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26. *Transformation of an Elastic Wave Solution related to Translation of the Origin of the Polar Coordinate System.*

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1. Introduction

When studying generation of seismic waves in a spherical earth, it is usually assumed that source function or the displacement potential is only a function of radial distance from a focus. Recent development of the dislocation theory, however, has shown that waves are radiated from the focus not uniformly but with some azimuthal and colatitudinal distribution. It is required to investigate behaviour of seismic waves radiated from a more realistic source model.

Let us assume that the displacement generated from the source can be expressed by

$$U=\mathrm{grad}\,\phi+\mathrm{rot}\,(r\psi,0,0)+\mathrm{rot}\,\mathrm{rot}\,(r\psi,0,0)\;, \tag{1}$$

where ϕ and ψ are generally given, in the polar coordinates (r,θ,φ) with the origin at the focus, by

$$\begin{aligned}\phi&=h_n^{(2)}(hr)\cdot P_n^m(\cos\theta)\cdot\exp(\pm im\varphi)\;,\\ \psi&=h_n^{(2)}(kr)\cdot P_n^m(\cos\theta)\cdot\exp(\pm im\varphi)\;,\end{aligned} \tag{2}$$

in which $h_n^{(2)}$ and P_n^m are a spherical Hankel function of the second kind and a Legendre function respectively, and h, k are wave numbers of the longitudinal and transverse waves respectively. To obtain seismic disturbances due to this generalized source model, we have to express the displacement in the other polar coordinates with the origin at the centre of the earth and to express these potentials by using variables in this new coordinates.

When $m=n=0$, *i.e.*, the potentials emitted from the focus are independent of the azimuth and colatitude, the complete transformation has been given in many textbooks of mathematical physics. The transformation is differently expressed according as distance of a referring point from the centre, R, is larger or smaller than that of the source from the centre, a. Satô (1950) gave the transformation for any m and n values without considering this restriction. That transformation should be valid for the case $R>a$, if referred to the solution $m=n=0$, so that the transformation is not complete. Jones (1964, p. 492) showed that a generalized solution multiplied by $j_n(ka)$ is expressible in terms of an integral representation involving the new coordinates, where j_n is a spherical Bessel function. Since the transformation becomes infinite for $j_n(ka)=0$, that integral is not always employed.

The transformation for $m=n=0$ can be obtained from the effect of the small variation of a on the solution, fixing the centre and the referring point (Jeffreys and Jeffreys, 1956, p. 659). By a similar treatment Edelstein (1963) gave the transformation of the solution $j_n(kr)\cdot P_n^m(\cos\theta)\cdot\exp(im\varphi)$. To discuss a progressive wave, however, it is necessary to use the solution expressed by spherical Hankel functions but not by spherical Bessel ones. In this paper, we derive the complete transformation, of the type (2), following those treatments. In terms of this transformation, each term of the displacement component (1) is expressed by the variables in the other coordinate system.

2. Mathematical preparations

For simplicity, the wave number is taken as unity, unless it is confused.

The solution which presents a diverging wave is expressed by (2). If we use a Gegenbauer function $C_n^{m+1/2}(\cos\theta)$ for a Legendre function $P_{m+n}^m(\cos\theta)$, the potential (2) can be written by an alternative form

$$\sin^m\theta\cdot h_{m+n}(r)\cdot C_n^{m+1/2}(\cos\theta)\cdot\exp(\pm im\varphi), \tag{3}$$

where h_n is a spherical Hankel function of order n and

$$C_n^{m+1/2}(\cos\theta)=\frac{P_{m+n}^m(\cos\theta)}{(2m-1)!!\cdot\sin^m\theta}. \tag{4}$$

We shall start with the solution (3).

Let us consider two coordinate systems where the origins are at points A and B, respectively, separated by distance a, as to say $(r_A, \theta_A, \varphi_A)$ and $(r_B, \theta_B, \varphi_B)$ as illustrated in Figure 1. Between these coordinates, the following relations hold, from the geometry ABP,

$$\varphi_B=\varphi_A \tag{5}$$

$$r_B \sin\theta_B = r_A \sin\theta_A\ ,$$
$$r_B \cos\theta_B = r_A \cos\theta_A - a\ , \tag{6}$$

$$r_B^2 = r_A^2 + a^2 - 2ar_A\cos\theta_A\ . \tag{7}$$

From Equation (5), a factor $\exp(\pm im\varphi)$, thus m, is not associated with the translation of the origin, so that we may neglect the factor $\exp(\pm im\varphi)$ from our considerations.

We introduce the notation

$$\{n, m\}_B^{(h)} = C_n^{m+1/2}(\cos\theta_B)\cdot h_n^m(r_B)\ , \tag{8}$$

where

$$h_n^m(r_B) = h_{m+n}(r_B)/r_B^m\ . \tag{9}$$

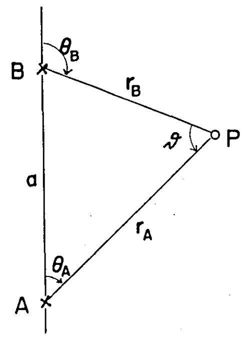

Fig. 1. Geometry of the coordinate origins. A is the centre of the earth, B is the focus, and P is a referring point.

Applying them to Gegenbauer's addition theorem for Bessel functions (Watson 1922, p. 363), we can write as

$$\{0, m\}_B^{(h)} = (2m-1)!!\cdot\sum_{l=0}^{\infty}(2l+2m+1)\cdot h_l^m(a)\cdot\{l, m\}_A^{(j)} \qquad \text{for} \qquad a > r_A\ , \tag{10}$$

$$= (2m-1)!!\cdot\sum_{l=0}^{\infty}(2l+2m+1)\cdot j_l^m(a)\cdot\{l, m\}_A^{(h)} \qquad \text{for} \qquad a < r_A\ , \tag{11}$$

where $j_l^m(r_A)$ and $\{l, m\}_A^{(j)}$ are given by taking $j_l(r_A)$ in place of $h_l(r_A)$ in Equations (9) and (8) respectively. The left hand side of Equation (10) or (11) is no other than the expression of $n=0$ in Equation (8). Thus for any n, let us assume, corresponding to these equations, that

$$\{n, m\}_B^{(h)} = \sum_{l=0}^{\infty}\varepsilon_l^{nm}(a)\cdot\{l, m\}_A^{(j)}\ , \qquad a > r_A\ , \tag{12}$$

$$= \sum_{l=0}^{\infty}\zeta_l^{nm}(a)\cdot\{l, m\}_A^{(h)}\ , \qquad a < r_A\ . \tag{13}$$

Before rigorous determination of the factors $\varepsilon_l^{nm}(a)$ and $\zeta_l^{nm}(a)$, we must know some characteristics of $\{n, m\}_B^{(h)}$. From (6), we have

and
$$\left.\begin{aligned} \frac{\partial r_B}{\partial a} &= -\cos\theta_B\ ,\\ \frac{\partial(\cos\theta_B)}{\partial a} &= -\frac{\sin^2\theta_B}{r_B}\ . \end{aligned}\right\} \tag{14}$$

Differentiating Equation (8) with respect to a and using these relations, we obtain

$$\frac{\partial}{\partial a}\{n, m\}_B = -C_n^{m+1/2}(\cos\theta_B)\cdot\cos\theta_B\cdot\frac{d}{dr_B}h_n^m(r_B)$$
$$-\sin^2\theta_B\cdot h_n^{m+1}(r_B)\cdot\frac{dC_n^{m+1/2}(\cos\theta_B)}{d(\cos\theta_B)} \quad . \tag{15}$$

Then application of the differential and recurrence relations of the spherical Hankel and Gegenbauer functions to Equation (15) yields the three-term recurrence relation

$$(2m+2n+1)\frac{\partial}{\partial a}\{n, m\}_B = (n+1)\{n+1, m\}_B$$
$$-(2m+n)\{n-1, m\}_B \; . \tag{16}$$

3. Case $a > r_A$

First we consider the case $a > r_A$. The transformation is written by the equation

$$\{n, m\}_B^{(h)} = \sum_{l=0}^{\infty} \varepsilon_l^{nm}(a)\cdot\{l, m\}_A^{(j)} \; . \tag{12}$$

Since the transformation in the case $n=0$ is given by Equation

$$\{0, m\}_B^{(h)} = \sum_{l=0}^{\infty} (2m-1)!!\cdot(2l+2m+1)\cdot h_l^m(a)\cdot\{l, m\}_A^{(j)} \; , \tag{10}$$

we assume that

$$\varepsilon_l^{nm}(a) = (2m-1)!!\cdot(2l+2m+1)\cdot F_n^m\left(\frac{d}{da}\right)h_l^m(a) \; , \tag{17}$$

where F_n^m is a functional operator to be determined. From Equations (10) and (12), denoting d/da by x, we have

$$F_{-1}^m(x) = 0 \qquad \text{and} \qquad F_0^m(x) = 1 \; . \tag{18}$$

After differentiation of Equation (12) with respect to a, application of Equations (16) and (17) yields

$$F_1^m(x) = (2m+1)x \; ,$$

$$F_2^m(x) = \frac{(2m+1)}{2}[(2m+3)x^2+1] \; ,$$

$$F_3^m(x) = \frac{(2m+1)(2m+3)}{3!}[(2m+5)x^2+3]x \; ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

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$$(n+1)F_{n+1}^m(x) = (2m+2n+1)xF_n^m(x) + (2m+n)F_{n+1}^m(x) \; . \tag{19}$$

This recurrence relation of $F_n^m(x)$ can be compared with that of Gegenbauer function, and we obtain

$$F_n^m(x) = i^n \cdot C_n^{m+1/2}(-ix) \ . \tag{20}$$

Therefore, we reach the crucial result, using Equation (6),

$$\begin{aligned}(2m-1)!! \cdot (r_B \sin \theta_B)^m \cdot \{n, m\}_B^{(h)} &= h_{n+m}(r_B) \cdot P_{n+m}^m(\cos \theta_B) \\ &= \sum_{l=0}^{\infty} \varepsilon_l^{nm}(a) \cdot j_{l+m}(r_A) \cdot P_{l+m}^m(\cos \theta_A) \ ,\end{aligned} \tag{21}$$

where symbolically,

$$\varepsilon_l^{nm}(a) = (2m-1)!! \cdot (2l+2m+1) \cdot i^n \cdot C_n^{m+1/2}(-id/da) h_{l+m}(a)/a^m \ . \tag{22}$$

As to the factor $\varepsilon_l^{nm}(a)$, we shall be discussed later.

4. Case $a < r_A$

In the case $a < r_A$ also, we can determine the transformation by analysis similar to that of the preceding section. Thus we write the equation

$$\{n, m\}_B^{(h)} = \sum_{l=0}^{\infty} \zeta_l^{nm}(a) \cdot \{l, m\}_A^{(h)} \ , \tag{13}$$

and for the factor $\zeta_l^{nm}(a)$, let us assume that

$$\zeta_l^{nm}(a) = (2m-1)!! \cdot (2l+2m+1) \cdot G_n^m(d/da) h_l^m(a) \ . \tag{23}$$

Functional operator $G_n^m(x)$ is determined similarly in the preceding section, and finally it is shown that

$$G_n^m(x) = F_n^m(x) = i^n \cdot C_n^{m+1/2}(-ix) \ . \tag{24}$$

Accordingly, we reach the result for this case

$$h_{m+n}(r_B) \cdot P_{m+n}^m(\cos \theta_B) = \sum_{l=0}^{\infty} \zeta_l^{nm}(a) \cdot h_{l+m}(r_A) \cdot P_{l+m}^m(\cos \theta_A) \ , \tag{25}$$

where symbolically

$$\zeta_l^{nm}(a) = (2m-1)!! \cdot (2l+2m+1) \cdot i^n \cdot C_n^{m+1/2}(-id/da) j_{l+m}(a)/a^m \ . \tag{26}$$

5. Factors $\varepsilon_l^{nm}(a)$ and $\zeta_l^{nm}(a)$

The factor $\varepsilon_l^{nm}(a)$ for the case $a > r_A$ is given by taking $h_{l+m}(a)$ for $j_{l+m}(a)$ in the factor $\zeta_l^{nm}(a)$ for the other case. If we denote by $E_l^{nm}(a)$

Table 1. Choice of $E_l^{nm}(ka)$ and $z_{l+m}(kr_A)$

Case	$E_l^{nm}(ka)$	$z_{l+m}(kr_A)$
$a > r_A$	$\varepsilon_l^{nm}(ka)$: *Eq.* (22)	$j_{l+m}(kr_A)$
$a < r_A$	$\zeta_l^{nm}(ka)$: *Eq.* (26)	$h_{l+m}(kr_A)$

the factor $\varepsilon_l^{nm}(a)$ or $\zeta_l^{nm}(a)$ and by $z_n(x)$ a spherical wave function of order $n, j_n(x)$ or $h_n(x)$, the transformation is written as

$$h_{m+n}(r_B)\cdot P_{m+n}^m(\cos\theta_B)=\sum E_l^{nm}(a)\cdot z_{l+m}(r_A)\cdot P_{l+m}^m(\cos\theta_A)\ , \tag{27}$$

and $E_l^{nm}(a)$ and $z_{l+m}(r_A)$ are chosen as in Table 1. In seismological applications, the cases $n=0, 1, 2, 3$ are important. The exact forms of the factors E_l^{nm} for these cases are easily obtained, by means of Equations (19), as follows:

$$\left.\begin{aligned}
&n=0 \qquad [\text{related to } h_m(r_B)\cdot P_m^m(\cos\theta_B)]\\
&\quad E_l^{0m}(a)=(2m-1)!!\cdot(2l+2m+1)z_{l+m}(a)/a_m\ .\\
&n=1 \qquad [\text{related to } h_{m+1}(r_B)\cdot P_{m+1}^m(\cos\theta_B)]\\
&\quad E_l^{1m}(a)=(2m+1)!!\cdot(2l+2m+1)\left\{l\frac{z_{l+m}(a)}{a^{m+1}}-\frac{z_{l+m+1}(a)}{a^m}\right\}\ ;\\
&n=2 \qquad [\text{related to } h_{m+2}(r_B)\cdot P_{m+2}^m(\cos\theta_B)]\\
&\quad E_l^{2m}(a)=\frac{(2m+3)!!}{2}\cdot(2l+2m+1)\left[l(l-1)\frac{z_{l+m}(a)}{a^{m+2}}\right.\\
&\qquad\qquad \left.+2(m+1)\left\{\frac{z_{l+m+1}(a)}{a^{m+1}}-\frac{z_{l+m}(a)}{(2m+3)a^m}\right\}\right]\ ;\\
&n=3 \qquad [\text{related to } h_{m+3}(r_B)\cdot P_{m+3}^m(\cos\theta_B)]\\
&\quad E_l^{3m}(a)=\frac{(2m+5)!!}{3!}\cdot(2l+2m+1)\left[l(l-1)(l-2)\frac{z_{l+m}(a)}{a^{m+3}}\right.\\
&\qquad\qquad -\{(l+m)(l+m+1)+3(m+1)(m+2)\}\frac{z_{l+m+1}(a)}{a^{m+2}}\\
&\qquad\qquad \left.+\frac{2(m+1)}{2m+5}\left\{(-l+2m+5)\frac{z_{l+m}(a)}{a^{m+1}}+\frac{z_{l+m+1}(a)}{a^m}\right\}\right].
\end{aligned}\right\} \tag{28}$$

In general, the factor $E_l^{nm}(a)$ can be expressed by the sum of a finite number of spherical wave functions, and it is proved as follows.

For $m=0$, we can write symbolically a spherical wave function as (Rayleigh, 1945, § 329)

$$\begin{aligned} z_n(a)&=(-i)^n\cdot P_n(-id/da)z_0(a)\\ &=(-i)^n\cdot C_n^{1/2}(-id/da)z_0(a)\ . \end{aligned} \tag{29}$$

Then, let us assume that

$$z_n^m(a) \equiv z_{m+n}(a)/a^m = f_n^m \cdot (-i)^n \cdot C_n^{m+1/2}(-id/da) z_0^m(a) \ , \tag{30}$$

whence because of $C_0^{m+1/2}(x) \equiv 1$ and Equation (29), we get

$$f_0^m = f_n^0 = 1 \ .$$

Putting $n+1$ for n in Equation (30), we have

$$z_{n+1}^m(a) = f_{n+1}^m \cdot (-i)^{n+1} \cdot C_{n+1}^{m+1/2}(-id/da) z_0^m(a) \ , \tag{31}$$

application of the recurrence formula of Gegenbauer function and Equation (30) yields

$$z_{n+1}^m(a) = \frac{(2m+n) f_{n+1}^m}{(n+1) f_{n-1}^m} z_{n-1}^m(a) - \frac{(2m+2n+1) f_{n+1}^m}{(n+1) f_n^m} \frac{d}{da} z_n^m(a) \ ,$$

and by using the recurrence formula and the derivative of the spherical wave function, we get

$$\left(\frac{2m+n}{f_{n-1}^m} - \frac{n}{f_n^m}\right) z_{m+n-1}(a) + \left(\frac{2m+n+1}{f_n^m} - \frac{n+1}{f_{n+1}^m}\right) z_{m+n+1}(a) = 0 \ .$$

This equation holds good identically, if

$$f_n^m = \frac{n}{(2m+n)} f_{n-1}^m = \frac{(2m)! \cdot n!}{(2m+n)!} \ . \tag{32}$$

Therefore, the following relation is obtained

$$z_n^m(a) = \frac{(2m)! \cdot n!}{(2m+n)!} \cdot (-i)^n \cdot C_n^{m+1/2}(-id/da) z_0^m(a) \ . \tag{33}$$

Moreover, if we assume

$$i^n \cdot C_n^{m+1/2}(-id/da) z_l^m(a) = \sum_{p=0}^{\infty} (-1)^{n+p} \cdot c_{nlp}^m \cdot z_{l+n-2p}^m(a) \ , \tag{34}$$

substitution of Equation (33) into $z_n^m(a)$ results symbolically in

$$C_n^{m+1/2}(x) \cdot C_l^{m+1/2}(x) = \sum_{p=0}^{\infty} \frac{(l+2m)! \cdot (l+n-2p)!}{l! \cdot (l+2m+n-2p)!} \cdot c_{nlp}^m \cdot C_{l+n-2p}^{m+1/2}(x) \ , \tag{35}$$

which is the equation determining the coefficients c_{nlp}^m; or application of the orthogonality of the associated Legendre function to Equation (35) and an integral formula involving three Gegenbauer functions (Hsü, 1938) yields

$$c_{nlp}^{m}=0\ , \qquad\qquad \text{for} \quad p>N\ ;$$

$$=\frac{(2l+2m+2n-4p+1)\cdot l!\cdot\{(2m-1)!!\}^2}{2\cdot(l+2m)!}$$

$$\times\int_{-1}^{1} C_l^{m+1/2}(x)\cdot C_n^{m+1/2}(x)\cdot C_{l+n-2p}^{m+1/2}(x)\cdot(1-x^2)^m dx$$

$$=\frac{(2l+2m+2n-4p+1)\cdot l!\cdot(l+2m+n-p)!\cdot A_{l-p}\cdot A_{n-p}\cdot A_p}{(2l+2m+2n-2p+1)\cdot(l+2m)!\cdot(l+n-p)!\cdot A_{l+n-p}}\ ,$$

$$\text{for} \quad p\leq N\ ; \tag{36}$$

where

$$A_p=\frac{(2p+2m-1)!!}{2^p\cdot p!\cdot(2m-1)!!}\ ,$$

and

$$N=\min(l,\ n)\ .$$

Therefore, the factor $E_l^{nm}(a)$ is expressed by the sum of $(N+1)$ spherical wave functions.

Satô (1950) obtained $\zeta_l^{nm}(a)$ in a power series of a without the restriction of $a<r_A$. In our calculation on the seismic wave the value a may be rather large, so that the solution may not be well convergent. It has been shown by Edelstein (1963) that Satô's solution is identical to $\zeta_l^{nm}(a)$ which involves the spherical Bessel functions.

6. Transformation of solutions of the elastic wave equation

In Figure 1, let the point B be the focus and A the centre of the earth. The seismic wave radiated from the focus B has the displacement $\boldsymbol{U}_B$ in the coordinates with the origin B

$$\boldsymbol{U}_B=\mathrm{grad}_B\,\phi_B+\mathrm{rot}_B\,\boldsymbol{F}_B^{(1)}+\frac{1}{k}\mathrm{rot\ rot}_B\,\boldsymbol{F}_B^{(2)}\ , \tag{37}$$

where suffix B signifies the differential operator and the potentials with respect to the coordinates B, and the vector potential $\boldsymbol{F}_B$ has the only radial component in the coordinates B

$$\boldsymbol{F}_B^{(j)}=(r_B\psi_B^{(j)},\ 0,\ 0)\ , \qquad j=1,2\ . \tag{38}$$

This displacement must be transformed into that in the coordinates with the origin A. The solution $\boldsymbol{U}_A$ would be written by a form

$$\boldsymbol{U}_A=\mathrm{grad}_A\,\phi_A+\mathrm{rot}_A\,\boldsymbol{F}_A^{(1)}+\frac{1}{k}\mathrm{rot\ rot}_A\,\boldsymbol{F}_A^{(2)}\ . \tag{39}$$

6.1. *Transformation of the longitudinal part.*

For the scalar potential ϕ_B, its transformation is obtained by that in the preceding sections, that is, if

$$\phi_B = h_{m+n}(hr_B)\cdot P_{m+n}^m(\cos\,\theta_B)\cdot\exp\,(\pm im\varphi_B)\ , \tag{40}$$

the transformed potential ϕ_A is expressed by

$$\phi_A = \sum_{l=0}^{\infty} E_l^{nm}(ha)\cdot z_{l+m}(hr_A)\cdot P_{l+m}^m(\cos\,\theta_A)\cdot\exp\,(\pm im\varphi_A)\ , \tag{41}$$

where $E_l^{nm}(ha)$ and $z_{l+m}(hr)$ are chosen as in Table 1.

6.2. *Transformation of the solenoidal parts.*

On the other hand, the vector potential $\boldsymbol{F}_B$ has not only the radial component but the colatitudinal component in the coordinates A. Putting ϑ for the angle $\angle APB$,

$$\vartheta = \theta_B - \theta_A\ , \tag{42}$$

and F_r and F_θ for the radial and colatitudinal components of $\boldsymbol{F}_B$ in the coordinates A respectively, we have

$$\begin{aligned} F_r &= r_B\cdot\psi_B\cdot\cos\vartheta\ ,\\ F_\theta &= r_B\cdot\psi_B\cdot\sin\vartheta\ . \end{aligned} \tag{43}$$

From Equations (42) and (6), we obtain

$$\begin{aligned} F_r &= (r_A - a\cos\theta_A)\cdot\psi_B\ ,\\ F_\theta &= a\cdot\sin\theta_A\cdot\psi_B\ . \end{aligned} \tag{44}$$

This vector can also be divided into two parts

$$\boldsymbol{F}_B = (r_A\cdot\psi_B,\ 0,\ 0) + (-a\cdot\psi_B\cdot\cos\theta_A,\ a\cdot\psi_B\cdot\sin\theta_A,\ 0)\ , \tag{45}$$

the second of which is a vector directed to the negative z-axis in the Cartesian coordinates.

If the fundamental solution is constructed from the radially directed vector, the solutions deduced from a vector in some other direction can be expressed by a superposition of the fundamental solutions of different order (Usami, Kano and Satô, 1962). When a vector $\boldsymbol{F}_z$ in the Cartesian coordinates has the components

$$\left.\begin{aligned} (\boldsymbol{F}_z)_x &= (\boldsymbol{F}_z)_y = 0\ ,\\ (\boldsymbol{F}_z)_z &= -\psi_n^m = -h_{m+n}(kr)\cdot P_{m+n}^m(\cos\theta)\cdot\exp\,(\pm im\varphi)\ , \end{aligned}\right\} \tag{46}$$

the rotation of this vector is expressed by the following

$$\begin{aligned}\operatorname{rot} \boldsymbol{F}_z = &-\frac{(n+1)k}{(2m+2n+1)(m+n+1)} \operatorname{rot} \boldsymbol{F}^m_{r,n+1} \\ &-\frac{(2m+n)k}{(2m+2n+1)(m+n)} \operatorname{rot} \boldsymbol{F}^m_{r,n-1} \\ &-\frac{1}{(m+n+1)(m+n)} \frac{\partial}{\partial\varphi} \operatorname{rot}\operatorname{rot} \boldsymbol{F}^m_{r,n}\end{aligned} \tag{47}$$

and

$$\begin{aligned}\operatorname{rot}\operatorname{rot} \boldsymbol{F}_z = &-\frac{(n+1)k}{(2m+2n+1)(m+n+1)} \operatorname{rot}\operatorname{rot} \boldsymbol{F}^m_{r,n+1} \\ &-\frac{(2m+n)k}{(2m+2n+1)(m+n)} \operatorname{rot}\operatorname{rot} \boldsymbol{F}^m_{r,n-1} \\ &-\frac{k^2}{(m+n+1)(m+n)} \frac{\partial}{\partial\varphi} \operatorname{rot} \boldsymbol{F}^m_{r,n} ,\end{aligned} \tag{48}$$

where, in the polar coordinates,

$$\boldsymbol{F}^m_{r,n} = (r \cdot \psi^m_n,\ 0,\ 0) . \tag{49}$$

6.2.1. *Transformation of* rot $\boldsymbol{F}_B$.

Since from Equation (45)

$$\boldsymbol{F}_B = \boldsymbol{F}_r + a \cdot \boldsymbol{F}_z ,$$

if we take

$$\psi_B = h_{m+n}(kr_B) \cdot P^m_{m+n}(\cos\theta_B) \cdot \exp(\pm im\varphi_B) , \tag{50}$$

the transformed solution is written, by using Equation (47),

$$\begin{aligned}\operatorname{rot} \boldsymbol{F}^{(1)}_B = \operatorname{rot} \boldsymbol{F}^m_{r,n} &-\frac{(n+1)ka}{(2m+2n+1)(m+n+1)} \operatorname{rot} \boldsymbol{F}^m_{r,n+1} \\ &-\frac{(2m+n)ka}{(2m+2n+1)(m+n)} \operatorname{rot} \boldsymbol{F}^m_{r,n-1} \\ &-\frac{a}{(m+n+1)(m+n)} \frac{\partial}{\partial\varphi} \operatorname{rot}\operatorname{rot} \boldsymbol{F}^m_{r,n} ,\end{aligned} \tag{51}$$

where

$$\begin{aligned}&\boldsymbol{F}^m_{r,n} = (r_A \cdot \psi^m_{An},\ 0,\ 0) , \\ &\psi^m_{An} = \sum_{l=0}^{\infty} E^{nm}_l(ka) \cdot z_{l+m}(kr_A) \cdot P^m_{l+m}(\cos\theta_A) \cdot \exp(\pm im\varphi_A) .\end{aligned} \tag{52}$$

6.2.2. *Transformation of* rot rot $\boldsymbol{F}^{(2)}_B$.

If we take

$$\psi_B = h_{m+n}(kr_B)\cdot P^m_{m+n}(\cos\theta_B)\cdot\exp(\pm im\varphi_B)\ ,$$

the transformed solution is written, by means of Equation (49),

$$\begin{aligned}\text{rot rot}\,\boldsymbol{F}^{(2)}_B = \text{rot rot}\,\boldsymbol{F}^m_{r,n} &- \frac{(n+1)ka}{(2m+2n+1)(m+n+1)}\,\text{rot rot}\,\boldsymbol{F}^m_{r,n+1}\\ &- \frac{(2m+n)ka}{(2m+2n+1)(m+n)}\,\text{rot rot}\,\boldsymbol{F}^m_{r,n-1}\\ &- \frac{k^2a}{(m+n+1)(m+n)}\,\frac{\partial}{\partial\varphi}\,\text{rot}\,\boldsymbol{F}^m_{r,n}\ ,\end{aligned}\qquad(53)$$

where $\boldsymbol{F}^m_{r,n}$ is formally given by Equation (52).

In Equation (52), $E^{nm}_l(ka)$ and $z_{l+m}(kr)$ are chosen as in Table 1.

7. Some numerical test of series

Convergence of the series (21) and (25) is examined in this section. Let us assume

$$H^m_n(ka_B,\ \theta_B,\ L) = \sum_{l=0}^{L} E^{nm}_l(ka)\cdot z_{l+m}(kr_A)\cdot P^m_{l+m}(\cos\theta_A)\ ,$$

which approaches to $h_{m+n}(kr_B)\cdot P^m_{m+n}(\cos\theta_B)$, as L goes to infinity. Figure 2 shows an example of this, when

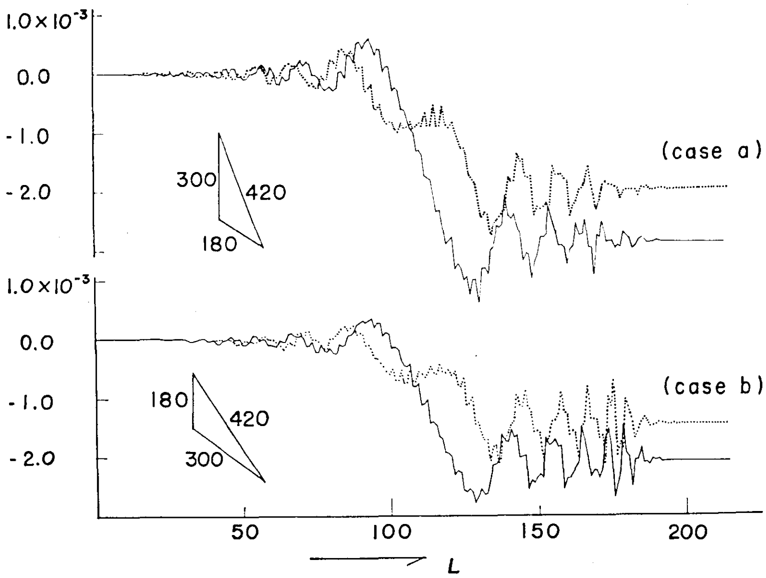

Fig. 2. Convergence of the series $H^1_1(kr_B, \theta_B, L)$. The solid and dotted lines denote the real and imaginary parts, respectively.

$$ka=300\,,\quad kr_A=180\,,\quad kr_B=420\,,$$
$$\theta_A=120.0^\circ\,,\quad \theta_B=158.2132^\circ \qquad \text{(case a)}\,;$$
$$ka=180\,,\quad kr_A=300\,,\quad kr_B=420\,,$$
$$\theta_A=120.0^\circ\,,\quad \theta_B=141.7868^\circ \qquad \text{(case b)}\,;$$

and $m=1, n=1$. It follows from this figure that the series apparently converge beyond a finite number L_s; the value L_s is related with the smaller one of a and r_A; the series for $L<L_s$ change with oscillation and for $L\geq L_s$ its variation becomes small; and both the real and imaginary parts behave similarly unless r_A is near a.

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26. 極座標の原点の移動に伴なう弾性方程式の解の変換

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地震波の生成・伝播の研究においては，震源函数が強い方向性をもつにもかかわらず，震源からの距離だけの函数と仮定している．これは，地球核や地表面での境界条件を適用するため，震源（点B）に原点をもつ図標系で表わした震源函数を地球中心（点A）に原点のある座標系で表わすこ

とが必要であるが，方向性をもつ震源函数に対するこの種の変換式が得られていなかつたためである．

本論文では，始めに座標系 $B(r_B, \theta_B, \varphi_B)$ において $h_n(kr_B)\cdot P_n^m(\cos\theta_B)\cdot\exp(\pm im\varphi_B)$ で表わされる一般のポテンシャルが，座標系$A(r_A, \theta_A, \varphi_A)$ に関して次の様に変換されることを示した:

$$h_{m+n}(kr_B)\cdot P_{m+n}^m(\cos\theta_B)\cdot\exp(\pm im\varphi_B)$$
$$=\sum_{l=0}^{\infty} E_l^{nm}(ka)\cdot z_{l+m}(kr_A)\cdot P_{l+m}^m(\cos\theta_A)\cdot\exp(\pm im\varphi_A)\,, \tag{27}$$

ただし，$z_{l+m}(kr_A)$ は $a>r_A$ のとき $j_{l+m}(kr_A)$, $a<r_A$ のとき $h_{l+m}(kr_A)$ をとるものとし，$E_l^{nm}(ka)$ は $a>r_A$ のとき $h_{l+m}(ka)$ を含み，$a<r_A$ のときは $h_{l+m}(ka)$ の代りに $j_{l+m}(ka)$ とおいた或る函数である．

次に，変位ポテンシャルが上記で与えられた場合の変換された変位の表現を求めた．

i）P波: 変位ポテンシャルは(27)で変換される．

ii）S波: $\psi_B=h_{m+n}(kr_B)\cdot P_{m+n}^m(\cos\theta_B)\cdot\exp(\pm im\varphi_B)$ のとき，

$$\mathrm{rot}\,(r_B\psi_B, 0, 0)=\mathrm{rot}\,\boldsymbol{F}_n^m-A_{n+1}\,\mathrm{rot}\,\boldsymbol{F}_{n+1}^m-B_{n-1}\,\mathrm{rot}\,\boldsymbol{F}_{n-1}^m$$
$$-\frac{1}{k}C_n\frac{\partial}{\partial\varphi}\,\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{F}_n^m \tag{51}$$

$$\mathrm{rot}\,\mathrm{rot}\,(r_B\psi_B, 0, 0)=\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{F}_n^m-A_{n+1}\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{F}_{n+1}^m$$
$$-B_{n-1}\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{F}_{n-1}^m-kC_n\frac{\partial}{\partial\varphi}\mathrm{rot}\,\boldsymbol{F}_n^m \tag{53}$$

ただし，

$$\boldsymbol{F}_n^m=(r_A\psi_A, 0, 0)\,,$$
$$\psi_A=\sum_{l=0}^{\infty} E_l^{mn}(ka)\cdot z_{l+m}(kr_A)\cdot P_{l+m}^m(\cos\theta_A)\cdot\exp(\pm im\varphi_A)\,,$$
$$A_{n+1}=\frac{(m+1)ka}{(2m+2n+1)(m+n+1)}\,,$$
$$B_{n-1}=\frac{(2m+n)ka}{(2m+2n+1)(m+n)}\,,$$
$$C_n=\frac{ka}{(m+n+1)(m+n)}\,.$$

で表わされる．