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FINITE ELEMENT ANALYSIS OF THERMO-VISCOELASTIC PROBLEMS

熱粘弾性問題の有限要素法解析

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Introduction

It has been well known that the success of the finite element method in continuum mechanics is due to the division of the continuum into the material elements. We can extend this division or discretization procedure to the deformation process in plasticity by adopting the incremental theory or the piecewise linear formulation of the problem. Further, for the viscoelastic materials which are concerned with in the present analysis, we can incorporate the discretization of the material properties by assuming the discontinuous spectra of mechanical elements which compose the viscoelastic models. Mathematically, this can be phrased that we express approximately the shear and bulk creep compliances (or relaxation moduli) of the materials by the exponential series (i.e. Prony series)[1)~5)]. It must be noted that the analysis easily allows for the thermal effects by assuming the thermo-rheologically simple nature of materials that obeys the temperature-time equivalence hypothesis.

The mechanical model of the present paper is the generalized Voigt (or Kelvin) type of Fig. 1. This is due to the fact, as will be shown in the last section, that we can determine reasonably by experiments the relevant shear and bulk creep compliances.

First, we attempt to obtain the constitutive equation of the linear viscoelastic materials in the form that[6)]

$$\left.\begin{aligned}&\{\dot{\sigma}\}=[D^e]\{\dot{\varepsilon}\}-\{\dot{\sigma}_a\}\\ \text{or }&\{d\sigma\}=[D^e]\{d\varepsilon\}-\{d\sigma_a\},\ \{d\sigma_a\}=\{\dot{\sigma}_a\}dt\end{aligned}\right\}\quad(1)$$

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where $\{\dot{\sigma}\}$ and $\{\dot{\varepsilon}\}$ are the stress and strain-rates respectively. $[D^e]$ denotes the elastic stiffness matrix, $\{\dot{\sigma}_a\}$ is the apparent stress-rate associated with the viscous and/or plastic components in the respective mechanical model. Eq. (1) may be

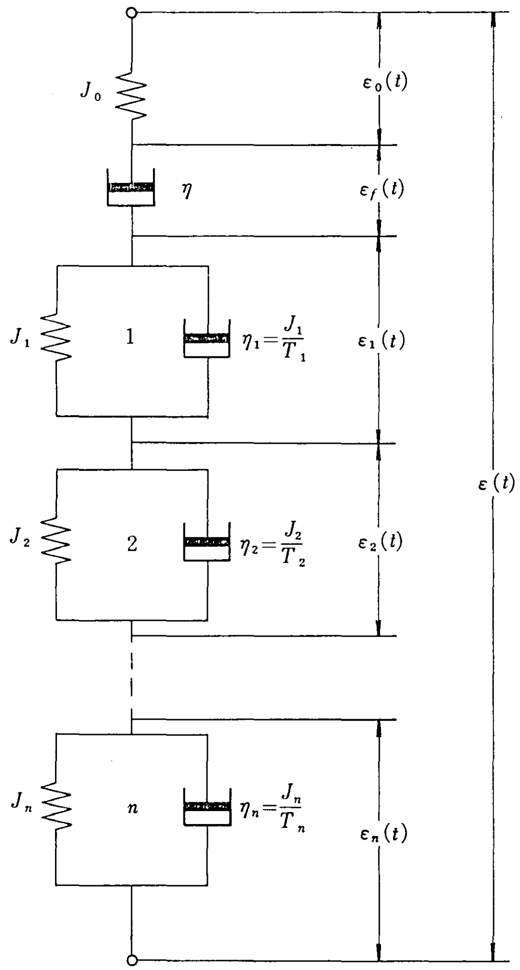


Fig. 1 The generalized Voigt (or Kelvin) model for uniaxial stress field. For deviatoric or bulk deformation, J_0, J_1, $T_1\cdots$ are replaced by D_0, D_1, $T_{D,1}\cdots$ and B_0, B_1, $T_{B,1}\cdots$.

used as a forward difference equation, but the accuracy and stability of the solution are expected to be improved considerably by assuming the linear change of stress (or strain) during each discrete time interval. The improvement is exemplified by a case study of the stress relaxation of the generalized Voigt materials.

Viscoelastic Stress-Strain Matrix

The constitutive equation of the viscoelastic material can be expressed by the hereditary integral, by assuming the uniaxial stress field:

$$\varepsilon(t)=\int_{0^-}^{t} J(t-\tau)\frac{d\sigma}{d\tau}d\tau \tag{2}$$

where $J(t)$ is the tensile creep compliance. The lower limit of integral 0^- is adopted to allow for the discontinuous change of stress at $t=0$.

By discretizing the element distribution function in the mechanical model and adopting the generalized Voigt model of Fig. 1, the compliance $J(t)$ can be expressed by the following exponential series of Prony:

$$J(t)=J_0+t/\eta+\sum_{i=1}^{n} J_i[1-\exp(-t/T_i)] \tag{3}$$

where J_0 and η are the glass compliance and viscosity of a Maxwell element. J_i and T_i represent the compliance and retardation time of the i th Voigt element. By substituting (3) into (2)

$$\varepsilon(t)=\varepsilon_0(t)+\varepsilon_f(t)+\sum_{i=1}^{n}\varepsilon_i(t) \tag{4}$$

where

$$\left.\begin{aligned}&\varepsilon_0(t)=J_0\sigma(t), \qquad \varepsilon_f(t)=\frac{1}{\eta}\int_{0^+}^{t}\sigma(\tau)d\tau\\ &\varepsilon_i(t)=\frac{J_i}{T_i}\int_{0^+}^{t}\sigma(\tau)\exp\{-(t-\tau)/T_i\}\,d\tau\end{aligned}\right\} \tag{5}$$

Note that ε_0, ε_f and ε_i can be interpreted physically as strains in the respective elements of the model of Fig. 1.

Differentiation of Eq. (4) yields

$$\dot{\varepsilon}(t)=J_0\dot{\sigma}(t)+\frac{\sigma(t)}{\eta}+\sum_{i=1}^{n}\left(\frac{J_i}{T_i}\sigma(t)-\frac{\varepsilon_i(t)}{T_i}\right) \tag{6}$$

Eq. (6) is the incremental relation of the type of Eq. (1), and may be used as a forward difference relation or as the first approximation in the predictor-corrector integration scheme. Accuracy of the numerical computation is expected, however, to be improved significantly by applying the integral form given by (5). We assume in the present analysis that the stress varies linearly within the time interval $t-h\leqslant\tau\leqslant t$ so that

$$\sigma(\tau)=\sigma(t-h)+\frac{\tau-t+h}{h}\Delta\sigma(t) \tag{7}$$

where $\Delta\sigma(t)$ is the stress increment for the time interval h. By substituting (7) into (4) and (5), we obtain the following difference equation:

$$\Delta\varepsilon(t)=\Delta\varepsilon_0(t)+\Delta\varepsilon_f(t)+\sum_{i=1}^{n}\Delta\varepsilon_i(t) \tag{8}$$

where

$$\left.\begin{aligned}&\Delta\varepsilon_0(t)=J_0\Delta\sigma(t)\\ &\Delta\varepsilon_f(t)=\frac{h}{2\eta}\Delta\sigma+\frac{h}{\eta}\sigma(t-h)\\ &\Delta\varepsilon_i(t)=\{1-\alpha_i(h)\}\{J_i\sigma(t-h)\\ &\qquad -\varepsilon_i(t-h)\}+J_i\beta_i(h)\Delta\sigma(t)\end{aligned}\right\} \tag{9}$$

and

$$\left.\begin{aligned}&\alpha_i(h)=\exp(-h/T_i)\\ &\beta_i(h)=1-\frac{T_i}{h}\{1-\alpha_i(h)\}\\ &\varepsilon_i(t)=\alpha_i(h)\varepsilon_i(t-h)+J_i[\beta_i(h)\Delta\sigma(t)\\ &\qquad +\{1-\alpha_i(h)\}\sigma(t-h)]\end{aligned}\right\} \tag{10}$$

By solving Eq. (8) for $\Delta\sigma(t)$

$$\Delta\sigma(t)=E\{\Delta\varepsilon(t)-\Delta I(t)\} \tag{11}$$

where E and $\Delta I(t)$ are

$$\left.\begin{aligned}&1/E=J_0+\frac{h}{2\eta}+\sum_{i=1}^{n}J_i\beta_i(h)\\ &\Delta I(t)=\frac{h}{\eta}\sigma(t-h)\\ &\quad+\sum_{i=1}^{n}\{1-\alpha_i(h)\}\{J_i\sigma(t-h)-\varepsilon_i(t-h)\}\end{aligned}\right\} \tag{12}$$

Thermo-viscoelastic Constitutive Equation

When we assume that the materials behave in a thermo-rheologically simple way[7], there exists the temperature-time equivalence which is expressed in terms of the reduced time $\xi(t)$

$$\xi(t)=\int_0^t\phi[T(\tau)]d\tau \tag{13}$$

$T(\tau)$ is the absolute temperature, and $\phi[T(\tau)]$ denotes the shift function which is determined experimentally. Then, extending Eq. (2) to the general three dimensional case, the constitutive equation in the varying temperature field can be written as[8)]

$$\left.\begin{aligned} e_{ij}(t)&=\frac{1}{2}\int_{0^-}^{t} D[\xi(t)-\xi(\tau)]\frac{ds_{ij}}{d\tau}d\tau \\ \varepsilon_{kk}(t)&-3\alpha_0\theta(t) \\ &=\frac{1}{3}\int_{0^-}^{t} B[\xi(t)-\xi(\tau)]\frac{d\sigma_{kk}}{d\tau}d\tau \end{aligned}\right\} \quad (14)$$

where

$$\varepsilon_{kk}=\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz},\quad \sigma_{kk}=\sigma_{xx}+\sigma_{yy}+\sigma_{zz}$$

and

$$e_{ij}=\varepsilon_{ij}-\frac{1}{3}\delta_{ij}\varepsilon_{kk},\quad s_{ij}=\sigma_{ij}-\frac{1}{3}\delta_{ij}\sigma_{kk}$$

In Eq. (14), the shear and bulk creep compliances $D(t)$ and $B(t)$ are defined with respect to the reference temperature T_0. Note that the effect of thermal expansion is included in Eq. (14), where α_0 denotes the coefficient of thermal expansion at temperature T_0 and $\theta(t)$ is the pseudo-temperature specified by

$$\theta(t)=\frac{1}{\alpha_0}\int_{T_0}^{T}\alpha(T')dT' \quad (15)$$

By assuming that the shift function ψ is constant within each time interval, we can obtain for the deviatoric and dilatational components the following reccurence formulas which are similar to (11). For deviatoric components:

$$\Delta s_{ij}(t)=2G\left[\Delta e_{ij}(t)-\frac{1}{2}\Delta I_{ij}(t)\right] \quad (16)$$

where

$$\left.\begin{aligned} 1/G&=D_0+h_\xi/(2\eta_D)+\sum_{m=1}^{p} D_m\beta_{D,m}(h_\xi) \\ \Delta I_{ij}(t)&=(h_\xi/\eta_D)s_{ij}(t-h) \\ &\quad+\sum_{m=1}^{p}\{1-\alpha_{D,m}(h_\xi)\} \\ &\qquad\cdot\{D_m s_{ij}(t-h)-e_{ij,m}(t-h)\} \\ e_{ij,m}(t)&=\alpha_{D,m}(h_\xi)e_{ij,m}(t-h) \\ &\quad+D_m[\beta_{D,m}(h_\xi)\Delta s_{ij}(t) \\ &\qquad+\{1-\alpha_{D,m}(h_\xi)\}s_{ij}(t-h)] \\ \alpha_{D,m}(h_\xi)&=\exp(-h_\xi/T_{D,m}) \\ \beta_{D,m}(h_\xi)&=1-(T_{D,m}/h_\xi)\{1-\alpha_{D,m}(h_\xi)\} \end{aligned}\right\} \quad (17)$$

For dilatational components:

$$\Delta\sigma_{kk}(t)=3K\left[\Delta\varepsilon_{kk}(t)-\frac{1}{3}\Delta I_B(t)-3\alpha_0\Delta\theta(t)\right] \quad (18)$$

where

$$\left.\begin{aligned} 1/K&=B_0+h_\xi/(2\eta_B)+\sum_{m=1}^{q} B_m\beta_{B,m}(h_\xi) \\ \Delta I_B(t)&=(h_\xi/\eta_B)\sigma_{kk}(t-h) \\ &\quad+\sum_{m=1}^{q}\{1-\alpha_{B,m}(h_\xi)\} \\ &\qquad\cdot\{B_m\sigma_{kk}(t-h)-\varepsilon_{kk,m}(t-h)\} \\ \varepsilon_{kk,m}(t)&=\alpha_{B,m}(h_\xi)\varepsilon_{kk,m}(t-h) \\ &\quad+B_m[\beta_{B,m}(h_\xi)\Delta\sigma_{kk}(t) \\ &\qquad+\{1-\alpha_{B,m}(h_\xi)\}\sigma_{kk}(t-h)] \\ \alpha_{B,m}(h_\xi)&=\exp(-h_\xi/T_{B,m}) \\ \beta_{B,m}(h_\xi)&=1-(T_{B,m}/h_\xi)\{1-\alpha_{B,m}(h_\xi)\} \end{aligned}\right\} \quad (19)$$

In Eqs. (17) and (19), h_ξ represents the reduced time interval, i.e.

$$h_\xi=\xi(t)-\xi(t-h) \quad (20)$$

We can take into account of the variation of the coefficient of thermal expansion α by putting $\Delta\theta(t)$ in Eq. (18) as

$$\Delta\theta=\{\alpha[T(t-h)]+\alpha[T(t)]\}\cdot\{T(t)-T(t-h)\}/(2\alpha_0) \quad (21)$$

By combining (16) with (18), the incremental stress-strain relation is given as

$$\begin{aligned}\Delta\sigma_{ij}&=2G\Delta\varepsilon_{ij}+\delta_{ij}\left(K-\frac{2}{3}G\right)\Delta\varepsilon_{kk} \\ &\quad-G\Delta I_{ij}-\delta_{ij}K\left(\frac{1}{3}\Delta I_B+3\alpha_0\Delta\theta\right)\end{aligned} \quad (22)$$

Alternatively, in the matrix form for which use is made of the engineering shear strain in place of the tensor component:

$$\{\Delta\sigma\}=[D]\{\Delta\varepsilon\}-\{\Delta\sigma_v\}-\{\Delta\sigma_\theta\} \quad (23)$$

$$[D]=\begin{bmatrix} K+\frac{4}{3}G & & & & & \text{SYM.} \\ K-\frac{2}{3}G & K+\frac{4}{3}G & & & & \\ K-\frac{2}{3}G & K-\frac{2}{3}G & K+\frac{4}{3}G & & & \\ 0 & 0 & 0 & G & & \\ 0 & 0 & 0 & 0 & G & \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \quad (24)$$

The apparent stress vectors $\{\Delta\sigma_v\}$ and $\{\Delta\sigma_\theta\}$, which respectively are due to the viscoelastic deformation and the thermal expansion, are

$$\{\Delta\sigma_v\}=\begin{bmatrix} G & 0 & 0 & 0 & 0 & 0 & \frac{K}{3} \\ 0 & G & 0 & 0 & 0 & 0 & \frac{K}{3} \\ 0 & 0 & G & 0 & 0 & 0 & \frac{K}{3} \\ 0 & 0 & 0 & G & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G & 0 \end{bmatrix}\begin{Bmatrix} \Delta I_x \\ \Delta I_y \\ \Delta I_z \\ \Delta I_{yz} \\ \Delta I_{zx} \\ \Delta I_{xy} \\ \Delta I_B \end{Bmatrix} \tag{25}$$

$$\{\Delta\sigma_\theta\}^T=3\alpha_0 K\Delta\theta \lfloor 1\ 1\ 1\ 0\ 0\ 0 \rfloor \tag{26}$$

or

$$\{\Delta\sigma_v\}=[H]\{\Delta I\}, \quad \{\Delta\sigma_\theta\}=[C]\Delta\theta \tag{27}$$

The stress-strain relation for the plane stress field can be obtained as a special case of Eq. (23).

Finite Element Solution Procedure

The explicit viscoelastic constitutive equation of (23) can be easily incorporated into the standard finite element procedure. The resulting stiffness equation for the element is

$$\{\Delta F\}=[K]\{\Delta d\}-\{\Delta F_v\}-\{\Delta F_\theta\} \tag{28}$$

where $[K]$, $\{\Delta d\}$ and $\{\Delta F\}$ are the stiffness matrix, the displacement and external load increments at nodes respectively. $\{\Delta F_v\}$ and $\{\Delta F_\theta\}$ arise from the apparent stresses $\{\Delta\sigma_v\}$ and $\{\Delta\sigma_\theta\}$ of Eq. (27). In evaluating $\{\Delta F_\theta\}$, the temperature increment $\Delta\theta$ at a generic point is interpolated in terms of the temperature increments at the nodal points.

Assemblage of the element stiffness gives the incremental stiffness equation of the system as

$$\{\Delta\bar{F}\}=[\bar{K}]\{\Delta\bar{d}\}-\{\Delta\bar{F}_v\}-\{\Delta\bar{F}_\theta\} \tag{29}$$

The computational algorithm for the present solution is summarised as follows:

(i) The first step is to solve the elastic problem for the instantaneous mechanical and thermal loads $\{\Delta\bar{F}\}$ and $\{\Delta\bar{F}_\theta\}$ at $t=0$.

(ii) Assign the appropriate real time interval h. The corresponding reduced time interval h_ξ for individual element can be determined according to the element temperature.

(iii) Solve the overall stiffness equation (29) for the unknown nodal displacement $\{\Delta\bar{d}\}$.

(iv) The stress $\{\sigma\}$ and strain $\{\varepsilon\}$ can be calculated from the solution $\{\Delta\bar{d}\}$. Compute and store the components $e_{ij,m}$ of Eq. (17) and $\varepsilon_{kk,m}$ of Eq. (19), then return to stage (ii) and proceed to the next cycle of calculation.

Numerical Example

(**a**) Relaxation of stress under constant strain condition

Fig. 2 shows the relaxation of stress of a Maxwell model under the constant strain input of $\varepsilon(t)=1(t)$. This problem was solved to test the finite difference equation (11) and consequently the validity of the underlying numerical procedure of the present paper. It can be seen that the assumption of linear change of stress expressed by Eq. (7) gives a satisfactory result in comparison with the one obtained by Zienkiewicz's method[3)]

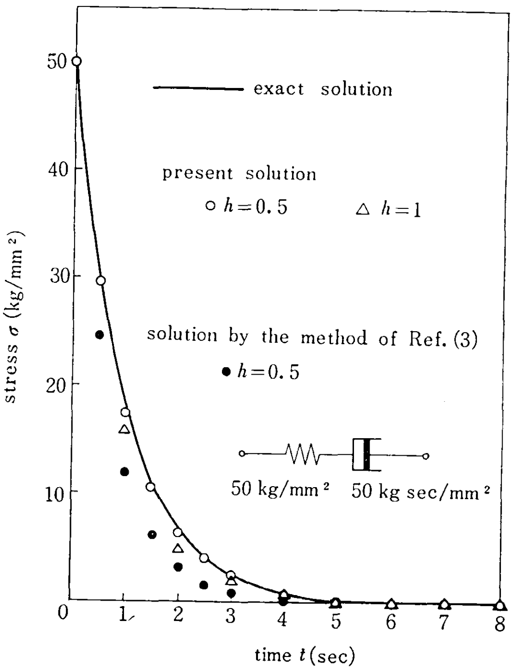


Fig. 2 Stress relaxation test of a Maxwell model.

where the stress is assumed constant within each time interval.

(**b**) Relaxation of stress under transient thermal load

Fig. 3 shows the stress relaxation in a viscoelastic bar element fixed at both ends and subjected to a uniform heat loading. The material is assumed to be represented by a three element Voigt model whose creep compliance is

$$J(\tau)=J_0\{2-\exp(-\tau)\} \quad \mathrm{mm^2/kg}$$

where τ is the nondimensional time scale normalized by the retardation time T of the Voigt model. The temperature input given for the problem of Fig. 3 is

$$\theta(\tau)=10\{1-\exp(-\tau/\lambda)\} \quad \mathrm{deg}$$

where λ denotes the parameter specifying the rate of heating. By comparing (A) and (B) of Fig. 3, it can be seen that the temperature dependency of material properties is a decisive factor in the stress relaxation. The WLF equation[9] used as the shift function is indicated in the figure.

The second example concerns with a semi-infinite plate subjected to the thermal shock on the edge surface. As the temperature is uniform in the direction of y, we can use the material strip division in the x direction of Fig. 4. The material property is assumed to be expressed by the following shear and bulk compliances:

$$D(t)=0.04-0.02\exp(-t/200) \quad \mathrm{mm^2/kg}$$

and $\quad B(t)=0.008 \quad \mathrm{mm^2/kg}$

The rapid stress relaxation at the heated edge surface in the case of temperature-dependent material (B) should be notified.

Determination of Shear and Bulk Creep Compliances

Uniaxial creep test is simplest and most convenient to obtain the shear and bulk properties of the viscoelastic materials. In the creep test, the axial stress is constant so that

$$\sigma_x(t)=\sigma_{x0}H(t)=\text{const.} \qquad (30)$$

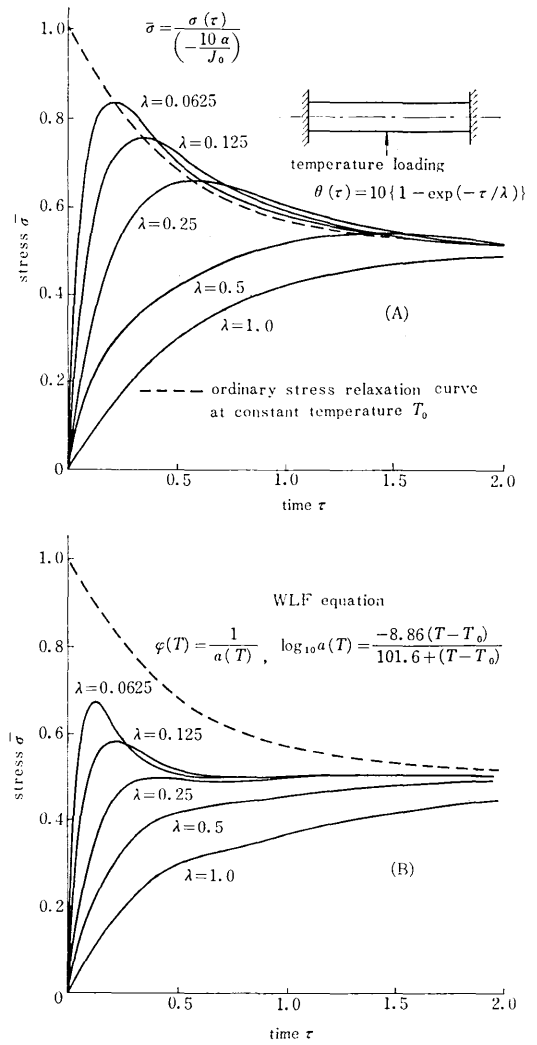


Fig. 3 Stress relaxation of a viscoelastic bar. Material property is temperature-independent in (A), and temperature-dependent in (B) according to the WLF postulate.

where σ_{x0} is the stress amplitude and $H(t)$ denotes the unit step function. The longitudinal and transverse strain ε_x, ε_y for the creep test are

$$\varepsilon_x(t)=\sigma_{x0}J(t) \qquad (31)$$

$$\varepsilon_y(t)=-\int_{0^-}^{t}\nu(t-\tau)\frac{d\varepsilon_x}{d\tau}d\tau \qquad (32)$$

$J(t)$ and $\nu(t)$ represent the axial creep compliance and the Poisson's ratio respectively. The Laplace transform of (31) and (32) are, with the transform variable s

$$\varepsilon_x{}^*=J^*\sigma_{x0} \qquad (33)$$

$$\varepsilon_y{}^*=-s\nu^*\varepsilon_x{}^*=-s\nu^*J^*\sigma_{x0} \qquad (34)$$

On the other hand, the following relations hold between the transformed compliances according to the well-known correspondence principle of viscoelasticity

$$D^*=2J^*(1+s\nu^*) \qquad (35)$$

$$B^*=3J^*(1-2s\nu^*) \qquad (36)$$

where D^* and B^* are the transformed shear and bulk compliances. By substituting J^* and ν^* from Eqs. (33) and (34), we obtain after the inverse transformation

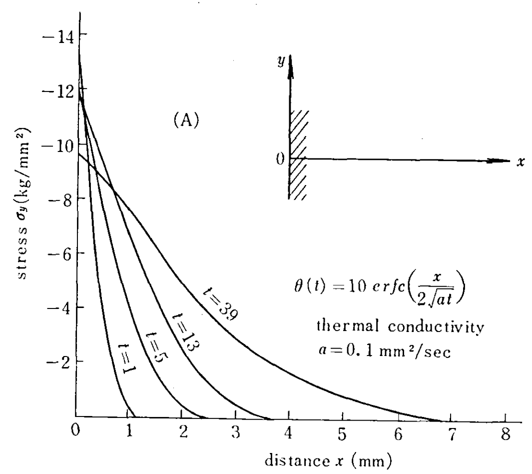
$$D(t)=\frac{2}{\sigma_{x0}}\{\varepsilon_x(t)-\varepsilon_y(t)\} \qquad (37)$$

$$B(t)=\frac{3}{\sigma_{x0}}\{\varepsilon_x(t)+2\varepsilon_y(t)\} \qquad (38)$$

It must be emphasized that the shear and bulk creep compliances $D(t)$ and $B(t)$ can be directly determined from the strain data $\varepsilon_x(t)$ and $\varepsilon_y(t)$ of the uniaxial creep test.

Concluding Remarks

Present paper exemplifies the power and versatility of the incremental approach for the problems of material nonlinearity. Test examples indicate that the assumption of the piecewise linear variation of the relevant variables is suitable for the accuracy and stability of the numerical solution. Temperature dependency of the material properties has considerable effects on the stress and/or strain solutions. Care should be taken for the appropriate choice of the time interval of the incremental computation, depending upon the degree of the temperature dependency. (Manuscript received Jan. 11, 1972)



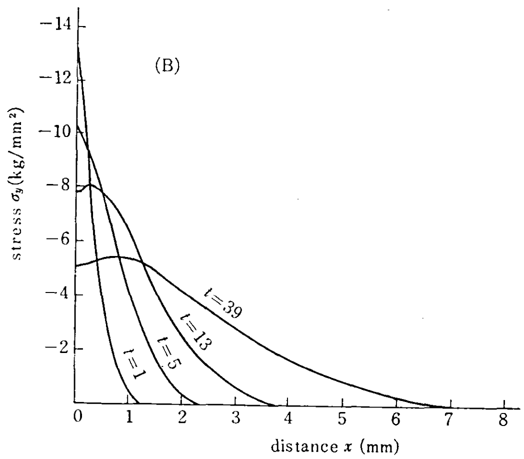


Fig. 4 Thermal stress in a semi-infinite plate. The material property is the same as in (A) and (B) of Fig. 3.

References

1) J. L. White, Proc. 2nd Conference on Matrix Methods in Structural Mechanics, AFFDL-TR-68-150, 489-516 (1968)
2) D. M. Malone and J. J. Conner, M. I. T. Research Report R69-11 (1969)
3) O. C. Zienkiewicz, W. Watson and I. P. King, Int. J. Mech. Sci., Vol. 10, 807-827 (1968)
4) R. Taylor, K. S. Pister and G. L. Goudreau, Int. J. Numerical Methods in Engineering, Vol. 2, 45-59 (1970)
5) E. Heer and J. C. Chen, J. P. L. Technical Report 32-1381 (1969)
6) Y. Yamada, "Plasticity and Viscoelasticity", (in Japanese) Baifukan (1972)
7) R. M. Christensen, "Theory of Viscoelasticity", Academic Press (1971)
8) L. W. Morland and E. H. Lee, Trans. Soc. of Rheology IV, 233-263 (1960)
9) M. L. Williams, AIAA Journal, Vol. 2, No. 5, 785-808 (1964)