)}{r})+\sum_{q=1}^{\infty}\frac{(-p)^q}{(q+1)!}$$

$$= -\frac{r-1}{(-p)}\sum_{q=1}^{\infty}\frac{(-p)^q}{(q+r-1)q!}+\frac{r-1}{r}+\sum_{q=1}^{\infty}\frac{(-p)^q}{(q+1)!}$$

Derivation 7

$$\begin{aligned}
b_n &= \lim_{p\to\infty}[\sum_{m=1}^{\infty}\frac{(n)!(-\tau p)^m}{m(m+n)!}(\sum_{r=1}^{n+m}\frac{1}{r})+\log p(\sum_{r=1}^{n}\frac{1}{r})] \\
&= \lim_{p\to\infty}[\sum_{m=1}^{\infty}\frac{(n)!(-\tau p)^m}{m(m+n)!}(\sum_{r=1}^{n}\frac{1}{r}+\sum_{r=n+1}^{n+m}\frac{1}{r})+\log p(\sum_{r=1}^{n}\frac{1}{r})] \\
&= (\sum_{r=1}^{n}\frac{1}{r})\lim_{p\to\infty}(\sum_{m=1}^{\infty}\frac{(n)!(-\tau p)^m}{m(m+n)!}+\log p)+\lim_{p\to\infty}[\sum_{m=1}^{\infty}\frac{(n)!(-\tau p)^m}{m(m+n)!}\sum_{r=n+1}^{n+m}\frac{1}{r}] \\
&= (\sum_{r=1}^{n}\frac{1}{r})(-C+\sum_{q=1}^{n}\frac{1}{q})+\lim_{p\to\infty}[(\sum_{m=1}^{\infty}\frac{(n)!(-\tau p)^m}{m(m+n)!}\sum_{r=1}^{\infty}(\frac{1}{n+r}-\frac{1}{n+m+r})] \\
&= (\sum_{r=1}^{n}\frac{1}{r})(-C+\sum_{q=1}^{n}\frac{1}{q})+\lim_{p\to\infty}[(\sum_{m=1}^{\infty}\frac{(n)!(-\tau p)^m}{(m+n)!}\sum_{r=1}^{\infty}(\frac{1}{(n+r)(n+m+r)})] \\
&= (\sum_{r=1}^{n}\frac{1}{r})(-C+\sum_{q=1}^{n}\frac{1}{q})+\lim_{p\to\infty}[\sum_{r=1}^{\infty}(\frac{1}{(n+r)})(\sum_{m=1}^{\infty}\frac{(n)!(-\tau p)^m}{(n+m+r)(m+n)!}] \\
&= (\sum_{r=1}^{n}\frac{1}{r})(-C+\sum_{q=1}^{n}\frac{1}{q})+\sum_{r=1}^{\infty}(\frac{1}{(n+r)})(-\frac{1}{(n+r)}) \\
&= (\sum_{r=1}^{n}\frac{1}{r})(-C+\sum_{q=1}^{n}\frac{1}{q})-\sum_{r=1}^{\infty}\frac{1}{r^2}+\sum_{r=1}^{n}\frac{1}{r^2} \\
&= (\sum_{r=1}^{n}\frac{1}{r})(-C+\sum_{q=1}^{n}\frac{1}{q})+\sum_{r=1}^{n}\frac{1}{r^2}-\frac{\pi^2}{6}
\end{aligned}$$

Derivation 8

$$\begin{aligned}
b_n &= \lim_{p\to\infty}[\sum_{q=1}^{\infty}\sum_{r=0}^{\infty}\frac{(-\tau)^r}{(n+r+q)r!}\frac{n!(-\tau p)^q}{q(n+q)!}+\sum_{r=0}^{\infty}\frac{(-\tau)^r}{(n+r)r!}\log p] \\
&= \lim_{p\to\infty}[\sum_{r=0}^{\infty}\{\sum_{q=1}^{\infty}\frac{n!(-\tau p)^q}{(n+r+q)q(n+q)!}+\frac{\log p}{(n+r)}\}\frac{(-\tau)^r}{r!}]
\end{aligned}$$

$$= \lim_{p\to\infty}[\sum_{r=0}^{\infty}\{\sum_{q=1}^{\infty}\frac{n!(-\tau p)^q}{q(n+q)!} - \sum_{q=1}^{\infty}\frac{n!(-\tau p)^q}{(n+r+q)(n+q)!} + \log p\ \}\frac{(-\tau)^r}{(n+r)r!}]$$

$$= \sum_{r=0}^{\infty}(e_n - g_r^n)\frac{(-\tau)^r}{(n+r)r!}$$

$$= \sum_{m=0}^{\infty}[e_n + \frac{1}{n+m}]\frac{(-\tau)^r}{(n+m)m!}$$

Derivation 9

We define a new function $W_n(\tau)$ as

$$W_n(\tau) = \int_0^1 \mu_3^n \exp(-\frac{\tau}{\mu_3})\frac{d\mu_3}{2}.$$

$W_0(\tau)$ is evaluated

$$W_0(\tau) = \int_0^1 \exp(-\frac{\tau}{\mu_3})\frac{d\mu_3}{2} = [\frac{\mu_3}{2}\exp(-\frac{\tau}{\mu_3})]_0^1 - \int_0^1 \frac{\mu_3}{2}\{\exp(-\frac{\tau}{\mu_3})\}'d\mu_3$$

$$= \frac{1}{2}[\exp(-\tau) - (-\tau)(C + \mathrm{eax}(\tau))].$$

The recurrence relation of $W_n(\tau)$ is given as

$$W_n(\tau) = \frac{\exp(-\tau)}{2(n+1)} + \frac{(-\tau)}{n+1}W_{n-1}(\tau).$$

We can evaluate $W_n(\tau)$ by the recurrence relation and the initial function $W_0(\tau)$. Using $W_n(\tau)$, we can derive the recurrence relation for $U_2^n(\tau,\mu)$

$$U_2^n(\tau,\mu_0) = \int_0^1\{(1 - \exp(-\frac{\tau}{\mu_3} - \frac{\tau}{\mu_0})\}\frac{\mu_0\mu_3^n d\mu_3}{(\mu_3+\mu_0)}$$

$$= \int_0^1\{\cdots\}\frac{\mu_0\mu_3^{n-1}(\mu_3+\mu_0) - \mu_3^{n-1}\mu_0^2}{(\mu_3+\mu_0)}d\mu_3 = \mu_0[\int_0^1 \mu_3^{n-1}\{\cdots\}d\mu_3 - U_2^{n-1}(\mu_0)]$$

$$= \mu_0[\frac{1}{n} - \exp(-\frac{\tau}{\mu_0})\int_0^1 \mu_3^{n-1}\exp(-\frac{\tau}{\mu_3})d\mu_3 - U_2^{n-1}(\mu_0)]$$

$$= \mu_0[\frac{1}{n} - \exp(-\frac{\tau}{\mu_0})W_{n-1} - U_2^{n-1}(\mu_0)]$$

Truncating the series expansion up to the second degree in $(-\tau)$, we obtain

$$U_2^1(\tau,\mu_0) = -\frac{(-\tau)}{2} - \frac{3(-\tau)^2}{8} + C\frac{(-\tau)^2}{4} - \frac{1}{4}\frac{(-\tau)^2}{\mu_0},$$

$$U_2^n(\tau,\mu_0) = -\frac{(-\tau)}{2n} - \frac{(-\tau)^2}{4(n-1)} - \frac{1}{4n}\frac{(-\tau)^2}{\mu_0}. \quad (n \geq 2)$$

Derivation 10

$$
\begin{array}{llllllll}
m = e, & l = e, & k = e, & P_l^m(\mu_3)P_k^m(\mu_3) = a_0 & & + a_2\mu_3^2 & & \\
m = e, & l = o, & k = o, & P_l^m(\mu_3)P_k^m(\mu_3) = & & a_2\mu_3^2 & & + a_4\mu_3^4 \\
m = e, & l = e, & k = o, & P_l^m(\mu_3)P_k^m(\mu_3) = & a_1\mu_3 & & + a_3\mu_3^3 & \\
m = e, & l = o, & k = e, & P_l^m(\mu_3)P_k^m(\mu_3) = & a_1\mu_3 & & + a_3\mu_3^3 & \\
m = o, & l = e, & k = e, & P_l^m(\mu_3)P_k^m(\mu_3) = & & a_2\mu_3^2 & & + a_4\mu_3^4 \\
m = o, & l = o, & k = o, & P_l^m(\mu_3)P_k^m(\mu_3) = a_0 & & + a_2\mu_3^2 & & \\
m = o, & l = e, & k = o, & P_l^m(\mu_3)P_k^m(\mu_3) = & a_1\mu_3 & & + a_3\mu_3^3 & \\
m = o, & l = o, & k = e, & P_l^m(\mu_3)P_k^m(\mu_3) = & a_1\mu_3 & & + a_3\mu_3^3 &
\end{array}
$$

END