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21. A Short Article on the Second and Lower Degree Terms in the Ellipsoidal Harmonic Expansion of the Gravitational Potential of an Ellipsoid.

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Abstract

The gravitational potential of an ellipsoid of revolution can be expressed by a sum of low degree terms in ellipsoidal harmonics. However, the derivation methods which have already been introduced are somewhat complicated. This short article introduces a simpler method using a series of the ellipsoidal harmonic expansions.

Picture an ellipsoid of revolution whose semimajor and semiminor axes are a and b, respectively. The gravitational potential of the ellipsoid at an external point P can be expressed as an integral over the total volume v of the ellipsoid:

$$V=G\iiint_v \frac{\mathrm{d}M}{l} \tag{1}$$

In the above equation, G is Newton's gravitational constant and l the distance between P and the variable mass element dM.

The ellipsoidal coordinates (u, β, λ) are related to the rectangular coordinates (x, y, z) by the equations:

$$\left.\begin{aligned} x&=\sqrt{u^2+\varepsilon^2}\ \cos\beta\ \cos\lambda \\ y&=\sqrt{u^2+\varepsilon^2}\ \cos\beta\ \sin\lambda \\ z&=u\sin\beta \end{aligned}\right\} \tag{2}$$

where

$$\varepsilon=\sqrt{a^2-b^2} \tag{3}$$

is the focal distance. The coordinate u is the semiminor axis, β the reduced latitude and λ the geocentric longitude. The equation $u=b$ represents an ellipsoid of revolution having semiaxes of a and b.

According to HOBSON (1931), the reciprocal of l is expanded into a series of ellipsoidal harmonics with the associated Legendre function of the first kind P_n^m and the second kind Q_n^m, of degree n and order m:

$$\frac{1}{l}=\frac{2i}{\varepsilon}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m(2n+1)}{\varepsilon_m}\left\{\frac{(n-m)!}{(n+m)!}\right\}^2 Q_n^m\left(\frac{iu}{\varepsilon}\right)$$

$$\cdot P_n^m\left(\frac{iu'}{\varepsilon}\right)P_n^m(\sin\beta)P_n^m(\sin\beta')\cos m(\lambda-\lambda'), \tag{4}$$

where (u', β', λ') are the ellipsoidal coordinates of the mass element dM and

$$\varepsilon_m=\begin{cases} 2 & \text{for } m=0 \\ 1 & \text{for } m\neq 0. \end{cases} \tag{5}$$

Substitution of (4) into (1) gives

$$V=\frac{GM}{b}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{Q_n^m(iu/\varepsilon)}{Q_n^m(ib/\varepsilon)}(C_n^m\cos m\lambda+S_n^m\sin m\lambda)P_n^m(\sin\beta), \tag{6}$$

where M is the mass of the ellipsoid and C_n^m and S_n^m are the constant coefficients to be determined from

$$\binom{C_n^m}{S_n^m}=\frac{2ib(-1)^m(2n+1)}{M\varepsilon\varepsilon_m}Q_n^m\left(\frac{ib}{\varepsilon}\right)\left\{\frac{(n-m)!}{(n+m)!}\right\}^2$$

$$\cdot\iiint_v\binom{\cos m\lambda'}{\sin m\lambda'}P_n^m\left(\frac{iu'}{\varepsilon}\right)P_n^m(\sin\beta')dM. \tag{7}$$

by using the orthogonality relation of spherical harmonics. This equation indicates the fact that these coefficients can be determined from the mass distribution inside the ellipsoid.

The shape of an ellipsoid of revolution is rotational symmetry, so that V does not depend on λ. Therefore, with all non-zonal terms of V removed, (6) then becomes

$$V=\frac{GM}{b}\sum_{n=0}^{\infty}\frac{Q_n(iu/\varepsilon)}{Q_n(ib/\varepsilon)}C_nP_n(\sin\beta). \tag{8}$$

Accordingly, (7) can be rewritten as

$$C_n=\frac{ib(2n+1)}{M\varepsilon}Q_n\left(\frac{ib}{\varepsilon}\right)\iiint_v P_n\left(\frac{iu'}{\varepsilon}\right)P_n(\sin\beta')\,dM. \tag{9}$$

In the above equations, the zonal terms are usually denoted by C_n, P_n and Q_n instead of C_n^0, P_n^0 and Q_n^0, respectively.

Zero-Degree Term

In the special case $n=m=0$, (9) becomes

$$C_0 = \frac{ib}{M\varepsilon} Q_0\left(\frac{ib}{\varepsilon}\right) \iiint_v \mathrm{d}M.$$

From the known relations

$$Q_0\left(\frac{ib}{\varepsilon}\right) = -i \tan^{-1}\frac{\varepsilon}{b}.$$

and

$$M = \iiint_v \mathrm{d}M,$$

we obtain

$$C_0 = \frac{b}{\varepsilon} \tan^{-1}\frac{\varepsilon}{b}. \tag{10}$$

Therefore, the zero-degree term of V

$$V_0 = \frac{GM}{\varepsilon} \tan^{-1}\frac{\varepsilon}{u} \tag{11}$$

is easily verified by straightforward computation from (8). Furthermore, it can be proved that V_0 converges to GM/u when ε/u is very small. Although (11) has already been introduced by MOLODENSKII *et al.* (1962) and HEISKANEN and MORITZ (1967), their derivation methods are more complicated.

First-Degree Terms

The rectangular coordinates of the gravity center of the ellipsoid (x_0, y_0, z_0) are expressed as

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \frac{1}{M} \iiint_v \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \mathrm{d}M$$

$$= \frac{1}{M} \iiint_v \begin{pmatrix} \sqrt{u'^2+\varepsilon^2} \cos\beta' \cos\lambda' \\ \sqrt{u'^2+\varepsilon^2} \cos\beta' \sin\lambda' \\ u' \sin\beta' \end{pmatrix} \mathrm{d}M. \tag{12}$$

Meanwhile, coefficients of three first-degree terms C_1, C_1^1 and S_1^1 are obtained from (7). Taking (12) into consideration, we get

$$\left.\begin{aligned} C_1 &= -\frac{3b}{\varepsilon^2} z_0 Q_1\left(\frac{ib}{\varepsilon}\right) \\ C_1^1 &= \frac{3b}{2\varepsilon^2} x_0 Q_1^1\left(\frac{ib}{\varepsilon}\right) \\ S_1^1 &= \frac{3b}{2\varepsilon^2} y_0 Q_1^1\left(\frac{ib}{\varepsilon}\right) \end{aligned}\right\} \tag{13}$$

These relations indicate that the first-degree terms correspond to the coordinates of the gravity center of the ellipsoid. When ε is very small compared to b, $Q_n^m(ib/\varepsilon)$ converges to

$$Q_n^m\left(\frac{ib}{\varepsilon}\right)\approx\frac{(-1)^m n!(n+m)!}{2(2n+1)!}\left(\frac{2\varepsilon}{ib}\right)^{n+1} \tag{14}$$

for a very large value of b/ε, so that (13) is approximated to

$$\left.\begin{aligned} C_1 &\approx \frac{z_0}{b} \\ C_1^1 &\approx \frac{x_0}{b} \\ S_1^1 &\approx \frac{y_0}{b} \end{aligned}\right\} \tag{15}$$

The gravity center of an ellipsoid, in general, is set at the origin of the coordinates. In other words, (x_0, y_0, z_0) are set zero, and hence the three first-degree terms become zero.

Second-Degree Terms

Moments of inertia with respect to the x, y and z-axes are denoted by A, B and C, respectively. They are given by

$$\begin{aligned}\begin{pmatrix} A \\ B \\ C \end{pmatrix} &= \iiint_v \begin{pmatrix} y'^2+z'^2 \\ z'^2+x'^2 \\ x'^2+y'^2 \end{pmatrix} \mathrm{d}M \\ &= \iiint_v \begin{pmatrix} (u'^2+\varepsilon^2)\cos^2\beta'\ \sin^2\lambda'+u'^2\ \sin^2\beta' \\ u'^2\ \sin^2\beta'+(u'^2+\varepsilon^2)\ \cos^2\beta'\ \cos^2\lambda' \\ (u'^2+\varepsilon^2)\ \cos^2\beta' \end{pmatrix} \mathrm{d}M\,. \end{aligned} \tag{16}$$

Meanwhile, C_2 and C_2^2 are obtained from (7) as follows:

$$C_2=\frac{5ib}{4M\varepsilon^3}\,Q_2\left(\frac{ib}{\varepsilon}\right)\iiint_v (3u'^2+\varepsilon^2)(1-3\sin^2\beta')\,\mathrm{d}M$$

and

$$C_2^2=\frac{5ib}{32M\varepsilon^3}\,Q_2^2\left(\frac{ib}{\varepsilon}\right)\iiint_v (u'^2+\varepsilon^2)\cos^2\beta'\ \cos 2\lambda'\,\mathrm{d}M\,.$$

Our purpose here is to describe these two coefficients in terms of A, B and C. Using algebra, it can be proved that C_2 and C_2^2 become

$$C_2=-\frac{5ib}{2M\varepsilon^3}\,Q_2\left(\frac{ib}{\varepsilon}\right)\left\{M\varepsilon^2+3\left(\frac{A+B}{2}-C\right)\right\} \tag{17}$$

and

$$C_2^2 = \frac{5ib}{32M\varepsilon^3}(B-A)Q_2^2\left(\frac{ib}{\varepsilon}\right), \tag{18}$$

respectively. For a very large value of b/ε, (17) and (18) can be approximated to

$$C_2 \approx \frac{A+B-2C}{2Mb^2} \tag{19}$$

and

$$C_2^2 \approx \frac{B-A}{4Mb^2} \tag{20}$$

As an ellipsoid of revolution is rotational symmetry, the moment of inertia of the x-axis is equal to that of the y-axis, *i. e.* $A=B$. In this case

$$C_2 = -\frac{5ib}{2M\varepsilon^3}Q_2\left(\frac{ib}{\varepsilon}\right)\{M\varepsilon^2+3(A-C)\} \tag{21}$$

$$\approx \frac{A-C}{Mb^2}$$

$$C_2^2 \approx 0 \tag{22}$$

Furthermore, the product of inertia with respect to the x, y and z-axes are defined as

$$\begin{pmatrix} D \\ E \\ F \end{pmatrix} = \iiint_v \begin{pmatrix} y'z' \\ z'x' \\ x'y' \end{pmatrix} \mathrm{d}M$$

$$= \iiint_v \begin{pmatrix} u'\sqrt{u'^2+\varepsilon^2}\sin\beta'\cos\beta'\sin\lambda' \\ u'\sqrt{u'^2+\varepsilon^2}\sin\beta'\cos\beta'\cos\lambda' \\ (u'^2+\varepsilon^2)\cos^2\beta'\sin\lambda'\cos\lambda' \end{pmatrix} \mathrm{d}M. \tag{23}$$

Then C_2^1 and S_2^1 can easily be expressed in terms of E and D, respectively, by means of (7) and (23), such as

$$\begin{pmatrix} C_2^1 \\ S_2^1 \end{pmatrix} = \frac{5ib}{2M\varepsilon^3}Q_2^1\left(\frac{ib}{\varepsilon}\right)\iiint_v \begin{pmatrix} \cos\lambda' \\ \sin\lambda' \end{pmatrix} u'\sqrt{u'^2+\varepsilon^2}\sin\beta'\cos\beta'\,\mathrm{d}M$$

$$= \frac{5ib}{2M\varepsilon^3}Q_2^1\left(\frac{ib}{\varepsilon}\right)\begin{pmatrix} E \\ D \end{pmatrix}. \tag{24}$$

Similarly, S_2^2 is expressed in terms of F, *i. e.*

$$S_2^2 = -\frac{5ib}{16M\varepsilon^3}FQ_2^2\left(\frac{ib}{\varepsilon}\right). \tag{25}$$

Using the approximate formula (14) for a very large value of b/ε, these three coefficients can be approximated to

$$\begin{pmatrix} C_2^1 \\ S_2^1 \\ S_2^2 \end{pmatrix} \approx \frac{1}{Mb^2} \begin{pmatrix} E \\ D \\ F/2 \end{pmatrix}. \tag{26}$$

D and E are zero if the coordinate axes coincide with the principal axes of inertia of the ellipsoid. F also becomes zero when $A=B$ holds on, *i. e.* the ellipsoid has rotational symmetry. Hence

$$C_2^1 = S_2^1 = S_2^2 = 0\,. \tag{27}$$

In the five second-degree coefficients, only C_2 remains non-zero as mentioned above. Accordingly, the second-degree term of V can be explicitly written in the form:

$$V_2 = -\frac{5iGM}{2\varepsilon^3} Q_2\left(\frac{iu}{\varepsilon}\right) P_2(\sin\beta) \left\{\varepsilon^2 + \frac{3(A-C)}{M}\right\}. \tag{28}$$

Principal Moments of Inertia

The normal gravity potential of an ellipsoid of revolution, which rotates around the z-axis with an angular velocity of ω, is the sum of the gravitational potential of the ellipsoid and the centrifugal potential. The shape of the ellipsoid of revolution, whose semiaxes are a and b, is an equipotential surface which is defined as the sum of these potentials to be constant. HEISKANEN and MORITZ (1967) formulated the normal gravity potential by using Stokes' constants a, b, GM and ω.

According to HEISKANEN and MORITZ's results, the gravitational potential of the ellipsoid is given by

$$V = \frac{GM}{\varepsilon} \tan^{-1}\frac{\varepsilon}{u} + \frac{\omega^2 a^2 Q_2(iu/\varepsilon)}{3Q_2(ib/\varepsilon)} P_2(\sin\beta) \tag{29}$$

The first term of the rigthand members of (29) is quite similar to the zero-degree term (11), and the second term corresponds to the second-degree term (28). A comparison of (28) with the second term of the righthand members of (29) gives

$$-\frac{5iGM}{2\varepsilon^3}\left\{\varepsilon^2 + \frac{3(A-C)}{M}\right\} Q_2\left(\frac{ib}{\varepsilon}\right) = \frac{\omega^2 a^2}{3}.$$

From the above relation we can derive an equation for the pricipal moments of inertia:

$$\frac{C-A}{M}=\frac{\varepsilon^2}{3}\left\{1+\frac{2me'}{15iQ_2(ib/\varepsilon)}\right\}, \tag{30}$$

where

$$m=\frac{\omega^2a^2b}{GM}$$

and the second eccentricity

$$e'=\frac{\varepsilon}{b}.$$

The equation (30) is also introduced by HEISKANEN and MORITZ (1967) in the mathematical process of expanding the normal gravity potential in spherical harmonics.

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21. 楕円体の引力ポテンシャル低次項に関するノート

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楕円体の引力ポテンシャルのスフェロイダル座標による取扱いは MOLODENSKII et al. (1962) や HEISKANEN and MORITZ (1967) に体系的記載がある．しかしこれらの数学的導出法はかなり繁雑である．ところが，スフェロイダル調和関数によって2点間の距離の逆数を展開した HOBSON (1931) の式（本文中（4）式）を用いると，数学的記述がかなり簡単化されるのみならず，その物理学的意味が判りやすい．