An approach to the Decision Theory under Knightian Uncertainty

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Contents

1 Introduction

1. In economic analyses dealing with uncertainty, it is generally necessary to use some utility representation functions. Probabilistic Decision Theory provides the foundation for such functions. Ramsey (1926) (32) provided the first attempt at an axiomatization of choice under uncertainty. His great contribution was to suggest a way of deriving a consistent theory of choice under uncertainty that could isolate beliefs from preferences while still maintaining subjective probabilities. His purpose was to construct the foundation of the logical inference rather than of the probabilistic one which was developed later by the mainstream proponents of the decision theory. The most important studies were provided by Neumann and Morgenstern (1947)(29), Savage(1954) (34), Anscombe and Aumann (1963)(2).

Neumann and Morgenstern(1947) (29) prepared three axioms for a set of functions which assign probabilistic distribution on prize set and for a preference relation between them. The first axiom is the one which ensures the weak order of preference relation. The second and key axiom is called the *independence axiom.* This axiom is such that if every two functions (say p, q), make a convex combination between themselves and another function (say r) respectively (i.e. make $ap + (1 - a)r$ and $aq + (1 - a)r$ with any $a \in (0,1)$), the preference relation between p and q is preserved . This axiom is crucial in many studies of Probabilistic Decision Theory since this induces *additivity* or *affinity* of the probabilistic representation. Such additivity observed in various representations is the main topic of the present dissertation. The final axiom is the one which guarantees the continuity of the representation function. Using the above three axioms, they obtained the representation function in the form of probabilistic expectation of utility of prizes, which is called *expected utility representation.* An important point to emphasize is the fact that the probability measure in their framework is exogenous and so should be regarded as *objective.*

The formulation appearing in Savage(1954) (34) consists of seven axioms. This axiom set was epoch-making as it provided the foundation of *subjective probability.* More precisely, the probability measure on the given state space and its additivity was induced endogenously. These properties are due to the additivity of the (Lebesgue) integral operator in the representational form. In Savage's framework, additivity of the representation function (integral operator) is derived from two axioms. One is the *Sure-Thing Principle* and the other is the *Comparative Probability.* To explain these axioms, we define by $[f, E; g, E^c]$ the function which coincides with f on E and with g on E^c. Sure-thing principle is such that for all f, g,h,h', and all events E, $[f, E; h, E^c] \succeq [g, E; h, E^c]$ iff $[f, E; h', E^c] \succeq [g, E; h', E^c]$. The preference between two functions which coincide with each other on off of event E is determined by only the relation on E. Comparative Probability is that for all constant functions f, g, f', g' such that $f \succeq g$, $f' \succeq g'$, and for all events E, F, $[f, E; g, E^c] \succeq [f, F; g, F^c]$ iff $[f', E; g', E^c] \succeq [f', F; g', F^c]$. This axiom provides the binary relation between events as "*more likely than*". By these axioms, it holds that if event E_1 is more likely than event E_2 and $E_1 \cap F = E_2 \cap F = \emptyset$ then $E_1 \cup F$ is more likely than event $E_2 \cup F$. This property makes it possible for the probability of any event to be assigned by the *inner measure* method, which is the approximation from the inside, and for that probability to be an additive one.

Although Savage's framework is good, it is somewhat complicated. Anscombe and Aumann(1963) (2) improved this point with an alternative axiom set for subjective inference. They extended the ideas of Neumann and Morgenstern by preparing a simple act set. The simple act set consists of all functions that are defined on a state space and which have values originating from a set of all probabilistic distributions on a prize set with a finite support. In other words, these functions take their values from a set of lotteries where the prizes are also lotteries. Their axiom set consists of five axioms; AA1(Ordering), AA2(Independence), AA3(Continuity), AA4(Monotonicity), and AA5(Non-degeneracy). AA1,AA2, AA3 correspond to the three axioms of Neumann and Morgenstern(1947) (29) respectively and AA4, AA5 are the axioms which concern the simple acts. This method developed by Anscombe and Aumann(2) has been used in various studies since it is very tractable, although in comparison to Savage ' s theory, rather artificial.

2. A series of studies have thus made it clear that the decision maker's behavior under uncertainty can be recognized as maximizing expected utility with respect to the additive probabilistic measure. However, several experimental facts against the expected utility theory have been found. The most famous one is the experiment reported in Ellsberg(1961) (10), which is as follows: suppose there are two urns, each one containing 100 balls. The balls may be either black or red. Urn A is known to contain 50 black balls and 50 red ones. But there is no information about the allocation of balls in urn B. The examinee is asked to choose an urn and a color, and then to draw a ball from the urn which she chose. If the ball she draws is of the color she has chosen, she will gain 10 dollars. What will be the examinee's choice? Many experiments reported that, while most examinees are indifferent with respect to the color they bet on, they are not indifferent with respect to the urn they choose. They strictly prefer the urn A, in which the probabilities are known. This fact cannot be explained by any (objective or subjective) probabilistic decision. It may be assumed that the examinee estimates the probability of urn A and red; $P(Ar)$, as being larger than that of urn B and red; $P(Br)$, i.e. $P(Ar) > P(Br)$ since she prefer the bed in urn A to in urn B. Similarly, she also estimates that $P(Ab) > P(Bb)$. Thus if she thinks her inference rule is *additive* then she must be sure that $1 = P(Ar \cup Ab) = P(Ar) + P(Ab) > P(Br) + P(Bb) = P(Br \cup Bb) = 1$. It is contradiction. More precisely, her preference violates Savage's sure-thing principle. (see Ellsberg (10) or Gilboa (12)). This fact is called *Ellsberg paradox.* The Ellsberg paradox is an evidence that a decision maker distinguishes the environment in which the probability is known from the one in which the probability is unknown. Almost all decision makers prefer the former to the latter. An explanation for such a difference is concerned with the question of whether decision makers use an additive or non-additive probability

This nature of decision makers is called the *uncertainty aversion*, the concept of which comes originally from F.Knight.(1964) (23). Knight pointed out the difference between the environment in which the probability is known and the one in which the probability is unknown calling the former *Risk* and the latter *uncertainty*. He stated that most economic phenomena are placed under *uncertainty*. Non-additive probability theory is often called *Knightian uncertainty*.

Recent studies by Schmeidler(1989) (36) and Gilboa(1989) (12) provide an axiomatic characterization for expected utility with non-additive probabilities. Both approaches employ the Choquet integral of expected utility with respect to non-additive probability or capacity.

Non-additive probability or capacity, v, on state space S is defined by the following three axioms: (i)$v(\emptyset) = 0, v(S) = 1$, (ii)$0 \leq v(A) \leq 1$ for all $A \subseteq S$, (iii) $A \subseteq B$ implies $v(A) \leq v(B)$. So, instead of additivity, only monotonicity is imposed. For a random variable $a(\omega)$ over S, Choquet integral is defined as follows:

$$I(a) = \int_0^{\infty} v(a \geq \alpha)d\alpha + \int_{-\infty}^{0} (v(a \geq \alpha) - 1)d\alpha$$

Since this is a Riemann integral, additivity of the measure is not necessary for the well-definedness. Especially, for a finite step function $a = \sum \alpha_i 1_{E_i}$ where $\alpha_1 > \alpha_2 > \cdots > \alpha_k$, $E_i \cap E_j = \emptyset$ for all i, j, and 1_A is an indicator function, its integral is rewritten as

$$I(a) = \sum_{i=1}^{k} (\alpha_i - \alpha_{i+1}) v(\bigcup_{j=1}^{i} E_j)$$

In the case of v being additive probability, the Choquet integral coincides with the probabilistic expectation (i.e, Lebesgue integral). Schmeidler(1989) (36) explained the Ellsberg paradox in this way; suppose that $S = \{r, b\}$ is the state space. The non-additive probability assigned by a decision maker is $v(\emptyset) = 0, v(\{r\}) = 0.4, v(\{b\}) = 0.4, v(\{r, b\}) = 1$. If one bets on red in urn B, her Choquet expected payoff is $10v(\{r\}) + 0v(\{r, b\}) = 4$ dollars. On the other hand, if she bets on black in urn B, her Choquet expected payoff is $10v(\{b\}) + 0v(\{r, b\}) = 4$ dollars. Thus, whether she bets on red or black in urn B, her expected payoff is less than that of betting on urn A which is 5 dollars. Hence she prefers urn A to urn B. Using non-additive probability and Choquet expected utility, Schmeidler succeeded in explaining the Ellsberg paradox as a rational decision behavior.

The axiom set for Choquet expected utility representaion which Schmeidler provided is almost the same as that in Anscombe and Aumann (2). Schmeidler replaced only the axiom of AA2(Independence) by $AA2_{como}$(comonotonic independence). *Comonotonicity* is the notion which characterizes Choquet integral. Two random variables $a(\omega)$ and $b(\omega)$ are said to be comonotonic when $(a(s)-a(t))(b(s)-b(t)) \geq 0$ for all states s, t. Schmeidler (1986)(35) proved that the following two statements for an operator I are equivalent: (i) I has an expression of Choquet integral and (ii) $I(a+b) = I(a)+I(b)$ whenever a and b are comonotonic. The condition of (ii) is called *comonotonic additive.* This theorem means that comonotonic additivity characterizes the Choquet integral. In Schmeidler(1989) (36), the axiom $AA2_{como}$(comonotonic independence) for the preference of the decision maker induces comonotonic additivity in the representation operator and plays the role in the construction of Choquet expected utility representation as seen in the following theorem:

Theorem (Schmeidler 1989)

A binary relation $\succeq$ satisfies $AA1, AA2_{como}, AA3, AA4, AA5$

if and only if there exist a unique finitely nonadditive probability(capacity) v and an affine real valued function u on Y such that for all f and g,

$$f \succeq g \Leftrightarrow \int_S u(f(s))dv(s) \geq \int_S u(g(s))dv(s)$$

where the integrals are Choquet integrals.

In the case of Ellsberg's experiment, let random variable a be such that $a(Br) = 10, a(Bb) = 0$ and b be such that $b(Br) = 0, b(Bb) = 10$. Then these a and b are not comonotonic. Thus, expected payoff for $a + b$ does not coincide with the sum of that for a and that for b, which results in non-additivity of the inference of decision makers.

While Schmeidler(1989) (36) adopts the axiom set of Anscombe and Aumann(1963)(2), Gilboa(1987) (12) uses the framework of Savage(1954) (34). He replaced the Sure-thing principle by some weaker axioms which allow for non-additivity of the measure.

3. However, a full framework of Choqet expected utility is not necessary to explain the Ellsberg paradox. With only a little non-additivity, we can induce an Ellsberg decision. For example, the decision criterion called *ε-contamination* is the decision maker's preference. It is expressed mostly with an additive probabilistic expectation but only sometimes with a non-additive one. The notion of ε-contamination is old; it is discussed in the literature of robust estimation since Huber(1964) (18). Given a random variable a, ε-contamination representasion is expressed as follows:

$$J(a) = (1-\varepsilon)\int_S a(s)d\mu(s) + \varepsilon \min_{s \in S} a(s) \tag{1}$$

where μ is a finitely additive probability on state space S and ε is a some small positive constant . This representation form consists of a large additive part (integral part) but a small non-additive part (minimum part).

Using ε-contamination, we can explain the Ellsberg paradox as follows; let random variable a be such that $a(Br) = 10, a(Bb) = 0$ and b be such that $b(Br) = 0, b(Bb) = 10$. Then the expected payoff with respect to a is $J(a) = (1 - \varepsilon) \times 10\mu(Br) + 0 = 10(1 - \varepsilon)\mu(Br)$, which is strictly less than the expected payoff when betting on Ar; $10\mu(Ar)$ under the natural estimation $\mu(Br) = \mu(Ar) = 0.5$. We produce similar conclusion in the case of b.

As seen above, ε-contamination has a much simpler expression than the general Choquet integral expression in explaining the Ellsberg paradox. So, it should be given proper axiomatization. This study was independently done by Eichberger and Kelsey(1999) (8) and by Nishimura and Ozaki (2006) (30). Both studies use the axiomatic system of Anscombe and Aumann(1963) (2). The difference between the two is whether ε in the representation is exogenous or endogenous. It is endogenous in the former and exogenous in the latter. Eichberger and Kelsey(1999) (8) developed a more general representation, which contains ε-contamination as a special case, for dealing with the Ellsberg experiment using three kinds of colored balls. Their representation (called *E-capacity*) for a given random variable a is as follows:

$$J(a) = (1 - \varepsilon) \int_S a(s)d\mu(s) + \varepsilon \sum_{k=1}^{K} \rho(E_k) \min_{s \in E_k} a(s) \tag{2}$$

where $E_1, E_2, \ldots, E_K$ is a given partition of S.

4. The second chapter, which is based on the paper of Kojima(2004) (24), studies the axiomatization of ε-contamination. Here, we have the same motivation as Eichberger and Kelsey(1999) (8) but adopt a different approach from that. While Eichberger and Kelsey(1999) (8) uses the Anscombe and Aumann (2) method directly, we derive the same property from the more general Schmeidler(1989) (36) method which utilizes a weaker independence axiom than Anscombe and Aumann(1963) (2). We focus attention on the non-additive part in the ε-contamination representation; $min_{s \in S} a(s)$. We notice that this term has the additivity if restricted within the random variables having a common minimizer in S. That is, $min_{s \in S}(a(s)+b(s)) = min_{s \in S} a(s) + min_{s \in S} b(s)$ if $argmin_{s \in S} a(s) \cap min_{s \in S} b(s) \neq \emptyset$. Therefore it is natural to predict that if we introduce the independence axiom only to the simple acts which have a common minimizer on S then we will obtain the expression of ε-contamination. We call two functions *cominimum* if these have a common minimizer on S. Our notion of cominimum is a weaker than that of comonotonic since comonotonic implies cominimum. We define operator I to be *cominimum additive* if $I(a + b) = I(a) + I(b)$ whenever a and b are cominimum. Cominimum additivity is stronger than comonotonic additivity since cominimum additive implies comonotonic additive. Using the notion of cominimum additivity, we characterize ε-contamination formula by the following lemma:

Lemma

If I is a Choquet Integral with respect to the capacity v on S, then the following four conditions are equivalent:

(i) $E \cup F \neq S$, $E \cap F = \emptyset$ implies $v(E \cup F) = v(E) + v(F)$
(ii) There exists an additive probability measure μ and a real number ε such that $E \neq S$ implies $v(E) = (1-\varepsilon)\mu(E)$
(iii) There exists an additive probability measure μ and a real number ε such that for any random variable a

$$I(a) = (1-\varepsilon)\int_S a d\mu(s) + \varepsilon \min_{s \in S} a$$

(iv) I is cominimum additive
This lemma shows the equivalence not only between cominimum additivity (iv) and ε-contamination expression (iii) but also between these and the local additivity of the capacity (i)(ii). Moreover, taking the equivalence between (iii) and (iv) into account, we can expect that a similar theorem of Schmeidler(1989) (36) will be corroborated . Exchanging Schmeidler's comonotonic independence axiom AA_{como} for our cominimum independence axiom AA_{comi}, we succeed in axiomatizing ε-contamination as explained in the following theorem:

Theorem

A binary relation $\succeq$ satisfies $AA1, AA2_{comi}, AA3, AA4, AA5$
if and only if there exist a unique finitely additive probability measure μ on S and an affine function u and a real number ε such that :

$$f \succeq g \Leftrightarrow (1-\varepsilon)\int_S u(f(s))d\mu(s) + \varepsilon \min_{s \in S} u(f(s)) \geq (1-\varepsilon)\int_S u(g(s))d\mu(s) + \varepsilon \min_{s \in S} u(g(s))$$

Our cominimum independence axiom AA_{comi} coincides with the extremal independence axiom which Eichberger and Kelsey(1999) (8) introduced with the finite state space. The advantages of our approach are the following: firstly, our approach is constructed on the set of simple acts which possibly contain infinite states. Secondly, our construction uses the local additivity of the operation developed by Schmeidler. Finally, since our notion of cominimumity is flexible, our characterization can be extended to more general near-additive representations. We shall introduce these results in the third chapter.

5. The third chapter, which is based on the joint paper of Kajii, Kojima, and Ui(2007) (20) addresses the extension of the notion of cominimumity. In the previous chapter, we considered the cominimum additivity only on the whole state space S. In this chapter, we extend it to the collections of the events; the subsets of S. Let $\mathcal{E} \subseteq 2^S$ be a collection of subsets of S. Two functions x and y on S are said to be $\mathcal{E}$-cominimum if, for every $E \in \mathcal{E}$, the set of minimizers of x restricted on E and that of y have a common element. An operator I is said to be $\mathcal{E}$-cominimum additive if $I(x+y) = I(x) + I(y)$ whenever x and y are $\mathcal{E}$-cominimum. The case in the previous section is a special case of $\mathcal{E} = \{S\}$. The main result of this chapter is a representation theorem for homogeneous operators satisfying $\mathcal{E}$-cominimum additivity.

We treat v not as a non-additive probability but as a game or non-additive signed measure for the generality. A set function $v : 2^S \to \mathbb{R}$ with $v(\emptyset) = 0$ is called a *game* or a *non-additive signed measure*, especially in the context of the cooperative game theory. For a subset $T \subseteq S$, let u_T be the *unanimity game* on T defined by the rule: $u_T(X) = 1$ if $T \subseteq X$ and $u_T(X) = 0$ otherwise. The following result is well known as the Möbius inversion in discrete and combinatorial mathematics (see Shapley(1953) (33)).

Lemma

The collection $\{u_T\}$ is a linear base for the set of all games. The unique collection of coefficients $\{\beta_T\}$ satisfying $v = \sum_T \beta_T u_T$, or equivalently $v(E) = \sum_{T \subseteq E} \beta_T$ for all $E \subseteq S$, is given by $\beta_T = \sum_{E \subseteq T} (-1)^{|T|-|E|} v(E)$.

This linear base decomposition by Möbius inversion plays a very important role in obtaining our result since the following property is suitable for our Choquet integral representation with the following lemma:

Lemma(Gilboa and Schmeidler 1994(15))

For a function $x \in \mathbb{R}^S$ and a game $v = \sum_T \beta_T u_T$,

$$\int x dv = \sum_T \beta_T \int x du_T = \sum_T \beta_T \min_T x, \tag{3}$$

This method of using the minimum operator is adequate for treating cominimum functions.

We shall sketch our main result in the following. Notice that since $\mathcal{E}$-cominimum additivity implies comonotonic additivity, a homogeneous $\mathcal{E}$-cominimum additive operator is represented by the Choquet integral with respect to a non-additive signed measure v by Schmeidler's theorem, a fortiori. Since v can be uniquely written as $v = \sum_{T \subseteq S} \beta_T u_T$, where u_T is the unanimity game on T, the characterization of the operator can be done in terms of coefficients $\{\beta_T\}_{T \subseteq S}$. We say that T is $\mathcal{E}$-complete if, for any two points $\omega, \omega' \in T$, there exists $E \in \mathcal{E}$ satisfying $\{\omega, \omega'\} \subseteq E \subseteq T$; that is, any two elements are "connected" within T by an element of $\mathcal{E}$. And $\mathcal{E}$ is said to be *complete* when $\mathcal{E}$ contains all $\mathcal{E}$-complete subsets. The main result shows that under the assumption of $\mathcal{E}$ being complete, a homogeneous operator is $\mathcal{E}$-cominimum additive if and only if $\beta_T = 0$ for every $T \notin \mathcal{E}$ That is,

Theorem

Let $\mathcal{E}$ be complete. The following two statements are equivalent;

(i) $v = \sum_T \beta_T u_T$ is $\mathcal{E}$-cominimum additive, (ii) $\int x dv = \sum_{T \in \mathcal{E}} \beta_T \min_T x$.

Moreover we can also show the local additivity of the game which satisfies $\mathcal{E}$-cominimum additivity as follows: v is modular on a suitably defined collection of pairs of events: $v(T_1 \cup T_2) + v(T_1 \cap T_2) = v(T_1) + v(T_2)$ whenever the pair (T_1, T_2) belongs to the collection. These results are an extension of our results in the previous chapter.

We can obtain similar properties for *comaximum* functions easily using the *conjugation* of the game. Let us define two functions x and y to be $\mathcal{E}$-comaximum if, for every $E \in \mathcal{E}$, the set of maximizers of x restricted on E and that of y have a common element. An operator I is said to be $\mathcal{E}$-comaxiimum additive if $I(x+y) = I(x) + I(y)$ whenever x and y are $\mathcal{E}$-comaximum. The *conjugate* of v, denoted by v', is defined as $v'(E) = v(S) - v(S \backslash E)$ for all $E \in 2^S$. Note that $(v')' = v$ and $(v+w)' = v' + w'$ for any two games v, w. Let u'_T be the conjugate of an unanimity game u_T. Then $u'_T(X) = 1$ if $T \cap X \neq \emptyset$ and $u'_T(X) = 0$ otherwise. Thus it holds as (3) that

$$\int x dv' = \sum_T \beta_T \int x du'_T = \sum_T \beta_T \max_T x. \tag{4}$$

Hence, by the above similarity in conjugation, we obtain the following theorem with the same proof:

Theorem

Let $\mathcal{E}$ be complete. The following two statements are equivalent:

(i) $v = \sum_T \beta_T u_T$ is $\mathcal{E}$-comaximum additive, (ii) $\int x dv = \sum_{T \in \mathcal{E}} \beta_T \max_T x$.

Our representation theorem has the possibility to be applied widely. We discuss two applications for decision models under uncertainty. The first is the E-capacity expected utility model of Eichberger and Kelsey(1999) (8) which was mentioned in the previous chapter. A collection of partitions on which they constructed their representation satisfies our completeness. Hence our theorem can directly be applied to E-capacity. Owing to the success of this extension to the collection of subsets, our results contain the E-capacity representation as a corollary. The second is the multi-period decision model of Gilboa(1989) (13). Gilboa(1989)(13) considered the following type of representation;

$$\sum_{i=1}^{n} p_i x(i) + \sum_{i=2}^{n} \delta_i |x(i) - x(i-1)| \tag{5}$$

where $p_1, \ldots, p_n$ and $\delta_2, \ldots, \delta_n$ are constants. Interpret $S = \{1, \ldots, n\}$ as a collection of time periods, and $x(1), \ldots, x(n)$ as a stream of income. The utility describes the value of the stream of income as a weighted average $\sum_{i=1}^{n} p_i x(i)$ plus an adjustment factor $\sum_{i=2}^{n} \delta_i |x(i) - x(i-1)|$ which measures the variations of the stream. For this decision model, we also provide an alternative proof for the axiomatic characterizations using our results directly.

6. The fourth chapter, which is based on the joint paper of Kajii, Kojima, and Ui(2007) (22) analyzes the operator which has *both* cominimum additivity and comaximum additivity. Let $\mathcal{E}$ be a collection of subsets of S. Two functions x and y on S are said to be $\mathcal{E}$-coextrema if, for each $E \in \mathcal{E}$, the set of minimizers of x restricted on E and that of y have a common element, and the set of maximizers of x restricted on E and that of y have a common element as well. An operator I on the set of functions on S is $\mathcal{E}$-coextrema additive if $I(x+y) = I(x) + I(y)$ whenever x and y are $\mathcal{E}$-coextrema. The main result shows that a homogeneous coextrema additive operator I

can be represented as:

$$I(x) = \sum_{E \in \mathcal{E}} \{\lambda_E \max_{\omega \in E} x(\omega) + \mu_E \min_{\omega \in E} x(\omega)\} \tag{6}$$

where λ_E and μ_E are constants, when the collection $\mathcal{E}$ satisfies a certain regularity condition. Separating all singleton sets, $\mathcal{F}_1$, from the Σ term, it is rewritten as:

$$I(x) = \int x dp + \sum_{E \in \mathcal{E} \setminus \mathcal{F}_1} \{\lambda_E \max_E x + \mu_E \min_E x\}. \tag{7}$$

Interpreting S as the set of states describing uncertainty and function x as a random variable over S. Then the class of operators which can be written as in (7) has a natural interpretation that the value of x is the sum of its expected value $\int xdp$ and a weighted average of the most optimistic outcome and the most pessimistic outcomes on events. This interpretation admits it to correspond to Hurwicz criterion for decision making under uncertianty. Alternatively, interpret S as a collection of individuals (i.e., a society), and $x(\omega)$ as the wealth allocated to individual ω. Then $\int xdp$ can be seen as the (weighted) average income of the society, and $\max_E x$ and $\min_E x$ correspond to the wealthiest and the poorest in group E, respectively. In particular, when p is the uniform distribution and $\lambda_E = -1$ and $\mu_E = 1$, then the problem of maximizing (7) subject to $\int xdp$ being held constant means that of reducing the sum of wealth differences in various groups in $\mathcal{E}$.

As a corollary, our result shows that for the special case where $\mathcal{E}$ consists of singletons and the whole set S, a homogeneous $\mathcal{E}$-coextrema operator is exactly the Choquet integral of a NEO-additive capacity, which is axiomatized by Chateaunuff, Eichberger, and Grant(2002) (6). NEO-additive capacity provides an explanation for the phenomena such as the same individual being observed to buy insurance against risk and lottery tickets, which is hard to explain by the ordinary expected utility maximization. Our result provides a natural, and important generalization of the NEO-additive capacity result. Eichberger, Kelsey, and Schipper(2006)(9) applied a NEO-additive capacity model to the Bertrand and Cournot competition models to study combined effects of optimism and pessimism in economic environments.

While in the NEO-additive capacity, optimism and pessimism are about the whole states of the world, our model can accommodate more delicate combinations of optimism and pessimism measured in a family of events. Thus our $\mathcal{E}$-coextrema additivity model provides a rich framework for analyzing effects optimism and pessimism in economic problems.

In the previous chapter, we considered the class of cominimum additive operators, and each cominimum additive operator is shown to be a weighted sum of minimums. The class of comaximum operators is defined and characterized similarly. However, the class of coextrema additive operators is *not* the intersection of the two, and the characterization result reported in this chapter cannot be done by adopting these results. To see this, notice that both $\{u_T\}$ and $\{w_T\}$

constitute linear bases. Hence if the collection of events $\mathcal{E}$ contains a sufficient variety of events, not only coextrema additive games but also many other games can be expressed as in (24). In other words, for these expressions to be interesting, it is important to establish the uniqueness. But the reader will see that the issue of uniqueness in our characterization is far more technically involved.

7. The fifth chapter, which is based on the joint paper of Kajii, Kojima, and Ui(2006) (21), applies our method to cooperative game theory. Cooperative game theory considers the problem of how the outcome of a given N-person game $v : 2^N \to \mathbb{R}$ should be allocated between the N players in the grand coalition. The Shapley value (see Shapley (33)) is a well known solution. The Shapley value of v is the vector of payoffs $\phi(v) \in \mathbb{R}^N$ given by the following formula:

$$\phi_i(v) = \sum_{S \in 2^N : i \in S} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} \delta_i v(S) \text{ for all } i \in N.$$

where $\delta_i v(S)$ is denoted the marginal contribution of player $i \in S$ to $v(S)$; that is, $\delta_i v(S) = v(S) - v(S \backslash \{i\})$. In particular, the Shapley value of an unanimity game u_T is given by $\phi_i(u_T) = 1/|T|$ if $i \in T$, 0 otherwise. Since the Shapley value is linear in games, we have an alternative formula for the Shapley value of a game expressed in Möbius inversion form; $v = \sum_{T \in 2^N} \beta_T u_T$, as follows:

$$\phi_i(v) = \sum_{T \in 2^N} \beta_T \phi_i(u_T) = \sum_{T \in 2^N : i \in T} \beta_T / |T|. \tag{8}$$

Through this formula, we can apply our theorem in the previous chapter to the Shapley value.

Myerson (1977(27), 1980(28)) augmented a cooperative game by a *conference structure.* A conference is defined as a set of two or more players and a collection of conferences is called a conference structure. Myerson defined another cooperative game where the conference structure determines which coalitions are feasible. Myerson showed that the Shapley value of the induced cooperative game can be characterized by two axioms: fairness and component efficiency. This allocation rule is referred to as the Myerson value in the subsequent literature. In his result, the feasible coalition is the one in which any pair of players are either *directly* or *indirectly* connected (i.e. path connected) by the conferences contained in the coalition. For example, in a 3-person game, suppose that a conference structure is $\{\{1,2\},\{2,3\}\}$. Then, in the solution of the Myerson value, all players are allocated equally $v(\{1,2,3\})/3$ from the outcome $v(\{1,2,3\})$ since player 1 and player 3 are connected indirectly by the intermediary of player 2.

This chapter proposes a refinement of the Myerson value which distinguishes direct and indirect connections. Similar to Myerson(1977(27), 1980(28)), we augment a cooperative game by a conference structure and define another cooperative game where the conference structure determines which coalitions are feasible. But different from Myerson(1977(27), 1980(28)), the feasible coalition is the one in which any pair of players are *directly* connected by the conferences contained in the coalition. In the main result, we show that the Shapley value of the induced cooperative game can be characterized by three axioms: fairness, complete component efficiency, and no contribution by unconnected players. The latter two new axioms describe the behavior of the allocation rule distinguishing direct and indirect connections. Examining the above example, a conference structure $\{\{1,2\},\{2,3\}\}$ is not feasible in our cooperative game since player 1 and player 3 are not connected directly. But both $\{\{1,2\},\{2,3\}\{1,2,3\}\}$ and $\{\{1,2\},\{2,3\},\{1,3\}\}$ are feasible. Thus our solution proposes a different allocation from that of the Myerson value. Our new value is more sensitive than the Myerson value since the two values coincide under some adequate manipulation of the conference structure. To establish the main result, we take advantage of the idea of potentials for cooperative games first proposed by Hart and Mas-Colell(1989) (17).

2 ε-contamination and Comonotonic Independence Axiom

2.1 Introduction

The *subjective* expected utility theory was explored by Savage(1954) (34). Savage's framework which consists of seven axioms is excellent but somewhat complicated. Anscombe and Aumann(1963) (2) improved this point with an alternative axiom set for subjective inference. They extended the ideas of Neumann and Morgenstern(1947) (29) by preparing a simple act set. Their axiom set consists of five axioms; AA1(Ordering), AA2(Independence), AA3(Continuity), AA4(Monotonicity), and AA5(Non-degeneracy). AA1,AA2, AA3 correspond to the three axioms of Neumann and Morgenstern(1947) (29) respectively and AA4, AA5 are the axioms which concern the simple acts. This method developed by Anscombe and Aumann(1954)(2) has been used in various studies since it is very tractable, although in comparison to Savage ' s theory, rather artificial. A series of studies have thus made it clear that the decision maker's behavior under uncertainty can be recognized as maximizing expected utility with respect to the additive probabilistic measure.

However, several experimental facts against the expected utility theory have been found. The most famous one is the experiment reported in Ellsberg(1961) (10). The Ellsberg paradox is an evidence that a decision maker distinguishes the environment in which the probability is known from the one in which the probability is unknown. Almost all decision makers prefer the former to the latter. An explanation for such a difference is concerned with the question of whether decision makers use an additive or non-additive probability.

Recent studies by Schmeidler(1989) (36) and Gilboa(1989) (12) provide an axiomatic characterization for expected utility with non-additive probabilities. Both approaches employ the Choquet integral of expected utility with respect to non-additive probability or capacity.

Non-additive probability or capacity, v, on state space S is defined by the following three axioms: (i)$v(\emptyset) = 0, v(S) = 1$, (ii)$0 \leq v(A) \leq 1$ for all $A \subseteq S$, (iii) $A \subseteq B$ implies $v(A) \leq v(B)$. So, instead of additivity, only monotonicity is imposed. For a random variable $a(\omega)$ over S, Choquet integral is defined as follows:

$$I(a) = \int_0^\infty v(a \geq \alpha)d\alpha + \int_{-\infty}^0 (v(a \geq \alpha) - 1)d\alpha$$

Since this is a Riemann integral, additivity of the measure is not necessary for the well-definedness. Especially, for a finite step function $a = \sum \alpha_i 1_{E_i}$ where $\alpha_1 > \alpha_2 > \cdots > \alpha_k$, $E_i \cap E_j = \emptyset$ for all i, j, and 1_A is an indicator function, its integral is rewritten as

$$I(a) = \sum_{i=1}^{k} (\alpha_i - \alpha_{i+1}) v(\bigcup_{j=1}^{i} E_j)$$

In the case of v being additive probability, the Choquet integral coincides with the probabilistic expectation (i.e, Lebesgue integral). Using non-additive probability and Choquet expected utility, Schmeidler succeeded in explaining the Ellsberg paradox as a rational decision behavior.

The axiom set for Choquet expected utility representaion which Schmeidler provided is almost the same as that in Anscombe and Aumann (2). Schmeidler replaced only the axiom of AA2(Independence) by $AA2_{como}$(comonotonic independence). *Comonotonicity* is the notion which characterizes Choquet integral. Two random variables $a(\omega)$ and $b(\omega)$ are said to be comonotonic when $(a(s)-a(t))(b(s)-b(t)) \geq 0$ for all states s,t. Schmeidler (1986)(35) proved that the following two statements for an operator I are equivalent: (i) I has an expression of Choquet integral and (ii) $I(a+b) = I(a)+I(b)$ whenever a and b are comonotonic. The condition of (ii) is called *comonotonic additive.* This theorem means that comonotonic additivity characterizes the Choquet integral. In Schmeidler(1989) (36), the axiom $AA2_{como}$(comonotonic independence) for the preference of the decision maker induces comonotonic additivity in the representation operator and plays the role in the construction of Choquet expected utility representation as seen in the following theorem:

Theorem (Schmeidler 1989)

A binary relation $\succeq$ satisfies $AA1, AA2_{como}, AA3, AA4, AA5$
if and only if there exist a unique finitely nonadditive probability(capacity) v and an affine real valued function u on Y such that for all f and g,

$$f \succeq g \Leftrightarrow \int_S u(f(s))dv(s) \geq \int_S u(g(s))dv(s)$$

where the integrals are Choquet integrals.

While Schmeidler(1989) (36) adopts the axiom set of Anscombe and Aumann(1963)(2), Gilboa(1989) (12) uses the framework of Savage(1954) (34). He replaced the Sure-thing principle by some weaker axioms which allow for non-additivity of the measure.

However, a full framework of Choqet expected utility is not necessary to explain the Ellsberg paradox. With only a little non-additivity, we can induce an Ellsberg decision. For example, the decision criterion called *ε-contamination* is the decision maker's preference. It is expressed mostly with an additive probabilistic expectation but only sometimes with a non-additive one. The notion of ε-contamination is old; it is discussed in the literature of robust estimation since Huber(1964) (18). Given a random variable a, ε-contamination representasion is expressed as follows:

$$J(a) = (1-\varepsilon)\int_S a(s)d\mu(s) + \varepsilon \min_{s\in S} a(s) \tag{9}$$

where μ is a finitely additive probability on state space S and ε is a some small positive constant . This representation form consists of a large additive part (integral part) but a small non-additive part (minimum part).

ε-contamination has a much simpler expression than the general Choquet integral expression in explaining the Ellsberg paradox. So, it should be given proper axiomatization. This study was independently done by Eichberger and Kelsey(1999) (8) and by Nishimura and Ozaki (2006) (30). Both studies use the axiomatic system of Anscombe and Aumann(1963) (2). The difference between the two is whether ε in the representation is exogenous or endogenous. It is endogenous in the former and exogenous in the latter. Eichberger and Kelsey(1999) (8) developed a more general representation, which contains ε-contamination as a special case, for dealing with the Ellsberg experiment using three kinds of colored balls. Their representation (called *E-capacity*) for a given random variable a is as follows:

$$J(a) = (1-\varepsilon)\int_S a(s)d\mu(s) + \varepsilon \sum_{k=1}^{K} \rho(E_k) \min_{s \in E_k} a(s) \tag{10}$$

where $E_1, E_2, \ldots, E_K$ is a given partition of S.

This paper studies the axiomatization of ε-contamination. Here, we have the same motivation as Eichberger and Kelsey(1999) (8) but adopt a different approach from that. While Eichberger and Kelsey(1999) (8) uses the Anscombe and Aumann(1963) (2) method directly, we derive the same property from the more general Schmeidler(1989) (36) method which utilizes a weaker independence axiom than Anscombe and Aumann(1963) (2). We focus attention on the non-additive part in the ε-contamination representation; $min_{s\in S}a(s)$. We notice that this term has the additivity if restricted within the random variables having a common minimizer in S. That is, $min_{s\in S}(a(s)+b(s)) = min_{s\in S}a(s) + min_{s\in S}b(s)$ if $argmin_{s\in S}a(s) \cap min_{s\in S}b(s) \neq \emptyset$. Therefore it is natural to predict that if we introduce the independence axiom only to the simple acts which have a common minimizer on S then we will obtain the expression of ε-contamination. We call two functions *cominimum* if these have a common minimizer on S. Our notion of cominimum is a weaker than that of comonotonic since comonotonic implies cominimum. We define operator I to be *cominimum additive* if $I(a+b) = I(a) + I(b)$ whenever a and b are cominimum. Cominimum additivity is stronger than comonotonic additivity since cominimum additive implies comonotonic additive. Using the notion of cominimum additivity, we characterize ε-contamination formula by the following lemma:

Lemma

If I is a Choquet Integral with respect to the capacity v on S, then the following four conditions are equivalent:

(i) $E \cup F \neq S$, $E \cap F = \emptyset$ implies $v(E \cup F) = v(E) + v(F)$

(ii) There exists an additive probability measure μ and a real number ε such that $E \neq S$ implies $v(E) = (1-\varepsilon)\mu(E)$

(iii) There exists an additive probability measure μ and a real number ε such that for any random variable a

$$I(a) = (1-\varepsilon)\int_S a d\mu(s) + \varepsilon \min_{s\in S} a$$

(iv) I is cominimum additive

This lemma shows the equivalence not only between cominimum additivity (iv) and ε-contamination expression (iii) but also between these and the local additivity of the capacity (i)(ii). Moreover, taking the equivalence between (iii) and (iv) into account, we can expect that a similar theorem of Schmeidler(1989) (36) will be corroborated . Exchanging Schmeidler's comonotonic independence axiom AA_{como} for our cominimum independence axiom AA_{comi}, we succeed in axiomatizing ε-contamination as explained in the following theorem:

Theorem

A binary relation $\succeq$ satisfies $AA1, AA2_{comi}, AA3, AA4, AA5$

if and only if there exist a unique finitely additive probability measure μ on S and an affine function u and a real number ε such that :

$$f \succeq g \Leftrightarrow (1-\varepsilon)\int_S u(f(s))d\mu(s) + \varepsilon \min_{s\in S} u(f(s)) \geq (1-\varepsilon)\int_S u(g(s))d\mu(s) + \varepsilon \min_{s\in S} u(g(s))$$

Our cominimum independence axiom AA_{comi} coincides with the extremal independence axiom which Eichberger and Kelsey(1999) (8) introduced with the finite state space. The advantages of our approach are the following: firstly, our approach is constructed on the set of simple acts which possibly contain infinite states. Secondly, our construction uses the local additivity of the operation developed by Schmeidler. Finally, since our notion of cominimumity is flexible, our characterization can be extended to more general near-additive representations. This extension has already accomplished in Kajii, Kojima, and Ui(2007) (22).

The organization of this paper is as follows. Section 2 quotes the notion of Cominimum additivity. Section 3 provides the main results.

2.2 Comonotonic additivity and Cominimum additivity

In this section, we prepare a notion for the measurable function which is crucial for our paper. Let Σ denote a nonempty algebra of subsets of a set S, let $\mathcal{F}$ be the set of functions from S to $\Re$ which is constant on each element in some finite measurable partition of S, i.e. the set of all finite step functions from S to $\Re$. Especially for $A \in \Sigma$, 1_A denote the indicator function in $\mathcal{F}$. I denote a set function or an operator from $\mathcal{F}$ to $\Re$.

Definition 1 *(i) Two functions a and b in $\mathcal{F}$ are said to be comonotonic if $(a(s)-a(t))(b(s)-b(t)) \geq 0$ for all s and t in S*

(ii) Two functions a and b in $\mathcal{F}$ are said to be cominumum if $argmin_{t\in S}a(t) \cap argmin_{t\in S}b(t) \neq \emptyset$ (i.e. the set of minimizers of a and that of b have a common element .)

A set $C(\subseteq \mathcal{F})$ is said to be *comonotonic set* if every two functions $a, b \in C$ are comonotonic. Denote by $como(\mathcal{F})$ the collection of all comonotonic sets. Similarly, a set $C(\subseteq \mathcal{F})$ is said to be *cominimum set* if every two functions $a, b \in C$ are cominimum. Denote by $comi(\mathcal{F})$ the collection of all cominimum sets.

Definition 2 *(i) I is said to be comonotonic additive if* $\{a, b\} \in como(\mathcal{F})$ *implies* $I(a+b) = I(a) + I(b)$
(ii) I is said to be cominimum additive if $\{a, b\} \in comi(\mathcal{F})$ *implies* $I(a+b) = I(a) + I(b)$

The notion of comonotonic additive was introduced by Schmeidler (35) to construct Choquet expected utility representation. Our definition of cominimum additive is an extension of comonotonic additive, though stronger. The following lemma gives the relation between cominimum additivity and comonotonic additivity.

Lemma 1 *(i)* $\{a, b\} \in como(\mathcal{F})$ *implies* $\{a, b\} \in comi(\mathcal{F})$
(ii) If I is cominimum additive then I is also comonotonic additive.
(iii) For any $a \in \mathcal{F}$ *and any constant function* $\lambda 1_S$, $\{a, \lambda 1_S\} \in como(\mathcal{F})$ *thus* $\{a, \lambda 1_S\} \in comi(\mathcal{F})$
(iv) For all $\alpha \in (0,1)$; $\{a, b\} \in como(\mathcal{F})$ *implies* $\{a, b, \alpha a + (1-\alpha)b\} \in como(\mathcal{F})$
(v) For all $\alpha \in (0,1)$; $\{a, b\} \in comi(\mathcal{F})$ *implies* $\{a, b, \alpha a + (1-\alpha)b\} \in comi(\mathcal{F})$

Proof. (i) Notice that any finite step function has a minimizer on S. Suppose that $\{a, b\} \in como(\mathcal{F})$. Let s be a minimizer of a. Then $a(t) \geq a(s)$ for all $t \in S$. Thus $b(t) \geq b(s)$ for all $t \in S$ since a and b are comonotonic. So, s is also a minimizer of b. Hence $\{a, b\} \in comi(\mathcal{F})$. (ii) Suppose that $\{a, b\} \in como(\mathcal{F})$. By (i), $\{a, b\} \in comi(\mathcal{F})$. Thus $I(a+b) = I(a) + I(b)$ since I is cominimum additive. Hence the function I must be comonotonic additive. (iii) Any constant function $\lambda 1_S$ satisfies both $\lambda 1_S(s) \geq \lambda 1_S(t)$ and $\lambda 1_S(t) \geq \lambda 1_S(s)$ for every $s, t \in S$. Thus $\{a, \lambda 1_S\} \in como(\mathcal{F})$. So, $\{a, \lambda 1_S\} \in comi(\mathcal{F})$ by (i). (iv) and (v) hold since $a(t) \geq a(s)$ and $b(t) \geq b(s)$ imply $\alpha a(t) + (1-\alpha)b(t) \geq \alpha a(s) + (1-\alpha)b(s)$ for all $\alpha \in (0,1)$. ■

Schmeidler(1986) (35) provided a sufficient condition for comonotonic additive as (i) of the following lemma. We shall extend this for cominimum additive as (ii) of the following lemma. The proof by Schmeidler depends on the comonotonicity between a constant function and any other function with Lemma 1(iii)(iv). Therefore we can easily apply this to the proof for cominimum additive easily using Lemma 1(iii)(v). Let us simultaneously prove for comonotonic additive and for cominimum additive.

Lemma 2 *Suppose that* $I(\lambda 1_S) = \lambda$
(i) If for all $\{a, b, c\} \in como(B)$ *and all* $\alpha \in (0,1)$; $I(a) > I(b)$ *implies* $I(\alpha a + (1-\alpha)c) > I(\alpha b + (1-\alpha)c)$*, then I is comonotonic additive.*
(ii) If for all $\{a, b, c\} \in comi(B)$ *and all* $\alpha \in (0,1)$;$I(a) > I(b)$ *implies* $I(\alpha a + (1-\alpha)c) > I(\alpha b + (1-\alpha)c)$*, then I is cominimum additive.*

Proof. To prove (i) and (ii) simultaneously, we use the common notation $c(\mathcal{F})$ both for *como*$(\mathcal{F})$ and for *comi*$(\mathcal{F})$. Suppose that for all $\{a,b,c\} \in c(\mathcal{F})$ and all $\alpha \in (0,1)$; $I(a) > I(b)$ implies $I(\alpha a + (1-\alpha)c) > I(\alpha b + (1-\alpha)c)$. First, let us prove the following claim; for all $\{x,y\} \in c(\mathcal{F})$ and all $\alpha \in (0,1)$; $I(\alpha x + (1-\alpha)y) = \alpha I(x) + (1-\alpha)I(y)$. Indeed, pick any $\varepsilon > 0$. Then $(I(x)+\varepsilon)1_S \in \mathcal{F}$ satisfies $I((I(x)+\varepsilon)1_S) > I(x)$ and $(I(y)+\varepsilon)1_S \in \mathcal{F}$ satisfies $I((I(y)+\varepsilon)1_S) > I(y)$ by the assumption $I(\lambda 1_S) = \lambda$. Hence, $\alpha I(x) + (1-\alpha)I(y) + \varepsilon = I(\alpha(I(x)+\varepsilon)1_S + (1-\alpha)(I(y)+\varepsilon)1_S) > I(\alpha x + (1-\alpha)(I(y)+\varepsilon)1_S) > I(\alpha x + (1-\alpha)y)$. First inequality holds since $\{(I(x)+\varepsilon)1_S, x, (I(y)+\varepsilon)1_S\} \in c(\mathcal{F})$ by Lemma 1 and second inequality holds since $\{(I(y)+\varepsilon)1_S, y, x\} \in c(\mathcal{F})$ by Lemma 1. Since ε is any positive number, we obtain that $\alpha I(x) + (1-\alpha)I(y) \geq I(\alpha x + (1-\alpha)y)$. Furthermore, using similar argument for $\varepsilon < 0$, we can show the contrary inequality. Therefore it is proved that$I(\alpha x + (1-\alpha)y) = \alpha I(x) + (1-\alpha)I(y)$. Then we conclude our claim. Next let us use this claim twice. First, let $\alpha = \frac{1}{2}$,$x = 2a$,$y = 0$ for $\{a,0\} \in c(\mathcal{F})$, then $I(2a) = 2I(a)$ for all $a \in \mathcal{F}$. Second, let $\alpha = \beta = \frac{1}{2}$, $\{x,y\} = \{2a,2b\} \in c(\mathcal{F})$, then $I(a+b) = \frac{1}{2}I(2a) + \frac{1}{2}I(2b) = I(a) + I(b)$. Now we can obtain the conclusion. ∎

Let v denote a monotonic real valued function on Σ with $v(\emptyset) = 0, v(S) = 1$. Monotonicity means that for any E and F in Σ, $E \subset F$ implies $v(E) \leq v(F)$. v is said to be *non-additive probability measure* or *capacity* . For the given capacity v and measurable function a, Choquet integral is defined as follows:

$$I(a) = \int_0^\infty v(a \geq \alpha)d\alpha + \int_{-\infty}^0 (v(a \geq \alpha) - 1)d\alpha$$

For finite step function $a = \sum \alpha_i 1_{E_i}$,$\alpha_1 > \alpha_2 > \cdots > \alpha_k$, $E_i \cap E_j = \emptyset$, for all i,j ,Choquet integral $I(a)$ is represented as follows:

$$I(a) = \sum_{i=1}^{k} (\alpha_i - \alpha_{i+1}) v(\bigcup_{j=1}^{i} E_j)$$

Schmeidler proved the following theorem which characterizes the Choquet integral.

Theorem 1 *(Schmeidler 1986(35))*
Let an operator $I : \mathcal{F} \to R$ satisfying $I(1_S) = 1$ be given. Suppose also that the function I satisfies
(i) comonotonic additive (ii) Monotonicity i.e. $a \geq b$ on S implies $I(a) \geq I(b)$.
then, defining $v(E) = I(1_E)$ on Σ, v(E) is a capacity and we have for all a in $\mathcal{F}$

$$I(a) = \int_0^\infty v(a \geq \alpha)d\alpha + \int_{-\infty}^0 (v(a \geq \alpha) - 1)d\alpha$$

Proof. See Schmeidler 1986 (35) ∎

This theorem means that comonotonic additivity characterizes the Choquet integral. Considering the analogy between comonotonic additivity and cominimum additivity, we can predict that cominimum additivity may also characterize some integral representation. The following lemma provides an answer to our question. Moreover this lemma plays a crucial role in our main result and gives the different characterization from Eichberger and Kelsey(1999) (8).

Lemma 3 *Suppose $I : \mathcal{F} \to R$ be Chiquet Integral with respect to the capacity v on Σ, the following four conditions are equivalent:*
(i) $E \cup F \neq S$, $E \cap F = \emptyset$ implies $v(E \cup F) = v(E) + v(F)$
(ii) There exists an additive probability measure μ and a real number ε such that $E \neq S$ implies $v(E) = (1-\varepsilon)\mu(E)$
(iii) There exists an additive probability measure μ and a real number ε such that for any $a \in \mathcal{F}$

$$I(a) = (1-\varepsilon)\int_S ad\mu(s) + \varepsilon \min_{s \in S} a$$

(iv) I is cominimum additive

Proof. (i) $\to$ (ii)
Fix two distinct nonempty sets S_1, S_2 arbitrarily such that $S_1 \cup S_2 = S, S_1 \cap S_2 = \emptyset$ and denote $\epsilon = 1 - (v(S_1) + v(S_2))$. First we assume $\varepsilon \neq 1$. We define μ as follows: $\mu(E) = 1$ for $E = S$, $\mu(E) = \frac{1}{1-\varepsilon}v(E)$ for $E \neq S$. Here we claim that μ is additive probability measure. Indeed, suppose that $A \cap B = \emptyset$. If $A \cup B \neq S$ then $\mu(A \cup B) = \frac{1}{1-\varepsilon}v(A \cup B) = \frac{1}{1-\varepsilon}(v(A) + v(B)) = \mu(A) + \mu(B)$ by assumption(i). For the case of $A \cup B = S$, if $A = S, B = \emptyset$ or $B = S, A = \emptyset$ then $\mu(A) + \mu(B) = 1 = \mu(A \cup B)$. Otherwise use the decomposition with S_1, S_2. $\mu(A) + \mu(B) = \frac{1}{1-\varepsilon}v((A \cap S_1) \cup (A \cap S_2)) + v((B \cap S_1) \cup (B \cap S_2)) = \frac{1}{1-\varepsilon}(v(A \cap S_1) + v(A \cap S_2) + v(B \cap S_1) + v(B \cap S_2)) = \frac{1}{1-\varepsilon}(v(S_1) + v(S_2)) = 1 = \mu(S)$. So, our claim holds. Next, suppose $\varepsilon = 1$, i.e. $v(S_1) + v(S_2) = 0$. Since v is non-negative, $v(S_1) = v(S_2) = 0$. Thus since for every $E \neq S$, $v(E) = v((E \cap S_1) \cup (E \cap S_2)) = v(E \cap S_1) + v(E \cap S_2)$ by assumption (i) and $v(E \cap S_1) \leq v(S_1), v(E \cap S_2) \leq v(S_2)$ by monotonicity, it holds that $v(E) = 0$. Hence $v(E) = (1-\varepsilon)\mu(E)$ with every additive measure $\mu(E)$.
(ii) $\to$ (iii)
Suppose that $a = \sum \alpha_i 1_{E_i}$; $\alpha_1 > \alpha_2 > \cdots > \alpha_k$,; $E_i \cap E_j = \emptyset$, for all i, j .
In the case $k \geq 2$, by assumption (ii), $I(a) =$
$\sum_{i=1}^{k-1}(\alpha_i - \alpha_{i+1})v(\cup_{j=1}^{i} E_j) + \alpha_k v(S) = \sum_{i=1}^{k-1}(\alpha_i - \alpha_{i+1})(1-\varepsilon)\mu(\cup_{j=1}^{i} E_j) + (1-\varepsilon)\alpha_k + \varepsilon\alpha_k = \sum_{i=1}^{k-1}(\alpha_i - \alpha_{i+1})(1-\varepsilon)\sum_{j=1}^{i}\mu(E_j) + (1-\varepsilon)\alpha_k\sum_{j=1}^{k}\mu(E_j) + \varepsilon\alpha_k = (1-\varepsilon)\sum_{i=1}^{k}\alpha_i\mu(E_i) + \varepsilon\alpha_k = (1-\varepsilon)\int_S ad\mu + \varepsilon \min_S a(s)$
In the case $k = 1$, i.e. $a =$ constant function $\lambda 1_S$, $I(a) = I(\lambda 1_S) = \lambda = (1-\varepsilon)\int_S ad\mu + \varepsilon \min_S a(s)$
(iii) $\to$ (iv)

Note that for all $\{a,b\} \in comi(B)$; $\min_{s\in S}(a(s)+b(s)) = \min_{s\in S} a(s) + \min_{s\in S} b(s)$. Therefore $I(a+b) = (1-\varepsilon)\int_S (a+b)d\mu + \varepsilon \min_{s\in S}(a(s)+b(s)) = (1-\varepsilon)\int_S a d\mu + (1-\varepsilon)\int_S b d\mu + \min_{s\in S} a(s) + \min_{s\in S} B(s) = I(a) + I(b)$

(iv) $\rightarrow$ (i)

Suppose that $E \cup F \neq S$, $E \cap F = \emptyset$. Let $a = 1_E, b = 1_F$. Then $a(s) = b(s) = 0$ for some $s \in E^c \cap F^c = (E \cup F)^c$, so $\{a,b\} \in comi(B)$. Therefore, $v(E \cup F) = I(a+b) = I(a) + I(b) = v(E) + v(F)$ ∎

Remark 1 *The following two conditions are equivalent: (i) v is convex (ii)$\varepsilon > 0$.*
Indeed, If $A \neq S, B \neq S, A \cup B = S$ then $v(A\cup B) + v(A \cap B) - v(A) - v(B) = 1 + (1-\varepsilon)\mu(A\cap B) - (1-\varepsilon)(\mu(A)+\mu(B)) = \varepsilon + (1-\varepsilon)(1+\mu(A\cap B) - (\mu(A)+\mu(B))) = \varepsilon$. If $A \neq S, B \neq S, A\cup B \neq S$ then $v(A \cup B) + v(A\cap B) - v(A) - v(B) = (1-\varepsilon)(\mu(A\cup B) + \mu(A \cap B) - \mu(A) - \mu(B)) = 0$. And if $A = S$ then $v(A\cup B) + v(A \cap B) - v(A) - v(B) = v(S) + v(B) - v(S) - v(B) = 0$.

2.3 Main Result

In this section, we apply the results obtained in the previous section to the representation theory of preference for a *simple lottery act*. Let X be a set and Y be the set of distributions over X with finite supports, i.e.; $Y = \{y : X \rightarrow [0,1] | y(x) \neq 0$ for finitely many x's in X and $\sum_{x\in X} y(x) = 1\}$ We can identify X with the subset $\{y \in Y | y(x) = 1$ for some x in $X\}$ of Y. Denote by L_0 the set of all Σ-measurable function from S to Y which is constant on each element in some finite measurable partition of S, i.e. the set of all finite step functions. Denote by L_c the constant functions in L_0. Let L be a convex subset of Y^S which includes L_c. We call an element of L_0 a simple lottery act, or more simply, an act. Then, Y will be a mixture space. Given $y, y' \in Y$ and $\lambda \in [0,1]$, we denote by $\lambda y + (1-\lambda)y'$ the compound lottery. Note that every $f \in L_0$ has minimizers on S since it is a step function.

In the neo-bayesian nomenclature, elements of X are (deterministic) outcomes, elements of Y are random outcomes or (roulette) lotteries, and elements of L are acts(or horse lotteries). Elements of S are states (of nature) and elements of Σ are events. Let us introduce the notion of cominimum to a simple lottery act in the same manner as Schmeidler for comonotonic.

Definition 3 *(i)Two acts f and g in Y^S are said to be comonotonic provided for any $s, t \in S$ if $f(s) \succeq f(t)$ then $g(s) \succeq g(t)$.*
(ii)Two acts f and g in Y^S are said to be cominimum if $\{s | \forall t; f(t) \succeq f(s)\} \cap \{s | \forall t; g(t) \succeq g(s)\} \neq \emptyset$

We define *como*(L) and *comi*(L) corresponding to *como*(B) and *comi*(B) respectively; we then obtain the following lemma similar to Lemma 1. To construct the Choqeut representation for the preference on a simple lottery, Schmeidler (36) provided the following five axioms.

$A1$ (Weak Order):
(a) for all f and g in $L : f \succeq g$ or $g \succeq f$.
(b) For all f, g and h in L: If $f \succeq g$ and $g \succeq h$, then $f \succeq h$.
The relation $\succeq$ on L induces a relation also denoted by $\succeq$ on Y: $y \succeq z$ iff $y^S \succeq z^S$ where y^S denotes the constant function y on S.
$A2_{como}$ (Comonotonic Independence):
$\{f, g, h\} \in como(L)$ and $\alpha \in (0,1)$: $f \succeq g$ implies $\alpha f + (1-\alpha)h \succeq \alpha g + (1-\alpha)h$.
$A3$ (Continuity):
For all f, g, and h in L : if $f \succeq g$ and $g \succeq h$, then there are α and $\beta \in (0,1)$ such that $\alpha f + (1-\alpha)h \succeq g$ and $g \succeq \beta f + (1-\beta)h$.
$A4$ (Monotonicity):
For all f and g in L : If $f(s) \succeq g(s)$ on S then $f \succeq g$.
$A5$ (Nondegeneracy):
Not for all f and g in L, $f \succeq g$.

Under above five axioms, Schmeidler (36) proved the following theorem.

Theorem 2 *(Schmeidler (36)) A binary relation $\succeq$ defined on L_0 satisfies $A1, A2_{como}, A3, A4, A5$ if and only if there exist a unique finitely nonadditive probability(capacity) v on Σ and an affine real valued function u on Y such that for all f and g in L_0*

$$f \succeq g \Leftrightarrow \int_S u(f(s))dv(s) \geq \int_S u(g(s))dv(s)$$

Taking the similarity between comonotonicity and cominimumity into account, we can expect that a similar theorem will hold by exchanging comonotonicity for cominimumity. Now let us introduce a new axiom $A2_{comi}$ instead of $A2_{como}$.

$A2_{comi}$ (*Cominimum Independence*):
For $\{f, g, h\} \in comi(L)$ and $\alpha \in (0,1)$: $f \succeq g$ implies $\alpha f + (1-\alpha)h \succeq \alpha g + (1-\alpha)h$.

This axiom coincides with the one which Eichberger and Kelsey (8) introduced with finite state space. Our main result, which gives a representation for ε-contamination, is the following.

Theorem 3 *(Main Theorem) A binary relation $\succeq$ defined on L_0 satisfies $A1, A2_{comi}, A3, A4, A5$ if and only if there exist a unique finitely additive probability measure μ on (S, Σ) and an affine function u and a real number ε such that :*

$$f \succeq g \Leftrightarrow (1-\varepsilon)\int_S u(f(s))d\mu(s) + \varepsilon \min_{s\in S} u(f(s)) \geq (1-\varepsilon)\int_S u(g(s))d\mu(s) + \varepsilon \min_{s\in S} u(g(s))$$

Our proof for this theorem is obtained by tracing over that of Schmeidler (36)

Proof. (if part). We give the proof only $A2_{comi}, A4$. Denote $J(f) = (1-\varepsilon)\int_S u(f(s))d\mu(s) + \varepsilon \min_{s\in S} u(f(s))$. Suppose $\{f,g,h\} \in comi(L)$ and $f \succeq g$ i.e. $J(f) \geq J(g)$. Then, by affinity of $u()$ and by additivity of integral and by additivity of minimum operator on the set contained by $comi(B)$,
$J(\alpha f + (1-\alpha)h) - J(\alpha g + (1-\alpha)h)$
$= (1-\varepsilon)\int_S u(\alpha f + (1-\alpha)h)d\mu(s) + \varepsilon \min_{s\in S} u(\alpha f + (1-\alpha)h)$
$- (1-\varepsilon)\int_S u(\alpha g + (1-\alpha)h)d\mu(s) - \varepsilon \min_{s\in S} u(\alpha g + (1-\alpha)h)$
$= (1-\varepsilon)\int_S u(\alpha f)d\mu + \varepsilon \min_{s\in S} u(\alpha f) - (1-\varepsilon)\int_S u(\alpha g)d\mu - \varepsilon \min_{s\in S} u(\alpha g)$
$= \alpha(J(f) - J(g)) \geq 0$
Thus $A2_{comi}$ holds. Moreover suppose that $f(s) \succeq g(s)$ for all $s \in S$. Then $u(f(s)) \geq u(g(s))$ for all $s \in S$. Thus $\int_S u(f(s))d\mu(s) \geq \int_S u(f(s))d\mu(s)$ and $\min_{s\in S} u(f(s)) \geq \min_{s\in S} u(g(s))$. Hence $A4$ holds.
(only if part) Restricting to L_c, $A2_{comi}$ means the *independence* axiom of von Neumann-Morgenstern since $L_c \in comi(L)$. Therefore there exists a function $u : L_c \to \Re$ that represents the preference on L_c(by von Neumann-Morgenstern theorem). By $A5$, there exist $y^* \succ y_*$ and set $u(y^*) = 1$ and $u(y_*) = -1$ without loss of generality(This u is unique). For an arbitrary $f \in L_0$ denote $M_f = \{\alpha f + (1-\alpha)y^S | y \in Y, a \in [0,1]\}$. Then, since $M_f \in comi(L)$, there exists a function $J_f : M_f \to \Re$ that represents the preference on M_f and coincide with u on L_c. For $f \in L_0$ define $J(f)$ by $J(f) = J_f(f)$. $J(f)$ is well-defined and represents the preference on L_0.
Let B denote the set of Σ-measurable finite step functions on S. Let $U : L_0 \to B$ be defined by $U(f)(s) = u(f(s))$ for s in S and f in L_0. U is onto and well-defined. We now define a real valued function I on B . Given $a \in B$, let $f \in L_0$ be such that $U(f) = a$. Then define $I(a) = J(f)$. Let us check four properties as follows:
(1) I is well-defined
Indeed, suppose $U(f) = U(g)$, then $u(f(s)) = u(g(s))$ for all $s \in S$. Therefore $f \sim g$, so $I(U(f)) = J(f) = J(g) = I(U(g))$.
(2) I is monotonic i.e. $a \geq b$ implies $I(a) \geq I(b)$
this is clear because $U(f)(s) \geq U(g)(s)$ for all s implies $f \succeq g$ by $A4$
(3) $I(\lambda 1_S) = \lambda$
because $U(\lambda y^*) = \lambda 1_S$ and $J(\lambda y^*) = u(\lambda y^*) = \lambda u(y^*) = \lambda$.
(4) I is cominimum additive.
To prove this, it is sufficient to show that the condition of lemma 2 (ii) holds. Suppose $\{f,g,h\} \in$

$comi(L)$ and let $a = U(f), b = U(g), c = U(h)$, then $\{a, b, c\} \in comi(B)$. The following two condition are equivalent ;(i) $f \succeq g$ implies $\alpha f + (1-\alpha)h \succeq \alpha g + (1-\alpha)h$ (ii) $J(f) \geq J(g)$ implies $J(\alpha f + (1-\alpha)h) \geq J(\alpha g + (1-\alpha)h)$. This means that $I(a) \geq I(b)$ implies $I(\alpha a + (1-\alpha)c) \geq I(\alpha b + (1-\alpha)c)$. Thus we obtain the conclusion.

Now, by (4) and lemma 1(ii), I is comonotonic additive. Thus, adding (2) and (3), by Theorem 1, there exists Choquet Integral representation of I with respect to the capacity $v(E) = I(1_E)$ as $I(a) = \int_S a dv(s)$. Here, under (4), let us use Lemma 3. Therefore I can be expressed as (iii) of Lemma 3, i.e.,

$$I(a) = (1-\varepsilon)\int_S a d\mu(s) + \varepsilon \min_{s \in S} a$$

Replace a by $U(f)$ and $I(a)$ by $J(f)$ then the desired expression is obtained. The proof is complete. ■

3 Cominimum Additive Operators

3.1 Introduction

Consider an operator I on the set of real valued functions on a finite set Ω. It is well known that an operator I is homogeneous (i.e. $I(\lambda x) = \lambda I(x)$ for a function x on Ω and $\lambda > 0$) and additive (i.e. $I(x + y) = I(x) + I(y)$ for functions x and y on Ω) if and only if it is represented as the integral with respect to a signed measure v on Ω; that is, $I(x) = \int xdv$ for a function x on Ω.

In his seminal paper,Schmeidler 1986 (35) considered a homogeneous operator that is additive on comonotonic functions. Two functions x and y on Ω are said to be comonotonic if $(x(\omega) - x(\omega'))(y(\omega) - y(\omega')) \geq 0$ for all ω, $\omega' \in \Omega$. He showed that an operator I is homogeneous and additive on comonotonic functions (i.e. $I(x+y) = I(x)+I(y)$ whenever x and y are comonotonic) if and only if it is represented as the Choquet integral with respect to a non-additive signed measure v on Ω; that is, $I(x) = \int xdv$ for a function x on Ω with the understanding that the integral is the Choquet integral. In the decision theory under uncertainty, the utility function representable as a Choquet integral now constitutes one of the important benchmarks.

In this paper, we propose a class of weak additivity concepts for an operator on the set of real valued functions, which include both additivity and comonotonic additivity as extreme cases. To be precise, let $\mathcal{E} \subseteq 2^\Omega$ be a collection of subsets of Ω. Two functions x and y on Ω are said to be $\mathcal{E}$-cominimum if, for every $E \in \mathcal{E}$, the set of minimizers of x restricted on E and that of y have a common element. An operator I is said to be $\mathcal{E}$-cominimum additive if $I(x + y) = I(x) + I(y)$ whenever x and y are $\mathcal{E}$-cominimum.

For example, if $\mathcal{E}$ is empty or contains only singletons, then any two functions are trivially $\mathcal{E}$-cominimum. In this case, $\mathcal{E}$-cominimum additivity coincides with additivity. If $\mathcal{E}$ consists of all subsets of Ω, then any two comonotonic functions are $\mathcal{E}$-cominimum and conversely any two $\mathcal{E}$-cominimum functions are comonotonic. In this case, $\mathcal{E}$-cominimum additivity coincides with comonotonic additivity. Thus, in general, $\mathcal{E}$-cominimum additivity is stronger than comonotonic additivity but weaker than additivity.

The main result of this paper is a representation theorem for homogeneous operators satisfying $\mathcal{E}$-cominimum additivity, which we shall sketch in the following. Notice that since $\mathcal{E}$-cominimum additivity implies comonotonic additivity, a homogeneous $\mathcal{E}$-cominimum additive operator is represented by the Choquet integral with respect to a non-additive signed measure v by Schmeidler's theorem, a fortiori. Since v can be uniquely written as $v = \sum_{T\subseteq\Omega} \beta_T u_T$, where u_T is the so called unanimity game on $T \subseteq \Omega$, the characterization of the operator can be done in terms of coefficients $\{\beta_T\}_{T\subseteq\Omega}$. We say that $T \subseteq \Omega$ is $\mathcal{E}$-complete if, for any two points $\omega, \omega' \in T$, there exists $E \in \mathcal{E}$ satisfying $\{\omega, \omega'\} \subseteq E \subseteq T$; that is, any two elements are "connected" within T by an element of $\mathcal{E}$. The main result shows that a homogeneous operator is $\mathcal{E}$-cominimum additive if and only if $\beta_T = 0$ for every T which is not $\mathcal{E}$-complete. It also shows that this condition is equivalent to the condition that v is modular on a suitably defined collection of pairs of events: $v(T_1 \cup T_2) + v(T_1 \cap T_2) = v(T_1) + v(T_2)$ whenever the pair (T_1, T_2) belongs to the collection.

We shall supply two applications to decision models under uncertainty. The first is the E-capacity expected utility model of Eichberger and Kelsey (8). The E-capacities include the so called ε-contamination as a special case. The second is the multi-period decision model of Gilboa (13). For both decision models, we provide alternative proofs for the axiomatic characterizations using our results directly.

The organization of this paper is as follows. Section 4.2 quotes some known results about the Choquet integrals and Schmeidler's theorem. Section 3.3 introduces $\mathcal{E}$-cominimum functions and studies properties of $\mathcal{E}$-complete events. Section 3.4 provides the main results and Section 3.5 discusses applications.

3.2 The Choquet integrals and Schmeidler's theorem

Let $\Omega = \{1, \dots, n\}$ be a finite set of states of the world. A subset $E \subseteq \Omega$ is called an event. Denote by $\mathcal{F}$ the collection of all non-empty subsets of Ω, and by $\mathcal{F}_k$ the collection of subsets with k elements.

A set function $v : 2^\Omega \to \mathbb{R}$ with $v(\emptyset) = 0$ is called a *game* or a *non-additive signed measure.* Since each game is identified with a point in $\mathbb{R}^\mathcal{F}$, we denote by $\mathbb{R}^\mathcal{F}$ the set of all games. For a game $v \in \mathbb{R}^\mathcal{F}$, we use the following definitions:

- v is *non-negative* if $v(E) \geq 0$ for all $E \in 2^\Omega$.
- v is *monotone* if $E \subseteq F$ implies $v(E) \leq v(F)$ for all $E, F \in 2^\Omega$. A monotone game is non-negative.
- v is *additive* if $v(E \cup F) = v(E) + v(F)$ for all $E, F \in 2^\Omega$ with $E \cap F = \emptyset$, which is equivalent to $v(E) + v(F) = v(E \cup F) + v(E \cap F)$ for all $E, F \in 2^\Omega$.
- v is *convex* (or *supermodular*) if $v(E) + v(F) \leq v(E \cup F) + v(E \cap F)$ for all $E, F \in 2^\Omega$.
- v is *normalized* if $v(\Omega) = 1$.
- v is a *non-additive measure* if it is monotone. A normalized non-additive measure is called a *capacity.*
- v is a *measure* if it is monotone and additive. A normalized measure is called a *probability measure.*
- The *conjugate* of v, denoted by v', is defined as $v'(E) = v(\Omega) - v(\Omega \backslash E)$ for all $E \in 2^\Omega$. Note that $(v')' = v$ and $(v + w)' = v' + w'$ for $v, w \in \mathbb{R}^\mathcal{F}$.

For $T \in \mathcal{F}$, let $u_T \in \mathbb{R}^{\mathcal{F}}$ be the *unanimity game* on T defined by the rule: $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise. Let u'_T be the conjugate of an unanimity gameu_T. Then $u'_T(S) = 1$ if $T \cap S \neq \emptyset$ and $u'_T(S) = 0$ otherwise. The following result is well known as the Möbius inversion in discrete and combinatorial mathematics (cf. 33).

Lemma 4 *The collection $\{u_T\}_{T\in\mathcal{F}}$ is a linear base for $\mathbb{R}^{\mathcal{F}}$. The unique collection of coefficients $\{\beta_T\}_{T\in\mathcal{F}}$ satisfying $v = \sum_{T\in\mathcal{F}} \beta_T u_T$, or equivalently $v(E) = \sum_{T\subseteq E} \beta_T$ for all $E \in \mathcal{F}$, is given by $\beta_T = \sum_{E\subseteq T}(-1)^{|T|-|E|}v(E)$.*

The collection of coefficients $\{\beta_T\}_{T\in\mathcal{F}}$ is referred to as the Möbius transform of v. If $v = \sum_{T\in\mathcal{F}} \beta_T u_T$, then the conjugate v' is given by $v' = \sum_{T\in\mathcal{F}} \beta_T u'_T$.

Denote by $\mathbb{R}^{\Omega} = \{x \,|\, x : \Omega \to \mathbb{R}\}$ the set of all real valued functions on Ω. Let $1_E \in \mathbb{R}^{\Omega}$ be the indicator function of an event $E \in \mathcal{F}$. We write $\min_E x = \min_{\omega\in E} x(\omega)$, $\arg\min_E x = \arg\min_{\omega\in E} x(\omega)$, $\max_E x = \max_{\omega\in E} x(\omega)$, $\arg\max_E x = \arg\max_{\omega\in E} x(\omega)$ for $E \in \mathcal{F}$ and $x \in \mathbb{R}^{\Omega}$.

Definition 4 For $x \in \mathbb{R}^{\Omega}$ and $v \in \mathbb{R}^{\mathcal{F}}$, the *Choquet integral* of x with respect to v is defined as

$$\int x dv = \int_{\underline{x}}^{\bar{x}} v(x \geq \alpha) d\alpha + \underline{x} v(\Omega), \tag{11}$$

where $\bar{x} = \max_{\Omega} x(\omega)$, $\underline{x} = \min_{\Omega} x(\omega)$, and $v(x \geq \alpha) = v(\{\omega \in \Omega \,|\, x(\omega) \geq \alpha\})$.

For example, the Choquet integral of an indicator function is $\int 1_E dv = \int_0^1 v(1_E \geq \alpha) d\alpha = v(E)$; the Choquet integral with respect to unanimity games and their conjugates are

$$\int x du_T = \int_{\underline{x}}^{\bar{x}} u_T(x \geq \alpha) d\alpha + \underline{x} u_T(\Omega) = \left[\min_T x - \min_\Omega x\right] + \min_\Omega x = \min_T x,$$
$$\int x du'_T = \int_{\underline{x}}^{\bar{x}} u'_T(x \geq \alpha) d\alpha + \underline{x} u'_T(\Omega) = \left[\max_T x - \min_\Omega x\right] + \min_\Omega x = \max_T x$$

because $u_T(x \geq \alpha) = 1$ if $\min_T x \geq \alpha$ and $u_T(x \geq \alpha) = 0$ otherwise, and $u'_T(x \geq \alpha) = 1$ if $\max_T x \geq \alpha$ and $u'_T(x \geq \alpha) = 0$ otherwise.

It is straightforward to see that the Choquet integral is linear in games:

$$\int x d(sv + tw) = s \int x dv + t \int x dw \text{ for all } x \in \mathbb{R}^{\Omega},\ v, w \in \mathbb{R}^{\mathcal{F}}, \text{ and } s, t \in \mathbb{R}.$$

An important implication of the linearity is the following additive representation of the Choquet integral (cf. 15).

Lemma 5 *For $x \in \mathbb{R}^\Omega$ and $v = \sum_{T \in \mathcal{F}} \beta_T u_T \in \mathbb{R}^{\mathcal{F}}$,*

$$\int x dv = \sum_{T \in \mathcal{F}} \beta_T \int x du_T = \sum_{T \in \mathcal{F}} \beta_T \min_T x, \tag{12}$$

$$\int x dv' = \sum_{T \in \mathcal{F}} \beta_T \int x du'_T = \sum_{T \in \mathcal{F}} \beta_T \max_T x. \tag{13}$$

Lemma 5 says that the Choquet integral of x with respect to v can be represented as a weighted sum of all minima of x with respect to some possibly negative weights.

Two functions $x, y \in \mathbb{R}^\Omega$ are said to be *comonotonic* if $(x(\omega) - x(\omega'))(y(\omega) - y(\omega')) \geq 0$ for all $\omega, \omega' \in \Omega$. Observe that two functions $x, y \in \mathbb{R}^\Omega$ are comonotonic if and only if $\arg\min_E x \cap \arg\min_E y \neq \emptyset$ for all $E \in \mathcal{F}$. Symmetrically, two functions $x, y \in \mathbb{R}^\Omega$ are comonotonic if and only if $\arg\max_E x \cap \arg\max_E y \neq \emptyset$ for all $E \in \mathcal{F}$.

If x and y are comonotonic then $\min_T(x+y) = \min_T x + \min_T y$ for all $T \in \mathcal{F}$. Thus, the Choquet integral is additive on comonotonic functions by Lemma 5:

$$\int (x+y) dv = \sum_{T \in \mathcal{F}} \beta_T \min_T (x+y) = \sum_{T \in \mathcal{F}} \beta_T \min_T x + \sum_{T \in \mathcal{F}} \beta_T \min_T y = \int x dv + \int y dv.$$

We say that an operator $I : \mathbb{R}^\Omega \to \mathbb{R}$ satisfies *comonotonic additivity* provided it is additive on comonotonic functions, i.e., $I(x+y) = I(x) + I(y)$ whenever x and y are comonotonic. Thus, the Choquet integral satisfies comonotonic additivity. We say that an operator $I : \mathbb{R}^\Omega \to \mathbb{R}$ is *homogeneous* (more precisely, positively homogeneous of degree one) provided $I(\lambda x) = \lambda I(x)$ for all $\lambda > 0$. It is easy to see that the Choquet integral is homogeneous. Schmeidler 1986(35) showed that a homogeneous operator which satisfies comonotonic additivity must be the Choquet integral. The following is a slightly different version of Schmeidler's theorem.[1]

Theorem 4 *An operator $I : \mathbb{R}^\Omega \to \mathbb{R}$ is homogeneous and satisfies comonotonic additivity if and only if $I(x) = \int x dv$ for all $x \in \mathbb{R}^\Omega$ where $v \in \mathbb{R}^{\mathcal{F}}$ is defined by the rule $v(E) = I(1_E)$.*

Proof. This can be shown by just a minor modification of Schmeidler's proof. ∎

3.3 Cominimum functions

We will study homogeneous operators satisfying a property stronger than comonotonic additivity and weaker than additivity. For this purpose, we generalize the notion of comonotonic functions.

[1]Schmeidler 1986(35) assumed monotonicity instead of homogeneity. But this can be readily shown adopting his proof. In fact, since homogeneity is a consequence of monotonicity in his proof, our statement is less elegant. But with monotonicity, the resulting game is necessarily a non-additive measure, which is inconvenient for us since we want to work with general games.

Remember that two functions $x, y \in \mathbb{R}^{\Omega}$ are comonotonic if and only if $\arg\min_E x \cap \arg\min_E y \neq \emptyset$ for all $E \in \mathcal{F}$. By replacing $\mathcal{F}$ with a collection of events $\mathcal{E} \subseteq \mathcal{F}$, we have a weaker notion of comonotonic functions.[2]

Definition 5 Let $\mathcal{E} \subseteq \mathcal{F}$ be a collection of events. Two functions $x, y \in \mathbb{R}^{\Omega}$ are said to be *$\mathcal{E}$-cominimum*, provided $\arg\min_E x \cap \arg\min_E y \neq \emptyset$ for all $E \in \mathcal{E}$. Two functions $x, y \in \mathbb{R}^{\Omega}$ are said to be *$\mathcal{E}$-comaximum*, provided $\arg\max_E x \cap \arg\max_E y \neq \emptyset$ for all $E \in \mathcal{E}$.

Note that x and y are $\mathcal{E}$-cominimum if and only if $-x$ and $-y$ are $\mathcal{E}$-comaximum. So in fact any result about $\mathcal{E}$-cominimum functions can be translated for $\mathcal{E}$-comaximum functions in a straightforward manner.

The following properties are immediate consequences of the definition:

- If two functions are $\mathcal{E}$-cominimum (resp. comaximum) then they are $\mathcal{E}'$-cominimum (resp. comaximum) for any $\mathcal{E}' \subseteq \mathcal{E}$.
- If two functions are both $\mathcal{E}$-cominimum (resp. comaximum) and $\mathcal{E}'$-cominimum (resp. comaximum) then they are $\mathcal{E} \cup \mathcal{E}'$-cominimum (resp. comaximum).
- Any two functions are $\mathcal{F}_1$-cominimum (comaximum) where $\mathcal{F}_1 = \{\{\omega\} \mid \omega \in \Omega\}$.
- Two functions are $\mathcal{E}$-cominimum (resp. comaximum) if and only if they are $\mathcal{E} \cup \mathcal{F}_1$-cominimum (resp. comaximum).
- The following statements are equivalent.
  - Two functions are comonotonic.
  - Two functions are $\mathcal{F}_2$-cominimum (comaximum) where $\mathcal{F}_2 = \{\{\omega, \omega'\} \mid \omega, \omega' \in \Omega\}$.
  - Two functions are $\mathcal{F}$-cominimum (comaximum).
  - Two functions are $\mathcal{E}$-cominimum (comaximum) for all $\mathcal{E} \subseteq \mathcal{F}$.

The last item above implies that even if $\mathcal{E} \neq \mathcal{E}'$, the collection of $\mathcal{E}$-cominimum pairs of functions may coincide with that of $\mathcal{E}'$-cominimum pairs. Among collections of events which induce the same pairs of cominimum functions, there is a special collection, the complete collection, which will play an important role in the main result of this paper.

Definition 6 Let $\mathcal{E} \subseteq \mathcal{F}$ be a collection of events. An event $T \in \mathcal{F}$ is *$\mathcal{E}$-complete* provided, for any two distinct points ω_1 and ω_2 in T, there is $E \in \mathcal{E}$ such that $\{\omega_1, \omega_2\} \subseteq E \subseteq T$. The collection of all $\mathcal{E}$-complete events is called the *$\mathcal{E}$-complete collection* and denoted by $\Upsilon(\mathcal{E})$. A collection $\mathcal{E}$ is said to be *complete* if $\mathcal{E} = \Upsilon(\mathcal{E})$.

[2]Kojima (24) was the first to consider a weaker notion of comonotonic functions in this direction. He introduced the notion of cominimum functions, which are $\{\Omega\}$-cominimum functions in this paper.

We adopt the term "complete" from an analogy to a complete graph.[3] For $T \in \mathcal{F}$, consider an undirected graph with a vertex set T where $\{\omega, \omega'\} \subseteq T$ is an edge if there is $E \in \mathcal{E}$ satisfying $\{\omega_1, \omega_2\} \subseteq E \subseteq T$. This is a complete graph if and only if T is $\mathcal{E}$-complete.[4]

As an operator, Υ is monotone in the sense that $\Upsilon(\mathcal{E}) \subseteq \Upsilon(\mathcal{E}')$ whenever $\mathcal{E} \subseteq \mathcal{E}'$. Note that any $E \in \mathcal{E}$ is $\mathcal{E}$-complete, i.e., $\mathcal{E} \subseteq \Upsilon(\mathcal{E})$, and any singleton is $\mathcal{E}$-complete trivially, i.e., $\mathcal{F}_1 \subseteq \Upsilon(\mathcal{E})$. The following results show that $\Upsilon(\mathcal{E})$ itself is complete and it serves as a canonical collection among collections which induce the same pairs of cominimum functions.

Lemma 6 *For any $\mathcal{E} \subseteq \mathcal{F}$, $\Upsilon(\mathcal{E})$ is complete, i.e., $\Upsilon(\mathcal{E}) = \Upsilon(\Upsilon(\mathcal{E}))$.*

Proof. Since $\Upsilon(\mathcal{E}) \subseteq \Upsilon(\Upsilon(\mathcal{E}))$ by the monotonicity of Υ, it is enough to show that $\Upsilon(\mathcal{E}) \supseteq \Upsilon(\Upsilon(\mathcal{E}))$. Let $T \in \mathcal{F}$ be $\Upsilon(\mathcal{E})$-complete, i.e., $T \in \Upsilon(\Upsilon(\mathcal{E}))$. Then, for any $\omega_1, \omega_2 \in T$, there is $E \in \Upsilon(\mathcal{E})$ such that $\{\omega_1, \omega_2\} \subseteq E \subseteq T$. Since $E \in \Upsilon(\mathcal{E})$ is $\mathcal{E}$-complete, there is $E' \in \mathcal{E}$ such that $\{\omega_1, \omega_2\} \subseteq E' \subseteq E \subseteq T$. This implies that T is $\mathcal{E}$-complete and thus $T \in \Upsilon(\mathcal{E})$, which completes the proof. ■

Lemma 7 *Two functions are $\mathcal{E}$-cominimum if and only if they are $\Upsilon(\mathcal{E})$-cominimum.*

Proof. Since $\mathcal{E} \subseteq \Upsilon(\mathcal{E})$, $\Upsilon(\mathcal{E})$-cominimum functions are $\mathcal{E}$-cominimum. Conversely, let two functions x_1 and x_2 be $\mathcal{E}$-cominimum. Seeking a contradiction, suppose that these are not $\Upsilon(\mathcal{E})$-cominimum; that is, there is an $\mathcal{E}$-complete event $T \in \mathcal{F}$ such that $\arg\min_T x_1 \cap \arg\min_T x_2 = \emptyset$. Pick $\omega_1 \in \arg\min_T x_1$ and $\omega_2 \in \arg\min_T x_2$. Since T is $\mathcal{E}$-complete, there is $E \in \mathcal{E}$ with $\{\omega_1, \omega_2\} \subseteq E \subseteq T$. Since x_1 and x_2 are $\mathcal{E}$-cominimum, there is $\omega^* \in \arg\min_E x_1 \cap \arg\min_E x_2$. But then $x_i(\omega^*) \leq x_i(\omega_i)$ for $i = 1, 2$, and thus $\omega^* \in \arg\min_T x_1 \cap \arg\min_T x_2$, which is a contradiction. ■

If two functions are indicator functions, the $\mathcal{E}$-cominimum relation naturally induces a relation on a pair of events. We shall pursue this idea in the following.

Definition 7 Let $\mathcal{E} \subseteq \mathcal{F}$ be a collection of events. A pair of events $(T_1, T_2) \subseteq \mathcal{F} \times \mathcal{F}$ with $T_1 \not\subseteq T_2$ and $T_2 \not\subseteq T_1$ are said to be an *$\mathcal{E}$-decomposition pair* for $T \in \mathcal{F}$, provided $T_1 \cup T_2 = T$ and, for any $E \in \mathcal{E}$, $E \subseteq T$ implies $E \subseteq T_1$ or $E \subseteq T_2$ (or both). Denote by $W(\mathcal{E})$ the collection of all the $\mathcal{E}$-decomposition pairs for some events:

$$\begin{aligned} W(\mathcal{E}) = \{(T_1, T_2) \in \mathcal{F} \times \mathcal{F} \mid\, & T_1 \not\subseteq T_2 \text{ and } T_2 \not\subseteq T_1, \\ & E \subseteq T_1 \cup T_2 \text{ implies } E \subseteq T_1 \text{ or } E \subseteq T_2 \text{ for all } E \in \mathcal{E}\}. \end{aligned}$$

[3]We can also regard $(\Omega, \mathcal{E})$ as a hypergraph. The theory of hypergraph has the concept of completeness, which is different from that in this paper.

[4]In addition to the two applications we discuss in Section 3.5, the notion of $\mathcal{E}$-completeness has an interesting application in cooperative game theory. See ?).

An event $T \in \mathcal{F}$ is *$\mathcal{E}$-decomposable* if there exists an $\mathcal{E}$-decomposition pair for T, i.e., $T = T_1 \cup T_2$ for some $(T_1, T_2) \in W(\mathcal{E})$.

The idea of $\mathcal{E}$-decomposition is exactly the $\mathcal{E}$-cominimum relation restricted to indicator functions, as is shown next.

Lemma 8 *Let $T_1, T_2 \in \mathcal{F}$ be such that $T_1 \not\subseteq T_2$ and $T_2 \not\subseteq T_1$. Indicator functions 1_{T_1} and 1_{T_2} are $\mathcal{E}$-cominimum if and only if $(T_1, T_2) \in W(\mathcal{E})$.*

Proof. Suppose that $(T_1, T_2) \in W(\mathcal{E})$. Pick any $E \in \mathcal{E}$. If $E \subseteq T_1 \cup T_2$, then $E \subseteq T_1$ or $E \subseteq T_2$ and thus $\arg\min_E 1_{T_1} = E$ or $\arg\min_E 1_{T_2} = E$ must hold. In both cases, $\arg\min_E 1_{T_1} \cap \arg\min_E 1_{T_2} \neq \emptyset$ holds. If $E \not\subseteq T_1 \cup T_2$, then $\arg\min_E 1_{T_1} \cap \arg\min_E 1_{T_2} = E \backslash T_1 \cap E \backslash T_2 = E \backslash (T_1 \cup T_2) \neq \emptyset$. Therefore, 1_{T_1} and 1_{T_2} are $\mathcal{E}$-cominimum.

Conversely, assume that 1_{T_1} and 1_{T_2} are $\mathcal{E}$-cominimum. Seeking a contradiction, suppose that $(T_1, T_2) \notin W(\mathcal{E})$. Then, there is $E \in \mathcal{E}$ with $E \subseteq T_1 \cup T_2$ but $E \not\subseteq T_1$ and $E \not\subseteq T_2$. Thus, $\arg\min_E 1_{T_1} = E \backslash T_1 \subseteq (T_1 \cup T_2) \backslash T_1$ and $\arg\min_E 1_{T_2} = E \backslash T_2 \subseteq (T_1 \cup T_2) \backslash T_2$, which implies $\arg\min_E 1_{T_1} \cap \arg\min_E 1_{T_2} = \emptyset$, contrary to the assumption. Therefore, such an event $E \in \mathcal{E}$ cannot exist and so $(T_1, T_2) \in W(\mathcal{E})$. ■

As is then easily expected, $\mathcal{E}$-decomposability of an event is closely related to $\mathcal{E}$-completeness. Note that any singleton is not $\mathcal{E}$-decomposable trivially, and that any $E \in \mathcal{E}$ is not $\mathcal{E}$-decomposable. The latter implies that any $\mathcal{E}$-complete event, which is necessarily an element of $\Upsilon(\mathcal{E})$ by definition, is not $\Upsilon(\mathcal{E})$-decomposable. In fact, $\mathcal{E}$-decomposability and $\Upsilon(\mathcal{E})$-decomposability are equivalent as the following lemma shows.

Lemma 9 *For any $\mathcal{E} \subseteq \mathcal{F}$, $W(\mathcal{E}) = W(\Upsilon(\mathcal{E}))$.*

Proof. Since $\mathcal{E} \subseteq \Upsilon(\mathcal{E})$, $W(\mathcal{E}) \supseteq W(\Upsilon(\mathcal{E}))$. We show $W(\mathcal{E}) \subseteq W(\Upsilon(\mathcal{E}))$. Suppose that $(T_1, T_2) \in W(\mathcal{E})$ and $(T_1, T_2) \notin W(\Upsilon(\mathcal{E}))$. The former implies that $T_1 \not\subseteq T_2$ and $T_2 \not\subseteq T_1$, and the latter implies that there exists $E \in \Upsilon(\mathcal{E})$ such that $E \subseteq T_1 \cup T_2$ but neither $E \subseteq T_1$ nor $E \subseteq T_2$. Thus, there exist $\omega_1, \omega_2 \in E$ such that $\omega_1 \in T_1 \backslash T_2$ and $\omega_2 \in T_2 \backslash T_1$. Since E is $\mathcal{E}$-complete, there exists $E' \in \mathcal{E}$ such that $\{\omega_1, \omega_2\} \subseteq E' \subseteq E$, which contradicts to the assumption that (T_1, T_2) is an $\mathcal{E}$-decomposition pair for T. ■

The next result shows that the decomposability is in fact the "complement" of the completeness.

Lemma 10 *An event $T \in \mathcal{F}$ is $\mathcal{E}$-complete if and only if T is not $\mathcal{E}$-decomposable. Consequently,*

$$\begin{aligned}\Upsilon(\mathcal{E}) &= \{T \in \mathcal{F} \mid T \neq T_1 \cup T_2 \text{ for any } (T_1, T_2) \in W(\mathcal{E})\} \\ &= \mathcal{F} \backslash \{T_1 \cup T_2 \mid (T_1, T_2) \in W(\mathcal{E})\}.\end{aligned}$$

Proof. The "only if" part is clear from the definition. We shall establish the "if" part. Assume that T is not $\mathcal{E}$-complete. Then there exists two distinct points $\omega_1, \omega_2 \in T$ such that there exists no $E \in \mathcal{E}$ satisfying $\{\omega_1, \omega_2\} \subseteq E \subseteq T$. Set $T_1 = T \backslash \{\omega_1\}$ and $T_2 = T \backslash \{\omega_2\}$. By construction, $T_1 \not\subseteq T_2$, $T_2 \not\subseteq T_1$, and $T_1 \cup T_2 = T$. Also, for any $E \in \mathcal{E}$, if $E \subseteq T_1 \cup T_2$ then $\{\omega_1, \omega_2\} \not\subseteq E$ and so $E \subseteq T_1$ or $E \subseteq T_2$ must hold by construction. Therefore, $(T_1, T_2) \in W(\mathcal{E})$ and thus T is $\mathcal{E}$-decomposable. ■

To conclude this section, we shall give a sufficient condition for completeness.

Lemma 11 *Suppose that $\mathcal{E} \subseteq \mathcal{F}$ contains all the singleton events and satisfies the following property: if $E, E_1, \ldots, E_n \in \mathcal{E}$ satisfy $E \subseteq \bigcup_{i=1}^{n} E_i$ then $E \cup E_i \in \mathcal{E}$ for at least one $i \in \{1, \ldots, n\}$. Then, $\mathcal{E}$ is complete.*

Proof. Let $T \notin \mathcal{E}$. We want to show that T is not $\mathcal{E}$-complete. By Lemma 10, it suffices to show that T is $\mathcal{E}$-decomposable. Fix $\bar{\omega} \in T$, and let $T_1 \subseteq T$ be a maximal set containing $\bar{\omega}$ and included in $\mathcal{E}$. Since $T \notin \mathcal{E}$, T_1 must be a proper subset of T.

If $T_1 = \{\bar{\omega}\}$, then there is no event $E \in \mathcal{E}$ such that $\{\bar{\omega}\} \subsetneq E \subseteq T$. Then it is readily verified that T_1 and $T \backslash T_1$ constitute an $\mathcal{E}$-decomposition pair for T.

If $T_1 \neq \{\bar{\omega}\}$, then let $\mathcal{E}' = \{E \in \mathcal{E} \mid E \subseteq T \text{ and } E \not\subseteq T_1\}$. It must be true that $T_1 \not\subseteq \bigcup_{E \in \mathcal{E}'} E$. To see this, suppose that $T_1 \subseteq \bigcup_{E \in \mathcal{E}'} E$. Then, there exists $E \in \mathcal{E}'$ such that $T_1 \cup E \in \mathcal{E}$ by the assumption on $\mathcal{E}$. Since $E \subseteq T$ and $E \not\subseteq T_1$, we have $T \supseteq T_1 \cup E \supsetneq T_1$, which contradicts to the maximality of T_1.

Let $T_2 = (T \backslash T_1) \cup (\bigcup_{E \in \mathcal{E}'} E)$. We claim that (T_1, T_2) is an $\mathcal{E}$-decomposition pair for T. By construction, $T_1 \cup T_2 = T$. As we noted above, $T_1 \subsetneq T$. Since $T_1 \not\subseteq \bigcup_{E \in \mathcal{E}'} E$, $T_2 \subsetneq T$, and hence $T_1 \not\subseteq T_2$ and $T_2 \not\subseteq T_1$. Finally, pick any $E \in \mathcal{E}$ with $E \subseteq T$ and suppose $E \not\subseteq T_1$. Then $E \in \mathcal{E}'$, and so $E \subseteq T_2$. Thus $(T_1, T_2) \in W(\mathcal{E})$, which completes the proof. ■

In practice, a stronger condition is also useful.[5]

Lemma 12 *Suppose that $\mathcal{E} \subseteq \mathcal{F}$ contains all the singleton events and satisfies the following property: for any $E_1, E_2 \in \mathcal{E}$, if $E_1 \cap E_2 \neq \emptyset$ then $E_1 \cup E_2 \in \mathcal{E}$. Then, $\mathcal{E}$ is complete.*

Proof. The condition above implies the property of Lemma 11; if $E \subseteq \bigcup_{i=1}^{n} E_i$, then for at least one i, $E \cap E_i \neq \emptyset$, and so $E \cup E_i \in \mathcal{E}$. ■

If $E \cap E' = \emptyset$ or $E \subseteq E'$ or $E' \subseteq E$ for all $E, E' \in \mathcal{E}$, then $\mathcal{E} \cup \mathcal{F}_1$ is complete. Especially, if $\mathcal{E}$ is a partition of Ω, then $\mathcal{E} \cup \mathcal{F}_1$ is complete.

[5] This stronger condition is applied in cooperative game theory. See, for instance, Myerson (28), van den Nouweland et al. (38), and Algaba et al. (1).

Lemma 11, however, does not provide a necessary condition for completeness. For instance, let $\Omega = \{1,2,3,4\}$ and $\mathcal{E} = \{\{1\},\{2\},\{3\},\{4\}, \{1,2\},\{2,3\},\{3,4\}\}$. Then it can be readily checked that $\mathcal{E}$ is complete. But $\mathcal{E}$ does not satisfy the condition of Lemma 11.

3.4 Cominimum additive operators

The notion of $\mathcal{E}$-cominimum (comaximum) functions induces the following additivity property of an operator $I : \mathbb{R}^\Omega \to \mathbb{R}$.

Definition 8 An operator $I : \mathbb{R}^\Omega \to \mathbb{R}$ is *$\mathcal{E}$-cominimum additive* (resp. *comaximum additive*) provided $I(x+y) = I(x) + I(y)$ whenever x and y are $\mathcal{E}$-cominimum (resp. comaximum).

Since $\mathcal{E}$-cominimum (comaximum) additivity implies comonotonic additivity, we have the following corollary of Theorem 12.

Corollary 5 *An operator $I : \mathbb{R}^\Omega \to \mathbb{R}$ is homogeneous and $\mathcal{E}$-cominimum (comaximum) additive for some $\mathcal{E} \subseteq \mathcal{F}$ if and only if $I(x) = \int x dv$ for all $x \in \mathbb{R}^\Omega$ where $v \in \mathbb{R}^\mathcal{F}$ is defined by the rule $v(E) = I(1_E)$.*

Therefore, a homogeneous, $\mathcal{E}$-cominimum (comaximum) additive operator is associated with a game v. As is easily expected, $\mathcal{E}$-cominimum (comaximum) additivity of an operator requires some further structure on the corresponding game v. To find the required structure, we shall focus on a game v, and say that v is $\mathcal{E}$-cominimum (comaximum) additive to mean that the corresponding operator is $\mathcal{E}$-cominimum (comaximum) additive.

Definition 9 A game v is said to be *$\mathcal{E}$-cominimum additive* (resp. *$\mathcal{E}$-comaximum additive*) provided $\int (x+y)dv = \int xdv + \int ydv$ whenever x and y are $\mathcal{E}$-cominimum (resp. $\mathcal{E}$-comaximum).

The following result gives a simple sufficient condition for $\mathcal{E}$-cominimum additivity.

Lemma 13 *Let $v = \sum_{T\in\mathcal{F}} \beta_T u_T \in \mathbb{R}^\mathcal{F}$ be a game. If $\beta_T = 0$ for all $T \notin \mathcal{E}$, then v is $\mathcal{E}$-cominimum additive.*

Proof. Let two functions x and y be $\mathcal{E}$-cominimum. Note that, for all $E \in \mathcal{E}$, $\arg\min_E x \cap \arg\min_E y \neq \emptyset$ and thus $\min_E(x+y) = \min_E x + \min_E y$. So using (12), we have

$$\begin{aligned}\int (x+y)dv &= \sum_{T\in\mathcal{F}} \beta_T \min_T(x+y) = \sum_{T\in\mathcal{E}} \beta_T \min_T(x+y) = \sum_{T\in\mathcal{E}} \beta_T(\min_T x + \min_T y) \\ &= \sum_{T\in\mathcal{E}} \beta_T \min_T x + \sum_{T\in\mathcal{E}} \beta_T \min_T y = \sum_{T\in\mathcal{F}} \beta_T \min_T x + \sum_{T\in\mathcal{F}} \beta_T \min_T y = \int xdv + \int ydv,\end{aligned}$$

which completes the proof. ■

A natural question then is whether the converse is true, i.e., $\mathcal{E}$-cominimum additivity implies $\beta_T = 0$ for any $T \notin \mathcal{E}$. But in general, this is not true. For example, $\mathcal{F}_2$-cominimum additivity does not imply $\beta_T = 0$ for any $T \notin \mathcal{F}_2$ (where $\mathcal{F}_2$ is the set of all two-point events). This is because $\mathcal{F}_2$-cominimum additivity is equivalent to comonotonic additivity and the Choquet integral with respect to any game satisfies comonotonic additivity. Now remember that two functions are $\mathcal{E}$-cominimum if and only if they are $\Upsilon(\mathcal{E})$-cominimum by Lemma 7. Thus, one can expect that $\mathcal{E}$-cominimum additivity implies $\beta_T = 0$ for any $T \notin \Upsilon(\mathcal{E})$, which we will formally show later.

To provide the complete characterization of $\mathcal{E}$-cominimum additivity, we introduce the notion of modularity for $\mathcal{E}$-decomposition pairs. If v is $\mathcal{E}$-cominimum additive then, by Lemma 8 and the definition of the Choquet integral,

$$v(T_1 \cup T_2) + v(T_1 \cap T_2) = \int (1_{T_1} + 1_{T_2}) dv = \int 1_{T_1} dv + \int 1_{T_2} dv = v(T_1) + v(T_2) \tag{14}$$

for all $(T_1, T_2) \in W(\mathcal{E})$. We call this property the modularity for $\mathcal{E}$-decomposition pairs.

Definition 10 A game v is said to be *modular for $\mathcal{E}$-decomposition pairs* provided

$$v(T_1 \cup T_2) + v(T_1 \cap T_2) = v(T_1) + v(T_2) \text{ for all } (T_1, T_2) \in W(\mathcal{E}).$$

We can show that $\mathcal{E}$-cominimum additivity and the modularity for $\mathcal{E}$-decomposition pairs are equivalent, which leads us to the following main result of this paper.

Theorem 6 *Let $v = \sum_{T \in \mathcal{F}} \beta_T u_T \in \mathbb{R}^{\mathcal{F}}$ be a game. The following three statements are equivalent:* (i) *v is $\mathcal{E}$-cominimum additive;* (ii) *v is modular for $\mathcal{E}$-decomposition pairs;* (iii) *$\beta_T = 0$ for any $T \notin \Upsilon(\mathcal{E})$. Therefore, if $\mathcal{E}$ is complete, v is $\mathcal{E}$-cominimum additive if and only if $\beta_T = 0$ for any $T \notin \mathcal{E}$.*

Proof. (iii) $\Rightarrow$ (i). By Lemma 13, v is $\Upsilon(\mathcal{E})$-cominimum additive. By Lemma 7, two functions are $\Upsilon(\mathcal{E})$-cominimum if and only if they are $\mathcal{E}$-cominimum. Thus, v must be $\mathcal{E}$-cominimum additive.

(i) $\Rightarrow$ (ii). This is true by Lemma 8 and the definition of the Choquet integral, as in (14).

(ii) $\Rightarrow$ (iii). We show by induction that, for all $k \geq 1$, if $|T| = k$ and $T \notin \Upsilon(\mathcal{E})$ then $\beta_T = 0$. Since $|T| \geq 2$ for all $T \notin \Upsilon(\mathcal{E})$, the statement is true when $k = 1$ vacuously. Let $k \geq 2$ and suppose as an induction hypothesis that $\beta_T = 0$ for any $T \notin \Upsilon(\mathcal{E})$ with $|T| \leq k - 1$.

Let $T \notin \Upsilon(\mathcal{E})$ with $|T| = k$. Then T is $\mathcal{E}$-decomposable by Lemma 10, and so there exists $(T_1, T_2) \in W(\mathcal{E})$ such that $T = T_1 \cup T_2$. Since $W(\mathcal{E}) = W(\Upsilon(\mathcal{E}))$ by Lemma 9, any $S \in \Upsilon(\mathcal{E})$ with $S \subsetneq T$ must satisfy either $S \subseteq T_1$ or $S \subseteq T_2$ (or both, i.e., $S \subseteq T_1 \cap T_2$). Therefore, if $S \subseteq T$ satisfies $S \not\subseteq T_1$ and $S \not\subseteq T_2$, then $S \notin \Upsilon(\mathcal{E})$ and so $\beta_S = 0$ by the induction hypothesis, unless

$S = T$. Now from the modularity for $\mathcal{E}$-decomposition pairs, we have

$$\begin{aligned} 0 &= v(T_1 \cup T_2) + v(T_1 \cap T_2) - v(T_1) - v(T_2) \\ &= \sum_{S \subseteq T} \beta_S + \sum_{S \subseteq T_1 \cap T_2} \beta_S - \sum_{S \subseteq T_1} \beta_S - \sum_{S \subseteq T_2} \beta_S \\ &= \sum_{S \subseteq T,\ S \not\subseteq T_1,\ S \not\subseteq T_2} \beta_S = \beta_T, \end{aligned}$$

which completes the proof. ■

The cominimum additivity is the conjugate of the comaximum additivity, and vice versa in the following sense.

Lemma 14 *A game v is $\mathcal{E}$-cominimum additive if and only if v' is $\mathcal{E}$-comaximum additive.*

Proof. Since $\min_{\omega \in T} -x(\omega) = -\max_{\omega \in T} x(\omega)$, we have $\int -x dv = -\int x dv'$ by (12) and (13). Thus, $\int (x+y) dv = \int x dv + \int y dv$ if and only if $\int ((-x)+(-y)) dv' = \int (-x) dv' + \int (-y) dv'$. So the result holds because x and y are $\mathcal{E}$-cominimum if and only if $-x$ and $-y$ are $\mathcal{E}$-comaximum. ■

Using the conjugate, an analogous characterization can be done for $\mathcal{E}$-comaximum additivity.

Corollary 7 *Let $v = \sum_{T \in \mathcal{F}} \gamma_T u'_T \in \mathbb{R}^{\mathcal{F}}$ be a game. The following three statements are equivalent:* (i) *v is $\mathcal{E}$-comaximum additive;* (ii) *$v(T_1 \cup T_2) + v(T_1 \cap T_2) = v(T_1) + v(T_2)$ for all $(T_1, T_2) \in \mathcal{F} \times \mathcal{F}$ with $(\Omega \backslash T_1, \Omega \backslash T_2) \in W(\mathcal{E})$;* (iii) *$\gamma_T = 0$ for any $T \notin \Upsilon(\mathcal{E})$. Therefore, if $\mathcal{E}$ is complete, v is $\mathcal{E}$-comaximum additive if and only if $\gamma_T = 0$ for any $T \notin \mathcal{E}$.*

Proof. Note that $v' = \sum_{T \in \mathcal{F}} \gamma_T u_T$. By Lemma 14, v is $\mathcal{E}$-comaximum additive if and only if v' is $\mathcal{E}$-cominimum additive. So the result follows from Theorem 13. ■

A slight modification of Theorem 13 shows that the completeness is tight for our characterization in the following sense.

Corollary 8 *The following statements are equivalent:* (i) *$\mathcal{E}$ is complete, i.e., $\Upsilon(\mathcal{E}) = \mathcal{E}$;* (ii) *For any game $v = \sum_{T \in \mathcal{F}} \beta_T u_T \in \mathbb{R}^{\mathcal{F}}$, v is $\mathcal{E}$-cominimum additive if and only if $\beta_T = 0$ for any $T \notin \mathcal{E}$.*

Proof. (i) $\Rightarrow$ (ii). This is a restatement of Theorem 13.

(ii) $\Rightarrow$ (i). Suppose that $\mathcal{E}$ is not complete. Then there is $T^* \notin \mathcal{E}$ which is $\mathcal{E}$-complete, i.e., $T^* \in \Upsilon(\mathcal{E})$. Consider a game $v = \sum_{T \in \mathcal{F}} \beta_T u_T = u_{T^*}$. Since $\beta_T = 0$ for every $T \notin \Upsilon(\mathcal{E})$, v is $\mathcal{E}$-cominimum additive by Theorem 13. On the other hand, if (ii) is true, v is not $\mathcal{E}$-cominimum additive because $\beta_{T^*} \neq 0$ and $T^* \notin \mathcal{E}$, which is a contradiction. ■

3.5 Applications

3.5.1 The E-capacity and ε-contamination

Denote by $\Delta(\Omega)$ the set of all probability measures and by Π_E the set of probability measures assigning probability one to an event $E \in \mathcal{F}$, i.e., $\Pi_E = \{p \in \Delta(\Omega) \,|\, p(E) = 1\}$.

Definition 11 For $\pi \in \Delta(\Omega)$, $0 \leq \varepsilon \leq 1$, and $E \in \mathcal{F}$, the set of probability measures $\{(1-\varepsilon)\pi + \varepsilon p \,|\, p \in \Pi_E\}$ is referred to as the *ε-contamination* of π on E.

The notion of ε-contamination is old; it is discussed in the literature of robust statistics since ?). In economic applications, the ε-contamination is used with the maximin decision rule (?) which evaluates a function x by the minimum of expected values with respect to the ε-contamination. The following result characterizes this decision rule,[6] which follows from a more general result we shall present later.

Proposition 9 *Let $v \in \mathbb{R}^{\mathcal{F}}$ be a convex capacity and $E \in \mathcal{F}$ be an event. Then the following three statements are equivalent:* (i) *$\int(x+y)dv = \int xdv + \int ydv$ whenever $\arg\min_E x \cap \arg\min_E y \neq \emptyset$;* (ii) *there exist $\pi \in \Delta(\Omega)$ and $\varepsilon \in [0,1]$ such that $v = (1-\varepsilon)\pi + \varepsilon u_E$;* (iii) *there exist $\pi \in \Delta(\Omega)$ and $\varepsilon \in [0,1]$ such that $\int xdv = \min\{\int xdq \,|\, q = (1-\varepsilon)\pi + \varepsilon p,\ p \in \Pi_E\}$ for any function $x \in \mathbb{R}^{\Omega}$, i.e., the Choquet integral of x is the minimum of expected values with respect to the ε-contamination of π on E.*

The maximin decision rule with the ε-contamination of π on E is represented by the Choquet integral with respect to $v = (1-\varepsilon)\pi + \varepsilon u_E$.[7] Thus, we also call this capacity the ε-contamination of π on E.

Eichberger and Kelsey (8) investigated the class of capacities which explain the Ellsberg paradox. They called these capacities the E-capacities, and the ε-contamination is a special case.

Definition 12 Let $E_1, \ldots, E_K$ be non-empty, disjoint subsets of Ω with $|E_k| \geq 2$ for each k. Let $\mathcal{E} = \{E_1, \ldots, E_K\}$. A capacity $v \in \mathbb{R}^{\mathcal{F}}$ is said to be an *E-capacity* with respect to $\mathcal{E}$ if there exists a probability π and a number $\varepsilon \in [0,1]$, and probability assignment ρ on $\mathcal{E}$ (i.e. $\rho(E_k) \geq 0$ for each k and $\sum_{k=1}^{K} \rho(E_k) = 1$) such that $v = (1-\varepsilon)\pi + \varepsilon \sum_{k=1}^{K} \rho(E_k) u_{E_k}$.

Eichberger and Kelsey (8) gave an axiomatic characterization of the E-capacity, and so that of the ε-contamination, a fortiori. The next result, which generalizes Proposition 9, is essentially Proposition 3.1 of Eichberger and Kelsey (8), but we give an alternative proof based on our main result.[8]

[6] Proposition 9 is a generalization of Kojima (24) who considered the case with $E = \Omega$.

[7] In fact, the core of $v = (1-\varepsilon)\pi + \varepsilon u_E$ coincides with the ε-contamination of π on E, which is a consequence of additivity of the core (cf. 7).

[8] ?) gave an alternative axiomatization of the ε-contamination. Their axioms are not directly comparable with Eichberger and Kelsey (8) or Kojima (24).

Proposition 10 *Let $v \in \mathbb{R}^{\mathcal{F}}$ be a convex capacity. Let $E_1, \ldots, E_K$ be non-empty, disjoint subsets of Ω with $|E_k| \geq 2$ for each k. Let $\mathcal{E} = \{E_1, \ldots, E_K\}$. Then the following three statements are equivalent:* (i) *v is $\mathcal{E}$-cominimum additive;* (ii) *v is an E-capacity with respect to $\mathcal{E}$;* (iii) *there exists a probability π and numbers $\varepsilon_1, \ldots, \varepsilon_K \in [0,1]$ with $\sum_{k=1}^{K} \varepsilon_k \leq 1$ such that $\int x dv = \min\{\int x dq \,|\, q = (1 - \sum_{k=1}^{K} \varepsilon_k)\pi + \sum_{k=1}^{K} \varepsilon_k p_k,\ p_k \in \Pi_{E_k}\}$ for any function $x \in \mathbb{R}^{\Omega}$.*

Proof. (i) $\Rightarrow$ (ii): Let $\mathcal{E}^* = \mathcal{E} \cup \mathcal{F}_1$. From Lemma 12, $\mathcal{E}^*$ is complete. So by Theorem 13, (i) implies that $\beta_T = 0$ for every $T \notin \mathcal{E}^*$ where $v = \sum_{T \in \mathcal{F}} \beta_T u_T$. Therefore, v must be of the form $v = \sum_{\omega \in \Omega} \beta_{\{\omega\}} u_{\{\omega\}} + \sum_k \beta_{E_k} u_{E_k}$, and this expression is unique. Since $v(\Omega) = 1$, we have $\sum_{\omega \in \Omega} \beta_{\{\omega\}} + \sum_{k=1}^{K} \beta_{E_k} = 1$. Since v is non-negative, for all $\omega \in \Omega$, $\beta_{\{\omega\}} = v(\{\omega\}) \geq 0$. We claim $\beta_{E_k} \geq 0$ for each k. To see this, write E_k as the union of non-empty disjoint sets, F_1 and F_2, which is possible because $|E_k| \geq 2$. Then by the convexity of v, and from the assumption that E_k's are disjoint, $\sum_{\omega \in E_k} \beta_{\{\omega\}} + \beta_{E_k} = v(E_k) \geq v(F_1) + v(F_2) = \sum_{\omega \in F_1} \beta_{\{\omega\}} + \sum_{\omega \in F_2} \beta_{\{\omega\}}$. Hence $\beta_{E_k} \geq 0$. Set $\varepsilon = \sum_{k=1}^{K} \beta_{E_k} = 1 - \sum_{\omega \in \Omega} \beta_{\{\omega\}}$. We show that $v = \sum_{\omega \in \Omega} \beta_{\{\omega\}} u_{\{\omega\}} + \sum_k \beta_{E_k} u_{E_k}$ is indeed the required expression. If $0 < \varepsilon < 1$, set $\rho(E_k) = \beta_{E_k}/\varepsilon$ for each k, and set $\pi = \frac{1}{1-\varepsilon} \sum_{\omega \in \Omega} \beta_{\{\omega\}} u_{\{\omega\}}$. If $\varepsilon = 0$, set $\pi = \sum_{\omega \in \Omega} \beta_{\{\omega\}} u_{\{\omega\}}$, and if $\varepsilon = 1$, set $\rho(E_k) = \beta_{E_k}$ for each k.

(ii) $\Rightarrow$ (iii): Assume $v = (1-\varepsilon)\pi + \varepsilon \sum_{k=1}^{K} \rho(E_k) u_{E_k}$. Using (12), for any function $x \in \mathbb{R}^{\Omega}$, we have $\int x dv = \int x d((1-\varepsilon)\pi + \varepsilon \sum_{k=1}^{K} \rho(E_k) u_{E_k}) = (1-\varepsilon) \int x d\pi + \varepsilon \sum_{k=1}^{K} \rho(E_k) \min_{E_k} x = (1-\varepsilon) \int x d\pi + \varepsilon \sum_{k=1}^{K} \rho(E_k) \min_{p_k \in \Pi_{E_k}} \int x dp_k$. Since E_k's are disjoint, this is equal to $\min\{\int x dq \,|\, q = (1-\varepsilon)\pi + \varepsilon \sum_{k=1}^{K} \rho(E_k) p_k,\ p_k \in \Pi_{E_k}\}$, so set $\varepsilon_k = \varepsilon \rho(E_k)$, and we have (iii) since $\sum_{k=1}^{K} \varepsilon_k = \varepsilon \sum_{k=1}^{K} \rho(E_k) = \varepsilon$.

(iii) $\Rightarrow$ (i): Let two functions x and y be $\mathcal{E}$-cominimum. Then $\min_{E_k}(x+y) = \min_{E_k} x + \min_{E_k} y$ for every k. Set $\varepsilon = 1 - \sum_{k=1}^{K} \varepsilon_k$. We have $\int (x+y) dv = \min\{\int (x+y) dq \,|\, q = (1-\varepsilon)\pi + \sum_{k=1}^{K} \varepsilon_k p_k,\ p_k \in \Pi_{E_k}\} = (1-\varepsilon) \int (x+y) d\pi + \sum_{k=1}^{K} \varepsilon_k \min_{E_k}(x+y) = (1-\varepsilon)(\int x d\pi + \int y d\pi) + \sum_{k=1}^{K} \varepsilon_k (\min_{E_k} x + \min_{E_k} y) = (1-\varepsilon) \int x d\pi + \sum_{k=1}^{K} \varepsilon_k \min_{E_k} x + (1-\varepsilon) \int y d\pi + \sum_{k=1}^{K} \varepsilon_k \min_{E_k} y = \int x dv + \int y dv$, establishing $\mathcal{E}$-cominimum additivity of v. ∎

Let us point out that although we started with a convex capacity for the sake of brevity, the results above can be translated to an "uncertainty averse preference based" axiomatization of the E-capacity and the ε-contamination in a straightforward manner. Indeed, replace Schmeidler (36)'s comonotonic independence axiom with the $\mathcal{E}$-cominimum additivity with $\mathcal{E} = \{E_1, \ldots, E_K\}$. Since $\mathcal{E}$-cominimum additivity implies comonotonic additivity, by Schmeidler's theorem, we have a utility function in the Choquet expected utility form with a convex capacity v. Then apply the result above to show that v is the E-capacity with respect to $\mathcal{E}$.

3.5.2 Multi-period decisions

We shall consider a multi-period decision model developed by Gilboa (13), which axiomatizes the following special form of utility:

$$\sum_{i=1}^{n} p_i x\,(i) + \sum_{i=2}^{n} \delta_i |x(i) - x(i-1)|, \tag{15}$$

where $p_1, \ldots, p_n$ and $\delta_2, \ldots, \delta_n$ are constants.[9] Interpret $\Omega = \{1, \ldots, n\}$ as a collection of time periods, and $x(1), \ldots, x(n)$ as a stream of income. The utility in (15) describes the value of the stream of income as a weighted average $\sum_{i=1}^{n} p_i x(i)$ plus an adjustment factor $\sum_{i=2}^{n} \delta_i |x(i) - x(i-1)|$ which measures the variations of the stream.

Let $\mathcal{E} = \{\{i, i+1\} \mid 1 \leq i < n\}$. Thus, $\mathcal{E}$ is the collection of adjacent time periods. Note that $\mathcal{E} \cup \mathcal{F}_1$ is complete since if $E \notin \mathcal{E} \cup \mathcal{F}_1$ then E must contain two points which are not adjacent.

Proposition 11 *Let* $v = \sum_{T \in \mathcal{F}} \beta_T u_T \in \mathbb{R}^{\mathcal{F}}$ *be a game, and define* $\mathcal{E}$ *as above. Then the following two statements are equivalent:* (i) v *is* $\mathcal{E}$*-cominimum additive;* (ii) *the Choquet integral with respect to* v *has the form* (15).

Proof. Note that $|a - b| = a + b - 2\min\{a, b\}$ for any $a, b \in \mathbb{R}$. So, (15) can be written as

$$\begin{aligned} \sum_{i=1}^{n} p_i x(i) + \sum_{i=2}^{n} \delta_i |x(i) - x(i-1)| &= \sum_{i=1}^{n} p_i x(i) + \sum_{i=2}^{n} \delta_i (x(i) + x(i-1) - 2\min\{x(i), x(i-1)\}) \\ &= \sum_{i=1}^{n} \beta_i x(i) + \sum_{i=2}^{n} \beta_{\{i-1,i\}} \min\{x(i), x(i-1)\}, \end{aligned} \tag{16}$$

where $\beta_1 = p_1 + \delta_2$, $\beta_i = p_i + \delta_i + \delta_{i+1}$ for $i \in \{2, \ldots, n-1\}$, $\beta_n = p_n + \delta_n$, and $\beta_{\{i-1,i\}} = -2\delta_i$ for $i \in \{2, \ldots, n\}$.

Since $\mathcal{E} \cup \mathcal{F}^1$ is complete, by Theorem 13, (i) is equivalent to the condition that $\beta_T = 0$ unless T is a singleton or $T \in \mathcal{E}$. This is true if and only if the Choquet integral with respect to v is of the form (16). ■

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[9]We thank I. Gilboa for suggesting this application. This is a simplified version of the model studied in Gilboa (13), which we adopted for ease of exposition.

4 Coextrema Additive Operators

4.1 Introduction

The purpose of this paper is to characterize operators on the set of real valued functions on a finite set which is *coextrema additive*: let Ω be a finite set and let $\mathcal{E} \subseteq 2^{\Omega}$ be a collection of subsets of Ω. Two functions x and y on Ω are said to be $\mathcal{E}$-coextrema if, for each $E \in \mathcal{E}$, the set of minimizers of function x restricted on E and that of function y have a common element, and the set of maximizers of x restricted on E and that of y have a common element as well. An operator I on the set of functions on Ω is $\mathcal{E}$-coextrema additive if $I(x+y) = I(x) + I(y)$ whenever x and y are $\mathcal{E}$-coextrema. Note that if two functions are comonotonic, then they are $\mathcal{E}$-extrema, *a fortiori*.

The main result shows that a homogeneous coextrema additive operator I can be represented as $I(x) = \sum_{E\in\mathcal{E}}\{\lambda_E \max_{\omega\in E} x(\omega) + \mu_E \min_{\omega\in E} x(\omega)\}$, where λ_E and μ_E are unique constants, when the collection $\mathcal{E}$ satisfies a certain regularity condition. This expression can also be written as the Choquet integral with respect to a certain non-additive (signed) measure. Therefore, a homogeneous coextrema additive operator corresponds to a special class of the Choquet integral, which is expressed as a weighted sum of "optimistic evaluation" $\max_{\omega\in E} x(\omega)$ and "pessimistic evaluation" $\min_{\omega\in E} x(\omega)$. For the case where $I(1) = 1$, we have $\sum_{E\in\mathcal{E}} (\lambda_E + \mu_E) = 1$, and then these weights can be interpreted as beliefs on events in $E \in \mathcal{E}$ if these are non-negative numbers.

As a corollary, our result shows that for the special case where $\mathcal{E}$ consists of singletons and the whole set Ω, a homogeneous $\mathcal{E}$-coextrema operator is exactly the Choquet integral of a NEO-additive capacity, which is axiomatized by Chateaunuff, Eichberger, and Grant (6). Thus, our result provides a natural, and important generalization of the NEO-additive capacity result. Eichberger, Kelsey, and Schipper (9) applied a NEO-additive capacity model to the Bertrand and Cournot competition models to study combined effects of optimism and pessimism in economic environments.

While in the NEO-additive capacity, optimism and pessimism are about the whole states of the world, our model can accommodate more delicate combinations of optimism and pessimism measured in a family of events. Thus our $\mathcal{E}$-coextrema additivity model provides a rich framework for analyzing effects optimism and pessimism in economic problems.

Kajii, Kojima, and Ui (20) considered the class of cominimum additive operators, and each cominimum additive operator is shown to be a weighted sum of minimums. The class of comaximum operators is defined and characterized similarly. However, the class of coextrema additive operators is *not* the intersection of the two, and the characterization result reported in this paper cannot be done by adopting these results. In fact, the reader will see that the issue of characterization is far more technically involved.

Ghirardato, Maccheroni, and Marinacci (2004) axiomatized the following class of operators called the α-MEU functional: $I(x) = \alpha \min_{q \in C} \int x dq + (1-\alpha) \max_{q \in C} \int x dq$ where C is a convex set of additive measures. It can be readily verified that the NEO-additive capacity model is a special class of the α-MEU functional, and so $\mathcal{E}$-coextrema additive operators are also α-MEU functionals, when $\mathcal{E}$ consists of singletons and the whole set Ω. But for general $\mathcal{E}$, there is no direct connection as far as we can tell.

The organization of this paper is as follows. After a summary of basic concepts and preliminary results in Section 2, a formal definition of the coextrema operator is given in Section 3. Section 3 also contains some discussions on the operator, including potential applications to economics and social sciences. The main result is stated in Section 4, and a proof is provided in Section 5.

4.2 The model and preliminary results

Let Ω be a finite set, whose generic element is denoted by ω. Denote by $\mathcal{F}$ the collection of all non-empty subsets of Ω, and by $\mathcal{F}_1$ the collection of singleton subsets of Ω. A typical interpretation is that Ω is the set of the states of the world and a subset $E \subseteq \Omega$ is an event.

We shall fix a collection $\mathcal{E} \subseteq \mathcal{F}$, $\mathcal{E} \neq \emptyset$, throughout the analysis. Write $\sigma(\mathcal{E})$ for the algebra of Ω generated by $\mathcal{E}$, i.e., the smallest σ-algebra containing each element of $\mathcal{E}$. Let $\Pi(\mathcal{E}) \subseteq \mathcal{F}$ be the collection of minimal elements of $\sigma(\mathcal{E})$, which constitutes a well defined *partition* of Ω, since Ω is a finite set. A generic element of partition $\Pi(\mathcal{E})$ will be denoted by S. For each $F \in \mathcal{F}$, let $\kappa(F) \in \sigma(\mathcal{E})$ denote the minimal $\sigma(\mathcal{E})$-measurable set containing F; that is, $\kappa(F) := \cap\{E \in \sigma(\mathcal{E}) : F \subseteq E\}$.

Remark 2 Note that every element of $\Pi(E)$ belongs to $\sigma(\mathcal{E})$ and that any element $E \in \sigma(\mathcal{E})$ is the union of some elements of $\Pi(E)$. So in particular, for every $E \in E$ and every $S \in \Pi(E)$, either $S \subseteq E$ or $S \subseteq E^c$ holds. By construction, $\kappa(F) = \cup\{S \in \Pi(\mathcal{E}) : S \cap F \neq \emptyset\}$, i.e., $\kappa(F)$ is the union of elements in partition $\Pi(\mathcal{E})$ intersecting F. It is readily verified that if $E \in \sigma(E)$, then $\kappa(E \cap F) = E \cap \kappa(F)$ holds for any $F \in \mathcal{F}$, and so in particular $\kappa(E) = E$.

Example 1 Let $\Omega = \{1, 2, \cdots, 8\}$ and $\mathcal{E} = \{E_1, E_2, E_3, E_4\}$ where $E_1 = \{1,2,3,4\}$, $E_2 = \{3,4,5,6\}$, $E_3 = \{1,2,5,6\}$, $E_4 = \{5,6,7,8\}$. Then, $\Pi(\mathcal{E}) = \{S_1, \ldots, S_4\}$, where $S_1 = \{1,2\}$, $S_2 = \{3,4\}$, $S_3 = \{5,6\}$, $S_4 = \{7,8\}$. In this case, $E_1 = S_1 \cup S_2$, $E_2 = S_2 \cup S_3$, $E_3 = S_1 \cup S_3$, $E_1 = S_3 \cup S_4$. For instance, for $R = \{1,3,5,7\}$, we have $\kappa(R) = \Omega$, because every $S \in \Pi(\mathcal{E})$ intersects R.

A set function $v : 2^{\Omega} \to \mathbb{R}$ with $v(\emptyset) = 0$ is called a *game* or a *non-additive signed measure*. Since each game is identified with a point in $\mathbb{R}^{\mathcal{F}}$, we denote by $\mathbb{R}^{\mathcal{F}}$ the set of all games. For a game $v \in \mathbb{R}^{\mathcal{F}}$, we use the following definitions:

- v is *non-negative* if $v(E) \geq 0$ for all $E \in 2^{\Omega}$.

- v is *monotone* if $E \subseteq F$ implies $v(E) \leq v(F)$ for all $E, F \in 2^{\Omega}$. A monotone game is non-negative.
- v is *additive* if $v(E \cup F) = v(E) + v(F)$ for for all $E, F \in 2^{\Omega}$ with $E \cap F = \emptyset$, which is equivalent to $v(E) + v(F) = v(E \cup F) + v(E \cap F)$ for all $E, F \in 2^{\Omega}$.
- v is *convex* (or *supermodular*) if $v(E) + v(F) \leq v(E \cup F) + v(E \cap F)$ for all $E, F \in 2^{\Omega}$.
- v is *normalized* if $v(\Omega) = 1$.
- v is a *non-additive measure* if it is monotone. A normalized non-additive measure is called a *capacity*.
- v is a *measure* if it is non-negative and additive. A normalized measure is called a *probability measure*.
- The *conjugate* of v, denoted by v', is defined as $v'(E) = v(\Omega) - v(\Omega \backslash E)$ for all $E \in 2^{\Omega}$. Note that $(v')' = v$ and $(v + w)' = v' + w'$ for $v, w \in \mathbb{R}^{\mathcal{F}}$.

For $T \in \mathcal{F}$, let $u_T \in \mathbb{R}^{\mathcal{F}}$ be the *unanimity game* on T defined by the rule: $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise. Let w_T be the conjugate of u_T. Then $w_T(S) = 1$ if $T \cap S \neq \emptyset$ and $w_T(S) = 0$ otherwise. Note that when $T = \{\omega\}$, i.e., T is a singleton set, $u_T = w_T$ and they are additive. The following result is well known as the Möbius inversion in discrete and combinatorial mathematics (cf. 33).

Lemma 15 *The collection $\{u_T\}_{T \in \mathcal{F}}$ is a linear base for $\mathbb{R}^{\mathcal{F}}$, so is the collection $\{w_T\}_{T \in \mathcal{F}}$. The unique collection of coefficients $\{\beta_T\}_{T \in \mathcal{F}}$ satisfying $v = \sum_{T \in \mathcal{F}} \beta_T u_T$ is given by $\beta_T = \sum_{E \subseteq T, E \neq \emptyset} (-1)^{|T|-|E|} v(E)$.*

By convention, we shall omit the empty set in the summation indexed by subsets of Ω. By the definition of u_T, we have $v(E) = \sum_{T \subseteq E} \beta_T$ for all $E \in \mathcal{F}$. The collection of coefficients $\{\beta_T\}_{T \in \mathcal{F}}$ is referred to as the Möbius transform of v. If $v = \sum_{T \in \mathcal{F}} \beta_T u_T$, then the conjugate v' is given by $v' = \sum_{T \in \mathcal{F}} \beta_T w_T$. Using the formula in Lemma 15, by direct computation, one can show that for each $E \in \mathcal{F}$:

$$w_E = \sum_{T \subseteq E} (-1)^{|T|-1} u_T. \tag{17}$$

Remark 3 If $v = \sum_{T \in \mathcal{F}} \beta_T u_T$, the game v is additive if and only if $\beta_T = 0$ unless $|T| = 1$. Obviously, $\sum_{\omega \in \Omega} \beta_{\{\omega\}} u_{\{\omega\}}$ is an additive game. So, we can also write $v = p + \sum_{T \in \mathcal{F}, |T| > 1} \beta_T u_T$ where p is an additive game.

By convention, a function $x : \Omega \to \mathbb{R}$ is identified with an element of $\mathbb{R}^{\Omega}$, and we denote by 1_E the indicator function of event $E \in \mathcal{F}$. For a function $x \in \mathbb{R}^{\Omega}$, and an event E, we write $\min_E x := \min_{\omega \in E} x(\omega)$ and $\arg\min_E x := \arg\min_{\omega \in E} x(\omega)$. Similarly, we write $\max_E x := \max_{\omega \in E} x(\omega)$ and $\arg\max_E x := \arg\max_{\omega \in E} x(\omega)$.

Definition 13 For $x \in \mathbb{R}^{\Omega}$ and $v \in \mathbb{R}^{\mathcal{F}}$, the *Choquet integral* of x with respect to v is defined as

$$\int x dv = \int_{\underline{x}}^{\bar{x}} v(x \geq \alpha) d\alpha + \underline{x} v(\Omega), \tag{18}$$

where $\bar{x} = \max_{\Omega} x$, $\underline{x} = \min_{\Omega} x$, and $v(x \geq \alpha) = v(\{\omega \in \Omega : x(\omega) \geq \alpha\})$.

By definition, $\int 1_E dv = v(E)$. A direct computation reveals that, for any two sets E and F in $\mathcal{F}$,

$$\int (1_E + 1_F) \, dv = v(E \cup F) + v(E \cap F). \tag{19}$$

Then for each event T, we see from (18) that $\int x du_T = \min_T x$ and $\int x dw_T = \max_T x$. Also it can be readily verified that the Choquet integral is additive in games. Recall that for a game v, there is a unique set of coefficients $\{\beta_T : T \in \mathcal{F}\}$ such that $v = \sum_T \beta_T u_T$ by Lemma 15. Using additivity, therefore, we have $\int x dv = \sum_T \beta_T \min_T x$, as is pointed out in Gilboa and Schmeidler (15).

Note that the additivity implies the following property: for any $T \in \mathcal{F}$ and real numbers λ and μ, $\int x d(\lambda w_T + \mu u_T) = \int x d(\lambda w_T) + \int x d(\mu u_T) = \lambda \max_T x + \mu \min_T x$, and so

$$\int x d \left(\sum_{E \in \mathcal{F}'} \lambda_E w_E + \mu_E u_E \right) = \sum_{E \in \mathcal{F}'} \{\lambda_E \max_E x + \mu_E \min_E x\}, \tag{20}$$

for any collection of events $\mathcal{F}' \subseteq \mathcal{F}$ and collections of real numbers $\{\lambda_E : E \in \mathcal{F}'\}$ and $\{\mu_E : E \in \mathcal{F}'\}$.

Definition 14 Let $\mathcal{E} \subseteq \mathcal{F}$ be a collection of events. Two functions $x, y \in \mathbb{R}^{\Omega}$ are said to be *$\mathcal{E}$-cominimum*, provided $\arg\min_E x \cap \arg\min_E y \neq \emptyset$ for all $E \in \mathcal{E}$. Two functions $x, y \in \mathbb{R}^{\Omega}$ are said to be *$\mathcal{E}$-comaximum*, provided $\arg\max_E x \cap \arg\max_E y \neq \emptyset$ for all $E \in \mathcal{E}$.

Remark 4 Clearly, x and y are $\mathcal{E}$-cominimum, if and only if $-x$ and $-y$ are $\mathcal{E}$-comaximum. Also, the $\mathcal{E}$-cominimum and the $\mathcal{E}$-comaximum relations are invariant of adding a constant. In particular, if two indicator functions 1_A and 1_B are $\mathcal{E}$-cominimum, $1_{\Omega \setminus A}$ $(=1 - 1_A)$ and $1_{\Omega \setminus B}$ $(=1 - 1_B)$ are $\mathcal{E}$-comaximum, and vice versa.

A function $I : \mathbb{R}^{\Omega} \to \mathbb{R}$ is referred to as an operator.

Definition 15 An operator I is said to be homogeneous if $\mathrm{I}(\alpha x) = \alpha I(x)$ for any $\alpha > 0$.

Kajii, Kojima, and Ui (20) studied $\mathcal{E}$-cominimum and $\mathcal{E}$-comaximum operators defined as follows:

Definition 16 An operator $I : \mathbb{R}^{\Omega} \to \mathbb{R}$ is $\mathcal{E}$-cominimum (resp. comaximum) additive provided $I(x+y) = I(x) + I(y)$ whenever x and y are $\mathcal{E}$-cominimum (resp. comaximum).

A pair of functions x and y are said to be *comonotonic* if $(x(\omega) - x(\omega'))(y(\omega) - y(\omega')) \geq 0$ for any $\omega, \omega' \in \Omega$. Notice that if $\mathcal{E} = \mathcal{F}$, a pair of functions x and y are comonotonic if and only if they are $\mathcal{E}$-cominimum, as well as $\mathcal{E}$-comaximum. So when $\mathcal{E} = \mathcal{F}$, the $\mathcal{E}$-cominimum additivity, as well as the $\mathcal{E}$-comaximum additivity, is equivalent to the comonotonic additivity which Schmeidler (1986) characterized. Then in general both the $\mathcal{E}$-cominimum and the $\mathcal{E}$-comaximum additivity imply the comonotonic additivity. Therefore, the following can be obtained from Schmeidler's theorem in a straightforward manner.[10]

Theorem 12 *If an operator $I : \mathbb{R}^{\Omega}_{+} \to \mathbb{R}$ is homogenous and satisfies $\mathcal{E}$-cominimum additivity (or $\mathcal{E}$-comaximum additivity), then there exists a unique game $v \in \mathbb{R}^{\mathcal{F}}$ such that $I(x) = \int x dv$ for all $x \in \mathbb{R}^{\Omega}$. Moreover, game v is defined by the rule $v(E) = I(1_E)$.*

We say that a game v is $\mathcal{E}$-cominimum additive (resp. $\mathcal{E}$-comaximum additive) if the operator $I(x) := \int x dv$ is $\mathcal{E}$-cominimum additive (resp. $\mathcal{E}$-comaximum additive). Since $\mathcal{E}$-cominimum additivity as well as $\mathcal{E}$-comaximum additivity implies comonotonic additivity, Theorem 12 assures that this is a consistent terminology.

Obviously, the properties of $\mathcal{E}$-cominimum additive or $\mathcal{E}$-comaximum additive operators depend on the structure of the family $\mathcal{E}$.

Definition 17 Let $\mathcal{E} \subseteq \mathcal{F}$ be a collection of events. An event $T \in \mathcal{F}$ is $\mathcal{E}$-complete provided, for any two distinct points ω_1 and ω_2 in T, there is $E \in \mathcal{E}$ such that $\{\omega_1, \omega_2\} \subseteq E \subseteq T$. The collection of all $\mathcal{E}$-complete events is called the $\mathcal{E}$-complete collection and denoted by $\Upsilon(\mathcal{E})$. A collection $\mathcal{E}$ is said to be complete if $\mathcal{E} = \Upsilon(\mathcal{E})$.

Note that a singleton set is automatically $\mathcal{E}$-complete, so is any $E \in \mathcal{E}$. For each T, consider the graph where the set of vertices is T and the set of edges consists of the pairs of vertices $\{\omega_1, \omega_2\}$ with $\{\omega_1, \omega_2\} \subseteq E \subseteq T$ for some $E \in \mathcal{E}$. This graph is a complete graph if and only if T is $\mathcal{E}$ -complete.

Remark 5 For $E, E' \in \mathcal{E}$, $E \cup E'$ is not necessarily $\mathcal{E}$-complete. However, by definition, for any $T \in \Upsilon(\mathcal{E})$ with $|T| > 1$, T coincides with the union of sets in $\mathcal{E}$ which are included in T, thus T is the union of (partition) elements in $\Pi(\mathcal{E})$ which are included in T. In particular, T must contain at least one element of $\Pi(\mathcal{E})$.

[10] Schmeidler 1986(35) assumes monotonicity instead of homogeneity of the operator, but the method of his proof can be adopted for this result with little modification.

It can be shown that for any $\mathcal{E} \subseteq \mathcal{F}$, $\Upsilon(\mathcal{E})$ is complete, i.e., $\Upsilon(\mathcal{E}) = \Upsilon(\Upsilon(\mathcal{E}))$. See Kajii, Kojima, and Ui (20) for further discussions on this concept, as well as for the proofs of the results shown in the rest of this section.

Example 2 In Example 1, $S = S_1 \cup S_2 \cup S_3 = \{1,2,3,4,5,6\}$ is $\mathcal{E}$-complete, but $S_2 \cup S_3 \cup S_4 = \{3,4,5,6,7,8\}$ is not $\mathcal{E}$-complete since there is no $E \in \mathcal{E}$ with $\{3,7\} \subseteq E \subseteq S_2 \cup S_3 \cup S_4$.

The completeness plays a crucial role in our analysis, as is indicated in the next result:

Lemma 16 *Two functions x and y are $\mathcal{E}$-cominimum (resp. $\mathcal{E}$-comaximum) if and only if they are $\Upsilon(\mathcal{E})$-cominimum (resp. $\Upsilon(\mathcal{E})$-comaximum).*

The idea of "cominimum" can be stated in terms of sets by looking at the indicator functions. Say that a pair of sets A and B is an $\mathcal{E}$-decomposition pair if for any $E \in \mathcal{E}$, $E \subseteq A \cup B$ implies that $E \subseteq A$ or $E \subseteq B$ or both. Then the following can be shown:

Lemma 17 *Two indicator functions 1_A and 1_B are $\mathcal{E}$-cominimum if and only if the pair of sets A and B constitutes an $\mathcal{E}$-decomposition pair.*

Remark 6 From Lemma 17 and Remark 4, we see that two indicator functions 1_A and 1_B are $\mathcal{E}$-comaximum if and only if for any $E \in \mathcal{E}$, $E \subseteq \Omega \backslash (A \cap B)$ implies that $E \subseteq \Omega \backslash A$ or $E \subseteq \Omega \backslash B$ or both.

Finally, a characterization of cominimum additive and comaximum additive operators is given below.

Theorem 13 *Let $v \in \mathbb{R}^{\mathcal{F}}$ be a game, and let $I(x) = \int x dv$. Write $v = \sum_{T \in \mathcal{F}} \beta_T u_T = \sum_{T \in \mathcal{F}} \eta_T w_T$. Then,*
(1) the following three statements are equivalent: (i) *operator I is $\mathcal{E}$-cominimum additive;* (ii) *$v(A) + v(B) = v(A \cup B) + v(A \cap B)$ for any $\mathcal{E}$-decomposition pair A and B;* (iii) *$\beta_T = 0$ for any $T \notin \Upsilon(\mathcal{E})$, and*
(2) the following three statements are equivalent: (i) *operator I is $\mathcal{E}$-comaximum additive;* (ii)*; $v(A^c) + v(B^c) = v(A^c \cup B^c) + v(A^c \cap B^c)$ for any $\mathcal{E}$-decomposition pair A and B;* (iii) *$\eta_T = 0$ for any $T \notin \Upsilon(\mathcal{E})$.*

4.3 Coextrema additive operators

In this paper we study pairs of functions which share *both* a minimizer and a maximizer for events in a given collection $\mathcal{E}$, which is fixed throughout.

Definition 18 Two functions $x, y \in \mathbb{R}^{\Omega}$ are said to be *$\mathcal{E}$-coextrema*, provided they are both *$\mathcal{E}$-cominimum and $\mathcal{E}$-comaximum;* that is, $\arg\min_E x \cap \arg\min_E y \neq \emptyset$ and $\arg\max_E x \cap \arg\max_E y \neq \emptyset$ for all $E \in \mathcal{E}$.

Analogous to the cases of cominimum and comaximum functions, the notion of $\mathcal{E}$-coextrema functions induces the following additivity property of an operator $I : \mathbb{R}^\Omega \to \mathbb{R}$.

Definition 19 An operator $I : \mathbb{R}^\Omega \to \mathbb{R}$ is $\mathcal{E}$-coextrema additive provided $I(x+y) = I(x) + I(y)$ whenever x and y are $\mathcal{E}$-coextrema.

The completion $\Upsilon(\mathcal{E})$ plays an important role here again: the following is an immediate consequence of the definition and Lemma 16.

Lemma 18 Two functions x and y are $\mathcal{E}$-coextrema if and only if they are $\Upsilon(\mathcal{E})$-coextrema.

By definition, the $\mathcal{E}$-coextrema additivity implies the comonotonic additivity. So by Theorem 12, we obtain the following result.

Lemma 19 If an operator $I : \mathbb{R}^\Omega \to R$ is homogeneous and $\mathcal{E}$-coextrema additive for some $\mathcal{E} \subseteq \mathcal{F}$, then there exists a unique game v such that $I(x) = \int x dv$ for any $x \in \mathbb{R}^\Omega$. Moreover, v is defined by the rule $v(E) = I(1_E)$.

Thus the following definition is justified:

Definition 20 A game v is said to be $\mathcal{E}$-coextrema additive provided $\int (x+y)dv = \int xdv + \int ydv$ whenever x and y are $\mathcal{E}$-coextrema.

Our goal is to establish that a game v is $\mathcal{E}$-coextrema additive if and only if v can be expressed in the form

$$v = \sum_{E \in \Upsilon(\mathcal{E})} \{\lambda_E w_E + \mu_E u_E\}. \tag{21}$$

Note that from (20), this is equivalent to say that the original operator I can be written as

$$I(x) = \sum_{E \in \Upsilon(\mathcal{E})} \{\lambda_E \max\nolimits_E x(\omega) + \mu_E \min\nolimits_E x(\omega)\}. \tag{22}$$

In addition, if $\mathcal{E}$ is complete, i.e., $\mathcal{E} = \Upsilon(\mathcal{E})$, we have the expression written in Introduction.

Remark 7 Note that by definition $u_{\{\omega\}} = w_{\{\omega\}}$, and they are the probability measure δ_ω which assigns probability one to $\{\omega\}$. Since $\Upsilon(E)$ contains all the singleton subsets of Ω, the (21) has a trivial redundancy for E with $|E| = 1$. Taking this into account, (21) can be written as:

$$v = p + \sum_{E \in \Upsilon(\mathcal{E}) \setminus \mathcal{F}_1} \{\lambda_E w_E + \mu_E u_E\}, \tag{23}$$

where p is an additive measure given by $p := \sum_{\omega \in \Omega} (\lambda_{\{\omega\}} + \mu_{\{\omega\}}\} \delta_\omega$. Similarly, (22) can be written as

$$I(x) = \int xdp + \sum_{E \in \Upsilon(\mathcal{E}) \setminus \mathcal{F}_1} \{\lambda_E \max\nolimits_E x + \mu_E \min\nolimits_E x\}. \tag{24}$$

We will also show that these expressions are unique under some conditions.

As we mentioned before, a leading case for our set up is to interpret Ω as the set of states describing uncertainty and function x as a random variable over Ω. Then the class of operators which can be written as in (24) with underlying capacity of the form (23) has a natural interpretation that the value of x is the sum of its expected value $\int xdp$ and a weighted average of the most optimistic outcome and the most pessimistic outcomes on events in $\Upsilon(\mathcal{E})$. That is, $I(x)$ is the expectation biased by optimism and pessimism conditional on various events in $\Upsilon(\mathcal{E})$.

Alternatively, interpret Ω as a collection of individuals (i.e., a society), and $x(\omega)$ as the wealth allocated to individual ω. Then $\int xdp$ can be seen as the (weighted) average income of the society, and $\max_E x$ and $\min_E x$ correspond to the wealthiest and the poorest in group E, respectively. In particular, when p is the uniform distribution and $\lambda_E = -1$ and $\mu_E = 1$, then the problem of maximizing (24) subject to $\int xdp$ being held constant means that that of reducing the sum of wealth differences in various groups in $\Upsilon(\mathcal{E})$.

An interesting special subclass of (24) is the class of NEO-additive capacities obtained by Chateaunuff, Eichberger, and Grant (6): a NEO-additive capacity is a capacity of the form $v = (1-\lambda-\mu)\,q + \lambda w_\Omega + \mu u_\Omega$, i.e., $\mathcal{E} = \{\Omega\}$ in (24) and $I(1_\Omega) = 1$.[11] More generally, let $\mathcal{E}$ be a partition of Ω, and write $\mathcal{E} = \{E_1, ..., E_K\}$. Then (24) is essentially $v = p + \sum \lambda_k w_{E_k} + \mu_k u_{E_k}$, where p is an additive game. Not only this is a generalization of the NEO-additive capacity, but also it is a generalization of the E-capacities of Eichberger and Kelsey (1999), which correspond to the case where $\lambda_k = 0$ for all k.

4.4 Main characterization result

One direction of the characterization can be readily established, as is shown below.

Lemma 20 *Let $v = \sum_{E\in\Upsilon(\mathcal{E})}\{\lambda_E w_E + \mu_E u_E\}$. Then v is $\mathcal{E}$-coextrema additive.*

Proof. Let x and y be $\mathcal{E}$-coextrema functions. Then by Lemma 18, x and y are $\Upsilon(\mathcal{E})$-coextrema. For every $E \in \Upsilon(\mathcal{E})$, let $\bar{\omega} \in \arg\max_E x \cap \arg\max_E y$ and $\underline{\omega} \in \arg\min_E x \cap \arg\min_E y$. Then, $\max_E(x+y) = (x+y)(\bar{\omega}) = x(\bar{\omega}) + y(\bar{\omega}) = \max_E x + \max_E y$, and $\min_E(x+y) = (x+y)(\underline{\omega}) = x(\underline{\omega}) + y(\underline{\omega}) = \min_E x + \min_E y$.

[11] When $\lambda = 0$, i.e., there is no part for optimism, this type of capacity is also referred to as an ε-contamination. See Kajii, Kojima, and Ui (20) for more discussions.

Using these relations, since the Choquet integral is additive in games (see (20)), we have

$$\begin{aligned}
\int (x+y)dv &= \int (x+y)d[\sum_{E\in\Upsilon(\mathcal{E})}\{\lambda_E w_E + \mu_E u_E\}], \\
&= \sum_{E\in\Upsilon(\mathcal{E})}\{\lambda_E \max\nolimits_E(x+y) + \mu_E \min\nolimits_E(x+y)\}, \\
&= \sum_{E\in\Upsilon(\mathcal{E})}\{\lambda_E (\max\nolimits_E x + \max\nolimits_E y) + \mu_E (\min\nolimits_E x + \min\nolimits_E y)\}, \\
&= \sum_{E\in\Upsilon(\mathcal{E})}\{\lambda_E \max\nolimits_E x + \mu_E \min\nolimits_E x\} + \sum_{E\in\Upsilon(\mathcal{E})}\{\lambda_E \max\nolimits_E y + \mu_E \min\nolimits_E y\}, \\
&= \int xdv + \int ydv,
\end{aligned}$$

which completes the proof. ■

The other direction is far more complicated. Observe first that since both $\{u_T : T \in \mathcal{F}\}$ and $\{w_T : T \in \mathcal{F}\}$ constitute linear bases, if the collection of events $\Upsilon(\mathcal{E})$ contains a sufficient variety of events, not only coextrema additive games but also many other games can be expressed as in (21) or (22). In other words, for these expressions to be interesting, it is important to establish the uniqueness, and one can easily expect that the collection $\Upsilon(\mathcal{E})$ should not contain too many elements for this purpose. On the other hand, $\mathcal{E}$ must be rich enough relative to Ω as the following example shows.

Example 3 Let $|\Omega| \geq 3$ and $\mathcal{E} = \{\{1,2,3\}\}$. Then $\Upsilon(\mathcal{E})\backslash\mathcal{F}_1 = \mathcal{E}$. Notice that in general when $|E| = 3$, if x and y are coextrema on E, then x and y are automatically comonotonic on E. So any non-game v of the form $v = \sum_{T\subseteq\{1,2,3\}} \beta_T u_T$ is $\mathcal{E}$-coextrema additive, in particular $u_{\{1,2\}}$ is $\mathcal{E}$-coextrema additive. But it can be shown that $u_{\{1,2\}}$ cannot be written in the form (21).

To exclude cases like Example 3, we need to guarantee that $\Upsilon(\mathcal{E})$ does not contain too many elements. The key condition formally stated below roughly says that the elements of $\mathcal{E}$, as well as their intersections, are not too small, i.e., the collection $\mathcal{E}$ are "coarse" enough:

Coarseness Condition $|E| \geq 4$ for every $E \in \mathcal{E}$ and $|S| \geq 2$ for every $S \in \Pi(\mathcal{E})$.

The Coarseness Condition is satisfied in Example 1, but it is violated in Example 3.

Remark 8 Obviously, if $\mathcal{E}$ is coarse, it contains no singleton set. However, as far as the representation result stated below is concerned, singletons are inessential since $\Upsilon(\mathcal{E})$ automatically contains all the singletons anyway. Put it differently, we could state the condition by first excluding singletons from $\mathcal{E}$ and then construct the relevant field and partition.

We are now ready to state the main result of this paper.

Theorem 14 *Let $\mathcal{E}$ be a collection of events which satisfies the coarseness condition. Let v be a game. Then the following two conditions are equivalent:*
(i) v is $\mathcal{E}$-coextrema additive; (ii) there exist an additive game p and two sets of real numbers, $\{\lambda_E : E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1\}$ and $\{\mu_E : E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1\}$, such that

$$v = p + \sum_{E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1} \{\lambda_E w_E + \mu_E u_E\}. \quad (25)$$

Moreover, (25) is unique; that is, if $v = p' + \sum_{E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1} \{\lambda'_E w_E + \mu'_E u_E\}$ where p' is additive, then $p = p'$, and $\lambda'_E = \lambda_E$ and $\mu'_E = \mu_E$ hold for every $E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1$.

We shall prove this result in the next section, but we note here that the coarseness condition is indispensable for Theorem 14. Recall that in Example 3 the coarseness condition is violated and there is a coextrema additive game which cannot be expressed in the form (25). The next example is also instructive for this point.

Example 4 *Let* $\Omega = \{1, 2, 3, 4\}$, $\mathcal{E} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{3, 4\}\}$. In this case, it is $\Upsilon(\mathcal{E}) \backslash \mathcal{F}_1 = \mathcal{E} \cup \{\Omega\}$. But if x and y are $\mathcal{E}$-coextrema, then it is comonotonic on both $\{1, 2, 3\}$ and $\{1, 2, 4\}$, and hence it is comonotonic on Ω. So any non-additive measure v is $\mathcal{E}$-coextrema additive.

Let us conclude this section with a couple of applications of Theorem 14. The first concerns a characterization of the generalized NEO-additive, E-capacities outlined before. Let $\mathcal{E}$ be a partition of Ω, and write $\mathcal{E} = \{E_1, ..., E_K\}$ as before. It can be readily verified that $\Upsilon(\mathcal{E}) = \mathcal{E} \cup \mathcal{F}_1$. Trivially, $\Pi(\mathcal{E}) = \mathcal{E}$. So if $|E_k| \geq 4$ for every $k = 1, ..., K$, by Theorem 14, $\mathcal{E}$ satisfies the coarseness condition and then v is $\mathcal{E}$-coextrema additive if and only if v can be written as $v = p + \sum \lambda_k w_{E_k} + \mu_k u_{E_k}$, where p is an additive game.

The second is a generalization of the variation averse operator proposed in Gilboa (1989). Let $T > 1$ and $M \geq 2$ be integers and set $\Omega = \{(m, t) : m = 1, ..., 2M, t = 1, ..., T\}$. The intended interpretation is that t is the time and at each time t there are m states representing some uncertainty. Let $\mathcal{E}$ be the collection of all sets of the following forms: $\{(m, t) : m = 1, ..., 2M\}$; $\{(m, t) : m = 1, ..., M\} \cup \{(m, t + 1) : m = 1, ..., M\}$; and $\{(m, t) : m = M + 1, ..., 2M\} \cup \{(m, t + 1) : m = M + 1, ..., 2M\}$. It can be readily verified that $\Upsilon(\mathcal{E}) = \mathcal{E} \cup \mathcal{F}_1$, and every set in $\Pi(\mathcal{E})$ contains M points. So the coarseness condition is met, and by Theorem 14, an $\mathcal{E}$-extrema additive capacity has the form in 24). Arguing analogously as in Kajii, Kojima, and Ui (20), the coefficients for the $\mathcal{E}$-events of the form $\{(m, t) : m = 1, ..., 2M\}$ represent measurements of optimism and pessimism about the uncertainty, whereas the coefficients for the $\mathcal{E}$-events of the other forms represent measurements of (conditional) degrees of variation loving and variation aversion.

4.5 The proof

This section is devoted to the proof of Theorem 14. Since Lemma 19 has already shown that (ii) implies (i), it suffices to establish the other direction. The proof consists of several steps: basically, starting with an $\mathcal{E}$-coextrema game v, we shall first show that a restriction of v is $\mathcal{E}$-comaximum. Then we show that this construction is invariant of the way the restriction is chosen as long as a certain condition is satisfied, which then implies the existence of a well-defined $\mathcal{E}$-comaximum additive game v_1. We then show that the game $v_2 := v - v_1$ is $\mathcal{E}$-cominimum additive. Theorem 13 can be applied to v_1 and v_2 to obtain the desired expression.

Let v be an $\mathcal{E}$-coextrema additive game with $v = \sum_{T \in \mathcal{F}} \beta_T u_T$. For any $R \in \mathcal{F}$, let $v_{|R}$ be the game defined by the rule $v_{|R}(E) = v(E \cap R)$ for all $E \in \mathcal{F}$, i.e., $v_{|R} = \sum_{T \subseteq R} \beta_T u_T$. Define $\mathcal{E}_{\cap R} = \{E \cap R \,|\, E \in \mathcal{E}, E \cap R \neq \emptyset\}$, which is the collection of intersections of elements of $\mathcal{E}$ and R, and also define $\mathcal{E}_{\subseteq R} = \{E \,|\, E \in \mathcal{E}, E \subseteq R\}$, which is the collection of elements of $\mathcal{E}$ contained in R. Note that $\mathcal{E}_{\subseteq R} \subseteq \mathcal{E}_{\cap R}$.

To construct the desired $\mathcal{E}$-comaximum additive game v_1, we first observe the following property.

Lemma 21 *Let v be $\mathcal{E}$-coextrema additive. Let $R \in \mathcal{F}$ be such that $\mathcal{E}_{\subseteq R} = \emptyset$ and $\mathcal{E}_{\cap R} \neq \emptyset$. Then, $v_{|R}$ is $\mathcal{E}_{\cap R}$-comaximum additive.*

Proof. Let 1_S and 1_T be $\mathcal{E}_{\cap R}$-comaximum. It is enough to show that $v_{|R}(S \cup T) + v_{|R}(S \cap T) = v_{|R}(S) + v_{|R}(T)$, which is rewritten as $v((S \cap R) \cup (T \cap R)) + v((S \cap R) \cap (T \cap R)) = v(S \cap R) + v(T \cap R)$. Therefore, it suffices to show that $1_{S \cap R}$ and $1_{T \cap R}$ are $\mathcal{E}$-coextrema because v is $\mathcal{E}$-coextrema additive.

Fix any $E \in \mathcal{E}$. Since $\mathcal{E}_{\subseteq R} = \emptyset$, either $E \cap R = \emptyset$, or $E \cap R \neq \emptyset$ and $E \backslash R \neq \emptyset$. If $E \cap R = \emptyset$, then $1_{S \cap R}$ and $1_{T \cap R}$ are 0 on E and thus have a common minimizer and maximizer on E. If $E \cap R \neq \emptyset$ and $E \backslash R \neq \emptyset$, then $1_{S \cap R}$ and $1_{T \cap R}$ have a common maximizer in $E \cap R \subseteq E$ since 1_S and 1_T are $\mathcal{E}_{\cap R}$-comaximum, and $1_{S \cap R}$ and $1_{T \cap R}$ have a common minimizer in $E \backslash R \subseteq E$ since $1_{S \cap R}$ and $1_{T \cap R}$ are 0 on R^c. Therefore, $1_{S \cap R}$ and $1_{T \cap R}$ are $\mathcal{E}$-coextrema. ■

By this lemma and Theorem 13, $v_{|R}$ has a unique expression

$$v_{|R} = \sum_{\omega \in R} \nu^R_{\{\omega\}} w_{\{\omega\}} + \sum_{E' \in \Upsilon(\mathcal{E}_{\cap R}) \backslash \mathcal{F}_1} \nu^R_{E'} w_{E'}. \tag{26}$$

To obtain the desired game v_1 which will constitute a part of the expression (25), we want the second part of the right hand side of (26) in the following form:$\sum_{E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1} \nu^R_{E \cap R} w_{E \cap R}$. Since each $E' \in \mathcal{E}_{\cap R} \backslash \mathcal{F}_1$ is written as $E' = E \cap R$ for some $E \in \mathcal{E}$, one way to proceed is to associate each E' with the corresponding E. Of course, this procedure is not well defined in general, since there may be many such E for candidates. So our next step is to find a condition on the set R

so that this procedure in fact unambiguously works. It turns out that the following property is suitable for this purpose.

Definition 21 A set $R \in \mathcal{F}$ is a *representation* of $\mathcal{E}$ if $\mathcal{E}_{\subseteq R} = \emptyset$, $\kappa(R) = \Omega$, and $|R \cap E| \geq 2$ for all $E \in \mathcal{E}$. Moreover we say that $R \in \mathcal{F}$ is a *minimal representation* of $\mathcal{E}$ if R is a representation of $E \in \mathcal{E}$ and any proper subset of R is not a representation.

In Example 1, the set R is a representation for $\mathcal{E}$. Another example follows below.

Example 5 Let $\Omega = \{1,2,3,4,5,6\}$, and set $\mathcal{E} = \{\{1,2,3,4\},\{3,4,5,6\}\}$. Then $\Pi(\mathcal{E}) = \{\{1,2\},\{3,4\},\{5,6\}\}$. $R = \{3,4\}$ is not a representation, since $\kappa(R) = \{3,4\} \neq \Omega$. $R = \{2,3,4,6\}$ is a representation but not minimal. $R = \{2,4,6\}$ is a minimal representation.

Lemma 22 *When $\mathcal{E}$ is coarse, if $T \in \mathcal{F}$ satisfies $\mathcal{E}_{\cap T} = \emptyset$, then there is a representation R such that $T \subseteq R$.*

Proof. Construct R by the following procedure: first set $R = T$ and then for each $S \in \Pi(\mathcal{E})$; if $S \in \mathcal{E}$ and $|T \cap S| \leq 1$, then add a point or two to R from $S \backslash T$ (recall that $|S| \geq 4$ if $S \in \mathcal{E}$ by the coarseness) so that two points from S are contained in R; if $S \in \mathcal{E}$ and $|T \cap S| \geq 2$, do nothing; if $S \notin \mathcal{E}$ and $T \cap S = \emptyset$, then add a point to R (note $S \backslash R \neq \emptyset$ by the coarseness); if $S \notin \mathcal{E}$ and $T \cap S \neq \emptyset$, do nothing. Then by construction, $\kappa(R) = \Omega$, and $|R \cap E| \geq 2$ for all $E \in \mathcal{E}$. Notice also that for any $E \in \mathcal{E}$, there is some point which is not added to R, so $\mathcal{E}_{\subseteq R} = \emptyset$ follows. ■

Note that if R is a representation of $\mathcal{E}$, $\kappa(R) = \Omega$ holds by definition and so every $S \in \Pi(\mathcal{E})$ must necessarily intersect R. Roughly speaking, a representation is obtained by choosing some representative elements from each S in $\Pi(\mathcal{E})$ when $\mathcal{E}$ is coarse. Formally, we have the following result:

Lemma 23 *If $\mathcal{E}$ is coarse, there exists a minimal representation, which can be constructed by the following rule: for each $S \in \Pi(\mathcal{E})$, choose two distinct elements from S if $S \in \mathcal{E}$, and one element if $S \notin \mathcal{E}$, and set R to be the set of chosen elements. Moreover, every minimal representation can be constructed in this way, and so in particular minimal representations contain exactly the same number of points.*

Proof. The set R constructed as above is well defined since by the coarseness condition every S has at least two elements. We claim that R is a representation. For all $E \in \mathcal{E}$, there is $S \in \Pi(\mathcal{E})$ with $S \subseteq E$. If $S = E$, R contains exactly two points belonging to E. If $S \subsetneq E$, then there is another $S' \neq S$ with $S' \subseteq E$ because E is the union of some elements in $\Pi(\mathcal{E})$. Since R contains one element of S and S', it contains at least two points belonging to E. Therefore, $|R \cap E| \geq 2$

for all $E \in \mathcal{E}$. Also, every $S \in \Pi(\mathcal{E})$ intersects with R and so $\kappa(R) = \Omega$. Finally, notice that $E \subseteq R$ is possible only if $E \in \Pi(\mathcal{E})$. But by the coarseness condition, $|E| \geq 4$ and so this case cannot occur in the construction, thus $\mathcal{E}_{\subseteq R} = \emptyset$.

Next we claim that R is minimal. Let R' be a proper subset of R and pick any $\omega \in R \backslash R'$. Let $S \in \Pi(\mathcal{E})$ be the set where ω is chosen from. If $S \in \mathcal{E}$, then R contains exactly two elements of S by construction. Then $|R' \cap S| = 1$, and so R' is not a representation. If $S \notin \mathcal{E}$, then ω is the only one element from S. Then $S \cap R' = \emptyset$ which implies $\kappa(R') \subseteq \Omega \backslash S$, and so R' is not a representation.

Finally, let R be a minimal representation. Then $S \cap R \neq \emptyset$ for every $S \in \Pi(\mathcal{E})$ so R contains at least one point from each S. If $S \in \Pi(\mathcal{E})$ and $S \in \mathcal{E}$, then $|R \cap S| \geq 2$ so at least two points from such S must be contained in R. Let R' the collection of all these points in the intersections, which is a minimum representation as we have shown above. Since $R' \subseteq R$, we conclude $R' = R$, which completes the proof. ■

Example 6 In Example 1, none of elements in $\Pi(\mathcal{E})$ belongs to $\mathcal{E}$. So to obtain a minimal representation one can choose exactly one point from each $S \in \Pi(\mathcal{E})$. For instance, $R = \{1, 3, 5, 7\}$ is a minimal representation.

When R constitutes a representation of $\mathcal{E}$, we can associate each $E \in \Upsilon(\mathcal{E}_{\cap R}) \backslash \mathcal{F}_1$ to some *unique* element in $\Upsilon(\mathcal{E}) \backslash \mathcal{F}_1$, as is shown in the next result.

Lemma 24 *Assume that $\mathcal{E}$ is coarse, and let $R \in \mathcal{F}$ be a representation of $\mathcal{E}$. Then $\kappa(F) \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1$ for any $F \in \Upsilon(\mathcal{E}_{\cap R}) \backslash \mathcal{F}_1$. Conversely, if $E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1$, then $E \cap R$ is a unique element of $\Upsilon(\mathcal{E}_{\cap R}) \backslash \mathcal{F}_1$ such that $\kappa(E \cap R) = E$. In short, given R, the restriction of κ, denoted by κ_R, constitutes a bijection between $\Upsilon(\mathcal{E}_{\cap R}) \backslash \mathcal{F}_1$ and $\Upsilon(\mathcal{E}) \backslash \mathcal{F}_1$ by the rule $\kappa_R(F) = \kappa(F)$ for all $F \in \Upsilon(\mathcal{E}_{\subseteq R}) \backslash \mathcal{F}_1$, and $\kappa_R^{-1}(E) = E \cap R$ for all $E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1$.*

Proof. Note that $\mathcal{E}$ and $\mathcal{E}_{\cap R}$ contain no singleton since $\mathcal{E}$ is coarse and R is a representation of $\mathcal{E}$. Also note that from the basic property of κ and $\kappa(R) = \Omega$ by the definition of representation, we have for each $E \in \mathcal{E}$, $E \cap R \in \mathcal{E}_{\cap R}$ and $\kappa(E \cap R) = \kappa(E) \cap \kappa(R) = \kappa(E) \cap \Omega = \kappa(E) = E$.

We first show that $\kappa(F) \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1$ for all $F \in \Upsilon(\mathcal{E}_{\cap R}) \backslash \mathcal{F}_1$. Fix any $F \in \Upsilon(\mathcal{E}_{\cap R}) \backslash \mathcal{F}_1$. Choose two distinct points $\omega_1, \omega_2 \in \kappa(F)$ arbitrarily, and we shall show that there is an $E \in \mathcal{E}$ such that $\{\omega_1, \omega_2\} \subseteq E \subseteq \kappa(F)$. By the construction of $\kappa(F)$, there are S_1, $S_2 \in \Pi(\mathcal{E})$ (possibly $S_1 = S_2$) such that $\omega_1 \in S_1$, $\omega_2 \in S_2$, and both $S_1 \cap F$ and $S_2 \cap F$ are non-empty. Suppose first that $S_1 \neq S_2$. Then we can select two distinct points $\omega_1' \in S_1 \cap F$ and $\omega_2' \in S_2 \cap F$. Since $F \in \Upsilon(\mathcal{E}_{\cap R}) \backslash \mathcal{F}_1$, there exists $F' \in \mathcal{E}_{\cap R}$ such that $\omega_1', \omega_2' \in F' \subseteq F$ by the definition of completeness. By the definition of $\mathcal{E}_{\cap R}$, there is $E \in \mathcal{E}$ with $F' = E \cap R$. Using the property of κ (see Remark 2), and the definition of a representation, $\kappa(F') = E \cap \kappa(R) = E$ and $\kappa(F') \subseteq \kappa(F)$. So we have $\{\omega_1, \omega_2\} \subseteq F' \subseteq \kappa(F') = E \subseteq \kappa(F)$, as we wanted. Suppose then $S_1 = S_2$ $(= \hat{S})$. Recall that $F \in \Upsilon(\mathcal{E}_{\cap R}) \backslash \mathcal{F}_1$

implies that F is the union of some elements in $\mathcal{E}_{\cap R}$. Since $\hat{S} \in \Pi(\mathcal{E})$, this means that there is at least one $E \in \mathcal{E}$ such that $\hat{S} \subseteq E$ and $E \cap R \subseteq F$. Then again by the definition of representation, $E = \kappa(E \cap R) \subseteq \kappa(F)$, and so this E has the desired property.

Next, we show that the restriction κ_R is a map from $\Upsilon(\mathcal{E}_{\cap R})\backslash\mathcal{F}_1$ onto $\Upsilon(\mathcal{E})\backslash\mathcal{F}_1$. Fix any $E \in \Upsilon(\mathcal{E})\backslash\mathcal{F}_1$. Since $E \in \sigma(\mathcal{E})$, $\kappa_R(E \cap R) = \kappa(E) \cap \kappa(R) = \kappa(E) \cap \Omega = \kappa(E) = E$; that is, $E \cap R$ is in the inverse image of κ_R. Thus, it is enough to show that $E \cap R \in \Upsilon(\mathcal{E}_{\cap R})\backslash\mathcal{F}_1$. By the definition of completeness, there exist $E_1, \ldots, E_K \in \mathcal{E}$ such that $E = \bigcup_{k=1}^K E_k$ and that for any pair of points $\omega, \omega' \in E$, $\omega, \omega' \in E_k$ holds for some k. So in particular, for any distinct points $\omega, \omega' \in E \cap R \subseteq E$, there exists k with $\omega, \omega' \in E_k$ and thus $\omega, \omega' \in E_k \cap R \in \mathcal{E}_{\cap R}$ since $\omega, \omega' \in R$. Therefore, κ_R is onto.

Finally we show that κ_R is one to one, i.e., $\kappa_R(F) = E$ occurs for $F \in \Upsilon(\mathcal{E}_{\cap R})\backslash\mathcal{F}_1$ only if $F = E \cap R$. Note that $F \in \Upsilon(\mathcal{E}_{\cap R})\backslash\mathcal{F}_1$ implies that there exist $E_1, \ldots, E_K \in \mathcal{E}$ such that $F = \bigcup_{k=1}^K (E_k \cap R) = (\bigcup_{k=1}^K E_k) \cap R$. Since R is a representation, R must intersect any $\Pi(\mathcal{E})$ -component of E_k for all k, and so $\kappa(F) = \kappa((\bigcup_{k=1}^K E_k) \cap R) = \bigcup_{k=1}^K E_k$. So $\kappa_R(F) = E$ implies $\bigcup_{k=1}^K E_k = E$ and so $F = E \cap R$ must hold. This completes the proof. ■

By Lemma 24, if R be a representation of $\mathcal{E}$, then, by rewriting (26), we have

$$v_{|R} = \sum_{\omega \in R} \lambda_{\{\omega\}}^R w_{\{\omega\}} + \sum_{E \in \Upsilon(\mathcal{E})\backslash\mathcal{F}_1} \lambda_E^R w_{E \cap R} \tag{27}$$

where $\lambda_E^R = \nu_{E \cap R}^R$ for each $E \in \Upsilon(\mathcal{E})$. By construction, the coefficients $\{\lambda_E^R : E \in \Upsilon(\mathcal{E})\}$ are uniquely determined with respect to a representation R except for singletons. It turns out that these do not depend upon the choice of representation R, which we shall demonstrate in the following in a few lemmas. Let R, $R' \in \mathcal{F}$ be representations of $\mathcal{E}$, and so there are corresponding expressions of the form (27). We write $R \doteq R'$ if $\lambda_{\{\omega\}}^R = \lambda_{\{\omega\}}^{R'}$ for all $\omega \in R \cap R'$ and $\lambda_E^R = \lambda_E^{R'}$ for all $E \in \Upsilon(\mathcal{E})\backslash\mathcal{F}_1$. Note that the first part holds vacuously if $R \cap R' = \emptyset$.

Lemma 25 *Assume that $\mathcal{E}$ is coarse and let v be $\mathcal{E}$-coextrema additive. Let $R, R', R'' \in \mathcal{F}$ be representations of $\mathcal{E}$. Suppose that $R \doteq R'$ and $R' \doteq R''$. Then, $R \doteq R''$ holds if $R \cap R'' \subseteq R'$.*

Proof. By definition, $\lambda_{\{\omega\}}^R = \lambda_{\{\omega\}}^{R''}$ for all $\omega \in R \cap R' \cap R''$ $(= R \cap R'')$ and $\lambda_E^R = \lambda_E^{R''}$ for all $E \in \Upsilon(\mathcal{E})\backslash\mathcal{F}_1$. ■

Lemma 26 *Assume that $\mathcal{E}$ is coarse and let v be $\mathcal{E}$-coextrema additive. Let $R, R' \in \mathcal{F}$ be representations of $\mathcal{E}$. Then $R \doteq R'$ holds if $R \cap R'$ is a representation. In particular, if $R \subseteq R'$, $R \doteq R'$ holds.*

Proof. Set $R^* = R \cap R'$. Note that by construction, for all $T \in \mathcal{F}$, $v_{|R^*}(T) = v_{|R}(T \cap R^*) = v_{|R'}(T \cap R^*)$. Using (27) on the other hand, we have

$$\begin{aligned} v_{|R}(T \cap R^*) &= \sum_{\omega \in R} \lambda^R_{\{\omega\}} w_{\{\omega\}}(T \cap R^*) + \sum_{E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1} \lambda^R_E w_{E \cap R}(T \cap R^*) \\ &= \sum_{\omega \in R^*} \lambda^R_{\{\omega\}} w_{\{\omega\}}(T) + \sum_{E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1} \lambda^R_E w_{E \cap R^*}(T) \end{aligned}$$

and

$$v_{|R'}(T \cap R^*) = \sum_{\omega \in R^*} \lambda^{R'}_{\{\omega\}} w_{\{\omega\}}(T) + \sum_{E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1} \lambda^{R'}_E w_{E \cap R^*}(T).$$

Thus, for all $T \in \mathcal{F}$,

$$\sum_{\omega \in R^*} \lambda^R_{\{\omega\}} w_{\{\omega\}}(T) + \sum_{E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1} \lambda^R_E w_{E \cap R^*}(T) = \sum_{\omega \in R^*} \lambda^{R'}_{\{\omega\}} w_{\{\omega\}}(T) + \sum_{E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1} \lambda^{R'}_E w_{E \cap R^*}(T).$$

Since R^* is also a representation by assumption, by Lemma 24, $\Upsilon(\mathcal{E}_{|R^*}) \backslash \mathcal{F}_1$ and $\Upsilon(\mathcal{E}) \backslash \mathcal{F}_1$ are isomorphic. Since $\{w_T : T \in \mathcal{F}\}$ are linearly independent, this means that the games in $\{w_{\{\omega\}}\}_{\omega \in R^*} \cup \{w_{E \cap R^*}\}_{E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1}$ are linearly independent. Therefore, the respective coefficients on the both sides of the above equation must coincide each other, which completes the proof. ■

Lemma 27 *Assume that $\mathcal{E}$ is coarse and let v be $\mathcal{E}$-coextrema additive. Let $R, R' \in \mathcal{F}$ be minimal representations of $\mathcal{E}$. Then $R \doteq R'$.*

Proof. If $R = R'$, then obviously $R \doteq R'$, and so let $R \neq R'$. By Lemma 23, $|R| = |R'|$ and so there is $\omega' \in R' \backslash R$. Let $S \in \Pi(\mathcal{E})$ be the unique element with $\omega' \in S$. Recall that a representation intersects every elements of $\Pi(\mathcal{E})$, and hence we can pick an $\omega \in R \cap S$. By construction $\omega \neq \omega'$. Set $R^1 = (R \backslash \{\omega\}) \cup \{\omega'\}$, i.e., R^1 is obtained by substituting ω with ω' both of which belong to S. So R^1 is also a minimal representation by Lemma 23.

We shall show that $R \doteq R^1$. For this, consider first $\hat{R} = R \cup \{\omega'\}$. Notice that $\hat{R}$ is a representation; since $R \subseteq \hat{R}$ and R is a representation, it is clear that $\kappa(\hat{R}) = \Omega$, and $\left|\hat{R} \cap E\right| \geq 2$ for all $E \in \mathcal{E}$. Since $\mathcal{E}$ is coarse and R is minimal, for all $E \in \mathcal{E}$, we have $|E \backslash R| \geq 2$ and so $\left|E \backslash \hat{R}\right| \geq 1$. Hence $\mathcal{E}_{\subseteq \hat{R}} = \emptyset$, which proves that $\hat{R}$ is a representation. By construction, both $R \cap \hat{R} = R$ and $R^1 \cap \hat{R} = R^1$ are representations, so by Lemma 26, $R \doteq \hat{R}$ and $\hat{R} \doteq R^1$. Note that $R \cap R^1 \subseteq \hat{R}$, which implies that $R \doteq R^1$ by Lemma 25.

Recall that both R and R' are finite and they can be obtained by the method described in Lemma 23, so repeating the argument above, i.e., replacing one ω in R with another $\omega' \in R' \backslash R$, we can construct a sequence of minimal representations R^0 $(= R)$ R^1, R^2, $\cdots$, $R^k = R'$ such that $R^{m-1} \doteq R^m$ for each $m = 1, .., k$. By definition, $\lambda^{R^{m-1}}_E = \lambda^{R^m}_E$ holds for all $E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1$ for every $m = 1, ..., k$, hence $\lambda^R_E = \lambda^{R'}_E$ holds for all $E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1$. For any $\omega \in R \cap R'$, since such ω is never replaced along the sequence above, we have $\lambda^{R^{m-1}}_{\{\omega\}} = \lambda^{R^m}_{\{\omega\}}$ for every $m = 1, ..., k$, and hence $\lambda^{R}_{\{\omega\}} = \lambda^{R'}_{\{\omega\}}$. Therefore, we conclude that $R \doteq R'$. ■

Lemma 28 *Assume that $\mathcal{E}$ is coarse and let v be $\mathcal{E}$-coextrema additive. Let $R, R' \in \mathcal{F}$ be representations of $\mathcal{E}$. Then $R \doteq R'$.*

Proof. Choose any two minimal representations Γ and Γ' such that $\Gamma \subseteq R$, $\Gamma' \subseteq R'$, and $\Gamma \cap \Gamma' \subseteq R \cap R'$. Notice that by Lemma 23 such minimal representations always exist and can be constructed as follows: for any $\omega \in R \cap R'$, then select this ω from $S \in \Pi(\mathcal{E})$ which contains ω. Now by Lemma 26, $R \doteq \Gamma$ and $\Gamma' \doteq R'$ hold. Also, by Lemma 27, $\Gamma \doteq \Gamma'$ holds. These imply that $\lambda^R_E = \lambda^{R'}_E$ for all $E \in \Upsilon(\mathcal{E})\backslash\mathcal{F}_1$ and that $\lambda^R_{\{\omega\}} = \lambda^{R'}_{\{\omega\}}$ for all $\omega \in \Gamma \cap \Gamma'$. Since the choice of $\Gamma \cap \Gamma' \subseteq R \cap R'$ is arbitrary as is pointed out above, we must have $\lambda^R_{\{\omega\}} = \lambda^{R'}_{\{\omega\}}$ for all $\omega \in R \cap R'$. Therefore, we conclude that $R \doteq R'$. ■

Since there is a representation containing any $\omega \in \Omega$, Lemma 28 implies that there exists a unique collection of constants $\{\lambda_E\}_{E\in\Upsilon(\mathcal{E})}$ such that, for *any* representation R of $\mathcal{E}$, $v_{|R} = \sum_{\omega\in R} \lambda_{\{\omega\}} w_{\{\omega\}} + \sum_{E\in\Upsilon(\mathcal{E})\backslash\mathcal{F}_1} \lambda_E w_{E\cap R}$. Using this collection, define two games v_1 and v_2 by the following rule:

$$v_1 = \sum_{E\in\Upsilon(\mathcal{E})} \lambda_E w_E \text{ and } v_2 = v - v_1. \tag{28}$$

By Theorem 13, v_1 is $\mathcal{E}$-comaximum additive. To show that v_2 is $\mathcal{E}$-cominimum additive, we use the following property of v_2.

Lemma 29 *Assume that $\mathcal{E}$ is coarse and let v be $\mathcal{E}$-coextrema additive. Then for any $T \in \mathcal{F}$,*

$$v_2(T) = v_2(\bigcup_{E\in\mathcal{E}_{\subseteq T}} E). \tag{29}$$

Proof. Case 1: $\mathcal{E}_{\subseteq T} = \emptyset$, i.e., no element in $\mathcal{E}$ is contained in T. Then, $v_2(\bigcup_{E\in\mathcal{E}_{\subseteq T}} E) = v(\emptyset) - v_1(\emptyset) = 0$, so we need to show that $v_2(T) = v(T) - v_1(T) = 0$. Note that there exists a representation R of $\mathcal{E}$ such that $T \subseteq R$ (see Lemma 22). Then, $v(T) = v(T \cap R) = v_{|R}(T) = \sum_{E\in\Upsilon(\mathcal{E}),\ E\cap T\neq\emptyset} \lambda_E = v_1(T)$, as claimed.

Case 2: $\mathcal{E}_{\subseteq T} \neq \emptyset$. Let $E^* = \bigcup_{E\in\mathcal{E}_{\subseteq T}} E$ and $T^* = T\backslash E^*$. We want to show that $v_2(T) = v_2(E^*)$. By construction, $E^* \in \sigma(\mathcal{E})$ is the union of some elements in $\Pi(\mathcal{E})$, choose one point from each of these elements and let A be the collection of these points. Note that $\kappa(A) = E^*$, and that $\mathcal{E}_{\subseteq A} = \mathcal{E}_{\subseteq T^*\cup A} = \emptyset$ follows from the coarseness. Thus, Case 1 applies to A and $T^* \cup A$, and we have

$$v_2(A) = v_2(T^* \cup A) = 0. \tag{30}$$

Now we claim that 1_{E^*} and $1_{T^*\cup A}$ are $\mathcal{E}$-coextrema. Note first that $E^* \cap (T^* \cup A) = A$ by construction. To see that they are $\mathcal{E}$-comaximum, recall Remark 6, and pick $F \in \mathcal{E}$ with $F \subseteq$

$\Omega\backslash A$. Then $F \cap E^* = \emptyset$ must follow, since both F and E^* are in $\sigma(\mathcal{E})$ and so for any $S \in \Pi(\mathcal{E})$ with $S \subseteq F$, $A \cap S \neq \emptyset$ would hold if $S \subseteq E^*$. Then $F \subseteq \Omega\backslash E^*$ as desired. To see that they are $\mathcal{E}$-cominimum as well, notice that if $F \in \mathcal{E}$ and $F \subseteq E^* \cup (T^* \cup A) = T$, then $F \subseteq E^*$ by construction. Thus E^* and $(T^* \cup A)$ are an $\mathcal{E}$-decomposition pair, and so apply Lemma 17.

By the coextrema additivity of v, $v(E^* \cup (T^* \cup A)) + v(E^* \cap (T^* \cup A)) = v(E^*) + v(T^* \cup A)$, which can be re-written as

$$v(E^* \cup T^*) + v(A) = v(E^*) + v(T^* \cup A). \tag{31}$$

On the other hand, since 1_{E^*} and $1_{T^* \cup A}$ are $\mathcal{E}$-comaximum and v_1 is $\mathcal{E}$-comaximum additive,

$$v_1(E^* \cup T^*) + v_1(A) = v_1(E^*) + v_1(T^* \cup A). \tag{32}$$

Subtracting (32) from (31), and using the definition of v_2, and the fact $T = E^* \cup T^*$, we have

$$v_2(T) + v_2(A) = v_2(E^*) + v_2(T^* \cup A).$$

Applying (30) here, we obtain the desired equation. ■

Now we are ready to show that v_2 is $\mathcal{E}$-cominimum additive.

Lemma 30 *Assume that $\mathcal{E}$ is coarse and let v be $\mathcal{E}$-coextrema additive. Then, v_2 is $\mathcal{E}$-cominimum additive and thus it has a unique expression*

$$v_2 = \sum_{E \in \Upsilon(\mathcal{E})} \mu_E u_E.$$

Proof. Let 1_A and 1_B be $\mathcal{E}$-cominimum, i.e., A and B constitute an $\mathcal{E}$-decomposition pair by Lemma 17. We need to show that $v_2(A \cup B) + v_2(A \cap B) = v_2(A) + v_2(B)$.

Note that for each $S \in \Pi(\mathcal{E})$ such that there is an $E \in \mathcal{E}$ with $S \subseteq E \subseteq A \cup B$, if $S \not\subseteq A \cap B$, then either $S \cap (B\backslash A) \neq \emptyset$ or $S \cap (A\backslash B) \neq \emptyset$, but not both; if both hold then A and B would not be an $\mathcal{E}$-decomposition pair.

For each $S \in \Pi(\mathcal{E})$ with $S \not\subseteq A \cap B$, choose a point ω_S from $S \cap (A\backslash B)$ if $S \cap (A\backslash B) \neq \emptyset$, or from $S \cap (B\backslash A)$ if $S \cap (B\backslash A) \neq \emptyset$. Let Ω^* be the set of chosen points. Finally, set $A^* = A \cup (B\backslash\Omega^*)$ and $B^* = B \cup (A\backslash\Omega^*)$. Notice that $A^* \cup B^* = A \cup B$ by construction.

We claim that if $E \in \mathcal{E}$ satisfies $E \subseteq A^*$, then $E \subseteq A$. Indeed, suppose that there is a point $\omega \in E \cap (A^*\backslash A)$. Since $E \in \mathcal{E}$, we can find (a unique) $S \in \Pi(\mathcal{E})$ with $\omega \in S \subseteq E$, and $\omega \in S \cap (B\backslash A)$. By the construction of Ω^*, this means that $S \cap ((B\backslash A) \cap \Omega^*) \neq \emptyset$ so $E \cap ((B\backslash A) \cap \Omega^*) \neq \emptyset$, which is impossible since $E \subseteq A^* = A \cup (B\backslash\Omega^*)$.

Similarly, if $E \in \mathcal{E}$ satisfies $E \subseteq B^*$, then $E \subseteq B$. To sum up, the collections of $\mathcal{E}$-elements contained in A^*, B^*, $A^* \cup B^*$ and $A^* \cap B^*$ coincide with those of A, B, $A \cup B$ and $A \cap B$, respectively. Therefore, by Lemma 29, we are done if $v_2(A^* \cup B^*) + v_2(A^* \cap B^*) = v_2(A^*) + v_2(B^*)$. For this, it suffices to show that 1_{A^*} and 1_{B^*} are $\mathcal{E}$-coextrema. Indeed, since v is $\mathcal{E}$-coextrema additive, we have $v(A^* \cup B^*) + v(A^* \cap B^*) = v(A^*) + v(B^*)$, and since v_1 is $\mathcal{E}$-comaximum additive, we have $v_1(A^* \cup B^*) + v_1(A^* \cap B^*) = v_1(A^*) + v_1(B^*)$. Since $v_2 = v - v_1$, the desired equation is established from these two equations.

To see 1_{A^*} and 1_{B^*} are $\mathcal{E}$-cominimum, notice that A and B constitutes a decomposition pair by assumption, and so do A^* and B^*; if $E \subseteq A^* \cup B^*$ with $E \in \mathcal{E}$, then $E \subseteq A \cup B$, which implies $E \subseteq A$ or $E \subseteq B$ and hence $E \subseteq A^*$ or $E \subseteq B^*$ as we have shown above. Thus 1_{A^*} and 1_{B^*} are $\mathcal{E}$-cominimum by Lemma 17.

It remains to show that 1_{A^*} and 1_{B^*} are $\mathcal{E}$-comaximum. Pick any $E \in \mathcal{E}$ with $E \subseteq \Omega \backslash (A^* \cap B^*)$. We need to show that $E \subseteq \Omega \backslash A^*$ or $E \subseteq \Omega \backslash B^*$ or both (see Remark 6). Suppose $E \cap (A^* \cup B^*) \neq \emptyset$ or else the implication holds trivially, and so it suffices to show that $E \cap (A^* \backslash B^*) = \emptyset$ or $E \cap (B^* \backslash A^*) = \emptyset$. If neither of these holds, then pick $\omega_A \in E \cap (A^* \backslash B^*)$ and $\omega_B \in E \cap (B^* \backslash A^*)$. Note that $A^* \backslash B^* = A \backslash B^*$ and $B^* \backslash A^* = B \backslash A^*$ holds, and thus ω_A and ω_B must belong to Ω^* by the construction of A^* and B^*. Since $E \in \mathcal{E}$, there must be $S_A \in \Pi(\mathcal{E})$ and $S_B \in \Pi(\mathcal{E})$ and $E_A \in \mathcal{E}$ and $E_B \in \mathcal{E}$ such that $\omega_A \in S_A \subseteq E_A \cap E \subseteq A \cup B$ and $\omega_B \in S_B \subseteq E_B \cap E \subseteq A \cup B$. But then, by the coarseness, both $S_A \cap B^*$ and $S_B \cap A^*$ are non-empty, which implies $E \cap (A^* \cap B^*) \neq \emptyset$, a contradiction. This completes the proof. ■

Since v_1 is $\mathcal{E}$-comaximum additive and v_2 is $\mathcal{E}$-cominimum additive, we have the desired expression $v = v_1 + v_2 = \sum_{E \in \Upsilon(\mathcal{E})} \lambda_E w_E + \sum_{E \in \Upsilon(\mathcal{E})} \mu_E u_E = p + \sum_{E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1} (\lambda_E w_E + \mu_E u_E)$ where $p = \sum_{\omega \in \Omega} p_{\{\omega\}} u_{\{\omega\}}$ and $p_{\{\omega\}} = \lambda_{\{\omega\}} + \mu_{\{\omega\}}$. It remains to show that this is a unique representation.

Lemma 31 *Assume that $\mathcal{E}$ is coarse and let v be $\mathcal{E}$-coextrema additive. Then, the expression $v = \sum_{\omega \in \Omega} p_{\{\omega\}} u_{\{\omega\}} + \sum_{E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1} (\lambda_E w_E + \mu_E u_E)$ is unique; that is, if $v = \sum_{\omega \in \Omega} p'_{\{\omega\}} u_{\{\omega\}} + \sum_{E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1} (\lambda'_E w_E + \mu'_E u_E)$ then $p_{\{\omega\}} = p'_{\{\omega\}}$ for all $\omega \in \Omega$, $\lambda_E = \lambda'_E$, and $\mu_E = \mu'_E$ for all $E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1$.*

Proof. Let $R \in \mathcal{F}$ be a representation of $\mathcal{E}$. Then,

$$v_{|R} = \sum_{\omega \in R} p_{\{\omega\}} u_{\{\omega\}} + \sum_{E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1} \lambda_E w_{E \cap R} = \sum_{\omega \in R} p'_{\{\omega\}} u_{\{\omega\}} + \sum_{E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1} \lambda'_E w_{E \cap R}.$$

By Lemma 24, $\Upsilon(\mathcal{E}_{\cap R}) \backslash \mathcal{F}_1$ and $\Upsilon(\mathcal{E}) \backslash \mathcal{F}_1$ are isomorphic and thus $\{w_{\{\omega\}}\}_{\omega \in R} \cup \{w_{E \cap R}\}_{E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1}$ are linearly independent. Therefore, $p_{\{\omega\}} = p'_{\{\omega\}}$ for all $\omega \in R$ and $\lambda_E = \lambda'_E$ for all $E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1$. Since the choice of R was arbitrary, $p_{\{\omega\}} = p'_{\{\omega\}}$ for all $\omega \in \Omega$. The linear independence also guarantees that the expression $v - \sum_{\omega \in \Omega} p_{\{\omega\}} w_{\{\omega\}} - \sum_{E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1} \lambda_E w_E = \sum_{E \in \Upsilon(\mathcal{E}) \backslash \mathcal{F}_1} \mu_E u_E$ must also be unique. ■

The proof of Theorem 14 is now complete.

5 A Refinement of the Myerson Value

5.1 Introduction

Myerson (27, 28) made a seminal contribution to describe how the outcome of a cooperative game might depend on which groups of players hold cooperative planning conferences. A conference is defined as a set of two or more players and a collection of conferences is called a conference structure.[12] Myerson (27, 28) augmented a cooperative game by a conference structure and defined another cooperative game where the conference structure determines which coalitions are feasible. The feasible coalition is the one in which any pair of players are either *directly* or *indirectly* connected (i.e. path connected) by the conferences contained in the coalition. Myerson (27, 28) showed that the Shapley value of the induced cooperative game can be characterized by two axioms: fairness and component efficiency. This allocation rule is referred to as the Myerson value in the subsequent literature.[13]

We emphasize that the Myerson value treats direct and indirect connections equally. For example, consider a conference structure $\{\{1,2\},\{2,3\}\}$. Player 1 and player 2 are directly connected in the sense that they have a chance of direct communication in $\{1,2\}$, and so are player 2 and player 3, whereas player 1 and player 3 are not directly connected but indirectly connected in the sense that they have a chance of indirect communication via an intermediary, i.e., player 2. In the construction of the Myerson value, this conference structure is identified with another conference structure $\{\{1,2\},\{2,3\},\{1,2,3\}\}$ where player 1 and player 3 are directly connected in $\{1,2,3\}$. That is, payoff allocations are the same in the two conference structures. This might be justified under the premise that indirect communication is costless and/or players' bargaining power is independent of directness of communication. This premise will be certainly plausible in some context, but not necessarily so in general. In the example above, if it is costly for player 2 to behave as an intermediary, say by some strategic reasons, then a conference $\{1,2,3\}$ may not function. In this case, it is more natural to distinguish the two conference structures $\{\{1,2\},\{2,3\}\}$ and $\{\{1,2\},\{2,3\},\{1,2,3\}\}$. We therefore contend that if an allocation rule does not distinguish them, it overlooks some interesting aspects of communication and bargaining.

This paper proposes a refinement of the Myerson value which distinguishes direct and indirect connections. Similar to Myerson (27, 28), we augment a cooperative game by a conference structure and define another cooperative game where the conference structure determines which coalitions are feasible. But different from Myerson (27, 28), the feasible coalition is the one in which any pair of players are *directly* connected by the conferences contained in the coalition.

[12] Myerson (27) considered special conferences with exactly two players and regarded a conference structure as a network, while Myerson (28) considered general conferences and nontransferable utility.

[13] The study of allocation rules with partial cooperation possibilities is well-documented since Aumann and Drèze (3). For other allocation rules, see Meesen (26), Borm et al. (5), Hamiache (16), Bilbao and López (4), and the review by Slikker and van den Nouweland (37), among others.

In the main result, we show that the Shapley value of the induced cooperative game can be characterized by three axioms: fairness, complete component efficiency, and no contribution by unconnected players. The latter two new axioms describe the behavior of the allocation rule distinguishing direct and indirect connections. To establish the main result, we take advantage of the idea of potentials for cooperative games originated by Hart and Mas-Colell (17). We prove that if an allocation rule satisfies the three axioms, then it is represented in terms of the marginal contributions of the potential for the induced cooperative game, which leads us to the main result. Also in the main result, we provide a characterization of the potential for the induced cooperative game, which extends the result of Hart and Mas-Colell (17).

The organization of the paper is as follows. Preliminary definitions and results are summarized in section 2. Conference structures and allocation rules are introduced in section 3. The main result is stated in section 4, which is proved in section 5. In section 6, we compare our result and that of Myerson (27, 28) and show that our allocation rule is in fact a refinement of the Myerson value. In the same section, we point out some connection of our result to the network games of Jackson and Wolinsky (19).

5.2 Preliminaries

Let $N = \{1, \ldots, n\}$ be a set of players. A subset $S \in 2^N$ is referred to as a coalition. A game v is a function from 2^N to $\mathbb{R}$ with $v(\emptyset) = 0$. The unanimity game on $T \in 2^N$ is denoted by u_T and defined as

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

A collection of coalitions $\mathcal{P} \subseteq 2^N$ is partially ordered with the set inclusion relation. Regard $[u_X(Y)]_{X,Y\in\mathcal{P}}$ as a $|\mathcal{P}| \times |\mathcal{P}|$ matrix and observe that it is non-singular and thus invertible. The *Möbius function* of $\mathcal{P}$ is defined as a function $\mu_{\mathcal{P}} : \mathcal{P} \times \mathcal{P} \to \mathbb{R}$ such that the matrix $[\mu_{\mathcal{P}}(X,Y)]_{X,Y\in\mathcal{P}}$ is the inverse matrix of $[u_X(Y)]_{X,Y\in\mathcal{P}}$;[14] that is, for $X, Y \in \mathcal{P}$, it holds that

$$\sum_{T\in\mathcal{P}} \mu_{\mathcal{P}}(X,T)u_T(Y) = \sum_{T\in\mathcal{P}} u_X(T)\mu_{\mathcal{P}}(T,Y) = \begin{cases} 1 & \text{if } X = Y, \\ 0 & \text{otherwise.} \end{cases} \tag{33}$$

[14] A function $\zeta_{\mathcal{P}} : \mathcal{P}\times\mathcal{P} \to \mathbb{R}$ such that $\zeta_{\mathcal{P}}(X,Y) = u_X(Y)$ for all $X, Y \in \mathcal{P}$ is called the *zeta function* of $\mathcal{P}$. The zeta function and the Möbius function are defined on any partially ordered set. See a textbook on combinatorics such as Lint and Wilson (25).

It is known that the Möbius function $\mu_{\mathcal{P}}$ is determined inductively by the following rule:[15]

$$\mu_{\mathcal{P}}(X,Y) = \begin{cases} 1 & \text{if } X = Y, \\ 0 & \text{if } X \not\subseteq Y, \\ -\displaystyle\sum_{T\in\mathcal{P}:X\subseteq T\subset Y} \mu_{\mathcal{P}}(X,T) & \text{if } X \subset Y. \end{cases} \tag{34}$$

For the special case of $\mathcal{P} = 2^N$, it holds that

$$\mu_{\mathcal{P}}(X,Y) = \begin{cases} (-1)^{|X|-|Y|} & \text{if } X \subseteq Y, \\ 0 & \text{otherwise.} \end{cases}$$

The following result is referred to as *the principle of Möbius inversion.*

Lemma 32 *For any function $v : \mathcal{P} \to \mathbb{R}$, if $f : \mathcal{P} \to \mathbb{R}$ is given by*

$$f(X) = \sum_{T\in\mathcal{P}} v(T)\mu_{\mathcal{P}}(T,X) \text{ for all } X \in \mathcal{P}, \tag{35}$$

then it holds that

$$v(X) = \sum_{T\in\mathcal{P}} f(T)u_T(X) \text{ for all } X \in \mathcal{P}. \tag{36}$$

Conversely, for any function $f : \mathcal{P} \to \mathbb{R}$, if $v : \mathcal{P} \to \mathbb{R}$ is given by (36)*, then* (35) *holds.*

The principle of Möbius inversion can be easily checked because (33) and (35) imply that

$$\begin{aligned}\sum_{T\in\mathcal{P}} f(T)u_T(X) &= \sum_{T\in\mathcal{P}} \left(\sum_{T'\in\mathcal{P}} v(T')\mu_{\mathcal{P}}(T',T)\right) u_T(X) \\ &= \sum_{T'\in\mathcal{P}} v(T') \left(\sum_{T\in\mathcal{P}} \mu_{\mathcal{P}}(T',T)u_T(X)\right) = v(X) \text{ for all } X \in \mathcal{P},\end{aligned}$$

and (33) and (36) imply that

$$\begin{aligned}\sum_{T\in\mathcal{P}} v(T)\mu_{\mathcal{P}}(T,X) &= \sum_{T\in\mathcal{P}} \left(\sum_{T'\in\mathcal{P}} f(T')u_{T'}(T)\right) \mu_{\mathcal{P}}(T,X) \\ &= \sum_{T'\in\mathcal{P}} f(T') \left(\sum_{T\in\mathcal{P}} u_{T'}(T)\mu_{\mathcal{P}}(T,X)\right) = f(X) \text{ for all } X \in \mathcal{P}.\end{aligned}$$

The principle of Möbius inversion for the special case of $\mathcal{P} = 2^N$ leads us to the well known fact that any game v is uniquely represented as a linear combination of unanimity games (33):

$$v = \sum_{T\in 2^N} \beta_T u_T \text{ where } \beta_T = \sum_{T\in 2^N:S\subseteq T} (-1)^{|T|-|S|}v(S).$$

[15] When $X \subset Y$ and $|Y| - |X| = 1$, this formula requires that $\mu_{\mathcal{P}}(X,Y) = -\mu_{\mathcal{P}}(X,X) = -1$, and once $\mu_{\mathcal{P}}(X,Y)$ is determined for $X \subset Y$ with $|Y| - |X| \leq k$, then the formula determines $\mu_{\mathcal{P}}(X,Y)$ for $X \subset Y$ with $|Y| - |X| = k + 1$, and so on.

Denote by $\delta_i v(S)$ the marginal contribution of player $i \in S$ to $v(S)$; that is,

$$\delta_i v(S) = v(S) - v(S \backslash \{i\}).$$

The Shapley value of v is the vector of payoffs $\phi(v) \in \mathbb{R}^N$ given by the following formula (33):

$$\phi_i(v) = \sum_{S \in 2^N : i \in S} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} \delta_i v(S) \text{ for all } i \in N.$$

In particular, the Shapley value of u_T is given by

$$\phi_i(u_T) = \begin{cases} 1/|T| & \text{if } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

Since the Shapley value is linear in games, we have an alternative formula for the Shapley value of $v = \sum_{T \in 2^N} \beta_T u_T$ as follows:

$$\phi_i(v) = \sum_{T \in 2^N} \beta_T \phi_i(u_T) = \sum_{T \in 2^N : i \in T} \beta_T / |T|. \tag{37}$$

A potential for a game v is a game p such that

$$\sum_{i \in S} \delta_i p(S) = v(S) \tag{38}$$

for all $S \in 2^N$. Hart and Mas-Colell (17) showed the following result.[16]

Proposition 15 *There exists a unique potential p for $v = \sum_{T \in 2^N} \beta_T u_T$, which is given by*

$$p = \sum_{T \in 2^N} \frac{\beta_T}{|T|} u_T.$$

Moreover, the vector of the marginal contributions $(\delta_i p(N))_{i \in N}$ coincides with the Shapley value of v; that is,

$$\delta_i p(N) = \phi_i(v) \textit{ for all } i \in N.$$

5.3 Conference structures and allocation rules

To describe how players organize their cooperation, we specify which groups of players are willing and able to confer together for the purpose of planning cooperative actions. Myerson (28) have used the term a *conference* to refer to any set of two or more players who might meet together to discuss their cooperative plans. So, we define a conference as a coalition with two or more

[16] Originally, Hart and Mas-Colell (17) defined a potential as a real-valued function over the space of games. The value assigned by the potential to the restriction of a game v to a coalition S corresponds to $p(S)$ in this paper.

players. A conference structure is then any collection of conferences. The collection of all possible conference structures is denoted by

$$\mathbf{CS} = \{\mathcal{H} \subseteq 2^N \mid |H| \geq 2 \text{ for all } H \in \mathcal{H}\}.$$

We write $\mathcal{H}_S = \{H \in \mathcal{H} \mid H \subseteq S\}$ and $\mathcal{H}_{-i} = \mathcal{H}_{N\setminus\{i\}}$ for $\mathcal{H} \in \mathbf{CS}$, $S \in 2^N$, and $i \in N$.

We consider two types of connections between players, direct and indirect ones.

Definition 22 Players $i, j \in N$ are said to be *directly $\mathcal{H}$-connected* in a coalition S if $i = j$ or there exists a conference $H \in \mathcal{H}_S$ with $\{i, j\} \subseteq H$. Players $i, j \in N$ are said to be *$\mathcal{H}$-connected* in a coalition S if there exist a sequence of players $i_1, ..., i_m$ with $i = i_1$ and $j = i_m$ such that i_k and i_{k+1} are directly $\mathcal{H}$-connected in S for $k = 1, \ldots, m-1$.

Thus, two players are directly $\mathcal{H}$-connected in S if they can be coordinated by direct communication; and two players are $\mathcal{H}$-connected in S if they can be coordinated either by direct communication or by indirect communication via intermediaries.[17] By definition, two players are (directly) $\mathcal{H}$-connected in S if and only if they are (directly) $\mathcal{H}_S$-connected in S. Also by definition, if two players are (directly) $\mathcal{H}$-connected in S then they are (directly) $\mathcal{H}$-connected in T with $S \subseteq T$.

The above notions of connectedness for players induce the corresponding notions for coalitions.[18]

Definition 23 A coalition $S \in 2^N$ is said to be *$\mathcal{H}$-complete* if any pair of players in S are directly $\mathcal{H}$-connected in S. A coalition $S \in 2^N$ is said to be *$\mathcal{H}$-connected* if any pair of players in S are $\mathcal{H}$-connected in S.

By definition, any singleton is $\mathcal{H}$-complete and $\mathcal{H}$-connected. Note that S is $\mathcal{H}$-complete if and only if it is $\mathcal{H}_S$-complete, and similarly, S is $\mathcal{H}$-connected if and only if it is $\mathcal{H}_S$-connected.

Let $\mathrm{cm}(\mathcal{H}) \in \mathbf{CS}$ denote the collection of all $\mathcal{H}$-complete conferences, and let $\mathrm{cn}(\mathcal{H}) \in \mathbf{CS}$ denote the collection of all $\mathcal{H}$-connected conferences (so, singletons are excluded). Both $\mathrm{cm}(\cdot)$ and $\mathrm{cn}(\cdot)$ are monotonic as operators on $\mathbf{CS}$ in the sense that $\mathrm{cm}(\mathcal{H}) \subseteq \mathrm{cm}(\mathcal{H}')$ and $\mathrm{cn}(\mathcal{H}) \subseteq \mathrm{cn}(\mathcal{H}')$ if $\mathcal{H} \subseteq \mathcal{H}'$. It follows that

$$\mathcal{H} \subseteq \mathrm{cm}(\mathcal{H}) \subseteq \mathrm{cn}(\mathcal{H}), \tag{39}$$

since any pair of players in $S \in \mathcal{H}$ are directly $\mathcal{H}$-connected in S and any pair of players in $S \in \mathrm{cm}(\mathcal{H})$ are (directly) $\mathcal{H}$-connected in S. Furthermore, we can show the properties below.[19]

[17] In Myerson (28), players $i, j \in N$ are said to be $\mathcal{H}$-connected in S if $i = j$ or there exists a sequence of conferences $H_1, \ldots, H_m \in \mathcal{H}_S$ such that $i \in H_1$, $j \in H_m$, and $H_k \cap H_{k+1} \neq \emptyset$ for $k = 1, \ldots, m-1$, which is equivalent to the above definition.

[18] The notion of $\mathcal{H}$-completeness is introduced by ?) for events, i.e., subsets of the set of states, and used for a characterization of the Choquet integral. The term "complete" is adopted from an analogy to complete graphs. For $S \in 2^N$, consider an undirected graph with a vertex set S such that $\{i, j\} \subseteq S$ is an edge if there is $H \in \mathcal{H}_S$ satisfying $\{i, j\} \subseteq H$. This is a complete graph if and only if S is $\mathcal{H}$-complete.

[19] ?) obtained results similar to Lemma 33.

Lemma 33 *Players $i, j \in N$ are directly $\mathcal{H}$-connected in a coalition S if and only if they are directly* $\mathrm{cm}(\mathcal{H})$*-connected in S. Thus, it holds that*

$$\mathrm{cm}(\mathcal{H}) = \mathrm{cm}(\mathrm{cm}(\mathcal{H})).$$

Proof. If $i = j$ then the above claim holds trivially. Suppose that $i, j \in N$ with $i \neq j$ are directly $\mathcal{H}$-connected in S. Then, there exists $H \in \mathcal{H}_S$ with $\{i, j\} \subseteq H$. Since $H \in \mathrm{cm}(\mathcal{H})_S$, they are also directly $\mathrm{cm}(\mathcal{H})$-connected in S. Conversely, suppose that $i, j \in N$ with $i \neq j$ are directly $\mathrm{cm}(\mathcal{H})$-connected in S. Then, there exists $H \in \mathrm{cm}(\mathcal{H})_S$ with $\{i, j\} \subseteq H$. Since H is $\mathcal{H}$-complete, there exists $T \in \mathcal{H}_H \subseteq \mathcal{H}_S$ with $\{i, j\} \subseteq T$. This implies that i and j are directly $\mathcal{H}$-connected in S. The equivalence of the direct $\mathcal{H}$-connected relation and the direct $\mathrm{cm}(\mathcal{H})$-connected relation implies the equivalence of $S \in \mathrm{cm}(\mathcal{H})$ and $S \in \mathrm{cm}(\mathrm{cm}(\mathcal{H}))$. ∎

Lemma 34 *Players $i, j \in N$ are $\mathcal{H}$-connected in a coalition S if and only if they are* $\mathrm{cn}(\mathcal{H})$*-connected in S. Thus, it holds that*

$$\mathrm{cn}(\mathcal{H}) = \mathrm{cm}(\mathrm{cn}(\mathcal{H})) = \mathrm{cn}(\mathrm{cn}(\mathcal{H})).$$

Proof. Suppose that $i, j \in N$ are $\mathcal{H}$-connected in S. Then, there exist a sequence of players $i_1, \ldots, i_m$ with $i = i_1$ and $j = i_m$ such that i_k and i_{k+1} are directly $\mathcal{H}$-connected in S for $k = 1, \ldots, m-1$. Since $\mathcal{H} \subseteq \mathrm{cn}(\mathcal{H})$, i_k and i_{k+1} are directly $\mathrm{cn}(\mathcal{H})$-connected in S for each k. This implies that i and j are $\mathrm{cn}(\mathcal{H})$-connected in S. Conversely, suppose that $i, j \in N$ are $\mathrm{cn}(\mathcal{H})$-connected in S. Then, there exist a sequence of players $i_1, \ldots, i_m$ with $i = i_1$ and $j = i_m$ such that i_k and i_{k+1} are directly $\mathrm{cn}(\mathcal{H})$-connected in S for $k = 1, \ldots, m-1$. Thus, there exists $S_k \in \mathrm{cn}(\mathcal{H})_S$ with $\{i_k, i_{k+1}\} \subseteq S_k$, which implies that i_k and i_{k+1} are $\mathcal{H}$-connected in S for each k. Since the $\mathcal{H}$-connected relation is transitive, i and j must be $\mathcal{H}$-connected in S. The equivalence of the $\mathcal{H}$-connected relation and the $\mathrm{cn}(\mathcal{H})$-connected relation implies the equivalence of $S \in \mathrm{cn}(\mathcal{H})$ and $S \in \mathrm{cn}(\mathrm{cn}(\mathcal{H}))$, establishing $\mathrm{cn}(\mathcal{H}) = \mathrm{cn}(\mathrm{cn}(\mathcal{H}))$. Since $\mathrm{cn}(\mathcal{H}) \subseteq \mathrm{cm}(\mathrm{cn}(\mathcal{H})) \subseteq \mathrm{cn}(\mathrm{cn}(\mathcal{H}))$ by (39), $\mathrm{cn}(\mathcal{H}) = \mathrm{cm}(\mathrm{cn}(\mathcal{H})) = \mathrm{cn}(\mathrm{cn}(\mathcal{H}))$ must follow. ∎

Note that the $\mathcal{H}$-connected relation in S is an equivalence relation, although the direct $\mathcal{H}$-connected relation in S might not be. For $S \in 2^N$ and $\mathcal{H} \in \mathbf{CS}$, let $S/\mathcal{H}$ denote the partition of S consisting of the equivalence classes induced by the $\mathcal{H}$-connected relation in S; that is,

$$S/\mathcal{H} = \{\{j \in S \mid i \text{ and } j \text{ are } \mathcal{H}\text{-connected in } S\} \mid i \in N\}.$$

It follows that $S/\mathrm{cn}(\mathcal{H}) = S/\mathcal{H} = S/\mathcal{H}_S$ by the equivalence of the $\mathrm{cn}(\mathcal{H})$-connected, $\mathcal{H}$-connected, and $\mathcal{H}_S$-connected relations in S. We call an element of $S/\mathcal{H}$ a component of S. A component of S is a maximal $\mathcal{H}$-connected coalition in S because any pair of players in a $\mathcal{H}$-connected coalition are $\mathcal{H}$-connected in the component to which they both belong.

An allocation rule assigns a vector of payoffs to each conference structure; that is, an allocation rule is a mapping $f : \mathbf{CS} \to \mathbb{R}^N$ where player i's payoff is $f_i(\mathcal{H})$ for $\mathcal{H} \in \mathbf{CS}$. Myerson (27, 28) considered the following axioms for an allocation rule f.

Component efficiency (CE)

$$\sum_{i \in S} f_i(\mathcal{H}) = v(S) \text{ if } S \in N/\mathcal{H}.$$

Fairness (F)

$$f_i(\mathcal{H}) - f_i(\mathcal{H}\backslash\{H\}) = f_j(\mathcal{H}) - f_j(\mathcal{H}\backslash\{H\}) \text{ if } i, j \in H \in \mathcal{H}.$$

Balanced contribution (BC)

$$f_i(\mathcal{H}) - f_i(\mathcal{H}_{-j}) = f_j(\mathcal{H}) - f_j(\mathcal{H}_{-i}) \text{ for all } i, j \in N.$$

Component efficiency (CE) says that if S is a component of N, i.e., a maximal $\mathcal{H}$-connected coalition, then the members of S ought to allocate to themselves the total wealth $v(S)$ available to them. Fairness (F) says that all players in a conference gain equally from their agreement to form the conference. Balanced contribution (BC) says that player j's contribution to i always equals i's contribution to j. The next result (28) shows that BC implies F.

Lemma 35 *If an allocation rule satisfies BC then it satisfies F.*

To characterize an allocation rule satisfying the axioms above, Myerson (27, 28) considered a game $r^{\mathcal{H}}$ determined by the collection of $\mathcal{H}$-connected coalitions, which is defined as follows:

$$r^{\mathcal{H}}(S) = \sum_{T \in S/\mathcal{H}} v(T) \text{ for all } S \in 2^N. \tag{40}$$

The game $r^{\mathcal{H}}$ is called the *restricted game* of v. The following result, originally due to Myerson (27, 28) and later elaborated by van den Nouweland et al. (38), is fundamental.

Proposition 16 *The following three statements about an allocation rule f are equivalent.*

(i) *f satisfies CE and F.*

(ii) *f satisfies CE and BC.*

(iii) *$f(\mathcal{H})$ is the Shapley value of the restricted game $r^{\mathcal{H}}$. That is, $f_i(\mathcal{H}) = \phi_i(r^{\mathcal{H}})$ for all $i \in N$ and $\mathcal{H} \in \mathbf{CS}$.*

Since the restricted game $r^{\mathcal{H}}$ is uniquely determined from v and $\mathcal{H}$ by (40), each statement in Proposition 16 identifies a unique allocation rule. Especially, this proposition shows that there exists a unique allocation rule satisfying CE and F. This allocation rule is referred to as the Myerson value.

Note that $r^{\mathcal{H}} = r^{\mathrm{cn}(\mathcal{H})}$ because $S/\mathcal{H} = S/\operatorname{cn}(\mathcal{H})$. This means that the Myerson value assigns the same vector of payoffs to different conference structures as far as the collections of $\mathcal{H}$-connected conferences are the same, even if those of $\mathcal{H}$-complete conferences are distinct. In this sense, the Myerson value treats direct and indirect connections equally. For example, let $N = \{1,2,3,4\}$ and

$$\begin{aligned} \mathcal{H}^1 &= \{\{1,2\},\{2,3\},\{3,4\},\{1,2,3,4\}\}, \\ \mathcal{H}^2 &= \{\{1,2\},\{2,3\},\{3,4\},\{1,2,3\},\{1,2,3,4\}\}, \\ \mathcal{H}^3 &= \{\{1,2\},\{2,3\},\{3,4\},\{1,2,3\},\{2,3,4\},\{1,2,3,4\}\}. \end{aligned} \tag{41}$$

Since $\operatorname{cn}(\mathcal{H}^1) = \operatorname{cn}(\mathcal{H}^2) = \operatorname{cn}(\mathcal{H}^3) = \mathcal{H}^3$, the payoff allocations by the Myerson value are identical for all the above conference structures. On the other hand, we have $\mathcal{H}^1 = \operatorname{cm}(\mathcal{H}^1)$, $\mathcal{H}^2 = \operatorname{cm}(\mathcal{H}^2)$, and $\mathcal{H}^3 = \operatorname{cm}(\mathcal{H}^3)$. In the next section, we propose an allocation rule which distinguishes conference structures with distinct collections of $\mathcal{H}$-complete conferences.

5.4 The main result

Our motivation is similar to Myerson's but we are interested in an allocation rule based upon direct connections. We formalize this idea in terms of the following new axioms and replace CE with them.

Complete component efficiency (CCE)

$$\sum_{i \in S} f_i(\mathcal{H}) = v(S) \text{ if } S \in N/\mathcal{H} \text{ and } S \text{ is } \mathcal{H}\text{-complete.}$$

No contribution by unconnected players (NCU)

$$f_i(\mathcal{H}) = f_i(\mathcal{H}_{-j}) \text{ if } i, j \in N \text{ are not directly } \mathcal{H}\text{-connected in } N.$$

Complete component efficiency (CCE) is in the same spirit as Myerson's component efficiency. However, since we regard direct connections as basic units for communication, a component $S \in N/\mathcal{H}$ can function and allocate the total wealth $v(S)$ if S is $\mathcal{H}$-complete. To put it differently, if S is not $\mathcal{H}$-complete, there are some pairs in S who cannot directly meet, and thus an agreement for cooperation may not occur. Clearly, CE implies CCE, but not vice versa.

No contribution by unconnected players (NCU) implies that player i's payoff remains the same when all conferences containing j, who are not directly $\mathcal{H}$-connected with i, are removed.

In other words, player j's contribution to i equals zero. Note by symmetry that $f_i(\mathcal{H})-f_i(\mathcal{H}_{-j}) = f_j(\mathcal{H}) - f_j(\mathcal{H}_{-i})(=0)$, which is the special case of BC. It can be readily seen that the Myerson value does not satisfy NCU because it treats direct and indirect connections equally.

To characterize an allocation rule satisfying the axioms above, we consider a game determined by the collection of $\mathcal{H}$-complete coalitions. Write $\text{cm}^*(\mathcal{H}) = \text{cm}(\mathcal{H}) \cup \{\{i\} \mid i \in N\}$ for the collection of all $\mathcal{H}$-complete coalitions and let $\mu_{\text{cm}^*(\mathcal{H})}$ be the Möbius function of $\text{cm}^*(\mathcal{H})$. Define the following game $v^{\mathcal{H}}$, which we call the *direct-connection restricted* (*d-restricted*) *game* of v:

$$v^{\mathcal{H}} = \sum_{T\in 2^N} \beta_T^{\mathcal{H}} u_T \text{ where } \beta_T^{\mathcal{H}} = \begin{cases} \sum\limits_{S\in \text{cm}^*(\mathcal{H})} \mu_{\text{cm}^*(\mathcal{H})}(S,T)v(S) & \text{if } T \in \text{cm}^*(\mathcal{H}), \\ 0 & \text{if } T \notin \text{cm}^*(\mathcal{H}). \end{cases} \tag{42}$$

We will see in section 5.6 that the construction of $v^{\mathcal{H}}$ generalizes that of the restricted game $r^{\mathcal{H}}$. The following lemma provides a simple characterization of $v^{\mathcal{H}}$.

Lemma 36 *Let $w = \sum_{T\in 2^N} \gamma_T u_T$ be a game. Then, $w = v^{\mathcal{H}}$ if and only if $w(S) = v(S)$ for all $S \in \text{cm}^*(\mathcal{H})$ and $\gamma_T = 0$ for all $T \notin \text{cm}^*(\mathcal{H})$.*

Proof. Assume that $w(S) = v(S)$ for all $S \in \text{cm}^*(\mathcal{H})$ and $\gamma_T = 0$ for all $T \notin \text{cm}^*(\mathcal{H})$. Then,

$$w(S) = \sum_{T\in \text{cm}^*(\mathcal{H})} \gamma_T u_T(S) = v(S) \text{ for all } S \in \text{cm}^*(\mathcal{H}). \tag{43}$$

By Lemma 32 with $f(X) = \gamma_X$ and $v(X) = w(X)$ restricted to $\text{cm}^*(\mathcal{H})$, (43) is equivalent to

$$\gamma_T = \sum_{S\in \text{cm}^*(\mathcal{H})} \mu_{\text{cm}^*(\mathcal{H})}(S,T)v(S) \text{ for all } T \in \text{cm}^*(\mathcal{H}). \tag{44}$$

By (42) and (44), $\gamma_T = \beta_T^{\mathcal{H}}$ for all $T \in 2^N$ and thus $w = v^{\mathcal{H}}$. Conversely, assume that $w = v^{\mathcal{H}}$ and thus $\gamma_T = \beta_T^{\mathcal{H}}$ for all $T \in 2^N$. Then, (42) implies that $\gamma_T = 0$ for all $T \notin \text{cm}^*(\mathcal{H})$ and (44), the latter of which is equivalent to (43). Therefore, $w(S) = v(S)$ for all $S \in \text{cm}^*(\mathcal{H})$ and $\gamma_T = 0$ for all $T \notin \text{cm}^*(\mathcal{H})$. ∎

Now we are ready to state our main result, which characterizes an allocation rule satisfying CCE, NCU, and F.

Proposition 17 *The following four statements about an allocation rule f are equivalent.*

(i) *f satisfies CCE, NCU, and F.*

(ii) *f satisfies CCE, NCU, and BC.*

(iii) *$f(\mathcal{H})$ is the vector of the marginal contributions of a game $p^{\mathcal{H}}$ satisfying the following two conditions:*

$$\sum_{i \in S} \delta_i p^{\mathcal{H}}(S) = v(S) \text{ if } S \text{ is } \mathcal{H}\text{-complete.} \tag{45}$$

$$\delta_i p^{\mathcal{H}}(S) = \delta_i p^{\mathcal{H}}(S \backslash \{j\}) \text{ if } i, j \in S \text{ are not directly } \mathcal{H}\text{-connected in } S. \tag{46}$$

That is, $f_i(\mathcal{H}) = \delta_i p^{\mathcal{H}}(N)$ for all $i \in N$ and $\mathcal{H} \in \mathbf{CS}$.

(iv) *$f(\mathcal{H})$ is the Shapley value of the d-restricted game $v^{\mathcal{H}}$. That is, $f_i(\mathcal{H}) = \phi_i(v^{\mathcal{H}})$ for all $i \in N$ and $\mathcal{H} \in \mathbf{CS}$.*

Since the d-restricted game $v^{\mathcal{H}}$ is uniquely determined from v and $\mathcal{H}$ by (42), each statement in Proposition 17 identifies a unique allocation rule. Especially, this proposition shows that there exists a unique allocation rule satisfying CCE, NCU, and F. We call this allocation rule the *direct-connection Myerson (d-Myerson) value.*

Notice the resemblance between $p^{\mathcal{H}}$ in (iii) and the potential for v. The latter satisfies (38) for all coalitions, whereas the former satisfies it for all $\mathcal{H}$-complete coalitions, which is the condition (45). The other condition (46) requires that the marginal contribution of player i to $p^{\mathcal{H}}(S)$ be determined by players who are directly $\mathcal{H}$-connected in S with i. In both of the conditions, the direct $\mathcal{H}$-connected relation is essential. Note that if $\mathcal{H}$ is the finest conference structure (hence any coalition is $\mathcal{H}$-complete), then (45) is identical to (38), and (46) holds trivially because any pair of players are directly $\mathcal{H}$-connected in any coalition containing them. Thus in this case, $p^{\mathcal{H}}$ coincides with the potential for v by Proposition 15. As will be shown in Lemma 40 in the next section, $p^{\mathcal{H}}$ is the potential for $v^{\mathcal{H}}$, which will explain why the allocation rule is uniquely determined.

5.5 The proof

This section provides the proof of Proposition 17. It proceeds in the following order: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), (iii) $\Leftrightarrow$ (iv), and (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

5.5.1 (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)

As the next result shows, F and NCU together imply BC. Thus, if an allocation rule satisfies CCE, NCU, and F, then it satisfies CCE, NCU, and BC, establishing (i) $\Rightarrow$ (ii).

Lemma 37 *If an allocation rule f satisfies F and NCU, then it satisfies BC.*

Proof. If $i, j \in N$ are not directly $\mathcal{H}$-connected in N, then NCU implies that $f_i(\mathcal{H}) - f_i(\mathcal{H}_{-j}) = f_j(\mathcal{H}) - f_j(\mathcal{H}_{-i}) = 0$. If $i, j \in N$ are directly $\mathcal{H}$-connected in N, then write $\{H \in \mathcal{H} \,|\, \{i, j\} \subseteq$

$H\} = \{H_1, \ldots, H_k\}$. By applying F repeatedly, we have

$$f_i(\mathcal{H}) - f_i(\mathcal{H}\backslash\{H_1, \ldots, H_k\}) = f_j(\mathcal{H}) - f_j(\mathcal{H}\backslash\{H_1, \ldots, H_k\}). \tag{47}$$

Note that i and j are not directly $\mathcal{H}\backslash\{H_1, \ldots, H_k\}$-connected in N since $\mathcal{H}\backslash\{H_1, \ldots, H_k\} = \{H \in \mathcal{H} \mid \{i,j\} \not\subseteq H\}$. Thus, NCU implies that

$$f_i(\mathcal{H}\backslash\{H_1, \ldots, H_k\}) = f_i((\mathcal{H}\backslash\{H_1, \ldots, H_k\})_{-j}) = f_i(\mathcal{H}_{-j}) \tag{48}$$

where the latter equality holds because $(\mathcal{H}\backslash\{H_1, \ldots, H_k\})_{-j} = \{H \in \mathcal{H} \mid \{i,j\} \not\subseteq H \text{ and } j \notin H\} = \{H \in \mathcal{H} \mid j \notin H\} = \mathcal{H}_{-j}$. Similarly, it follows that $f_j(\mathcal{H}\backslash\{H_1, \ldots, H_k\}) = f_j(\mathcal{H}_{-i})$. By plugging this and (48) into (47), we have established BC. ∎

As noted by Hart and Mas-Colell (17), BC is a finite difference analogue of the Frobenious integrability condition, i.e., the symmetry of the cross partial derivatives, which suggests that the solution admits a potential. In fact, BC assures the existence of a "potential" in the following sense.[20]

Lemma 38 *If an allocation rule f satisfies BC, then, for each $\mathcal{H} \in \mathbf{CS}$, there exists a game $p^{\mathcal{H}}$ such that $f_i(\mathcal{H}_S) = \delta_i p^{\mathcal{H}}(S)$ for all $i \in S$ and $S \in 2^N$.*

Proof. Define a game $p^{\mathcal{H}}$ by the following rule: for each $S = \{i_1, \ldots, i_k\} \in 2^N$ with $i_1 < \cdots < i_k$, $p^{\mathcal{H}}(S) = \sum_{l=1}^{k} f_{i_l}(\mathcal{H}_{\{i_1,\ldots,i_l\}})$. Note that, by construction, if $i = \max S$ then $f_i(\mathcal{H}_S) = p^{\mathcal{H}}(S) - p^{\mathcal{H}}(S\backslash\{i\}) = \delta_i p^{\mathcal{H}}(S)$.

We show by induction that $f_i(\mathcal{H}_S) = \delta_i p^{\mathcal{H}}(S)$ for all $i \in S$ and $S \in 2^N$. If $|S| = 1$ and $S = \{i\}$, then $f_i(\mathcal{H}_{\{i\}}) = p^{\mathcal{H}}(\{i\}) - p^{\mathcal{H}}(\emptyset) = \delta_i p^{\mathcal{H}}(\{i\})$. Suppose as an induction hypothesis that $f_i(\mathcal{H}_S) = \delta_i p^{\mathcal{H}}(S)$ for all $i \in S$ and $S \in 2^N$ with $|S| \leq k < n$. Let $S = \{i_1, \ldots, i_{k+1}\} \in 2^N$ with $i_1 < \cdots < i_{k+1}$. For every $i \in S$, by applying BC (with $\mathcal{H}_S$ instead of $\mathcal{H}$), we have

$$\begin{aligned} f_i(\mathcal{H}_S) &= f_{i_{k+1}}(\mathcal{H}_S) - f_{i_{k+1}}((\mathcal{H}_S)_{-i}) + f_i((\mathcal{H}_S)_{-i_{k+1}}) \\ &= f_{i_{k+1}}(\mathcal{H}_S) - f_{i_{k+1}}(\mathcal{H}_{S\backslash\{i\}}) + f_i(\mathcal{H}_{S\backslash\{i_{k+1}\}}). \end{aligned} \tag{49}$$

By the construction of $p^{\mathcal{H}}$,

$$f_{i_{k+1}}(\mathcal{H}_S) = p^{\mathcal{H}}(S) - p^{\mathcal{H}}(S\backslash\{i_{k+1}\}). \tag{50}$$

By the induction hypothesis,

$$f_{i_{k+1}}(\mathcal{H}_{S\backslash\{i\}}) = p^{\mathcal{H}}(S\backslash\{i\}) - p^{\mathcal{H}}(S\backslash\{i, i_{k+1}\}), \tag{51}$$

$$f_i(\mathcal{H}_{S\backslash\{i_{k+1}\}}) = p^{\mathcal{H}}(S\backslash\{i_{k+1}\}) - p^{\mathcal{H}}(S\backslash\{i, i_{k+1}\}). \tag{52}$$

[20]Consider a vector-valued mapping $F : \mathbb{R}^n \to \mathbb{R}^n$. In vector analysis, a function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be a potential of F if $F = (\partial f/\partial x_i)_{i=1}^n$.

Plugging (50), (51), and (52) into (49), we have

$$f_i(\mathcal{H}_S) = p^{\mathcal{H}}(S) - p^{\mathcal{H}}(S\backslash\{i\}) = \delta_i p^{\mathcal{H}}(S),$$

which completes the proof. ∎

A "potential" in Lemma 38 is shown to satisfy (45) and (46) if an allocation rule satisfies CCE and NCU in addition.

Lemma 39 *Let an allocation rule f satisfy CCE and NCU. Suppose that there exists a game $p^{\mathcal{H}}$ such that $f_i(\mathcal{H}_S) = \delta_i p^{\mathcal{H}}(S)$ for all $i \in S$ and $S \in 2^N$. Then, $p^{\mathcal{H}}$ satisfies* (45) *and* (46).

Proof. Suppose that S is $\mathcal{H}$-complete. Then $S \in N/\mathcal{H}_S$ and S is $\mathcal{H}_S$-complete. By CCE, $\sum_{i\in S} f_i(\mathcal{H}_S) = \sum_{i\in S}\delta_i p^{\mathcal{H}}(S) = v(S)$. Therefore, $p^{\mathcal{H}}$ satisfies (45). To show that $p^{\mathcal{H}}$ satisfies (46), suppose that $i, j \in S$ are not directly $\mathcal{H}$-connected in S. Then, they are not directly $\mathcal{H}_S$-connected in N. Thus, by NCU, $\delta_i p^{\mathcal{H}}(S) = f_i(\mathcal{H}_S) = f_i((\mathcal{H}_S)_{-j}) = f_i(\mathcal{H}_{S\backslash\{j\}}) = \delta_i p^{\mathcal{H}}(S\backslash\{j\})$. Therefore, $p^{\mathcal{H}}$ satisfies (46). ∎

By Lemma 38 and Lemma 39, if an allocation rule f satisfies CCE, NCU, and BC, then there exists a game $p^{\mathcal{H}}$ satisfying (45) and (46) such that $f_i(\mathcal{H}) = f_i(\mathcal{H}_N) = \delta_i p^{\mathcal{H}}(N)$, which establishes (ii) $\Rightarrow$ (iii).

5.5.2 (iii) $\Leftrightarrow$ (iv)

We shall show below that a game $p^{\mathcal{H}}$ which satisfies the conditions in (iii) must be the potential for the d-restricted game $v^{\mathcal{H}}$. This suffices to establish (iii) $\Leftrightarrow$ (iv) since $\delta_i p^{\mathcal{H}}(N) = \phi_i(v^{\mathcal{H}})$ holds by Proposition 15.

Lemma 40 *There exists a unique game $p^{\mathcal{H}}$ satisfying* (45) *and* (46). *The game $p^{\mathcal{H}}$ coincides with the potential for the d-restricted game $v^{\mathcal{H}}$.*

Proof. We first show that the potential for $v^{\mathcal{H}}$ does satisfy (45) and (46). So let $p^{\mathcal{H}}$ be the potential for $v^{\mathcal{H}} = \sum_{T\in 2^N}\beta_T^{\mathcal{H}} u_T$. Then by Proposition 15, $p^{\mathcal{H}} = \sum_{T\in 2^N}(\beta_T^{\mathcal{H}}/|T|)u_T$. Observe that $\sum_{i\in S}\delta_i p^{\mathcal{H}}(S) = v^{\mathcal{H}}(S) = v(S)$ if S is $\mathcal{H}$-complete, where the first equality holds since $p^{\mathcal{H}}$ is the potential for $v^{\mathcal{H}}$ and the second equality holds by Lemma 36. This is the condition (45). Next, observe that, since $\beta_T^{\mathcal{H}} = 0$ for all $T \notin \mathrm{cm}^*(\mathcal{H})$,

$$\begin{aligned}
\delta_i p^{\mathcal{H}}(S) &= p^{\mathcal{H}}(S) - p^{\mathcal{H}}(S\backslash\{i\}) \\
&= \sum_{T\in \mathrm{cm}^*(\mathcal{H})_S} \beta_T^{\mathcal{H}}/|T| - \sum_{T\in \mathrm{cm}^*(\mathcal{H})_{S\backslash\{i\}}} \beta_T^{\mathcal{H}}/|T| \\
&= \sum_{T\in \mathrm{cm}^*(\mathcal{H})_S : i\in T} \beta_T^{\mathcal{H}}/|T|, \qquad (53)
\end{aligned}$$

and similarly,

$$\delta_i p^{\mathcal{H}}(S\backslash\{j\}) = \sum_{T\in \mathrm{cm}^*(\mathcal{H})_{S\backslash\{j\}}:i\in T} \beta_T^{\mathcal{H}}/|T|. \tag{54}$$

Now suppose that $i, j \in S$ are not directly $\mathcal{H}$-connected in S. Then, there is no $T \in \mathrm{cm}^*(\mathcal{H})_S$ such that $\{i,j\} \subseteq T$ because any pair of players in $T \in \mathrm{cm}^*(\mathcal{H})_S$ are directly $\mathcal{H}$-connected in T and thus in S. This implies that $\{T \in \mathrm{cm}^*(\mathcal{H})_S \mid i \in T\} = \{T \in \mathrm{cm}^*(\mathcal{H})_{S\backslash\{j\}} \mid i \in T\}$ and thus $\delta_i p^{\mathcal{H}}(S) = \delta_i p^{\mathcal{H}}(S\backslash\{j\})$ by (53) and (54). This is the condition (46).

To complete the proof, we show that a game $p^{\mathcal{H}}$ satisfying (45) and (46) is unique, by constructing $p^{\mathcal{H}}$ recursively such that in the k-th step we determine the unique value of $p^{\mathcal{H}}(S)$ with $|S| = k$ from $p^{\mathcal{H}}(S')$ with $|S'| \leq k-1$. Start with $p^{\mathcal{H}}(\emptyset) = 0$ since $p^{\mathcal{H}}$ is a game. Consider the k-th step with $k \geq 1$ and pick any S with $|S| = k$. Suppose that S is $\mathcal{H}$-complete. Then, (45) is rewritten as

$$p^{\mathcal{H}}(S) = n^{-1}\left(v(S) + \sum_{i\in S} p^{\mathcal{H}}(S\backslash\{i\})\right).$$

Since $p^{\mathcal{H}}(S\backslash\{i\})$ on the right hand side is uniquely calculated for each $i \in N$ in the previous step, so is $p^{\mathcal{H}}(S)$ on the left hand side. Suppose that S is not $\mathcal{H}$-complete. Then, there exist two distinct players $i, j \in S$ who are not directly $\mathcal{H}$-connected in S. So, by (46),

$$p^{\mathcal{H}}(S) = p^{\mathcal{H}}(S\backslash\{i\}) + p^{\mathcal{H}}(S\backslash\{j\}) - p^{\mathcal{H}}(S\backslash\{i,j\}). \tag{55}$$

Since the terms on the right hand side are uniquely calculated in the earlier steps, so is $p^{\mathcal{H}}(S)$ on the left hand side. Note that $p^{\mathcal{H}}(S)$ in (55) does not depend upon the choice of i and j because (55) holds for any $i, j \in S$ who are not directly $\mathcal{H}$-connected in S. By the above procedure, we can uniquely determine $p^{\mathcal{H}}$ recursively, which establishes the uniqueness. ∎

5.5.3 (iii) ⇒ (ii) ⇒ (i)

Recall that BC implies F by Lemma 35, which establishes (ii) ⇒ (i). To prove (iii) ⇒ (ii), we use the following lemma.

Lemma 41 *Let $p^{\mathcal{H}}$ be a game satisfying* (45) *and* (46) *for each $\mathcal{H} \in$* **CS**. *Then, $\delta_i p^{\mathcal{H}_S}(S) = \delta_i p^{\mathcal{H}}(S)$ for all $i \in S$ and $S \in 2^N$.*

Proof. By Lemma 40 and Proposition 15, $p^{\mathcal{H}} = \sum_{T\in 2^N} (\beta_T^{\mathcal{H}}/|T|) u_T$ where

$$\beta_T^{\mathcal{H}} = \begin{cases} \displaystyle\sum_{S\in \mathrm{cm}^*(\mathcal{H})} \mu_{\mathrm{cm}^*(\mathcal{H})}(S,T)v(S) & \text{if } T \in \mathrm{cm}^*(\mathcal{H}), \\ 0 & \text{if } T \notin \mathrm{cm}^*(\mathcal{H}). \end{cases}$$

Observe that if $T \subseteq R$ then $\mu_{\mathrm{cm}^*(\mathcal{H})}(S,T) = \mu_{\mathrm{cm}^*(\mathcal{H}_R)}(S,T)$. This is because the recursive construction of $\mu_{\mathrm{cm}^*(\mathcal{H})}(S,T)$ in (34) implies $\mu_{\mathrm{cm}^*(\mathcal{H})}(S,T) = \mu_{\mathrm{cm}^*(\mathcal{H})_R}(S,T)$ and the definition of $\mathcal{H}$-completeness implies $\mathrm{cm}^*(\mathcal{H})_R = \mathrm{cm}^*(\mathcal{H}_R)$. Therefore, $\beta_T^{\mathcal{H}} = \beta_T^{\mathcal{H}_R}$ if $T \subseteq R$ and thus $p^{\mathcal{H}}(S) = \sum_{T\subseteq S} \beta_T^{\mathcal{H}}/|T| = \sum_{T\subseteq S} \beta_T^{\mathcal{H}_R}/|T| = p^{\mathcal{H}_R}(S)$ if $S \subseteq R$. This implies that $\delta_i p^{\mathcal{H}_S}(S) = \delta_i p^{\mathcal{H}}(S)$. ∎

We are ready to establish (iii) $\Rightarrow$ (ii).

Lemma 42 *Let f be an allocation rule stated in* (iii). *Then, f satisfies CCE, NCU, and BC.*

Proof. If $i \neq j$, then they are not directly $\mathcal{H}_{-j}$-connected in N. Thus $f_i(\mathcal{H}_{-j}) = \delta_i p^{\mathcal{H}_{-j}}(N) = \delta_i p^{\mathcal{H}_{-j}}(N\backslash\{j\})$ by (46). By setting $S = N\backslash\{j\}$ in Lemma 41, we have $\delta_i p^{\mathcal{H}_{-j}}(N\backslash\{j\}) = \delta_i p^{\mathcal{H}}(N\backslash\{j\})$. Therefore, $f_i(\mathcal{H}_{-j}) = \delta_i p^{\mathcal{H}}(N\backslash\{j\})$, which implies BC because

$$\begin{aligned} f_i(\mathcal{H}) - f_i(\mathcal{H}_{-j}) &= \delta_i p^{\mathcal{H}}(N) - \delta_i p^{\mathcal{H}}(N\backslash\{j\}) \\ &= p^{\mathcal{H}}(N) - p^{\mathcal{H}}(N\backslash\{i\}) - p^{\mathcal{H}}(N\backslash\{j\}) + p^{\mathcal{H}}(N\backslash\{i,j\}) \\ &= \delta_j p^{\mathcal{H}}(N) - \delta_j p^{\mathcal{H}}(N\backslash\{i\}) \\ &= f_j(\mathcal{H}) - f_j(\mathcal{H}_{-i}). \end{aligned}$$

If $i, j \in N$ are not directly $\mathcal{H}$-connected in N, then $\delta_i p^{\mathcal{H}}(N) = \delta_i p^{\mathcal{H}}(N\backslash\{j\})$ by (46), and the above equation is reduced to $f_i(\mathcal{H}) - f_i(\mathcal{H}_{-j}) = 0$, which is NCU.

It remains to prove that f satisfies CCE. Let $S \in N/\mathcal{H}$ be $\mathcal{H}$-complete. We first show that $\delta_i p^{\mathcal{H}}(N) = \delta_i p^{\mathcal{H}}(S)$ for $i \in S$. Let $N\backslash S = \{j_1, \dots, j_m\}$ and $T_k = N\backslash\{j_1, \dots, j_k\}$ for $k = 1, \dots, m$. Since $i, j_1 \in N$ are not directly $\mathcal{H}$-connected in N, it holds that $\delta_i p^{\mathcal{H}}(N) = \delta_i p^{\mathcal{H}}(N\backslash\{j_1\}) = \delta_i p^{\mathcal{H}}(T_1)$ by (46). Similarly, since $i, j_k \in T_{k-1}$ are not directly $\mathcal{H}$-connected in T_{k-1}, it holds that $\delta_i p^{\mathcal{H}}(T_{k-1}) = \delta_i p^{\mathcal{H}}(T_{k-1}\backslash\{j_k\}) = \delta_i p^{\mathcal{H}}(T_k)$ by (46) for $k = 2, \dots, m$. Therefore, $\delta_i p^{\mathcal{H}}(N) = \delta_i p^{\mathcal{H}}(T_1) = \cdots = \delta_i p^{\mathcal{H}}(T_m) = \delta_i p^{\mathcal{H}}(S)$. Then, we have $\sum_{i\in S} f_i(\mathcal{H}) = \sum_{i\in S} \delta_i p^{\mathcal{H}}(N) = \sum_{i\in S} \delta_i p^{\mathcal{H}}(S) = v(S)$ by (45), which is CCE. ∎

5.6 Discussions

5.6.1 Characterization of $v^{\mathcal{H}}$

The d-restricted game $v^{\mathcal{H}}$, which is fundamental to our allocation rule, can be characterized in various ways, as stated in the following result.

Lemma 43 *Fix a game v and $\mathcal{H} \in$* **CS**. *The following four statements about a game $w = \sum_{T\in 2^N} \gamma_T u_T$ are equivalent.*

(i) $w = v^{\mathcal{H}}$, *i.e., w is the d-restricted game.*

(ii) $w(S) = v(S)$ *if S is $\mathcal{H}$-complete and $\gamma_T = 0$ if T is not $\mathcal{H}$-complete.*

(iii) $\{\gamma_T\}_{T\in 2^N}$ *is determined recursively by the following rule:*

1. $\gamma_{\{i\}} = v(\{i\})$ *for all* $i \in N$.
2. *For* $T \in 2^N$ *with* $|T| \geq 2$,
   - $\gamma_T = v(T) - \sum_{S\subset T}\gamma_S$ *if* T *is* $\mathcal{H}$*-complete,*
   - $\gamma_T = 0$ *if* T *is not* $\mathcal{H}$*-complete.*

(iv) w *satisfies the following two conditions:*

$$w(S) = v(S) \text{ if } S \text{ is } \mathcal{H}\text{-complete.} \tag{56}$$

$$\delta_i w(S) = \delta_i w(S\backslash\{j\}) \text{ if } i,j \in S \text{ are not directly } \mathcal{H}\text{-connected in } S. \tag{57}$$

Proof. Lemma 36 established (i) $\Leftrightarrow$ (ii). So we prove (ii) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iv).

(ii) $\Leftrightarrow$ (iii): The rule in (iii) is rewritten as the condition that if S is $\mathcal{H}$-complete then $v(S) = \sum_{T\subseteq S}\gamma_T = w(S)$ and if T is not $\mathcal{H}$-complete then $\gamma_T = 0$, which is (ii).

(ii) $\Leftrightarrow$ (iv): Let w be as stated in (ii). Then, the condition (56) is obviously satisfied. If $i,j \in S$ are not directly $\mathcal{H}$-connected in S, then, as shown in the proof of Lemma 40, we have $\{T \in \mathrm{cm}^*(\mathcal{H})_S \,|\, i \in T\} = \{T \in \mathrm{cm}^*(\mathcal{H})_{S\backslash\{j\}} \,|\, i \in T\}$. Since $\gamma_T = 0$ for all $T \notin \mathrm{cm}^*(\mathcal{H})$, a calculation similar to (53) and (54) shows that

$$\delta_i w(S) = \sum_{T\in \mathrm{cm}^*(\mathcal{H})_S : i\in T}\gamma_T = \sum_{T\in \mathrm{cm}^*(\mathcal{H})_{S\backslash\{j\}} : i\in T}\gamma_T = \delta_i w(S\backslash\{j\}),$$

which is the condition (57).[21] Thus, (ii) implies (iv).

Suppose that w satisfies the conditions in (iv). To prove that (iv) implies (ii), it suffices to show that w is uniquely determined because $v^{\mathcal{H}}$ is the unique game that satisfies the conditions in (ii) by Lemma 36, which also satisfies the conditions in (iv) as discussed above. To show the uniqueness, we construct w recursively such that in the k-th step we determine the unique value of $w(S)$ with $|S| = k$ from $w(S')$ with $|S'| \leq k-1$. Start with $w(\emptyset) = 0$. Consider the k-th step with $k \geq 1$ and pick any S with $|S| = k$. If S is $\mathcal{H}$-complete, then $w(S) = v(S)$ by (56). If S is not $\mathcal{H}$-complete, then there exist $i,j \in S$ who are not directly $\mathcal{H}$-connected in S and so, by (57),

$$w(S) = w(S\backslash\{i\}) + w(S\backslash\{j\}) - w(S\backslash\{i,j\}). \tag{58}$$

Since the terms on the right hand side are uniquely calculated in the earlier steps, so is $w(S)$ on the left hand side. Note that $w(S)$ in (58) does not depend upon the choice of i and j because (58) holds for any $i,j \in S$ who are not directly $\mathcal{H}$-connected in S. By the above procedure, we can uniquely determine w recursively, which establishes the uniqueness. ∎

[21]?) considered a condition similar to (57) and called it modularity for $\mathcal{H}$-decomposition pairs. They showed that $\gamma_T = 0$ for all $T \notin \mathrm{cm}^*(\mathcal{H})$ if and only if w is modular for $\mathcal{H}$-decomposition pairs. It can be readily shown that (57) and modularity for $\mathcal{H}$-decomposition pairs are equivalent.

5.6.2 The Myerson value and the d-Myerson value

We shall first relate the d-restricted game $v^{\mathcal{H}}$ to the restricted game $r^{\mathcal{H}}$, which is done in the following result.

Lemma 44 *For each $\mathcal{H} \in \mathbf{CS}$, it holds that $v^{\mathrm{cn}(\mathcal{H})} = r^{\mathcal{H}}$.*

Proof. We prove that

$$v^{\mathrm{cn}(\mathcal{H})}(S) = \sum_{T \in 2^N} \beta_T^{\mathrm{cn}(\mathcal{H})} u_T(S) = \sum_{T \in S/\mathcal{H}} v(T) \text{ for all } S \in 2^N. \tag{59}$$

Let us write $\mathrm{cn}^*(\mathcal{H}) = \mathrm{cn}(\mathcal{H}) \cup \{\{i\} \mid i \in N\}$, which is the collection of all $\mathcal{H}$-connected coalitions. Note that $\beta_T^{\mathrm{cn}(\mathcal{H})} = 0$ for all $T \notin \mathrm{cm}^*(\mathrm{cn}(\mathcal{H}))$ by Lemma 36. Since $\mathrm{cm}(\mathrm{cn}(\mathcal{H})) = \mathrm{cn}(\mathcal{H})$ by Lemma 34, it follows that $\mathrm{cm}^*(\mathrm{cn}(\mathcal{H})) = \mathrm{cm}(\mathrm{cn}(\mathcal{H})) \cup \{\{i\} \mid i \in N\} = \mathrm{cn}(\mathcal{H}) \cup \{\{i\} \mid i \in N\} = \mathrm{cn}^*(\mathcal{H})$. Thus, for each $S \in 2^N$,

$$v^{\mathrm{cn}(\mathcal{H})}(S) = \sum_{T \subseteq S} \beta_T^{\mathrm{cn}(\mathcal{H})} = \sum_{T \in \mathrm{cm}^*(\mathrm{cn}(\mathcal{H}))_S} \beta_T^{\mathrm{cn}(\mathcal{H})} = \sum_{T \in \mathrm{cn}^*(\mathcal{H})_S} \beta_T^{\mathrm{cn}(\mathcal{H})}. \tag{60}$$

Observe that each $T \in \mathrm{cn}^*(\mathcal{H})_S$ is a $\mathcal{H}$-connected coalition contained in S and thus there exists $R \in S/\mathcal{H}$ such that $T \subseteq R$ because any pair of players in T are $\mathcal{H}$-connected in S and thus they are $\mathcal{H}$-connected in the component of S to which they both belong. Note that such $R \in S/\mathcal{H}$ is unique. Hence we have

$$\sum_{T \in \mathrm{cn}^*(\mathcal{H})_S} \beta_T^{\mathrm{cn}(\mathcal{H})} = \sum_{R \in S/\mathcal{H}} \left(\sum_{T \subseteq R} \beta_T^{\mathrm{cn}(\mathcal{H})} \right) = \sum_{R \in S/\mathcal{H}} v^{\mathrm{cn}(\mathcal{H})}(R). \tag{61}$$

Observe that $R \in S/\mathcal{H}$ is $\mathrm{cn}(\mathcal{H})$-complete because if R is a singleton then it is so by definition and if $|R| \geq 2$ then any $i, j \in R$ are $\mathcal{H}$-connected in R and thus $R \in \mathrm{cn}(\mathcal{H}) = \mathrm{cm}(\mathrm{cn}(\mathcal{H}))$ by Lemma 34, which with Lemma 36 implies that

$$v^{\mathrm{cn}(\mathcal{H})}(R) = v(R). \tag{62}$$

By (60), (61), and (62), we have (59), completing the proof. ∎

This lemma implies that the Shapley value of $r^{\mathcal{H}}$ and that of $v^{\mathrm{cn}(\mathcal{H})}$ coincide. Therefore, by Proposition 16 and Proposition 17, a payoff vector of the Myerson value for $\mathcal{H}$ coincides with that of the d-Myerson value for the collection of all $\mathcal{H}$-connected conferences.

Lemma 45 *Let f^M be the Myerson value and f^{dM} be the d-Myerson value. Then,*

$$f^M(\mathcal{H}) = f^{dM}(\mathrm{cn}(\mathcal{H})) \textit{ for all } \mathcal{H} \in \mathbf{CS}.$$

By this lemma, the d-Myerson value can be regarded as a refinement of the Myerson value. To illustrate this, we study a simple example. Consider again $\mathcal{H}^1, \mathcal{H}^2$, and $\mathcal{H}^3$ specified in (41) with $N = \{1,2,3,4\}$ and define a game v by

$$v = 3u_{\{1,2,3\}} - 3u_{\{2,3,4\}} + 4u_{\{1,2,3,4\}}.$$

Using the construction method (iii) of Lemma 43, we have $v^{\mathcal{H}^1} = 4u_{\{1,2,3,4\}}$, $v^{\mathcal{H}^2} = 3u_{\{1,2,3\}} + u_{\{1,2,3,4\}}$, and $v^{\mathcal{H}^3} = v$. The payoff vectors given by the d-Myerson value can be found by calculating the Shapley value of these games, and using the formula (37), we have them in the following table.

	player 1	players 2 and 3	player 4
$f^{dM}(\mathcal{H}^1)$	1	1	1
$f^{dM}(\mathcal{H}^2)$	5/4	5/4	1/4
$f^{dM}(\mathcal{H}^3)$	2	1	0

Note that $f^M(\mathcal{H}^1) = f^M(\mathcal{H}^2) = f^M(\mathcal{H}^3) = f^{dM}(\mathcal{H}^3)$ by Lemma 45. To appreciate the implication of these numbers, consider the following story. For each conference in $\mathcal{H}^1$, there are hotlines connecting its members, but indirect communication is very costly initially. Then the cost of indirect communication among players in $\{1,2,3\}$ is drastically reduced but not that in $\{2,3,4\}$. The resulting conference structure is thus $\mathcal{H}^2$. Under the Myerson value, players' shares are unchanged. But note that $f_i^{dM}(\mathcal{H}^2) > f_i^{dM}(\mathcal{H}^1)$ for $i \in \{1,2,3\}$; that is, addition of a conference $\{1,2,3\}$ to $\mathcal{H}^1$, i.e., reduction of communication cost among players in $\{1,2,3\}$, increases the payoffs of these players. Reduction of communication cost should not necessarily lead to improvement of payoffs in general, which is the case in this example too. Note that $f_i^{dM}(\mathcal{H}^2) > f_i^{dM}(\mathcal{H}^3)$ for $i \in \{2,3,4\}$; that is, deletion of a conference $\{2,3,4\}$ from $\mathcal{H}^3$ increases the payoffs of players in $\{2,3,4\}$.

Let us conclude with a final remark on the comparison of the two allocation rules. In some applications, it may make sense to require $\sum_{i\in S} f_i(\mathcal{H}) = v(S)$ to hold for all $S \in N/\mathcal{H}$. The d-Myerson value, however, do not satisfy CE and thus $\sum_{i\in S} f_i(\mathcal{H}) > v(S)$ is certainly possible for $S \in N/\mathcal{H}$ which is not $\mathcal{H}$-complete. Our suggestion to avoid this difficulty is simple: adopt $f(\mathcal{H} \cup (N/\mathcal{H}))$ as the payoff vector for $\mathcal{H}$ instead of $f(\mathcal{H})$. That is, every element of $N/\mathcal{H}$ is considered to be directly connected. Since each $S \in N/\mathcal{H}$ is $(\mathcal{H} \cup (N/\mathcal{H}))$-complete and $S \in N/(\mathcal{H} \cup (N/\mathcal{H}))$, it holds that $\sum_{i\in S} f_i(\mathcal{H} \cup (N/\mathcal{H})) = v(S)$ for all $S \in N/\mathcal{H}$ by CCE. We believe that this is not an ad hoc treatment because $\sum_{i\in S} f_i(\mathcal{H}) = v(S)$ implies that players in S can cooperate and thus it is natural to add S to $\mathcal{H}$. Note that, for the Myerson value f^M, it holds that $f^M(\mathcal{H} \cup (N/\mathcal{H})) = f^M(\mathcal{H})$ for all $\mathcal{H} \in \mathbf{CS}$ because $N/\mathcal{H} \subseteq \mathrm{cn}(\mathcal{H})$ and thus $\mathrm{cn}(\mathcal{H} \cup (N/\mathcal{H})) = \mathrm{cn}(\mathcal{H})$. So, it is also of interest to compare $f^M(\mathcal{H})$ and $f(\mathcal{H} \cup (N/\mathcal{H}))$. In the above example, we have $f(\mathcal{H}^k \cup (N/\mathcal{H}^k)) = f(\mathcal{H}^k)$ for each k because $\mathcal{H}^k = \mathcal{H}^k \cup (N/\mathcal{H}^k)$ holds.

5.6.3 Network games and d-restricted games

Let $\mathbf{G}$ be the collection of conference structures each conference of which contains exactly two players:

$$\mathbf{G} = \{\mathcal{G} \in \mathbf{CS} \mid |L| = 2 \text{ for all } L \in \mathcal{G}\}.$$

Each $\mathcal{G} \in \mathbf{G}$ is regarded as a network because $(N, \mathcal{G})$ is an undirected graph with a vertex set N and an edge set $\mathcal{G}$.

Jackson and Wolinsky (19) called a function $V : \mathbf{G} \to \mathbb{R}$ a *network game* where $V(\mathcal{G})$ is the total wealth when the network $\mathcal{G} \in \mathbf{G}$ is formed. They considered an allocation rule $f : \mathbf{G} \to \mathbb{R}^N$ given by the Shapley value of a game w satisfying $w(S) = V(\mathcal{G}_S)$ for all $S \in 2^N$. They called this allocation rule the Myerson value for network games and gave a characterization similar to that of Myerson (27).

The following result shows that a special class of network games are represented in terms of the d-restricted game $v^{\mathcal{G}}$.

Lemma 46 *Let $V : \mathbf{G} \to \mathbb{R}$ be a network game with $V(\emptyset) = 0$. Suppose that, for each $\mathcal{G} \in \mathbf{G}$, it holds that*

$$V(\mathcal{G}_S) - V(\mathcal{G}_{S\setminus\{i\}}) = V(\mathcal{G}_{S\setminus\{j\}}) - V(\mathcal{G}_{S\setminus\{i,j\}}) \text{ if } \{i,j\} \subseteq S \text{ and } \{i,j\} \notin \mathcal{G}. \tag{63}$$

Then, there exists a game v such that

$$v^{\mathcal{G}}(S) = V(\mathcal{G}_S) \text{ for all } S \in 2^N \text{ and } \mathcal{G} \in \mathbf{G}. \tag{64}$$

Proof. Let v be a game such that

$$v(S) = V(\{\{i,j\} \mid \{i,j\} \subseteq S\}) \text{ for all } S \in 2^N. \tag{65}$$

Fix $\mathcal{G} \in \mathbf{G}$. Let w be a game such that $w(S) = V(\mathcal{G}_S)$. If S is $\mathcal{G}$-complete, then $\mathcal{G}_S = \{\{i,j\} \mid \{i,j\} \subseteq S\}$ and thus $w(S) = V(\{\{i,j\} \mid \{i,j\} \subseteq S\}) = v(S)$. If $i, j \in S$ are not directly $\mathcal{G}$-connected in S, then $\{i,j\} \notin \mathcal{G}$ and $\delta_i w(S) = \delta_i w(S\setminus\{j\})$ by (63). Thus w satisfies the conditions (56) and (57). The equivalence of (i) and (iv) in Lemma 43 implies $w = v^{\mathcal{G}}$, which completes the proof. ∎

The condition (63) says that if players i and j contained in S are not linked in the network $\mathcal{G}$ then the marginal contribution of i to $V(\mathcal{G}_S)$ equals that to $V(\mathcal{G}_{S\setminus\{j\}})$. The above lemma implies that if a network game V satisfies (63) for all $\mathcal{G} \in \mathbf{G}$ then the Myerson value for V coincides with the d-Myerson value of v give by (65). For example, consider a network game V defined by

$$V(\mathcal{G}) = \sum_{L \in \mathcal{G}} w_L \text{ for all } \mathcal{G} \in \mathbf{G}$$

where $w_L \in \mathbb{R}$ is a constant. It is easy to check that V satisfies (63) for all $\mathcal{G} \in \mathbf{G}$ and that (64) holds for a game v such that

$$v(S) = \sum_{L \subseteq S} w_L \text{ for all } S \in 2^N.$$

5.7 Concluding remarks

This paper has proposed and axiomatized the d-Myerson value as a refinement of the Myerson value. In so doing, we have introduced the two new axioms, CCE and NCU, and the d-restricted game; the axiomatization of the d-Myerson value is done by replacing the CE axiom in that of the Myerson value with the CCE and NCU axioms, and the d-Myerson value is shown to coincide with the Shapley value of the d-restricted game in place of the restricted game in the Myerson value. As concluding remarks, we point out other possible applications of the CCE and NCU axioms and the d-restricted game.

The position value (26; 5) and the Hamiache value (16) are allocation rules defined on the collection of networks **G**. Later, these allocation rules are extended to those on the collection of conference structures **CS** (38; 4). Both of them also treat direct and indirect connections equally because the position value is defined in terms of the restricted game and the Hamiache value is axiomatized in terms of the CE axiom. So, it is natural to consider refinements of these allocation rules respecting differences of direct and indirect connections, as we did for the Myerson value. We speculate that the CCE and NCU axioms and the d-restricted game might be employed instead of the CE axiom in the Hamiache value and the restricted game in the position value.

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