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49. *Note on the Second Derivative of Time-Distance Curve.*

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The second derivative d^2t/dx^2 of the time-distance curve of a surface origin is considered to tend to zero with the distance x, when the velocity of propagation is "ordinary".[1] A theoretical conclusion of this kind may naturally give some restriction to the mathematical procedure of the curve fitting in the studies of a time-distance curve.

It is clear that the second derivative is closely related to the gradient of the velocity of propagation,[2] and what is meant by "ordinary" is changed according to our knowledge. As far as the theory of a time-distance curve is concerned, a minus gradient of the velocity to the depth is excluded from what is meant by the ordinary structure. However, a conclusion based on an assumption that the gradient is finite or moderate is not satisfactory from, at least, the stand point of applied mathematics in seismology, because even the effect of a discontinuity of the velocity on the Wiechert-Herglotz formula has already been studied, and effects of the discontinuity on the seismic waves have also been thoroughly studied; the velocity gradient of infinite magnitude would not necessarily yield an unnatural behavior of the wave. In this paper, d^2t/dx^2 at $x=0$ is studied in relation to the various surface conditions of the medium.

2. For the sake of an easy understanding of the fundamentals, we may start with an assumption of velocity model simple and familiar to seismologists.

Suppose a surface layered model of the medium: V_1 the velocity in the surface layer and V_2 in the top face of the basic structure, the velocity of the interior parts having nothing to do with the following procedure. Then, distance x_0 where the velocity V_2 comes out in the time-distance curve is given by the following well known formula,

1) BULLEN, *An Introduction to the Theory of Seismology*, § 7.3.
2) R. YOSHIYAMA, *Zisin*, [i], **8** (1936) 325-330, (in Japanese).

$$x_0=2\sqrt{V_2+V_1}\,\frac{h}{\sqrt{\delta V}}\;, \tag{1}$$

where $\delta V=V_2-V_1$, and $\delta V \nless 0$.

When the velocity of propagation is a gradually increasing function of the depth z, the formula that corresponds to (1) is derived from the Wiechert-Herglotz formula, being, however, transformed for a plane structure,

$$\pi z=\int_0^{x_0}\cosh^{-1}\frac{dt/dx}{(dt/dx)_{x=x_0}}\,dx\;. \tag{2}$$

As we are now concerned with a problem near the origin, $x_0=dx$, $z=dz$ and, instead of (2), the following formula is applicable.

$$\pi dz=\cosh^{-1}\frac{V}{a}\,dx\;, \tag{3}$$

where a is the velocity of propagation at $z=0$, and, at the same time, the apparent velocity at $x=0$ in the time-distance curve; V is that at $z=dz$, being apparent at $x=dx$ in the time-distance curve. When $V-a=\delta V$ is small, we obtain from (3),

$$dx=\frac{\pi}{2}\sqrt{V+a}\,\frac{dz}{\sqrt{\delta V}}\;. \tag{4}$$

Comparing (1) and (4) with each other, V_1 corresponds to a and V_2 to V. The only difference between (1) and (4) is that between 2 and $\pi/2$, so that, in the following solution, there may be an error of factor $\pi/4$ or $4/\pi$, but none other. Using the formula (1), mean value of the second derivative of the time-distance curve near the origin, $x=0\sim x_0$, is calculated according to its definition:

$$\begin{aligned}\left(\frac{d^2t}{dx^2}\right)_{\mathrm{m}}&=\frac{1}{x_0}\left\{\left(\frac{dt}{dx}\right)_{x=x_0}-\left(\frac{dt}{dx}\right)_{x=0}\right\}\\&=\frac{1}{x_0}\left\{\frac{1}{V_2}-\frac{1}{V_1}\right\}\\&=-\frac{\delta V}{V_2V_1}\,\frac{1}{2\sqrt{V_2+V_1}}\,\frac{\sqrt{\delta V}}{h}\;.\end{aligned} \tag{5}$$

If $x_0\to 0$ as $h\to 0$, then (5) gives the second derivative at $x=0$. The "ordinary" chart of the velocity would mean that $\delta V\to 0$ as $h\to 0$, but not necessarily any more: necessarily neither $\delta V/h=$ finite nor $\to 0$. So that the second derivative at $x=0$ is given by the following formula, being deduced from (5),

$$\left(\frac{d^2t}{dx^2}\right)_{x=0}=\left[-\frac{1}{2\sqrt{2}\,V_0^{5/2}}\frac{(\delta V)^{3/2}}{h}\right]_{h\to 0,\ \delta V\to 0}, \tag{6}$$

where V_0 is the velocity at the surface. If we put $\delta V=C\cdot h^n$ as $h\to 0$, where $n>0$ and C is a certain constant, there are three cases according to the n-value left for studies in this paper.

1. $2>n>2/3$ 2. $n=2/3$ 3. $0<n<2/3$

The condition $n<2$ is set from $x_0\to 0$ as $h\to 0$, but it has but a little significance in our studies.

When

$$\begin{aligned} &2>n>2/3, \quad \frac{d^2t}{dx^2}=0\,; \\ &n=2/3, \quad \frac{d^2t}{dx^2}=-\frac{1}{2\sqrt{2}\,V_0^{5/2}}C^{3/2}\,; \\ &0<n<2/3, \quad \frac{d^2t}{dx^2}\to -\infty \end{aligned} \tag{7}$$

at $x=0$.

Theoretically speaking, the solution of the problem is completely described by (7), and our interest at present is in the case $n\leqslant 2/3$. However, formulas when the velocity of propagation is a continuous function of the depth z are also necessary for convenience of mathematical application: those formulas are also derived from (6) or (7) as follows.

Since $\delta V=C\cdot h^n$, mean gradient of the velocity $\delta V/h=C'/(\delta V)^m$, where $C'=C^{1/n}$ and $m=1/n-1$; $n>2/3$, $=2/3$ and $<2/3$ corresponds to $m<1/2$, $=1/2$ and $>1/2$ respectively. So that, if we put, taking a term of the lowest power, near the surface,

$$\frac{dV}{dz}=\frac{K}{(V-V_0)^m}, \qquad z\to 0\,, \tag{8}$$

and $V=V_0$ at $z=0$, it is deduced from (6) or (7) that

$$\left(\frac{d^2t}{dx^2}\right)_{x=0}=\left[-\frac{1}{2\sqrt{2}}\frac{1}{V_0^{5/2}}\frac{K}{(V-V_0)^{m-1/2}}\right]_{V\to V_0}, \tag{9}$$

or

$$\left.\begin{aligned} m<1/2\,, &\quad \left(\frac{d^2t}{dx^2}\right)_{x=0}=0\,; \\ m=1/2\,, &\quad \left(\frac{d^2t}{dx^2}\right)_{x=0}=-\frac{1}{2\sqrt{2}}\,\frac{K}{V_0{}^{5/2}}\,; \\ m>1/2\,, &\quad \left(\frac{d^2t}{dx^2}\right)_{x=0}\to-\infty\,. \end{aligned}\right\} \quad (10)$$

As already pointed out, the solution (10) may have an error of the factor $4/\pi$. Therefore, if we deduce by the formula (10) the surface condition (8) and calculate m of the structure from the second derivative of a time-distance curve at $x=0$, there may be an error of the factor $\pi/4$ in $\frac{dV}{dz}$.

Three examples are studied in the following for the sake of understanding the significance of these formulas.

3. When the time-distance curve is given by

$$t=ax-bx^2\,, \tag{11}$$

$$V_0=1/a\,, \qquad \frac{d^2t}{dx^2}=-2b\,.$$

Making reference to (10), we obtain $K=4\sqrt{2}\cdot b/a^{5/2}$ and $m=1/2$. So that, putting K into (8), the surface condition of the structure fit for the time-distance curve (11) is from the result in the preceding section,

$$\left(\frac{dV}{dz}\right)_{z=0}=\left(\frac{4\sqrt{2}\,ba^{-5/2}}{\sqrt{V-1/a}}\right)_{V\to1/a}. \tag{12}$$

On the other hand, for the time-distance curve (11), we have an exact solution of the structure from the formula (2),

$$2\pi bz=-\frac{\sqrt{a^2V^2-1}}{V}+a\cosh^{-1}aV\,, \tag{13}$$

$$V=1/a \qquad \text{at} \qquad z=0\,.$$

Therefore,

$$\left(\frac{dV}{dz}\right)_{z=0}=\left(\frac{\sqrt{2}\,\pi ba^{-5/2}}{\sqrt{V-1/a}}\right)_{V\to1/a} \tag{14}$$

and we can see that (12) differs from the exact solution (14) by the factor $\pi/4$.

4. When the velocity is given by

$$\frac{dV}{dz}=\frac{\beta}{\sqrt{V^2-\alpha^2}}\,, \qquad V_0=\alpha\,, \tag{15}$$

$m=1/2$ and $K=\beta/\sqrt{2}\alpha$ at $z\to 0$ from (15) and (8). And by (10)

$$\left(\frac{d^2t}{dx^2}\right)_{x=0}=-\frac{1}{4}\frac{\beta}{\alpha^3}\,, \tag{16}$$

while an exact formula of the time-distance curve in the structure expressed by (15) is

$$x=\alpha t+\frac{\beta}{2\pi}t^2\,, \tag{17}$$

or

$$t=\frac{x}{\alpha}-\frac{\beta}{2\pi\alpha^3}x^2+\frac{\beta^2}{2\pi^2\alpha^5}x^3-\cdots\,. \tag{18}$$

So that the exact formula of the second derivative at $x=0$ is

$$\left(\frac{d^2t}{dx^2}\right)_{x=0}=-\frac{\beta}{\pi\alpha^3} \tag{19}$$

and (16) differs from (19) by the factor $4/\pi$ as might have been expected.

The velocity-depth relation given by (15) is

$$z=\frac{1}{2\beta}\left(V\sqrt{V^2-\alpha^2}-\alpha^2\cosh^{-1}\frac{V}{\alpha}\right).$$

5. When the velocity, instead of (15), is given by

$$\left.\begin{aligned}&\frac{dV}{dz}=\frac{\beta}{V-\alpha}\,,\\ &V_0=\alpha\,,\end{aligned}\right\} \tag{20}$$

$m=1$, $K=\beta$ and $V=\alpha+\sqrt{2\beta z}$. Therefore, from (9)

$$\left(\frac{d^2t}{dx^2}\right)_{x\to 0}=-\frac{1}{2\sqrt{2}}\frac{\beta}{\alpha^{5/2}}\left(\frac{1}{\sqrt{V-\alpha}}\right)_{V\to\alpha}\,. \tag{21}$$

While the time-distance curve for the structure (20) is given by the following equations:

$$\left.\begin{aligned} x&=\frac{1}{\beta}\left\{\frac{\sin^{-1}\sqrt{1-\alpha^2\kappa^2}}{\kappa^2}-\frac{\alpha\sqrt{1-\alpha^2\kappa^2}}{\kappa}\right\},\\ t&=\frac{2}{\beta}\left\{\frac{\sin^{-1}\sqrt{1-\alpha^2\kappa^2}}{\kappa}-\alpha\sinh^{-1}\frac{\sqrt{1-\alpha^2\kappa^2}}{\alpha\kappa}\right\},\end{aligned}\right\} \tag{22}$$

$$\kappa=\frac{dt}{dx}=\frac{1}{V}.$$

So that,

$$\begin{aligned}\frac{d^2t}{dx^2}&=-\frac{\beta\kappa^3}{2\cos^{-1}\alpha\kappa}\\ &\fallingdotseq-\frac{\beta\kappa^3}{2\sqrt{1-\alpha^2\kappa^2}}, \qquad \alpha\kappa\fallingdotseq 1,\end{aligned}$$

and, since $1/\kappa=V$ and $V\to\alpha$ as $x\to 0$,

$$\left(\frac{d^2t}{dx^2}\right)_{x\to 0}=-\frac{1}{2\sqrt{2}}\frac{\beta}{\alpha^{5/2}}\left(\frac{1}{\sqrt{V-\alpha}}\right)_{V\to\alpha}. \tag{23}$$

(21) is exactly equal to the rigorous formula (23), and the compensating factor $4/\pi$ is unnecessary. Since the order of divergency of dV/dz at $z=0$ becomes high with increasing m in (8), it seems natural that the behavior of the solution for large m approaches to that in the case of the surface layered model of discontinuous variation of the velocity.

6. Anyway, $\dfrac{dV}{dz}\to\infty$ at $z=0$ is a necessary condition for $\left(\dfrac{d^2t}{dx^2}\right)_{x=0}\neq 0$, and that may appear unnatural to some seismologists. It seems to the present writer that $\dfrac{dV}{dz}\to\infty$ in effect at $z=0$ is one of the possible cases, when, for example, the surface is affected by rapid erosion. Moreover, the space gradient of the velocity has nothing to do with the stress-strain relation in the theory of elastic waves: velocity of propagation may come into our observations directly, but, certainly, its space gradient does not. Therefore, there is no theoretical reason at present to stick to that $\dfrac{dV}{dz}=$finite, neither at the surface nor at the inner part of the medium. Study in the following of the behavior of a wave may clarify those circumstances more or less concretely.

Suppose a wave of a sufficiently short wave length incident normally to a surface with a structure given by (8). The behavior of the wave is obtained by solving the following equation,

$$\frac{\partial^2 u}{\partial T^2}=\frac{\partial}{\partial z}\left(V^2\frac{\partial u}{\partial z}\right), \qquad T: \text{ time coordinate,}$$

with a requisite boundary condition. Changing the space coordinate from z to t by $t=\int_0^z dz/V$, and putting

$$V^2\frac{\partial u}{\partial z}=\sqrt{V}\Phi(t)\exp(ipT)\,, \tag{24}$$

the problem is to solve

$$\frac{d^2\Phi}{dt^2}+[p^2-f(t)]\Phi=0\,, \tag{25}$$

where

$$f(t)=-V^{3/2}\frac{d^2\sqrt{V}}{dz^2}\,.$$

From (8),

$$-V^{3/2}\frac{d^2\sqrt{V}}{dz^2}=C_1z^{-(2m+1)/(m+1)}+C_2z^{-2m/(m+1)}\,,$$

$$C_1=\frac{V_0}{2}m(m+1)^{-(2m+1)/(m+1)}K^{1/(m+1)}\,,$$

$$C_2=\frac{1}{4}(2m+1)(m+1)^{-2m/(m+1)}K^{2/(m+1)}\,.$$

When $m=0$, $C_1=0$ and $f(t)=K^2/4$; the behavior of a wave ruled by the equation simplified in that way was already studied[3]. In the problem concerned at present $\infty>m\geqslant 1/2$. Since, at $z\to\infty$, $f(t)\to 0$, the solution of (25) is $\sin pt$ or $\cos pt$ at $z\to\infty$; any particular remarks to the solution at $z\to\infty$ will not be necessary. The debatable point is in the behavior of the solution at $z\to 0$. Of the two terms in $f(t)$, C_1's term predominates at $z\to 0$, and then $z\fallingdotseq V_0t$: the equation (25) is written in the form of

$$\frac{d^2\Phi}{dt^2}+\left\{p^2-\frac{\alpha\varepsilon(t)}{t^2}\right\}\Phi=0\,, \tag{26}$$

3) R. YOSHIYAMA, *Bull. Earthq. Res. Inst.*, **18** (1940), 41.

where $\varepsilon(t)=O(t^{1/(m+1)})$.

When t is small, but p is large and pt is not small, comparing (26) with the well-known Bessel's equation, the solution of (26) is approximately, or, say symbolically, expressed by,

$$\Phi=\sqrt{pt}\,J_n(pt)\,; \qquad \sqrt{pt}\,Y_n(pt)\ ,$$

where $n\doteqdot 1/2+\alpha\varepsilon(t)$, and, at $t\to 0$, $n\to 1/2$.

When t is small, p finite and, therefore, pt is also small, the solution of (26) is approximately

$$\Phi=\sqrt{x}\,I_{m+1}(\beta x^{1/2(m+1)})\,; \qquad \sqrt{x}\,K_{m+1}(\beta x^{1/2(m+1)})\ , \qquad 0\leqslant x\ll 1\ ,$$

where I and K are Bessel's functions of purely imaginary argument, $x=pt$ and $\beta=2(m+1)\alpha^{1/2}p^{-1/2(m+1)}$. As $x\to 0$, the one, $\sqrt{x}\,I_{m+1}$, tends to zero, and the other, $\sqrt{x}\,K_{m+1}$, to a certain finite value.

After all, it is deduced that (26) has two solutions such that either of which are linearly independent of each other, convergent at $t=0$ and tend to $\sin pt$ or $\cos pt$ at $t\to\infty$ as far as $\infty>m\geqslant 0$, and, therefore, that (25) has at $t=0$ an appropriate solution for a requisite boundary condition. Such a result above stated leads us to a conclusion that the surface condition (8), though it may appear unnatural, cannot be rejected by mere theory of elastic waves.

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49. 走時曲線の二次微係数について

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地表に震源があるとき走時曲線の二次微係数は之を通常ゼロとするのが理論的には正しいとされているが必ずしもそうとは云えない. 結局は通常と異常とを区別する規準の問題であり, それを論ずるのも理論的研究の役目であるが, この際, 理論をまた観測資料整理の指針と見るならば, 少くとも現在の観測技術とその精度のもとではむしろその様な従来の結論に執着することによって失うところの方が大きいであろう.