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31. *Propagation of Elastic Waves through a Heterogeneous Medium with Periodic Structures.—Application to Certain Seismological Problems.**

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1. Introduction

In early investigations (Onda, 1964, 1965, 1966 *a*, *b*, 1967), several characteristics of an elastic wave propagated through a heterogeneous medium with certain periodic structures are deduced by means of Hill's equation. An independent spatial variable is first transformed into a corresponding travel time and a term involving frequency is separated from the variable in the wave equation. The cases with periodic variation in velocity and in elasticity for which the variable is expressed in terms of an explicit analytic function are calculated, and we find that the characteristics of these solutions are similar to each other. An approximate solution for the more general heterogeneity of structures is also obtained.

The solution of the wave equation is associated with either an unstable or stable region, following Floquet's theorem. The wave in a periodic structure cannot be separated into a progressive and a retrogressive wave. The solution is not an explicit form of a progressive

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wave, but an expression of superposition of higher modes corresponding to scattered waves as suggested by Floquet's theorem. The stable and unstable regions exist alternately on the frequency axis. The region which plays the most important role is the first unstable region, that is, the lowest frequency domain among the unstable regions. Other frequency domains can be regarded as appearing in stable regions under the numerical accuracy treated in geophysical problems, even if they belong to one of the unstable regions defined by Floquet's theorem. If the undulatory structure covers a wide area, therefore, wave motions expressed by the first unstable solution probably predominate and may be interpreted as a type of standing wave with scattered waves of higher modes resonating in the periodic structure.

The stability of progressive waves is studied by inserting a heterogeneous medium with a certain periodic structure between two homogeneous media, in one of which an original wave is incident. In the result, we can see that the wave belonging to the first unstable region on the spectrum of a transmitted wave is attenuated. Moreover, from a study of the transmission of certain pulses, this apparent attenuation turns out to be caused by multiple reflections in this medium: a portion of the energy of the incident wave is stored in this medium, and is spread out after the transmission of the main part of the wave. If we consider multiple reflections of a wave passing through a periodic structure, an interpretation that a resonance develops in this medium is possible. There are many phenomena in which an apparently sinusoidal oscillation is observed for rather long duration in seismology: oscillatory seismograms of near earthquakes, microseisms, wave guide phenomena in seismic exploration and so on. These phenomena will be attributed to the nature of the above-mentioned resonance.

The flow of seismic energy from an earthquake in the near range was estimated by using the diffusion equation with a diffusivity varying directly as frequency by Wesley (1965). In that paper, the effect of dispersion of surface waves, the study of which has already been performed fairly well, had not been taken into consideration, and the physical meaning of each term of the equation had not been stated clearly. Accordingly, the quantitative argument seems to be indefinite. A similar equation had been derived by Dr. Takahasi (1937). In that study, the crust is assumed to consist of block structures, and conservation of wave energy coming into and out of a block yields a diffusion equation of the wave energy. Many phenomena, such as a proper oscillation of the ground, a decay of the maximum amplitudes and total duration of seismic waves he attempted to explain by means of multiple reflections of waves in the block structure. As for the block structure, a treatment by Hill's equation was tried by Prof. Yoshiyama (1941).

2. Wave passing along an uneven surface

Many investigations have been published on the elastic wave propagation along an uneven surface, but the problem has not been successfully solved because of mathematical difficulties. These can be classified by mathematical and physical treatments as follows:

1) *Perturbation method.* Stress on an uneven surface, the mean level of which is $z=0$, is approximately calculated in terms of the value of stress on $z=0$ by means of the first two terms of a Taylor series. It is well-known that the equation of motion with a periodically restitutive force is a type of Hill's equation. If the boundary surface is undulatory, the additional stress on the mean level is distributed regularly, and then it is suggested that the equation of motion concerned may be a Hill's equation. But if Mathieu's equation is solved by means of a general perturbation method, Floquet's theorem cannot be introduced; the solution in any unstable region is not defined. From this fact, it is doubtful whether the solution derived from the perturbation method is reasonable or not (Homma, 1941, 1942*a*, *b*; Sato, 1955*a*; Brekhovskikh, 1959; Gilbert and Knopoff, 1960; Kuo and Nafe, 1962; *etc.*).

2) *Rayleigh's method.* This is the procedure that was first applied to the theory of diffraction grating by Lord Rayleigh (1907). As Floquet's theorem is expressed by means of a superposition of higher mode scattered waves, the calculation is very laborious to establish results. From a view-point of numerical analysis, this procedure will be of interest (Sato, 1955*b*; Asano, 1960, 1961, Abubakar, 1962; Miller, 1964; Bose, 1966; *etc.*).

3) *Conformal mapping.* An orthogonal curvilinear coordinate system, in which a boundary condition is satisfied on one or two coordinates, is selected and the wave equation is transformed from Cartesian coordinates into these coordinates. It is, in general, unavoidable that the equation in this coordinate system is fairly complicated. However, if the type of solution can be deduced by means of some treatment, the nature of the wave propagation can be estimated accurately. From a

view-point of mathematical analysis, this method is of interest (Obukhov, 1963; *etc.*).

4) *Others.* Statistics in which physical quantities were averaged over one cycle of an undulatory surface have been studied (de Noyer, 1961), but we can never estimate the accuracy of results by that method.

In this section, we study SH wave propagation along an uneven surface with the same characteristics as an SH wave passing along a plane surface possesses by the methods of the conformal mapping technique.

The wave equation of SH motion is expressed as

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial z^2}+k_0^2 U=0 , \tag{2.1}$$

where

$$k_0=\omega/c_0 ,$$

U is the y-component of displacement, and c_0 is the shear wave velocity. The solution for which a surface specified by $z=0$ is a free surface is well-known:

$$U=U_0 \exp(i\omega t \pm ik_0 x \cos\theta) \cos(k_0 z \sin\theta) , \tag{2.2}$$

where θ is the angle of propagation direction measured from the x-axis and U_0 is the amplitude of the incident wave. If a wave is propagated along the x-axis, θ goes to zero and the wave is expressed by the form

$$U=U_0 \exp(i\omega t \pm ik_0 x) , \tag{2.3}$$

which, of course, fulfills the boundary condition at a free surface. The stress $\widehat{yz}$ introduced by the wave vanishes at all points in the medium.

Now, let us take a complex variable

$$\zeta=-\gamma' z+i\gamma' x , \tag{2.4}$$

and introduce a conformal mapping function

$$w(\zeta)=u+iv , \tag{2.5}$$

where γ' is a constant with the dimension of inverse length. If the

(z, x) plane is transformed into the (u, v) plane, equation (2.1) becomes

$$\frac{\partial^2 U}{\partial u^2}+\frac{\partial^2 U}{\partial v^2}+\frac{k_0^2}{\gamma'^2 h^2}U=0\ , \tag{2.6}$$

where

$$h^{-2}=\left\{\frac{\partial}{\partial u}(\gamma' x)\right\}^2+\left\{\frac{\partial}{\partial v}(\gamma' x)\right\}^2=\left\{\frac{\partial}{\partial u}(\gamma' z)\right\}^2+\left\{\frac{\partial}{\partial v}(\gamma' z)\right\}^2 . \tag{2.7}$$

If a free surface which is slightly uneven is specified by $u=u_0$, the above-mentioned condition may be equivalent to one in which the stress $\widehat{yu}$ and its gradient vanish. Therefore, the conditions are given, respectively, as

$$\frac{\widehat{yu}}{\mu}=\frac{\partial U}{\partial u}=0\ , \qquad \frac{\partial^2 U}{\partial u^2}=0\ . \tag{2.8}$$

If φ is taken as U/h, substitution of the conditions (2.8) into equation (2.6) yields

$$\frac{d^2\varphi}{dv^2}+\left(\frac{2}{h}\frac{\partial h}{\partial v}\right)_{u_0}\frac{d\varphi}{dv}+\frac{k_0^2}{\gamma'^2 h^2(u_0)}\varphi\{1+G(u_0, v)\}=0\ , \tag{2.9}$$

where

$$G(u, v)=\frac{\gamma'^2 h}{k_0^2}\frac{\partial^2 h}{\partial v^2}\ .$$

For u_0 such that

$$1\gg G(u_0, v)\ , \tag{2.10}$$

equation (2.9) is formally equivalent to a wave equation in a heterogeneous medium of one dimension:

$$\frac{d^2U}{dx^2}+\frac{2}{c}\frac{dc}{dx}\frac{dU}{dx}+\frac{\omega^2}{c^2}U=c\ . \tag{2.11}$$

The analogy is described in Table 1.

Table 1. Correspondence of elements in the problems of uneven surface and of a heterogeneous medium.

Uneven surface in a homogeneous medium	A heterogeneous medium
c_0h	$c(x)$
$\varphi=U/h$	U
v	$\gamma\cdot x$
$c_0k_0=\omega$	ω

Now, let us consider the following mapping function:

$$\zeta = w - e^{-w} , \qquad \gamma' = \gamma , \tag{2.12}$$

or

$$\gamma x = v + e^{-u} \sin v ,$$

$$\gamma z = -u + e^{-u} \cos v .$$

The mapping represents an epi-trochoid for $u<0$, a cycloid for $u=0$ and a hypo-trochoid for $u>0$, as shown in Fig. 1.

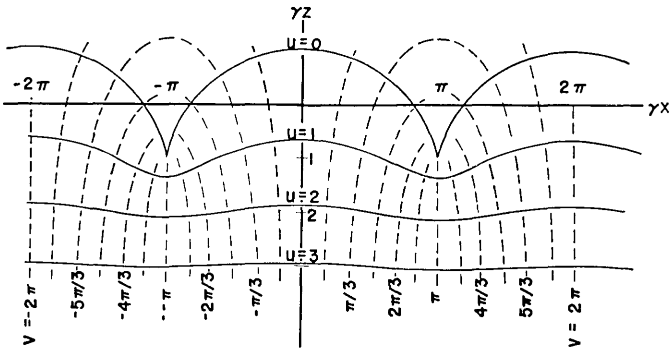

Fig. 1. Conformal mapping of the (u, v) plane on the (z, x) plane: $\zeta = w - e^{-w}$, where $w = u + iv$ and $\zeta = -\gamma z + i\gamma x$.

If u_0 is positive and large, a plane $u = u_0$ is regarded as a sinusoidal plane in the (z, x) plane with a wave length of undulation, $2\pi/\gamma$, and an undulation amplitude, $\exp(-u_0)/\gamma$. Under this consideration, calculation is carried out. The scale factor of mapping is

$$h^{-2} = 1 + 2e^{-u} \cos v + e^{-2u} , \tag{2.13}$$

and a measure of approximation (2.10) is

$$G(u_0, v) = \frac{\gamma^2 e^{-u_0}\{\cos v + e^{-u_0}(3 - \cos^2 v) + e^{-2u_0} \cos v\}}{k_0^2(1 + 2e^{-u_0} \cos v + e^{-2u_0})^3} \leq G(u_0, 0) . \tag{2.14}$$

These expressions are written by means of the wave length of the wave and the undulation, Λ and L, respectively, and an undulation amplitude Γ as

$$\left.\begin{aligned} \gamma/k_0 &= \Lambda/L \ , \\ \exp(-u_0) &= \Gamma\cdot\gamma = 2\pi\Gamma/L \ . \end{aligned}\right\} \tag{2.15}$$

Then

$$G(u_0, 0) = \left(\frac{\Lambda}{L}\right)^2 \frac{e^{-u_0}}{(1+e^{-u_0})^4} \ . \tag{2.16}$$

According to the results of the previous study (Onda, 1966*a*), waves most affected by a periodic structure have a wave length nearly equal to twice that of the structural wave length, so that, if an approximation

$$\exp(-u_0) \ll 1 \tag{2.17}$$

is permitted, the wave behaviour along an undulatory surface may be described by means of a wave equation in the heterogeneous medium with a periodic structure.[1)]

Since waves of maximum amplitudes observed in near earthquakes are surface waves with a wave length of 10 to 20 km, the structural wave length that has the greatest effect on the wave propagation is 5 to 10 km. On the other hand, since an elevation from trough to peak in a mountain range where adjacent peaks are distant by 10 km will be less than a few hundred meters, Γ/L is of the order 10^{-2}. Accordingly, the approximate calculation treated in this section will be applicable to the waves concerned.

From Table 1, the scale factor of mapping, h, corresponds to the velocity variation in a heterogeneous medium:

$$c_0 h = \frac{c_0}{\sqrt{1+2e^{-u_0}\cos v + e^{-2u_0}}} = c_0\{1 - e^{u_0}\cos v + O(e^{-2u_0})\} \ . \tag{2.18}$$

Therefore, this problem is equivalent to that in which velocity fluctuation is $\exp(-u_0) = 2\pi\Gamma/L$. Referring to the results in the previous study (Onda, 1966*a*), we see that an apparent attenuation in the spectrum of a progressive wave appears near the specified wave length Λ which

1) *Notes added in Proof*: A term in Eq. (2.6), $\partial^2 U/\partial u^2$, is proportional to the undulation, whereas $\partial^2 U/\partial v^2$ is proportional to the square of the wave number, so that for waves with short wave length $\partial^2 U/\partial u^2$ can be neglected in comparison with $\partial^2 U/\partial v^2$. For waves with a long wave length the effect of the undulation is very little, so that it is reasonable that $\partial^2 U/\partial u^2$ is neglected in comparison with $\partial^2 U/\partial v^2$, likely in the case of short waves. Although further discussions have not been made, this solution will certainly give the most important characteristics of the wave. (Also see the succeeding one of this paper).

is twice the wave length L of a structural heterogeneity, $\Lambda=2\cdot L$, and that waves with this wave length follow as a tail of the principal wave.

The solution is written as follows, with reference to the results obtained in the previous study (Onda, 1964):

$$U=U_1/h^{1/2}\ , \tag{2.19}$$

where, from relations (2.12) and (2.13),

$$h^{-1/2}=1+\frac{1}{2}e^{-u_0}\cos v+O(e^{-2u_0})=1+\frac{1}{2}e^{-u_0}\cos \gamma x+O(e^{-2u_0})\ , \tag{2.20}$$

and U_1 is expressed by two forms, one in an unstable region and one in a stable region. The final form of U_1 in an unstable region is

$$U_1=Ae^{\eta\xi}Y(\xi,\ \sigma)+Be^{-\eta\xi}Y(\xi,\ -\sigma)\ , \tag{2.21}$$

where

$$Y(\xi,\ \pm\sigma)=\sin(\xi\mp\sigma)+\frac{e^{-u_0}}{8}\sin(3\xi\mp\sigma)+O(e^{-2u_0})\ ,$$

$$\eta=\frac{e^{-u_0}}{2}\sin 2\sigma+O(e^{-2u_0})\ ,$$

$$\frac{2\omega}{\gamma c_0}=1+\frac{e^{-u_0}}{2}\cos 2\sigma+O(e^{-2u_0})\ .$$

The form for the stable region is

$$U_1=A'\varphi_s(\xi,\ \nu)+B'\varphi_c(\xi,\ \nu)\ , \tag{2.22}$$

where

$$\varphi_{\substack{s\\c}}(\xi,\ \nu)=\frac{\sin}{\cos}\nu\xi+\frac{e^{-u_0}}{8}\left\{\frac{1}{\nu+1}\frac{\sin}{\cos}(\nu+2)\xi-\frac{1}{\nu-1}\frac{\sin}{\cos}(\nu-2)\xi\right\}+O(e^{-2u_0})\ ,$$

$$\nu=\frac{2\omega}{\gamma c_0}\{1+O(e^{-2u_0})\}\ .$$

A common variable in both solutions is

$$\xi=\frac{1-\varepsilon^2}{2}\int\frac{dv}{h}=\frac{1}{2}\{v+e^{-u_0}\sin v+O(e^{-2u_0})\}=\frac{\gamma x}{2}\{1+O(e^{-2u_0})\}\ . \tag{2.23}$$

It is interesting that the independent variable of solution, ξ, is linearly associated with x.

The wave treated here is similar to a surface wave, *e.g.*, propagated along the surface, but it has not the typical character of concentration of wave energy near the surface. Therefore, notwithstanding that this wave is a particular kind of waves, the main feature of surface wave propagation along an uneven surface must be constituted by the characteristics obtained from this section as the first approximation.

3. A consideration on growth of microseisms and wave guide phenomena

Microseisms are apparently random oscillations of the ground. Most of the investigations in the subject have been directed toward the clarification of the following points: origin of microseisms, nature of microseismic waves, mode of propagation and direction of approach, and statistical properties of microseisms. This section deals with the nature of microseismic wave propagation. Microseisms with periods of several seconds were interpreted as channel waves, such as Lg or Rg waves, or as higher mode surface waves (Gutenberg, 1958). From many observations, however, the direction of approach is hardly unidirectional, so that the validity of such an interpretation is doubtful. Microseisms with periods less than about one second are interpreted as multiple reflections in superficial layered structures (Shima, 1962).

Another phenomenon similar to microseisms is known as a seismic wave guide phenomenon or offshore singing in seismic exploration in shallow-water sea, which has the following characteristics (Ewing, Jardetzky and Press, 1957, p. 184):

a) Large amplitude and long duration;

b) Almost constant frequency train of waves in some cases, fairly simple pattern of beats in others, apparent mixture of several discrete frequencies in others, characterized in all cases by numerous repetitions of a pattern of waves;

c) Occurrence usually when a hard stratum is found at or near the sea floor.

Burg, Ewing, Press and Stulken (1951) stated that waves are propagated by multiple reflections at angles of incidence between the normal and the critical angle for total reflection, under the condition of constructive interference. Therefore, the predominant frequency f_c is expressed by the value

$$f_c = (2n-1)c_p/4H\ , \tag{3.1}$$

where c_p is the sound velocity in water, H is the depth of water, and n is an integer. Although tide causes a depth of water change by about two meters, the predominant frequency in the field experiments at Ariake Sea remained unchanged, after Messrs. Kuroda and Kimura (1962). It is doubtful, therefore, if the predominant frequency is related only with depth of water by means of (3.1). From another experiment at Ariake Sea by Dr. Chujo and Mr. Kimura (1959), such a phenomenon was developed within a certain limited area, and these results also were in agreement with the area generated by the strongly scattered waves in the records of SPARKER. The sea bottom in that area was estimated as a gravel or fine sand layer.

Recently, Bose (1966) calculated the effect of an undulatory sea bottom for very long and short waves by means of Rayleigh's method cited in section 2, and pointed out that this effect cannot be negligible for very short waves and large undulations. It seems to be an unsatisfactory interpretation, for a small undulation was assumed in that calculation, and the wave most affected will be one with twice the wave length of undulation, as discussed in the previous study (Onda, 1964).

A reason why such experiments have not been carried out on microseisms may be that to change a condition from one to another at one place is very difficult.

Now, waves propagated along a homogeneous medium with an undulatory surface are equivalent to ones passing through the heterogeneous medium with a periodic structure, in which the characteristic wave can resonate. It is expected, therefore, that some mode of waves exists in the medium bounded by an undulatory surface, and that this mode is that in a horizontal heterogeneity of the medium. On the other hand, a surface wave in a layered medium has an amplitude characteristic (Satô, 1952; Tazime, 1957; *etc.*). This mode is that in a vertical heterogeneity of the medium.

If the interference between horizontal and vertical resonance can be neglected, the following consideration will be reasonable. The amplitude of surface waves in a two-layered medium depends on the thickness of the superficial layer, H, the wave length of the wave, Λ, the elasticity in the superficial layer and substratum, E_1 and E_2 and the densities, ρ_1 and ρ_2. If a surface or interface is undulatory, the amplitude also depends on the wave length of undulation, L, and its amplitude Γ. From the aspect of dimension analysis, the resulting amplitude can be writen as

$$U(H/\Lambda, E_1/E_2, \rho_1/\rho_2, \Lambda/L, \Gamma/L) . \tag{3.2}$$

Here, the first three terms involve no factors originating from the undulatory boundary, and the last two terms contain no functions of thickness.

On the assumption that the mode in a vertical heterogeneity does not interfere with that in a horizontal one, it is suggested from the vibration of a rectangular membrane that the resultant wave form is expressed by means of the superposition of those modes:

$$U(X_1, X_2) = U_1(X_1) \cdot U_2(X_2) , \tag{3.3}$$

where

$$X_1 = X_1(H/\Lambda, E_1/E_2, \rho_1/\rho_2) ,$$

and

$$X_2 = X_2(\Lambda/L, \Gamma/L) .$$

Here, U_1 is the amplitude characteristic of a mode in a vertical heterogeneity. In a structure where the velocity difference between two media is large, the wave length of the predominant wave is approximately equal to four times the thickness of the layer. On the other hand, U_2 is the amplitude of a mode in a horizontal heterogeneity, and predominates at a wave length nearly equal to twice the wave length of the surface undulation. Strictly speaking, the behaviour of the wave cannot be elucidated without the study of the interference between the two effects, one from the undulatory surface and the other from the vertical heterogeneity of the medium. However, if the surface undulation is small, its effect is small, so that the total effect will be expressed by the sum of each effect. As a cause of generation of a wave with the particular frequency, two effects of an undulatory surface and a vertical heterogeneity are taken into consideration, but the same result as obtained above can be expected under the assumption that an undulatory surface or block structure of the crust is distributed regularly from the neighbourhood of sources to the observatory.

4. Apparent attenuation of elastic waves

There are many investigations on attenuation of elastic wave propagation, and these have been reviewed by Knopoff (1964) and others. The attenuation is measured through Q or Q^{-1}, and the problem concerned is to study the behaviour of Q associated with frequency. Waves measured by many investigators are free oscillations of the earth, body and surface waves from earthquakes, body waves from artificial explosions, or free oscillations and wave propagations excited by ultrasonics. The net result is that the observed Q seems to vary from 100 to 1000 over a frequency range between 10^{-3} and 10^{5} cycles per second as shown in Fig. 2.

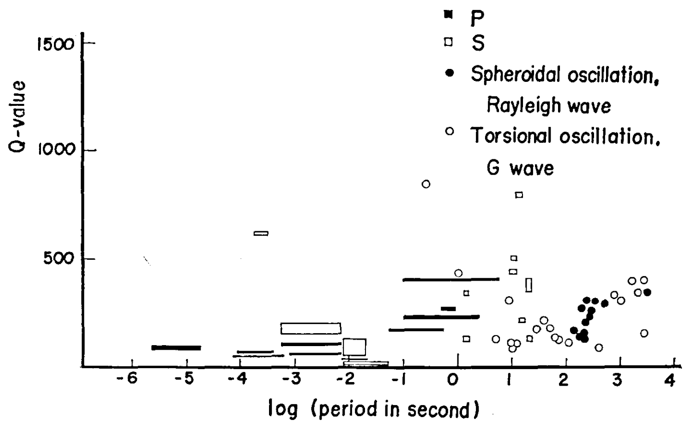

Fig. 2. Q-values taken from various observations.

The elastic wave propagation and attenuation through a viscoelastic medium has been calculated by Prof. Nagaoka, Hosali, Prof. Sezawa, *etc.* Prof. Nagaoka (1906) treated a model taking a resistance proportional to the particle velocity and to the displacement in the wave equation, which seems unacceptable from the rheological viewpoint, and solved it in correspondence to the equation of telegraphy. Hosali (1923) calculated the attenuation, by means of a Voigt model, taking the time derivative of the strain in the wave equation, and Prof. Sezawa (1927) discussed it in further detail. In the Voigt model of a viscoelastic medium, the factor Q is inversely proportional to frequency, and solid viscosity or retardation time is not dependent on frequency. On the other hand, in the Maxwell model of a viscoelastic medium, taking the time derivative of stress, the factor Q is proportional to frequency, and viscosity or relaxation time is not dependent on it.

In theoretical investigations, the attenuation mechanisms which have been proposed can be classified by the following limited list of references:

1) Complex elasticity, Yamakawa and Satô 1964; Anderson and Archambeau, 1964;

2) Creep functions, Lomnitz, 1957; MacDonald, 1961;

3) Non-linear friction, Förtsch, 1956; Knopoff and MacDonald, 1958;

4) Effect of scattering, Yamakawa, 1962; Knopoff and Hudson, 1964;

5) Thermal relaxation, Savage, 1965.

If the complex elasticity is considered fixed, viscosity must be inversely proportional to frequency. It is doubtful whether this dependence is valid, for it is presumed that the attenuation of waves can be expressed by using the invariable complex elasticity, and the attenuated waves have not been analyzed experimentally from any other stand points. Dr. Yamakawa and Prof. Satô (1964) pointed out that the complex elasticity, which depends on frequency, is connected with the creep function. However, the creep functions have generally been determined experimentally under the large strain of the order of 10^{-5} or more. Messrs. Kataoka and Oguri (1959) stated from their experiments that the linear strain theory should be applied for strain as low as the order 10^{-7}, and Peselnick and Outerbridge (1961) found that the critical strain is of the order 10^{-6}. From these experimental facts, the attenuation should be interpreted by means of other mechanisms than creep functions and non-linear friction, since the fundamental equation of motion is based on the linear strain theory, and the strain induced from seismic waves is of the order smaller than 10^{-6}.

Rayleigh pulses on fine grain granite are attenuated similarly to Rayleigh scattering in optics for frequencies as high as 400 kc/sec (Knopoff and Porter, 1963). In the high frequency range, the effect of scattering certainly predominates.

A thermoelastic internal friction between two grains, such as the Zener effect in polycrystalline metals, was considered by Savage (1965), and a relation in which the factor Q^{-1} is constant over a frequency range of 10 cycles to several mega-cycles per second was derived. If ordinary values for physical constants in that relation are taken, the factor Q^{-1} is of the order 10^{-1}, and this order of Q^{-1} seems to be too large compared with the experimental evidence.

Now, the observed spectrum $\bar{U}$ can be represented in the following terms:

$$\bar{U}=\bar{U}_F\cdot\bar{U}_R\cdot\bar{U}_D\cdot\bar{U}_T\cdot\bar{U}_V\cdot\bar{U}_H\cdot\bar{U}_S\ , \tag{4.1}$$

where

$\bar{U}_F$ = spectrum at the hypocentre, involving a focal mechanism,
$\bar{U}_R$ = frequency response of the recorder,
$\bar{U}_D$ = divergence factor of waves,
$\bar{U}_T$ = layering effect of the crust, mantle, core, etc. (large scale heterogeneity),
$\bar{U}_V$ = effect of viscous behaviour,
$\bar{U}_H$ = effect of small scale heterogeneity,
$\bar{U}_S$ = effect of Rayleigh scatterings.

Here, large scale heterogeneity corresponds to the earth model proposed by Jeffreys and Bullen, Gutenberg and others, while small scale heterogeneity is one corresponding to a fluctuation superposed on the large scale one. An influence of thermal losses may be involved in the $\bar{U}_V$ factor.

A paper concerning an influence of large and small scale heterogeneities was recently reported by Dr. Emura (1965). In that paper, it was assumed that the velocity gradient in a medium is sufficiently smaller than the frequency. However, a calculation was made for a medium with a transition zone of gradually increasing velocity, with an extreme velocity difference of 0.8 km/sec and a thickness of 1 km, so that waves with frequencies of several tens of cycles per second should be taken into account; however, those waves are of less interest in seismology. In the other calculation, for a structure with a transition zone including a low velocity layer, the velocity gradient is estimated to be of the order 10^{-2} (per second) so that the calculation can be applied to the analysis of seismometrical observations. It was shown that the effect of a low velocity layer on the transmitted and the reflected waves should not be neglected.

The influence of small scale fluctuating heterogeneity was not calculated because of mathematical complexity until Prof. Yoshiyama (1960) treated it. From the result in the previous studies (Onda, 1964, 1967) waves propagated through the small scale heterogeneity give rise to attenuation. Special interest in the problem of wave propagation through the heterogeneous medium with velocity distribution

$$c(x)=c_0(1+\varepsilon \cos \gamma x) \tag{4.2}$$

is shown for the unstable region, from the result obtained in the previous studies (Onda, 1964, 1967). The transmission coefficient in the unstable region is

$$|T|=|\cos \psi|/\cosh (\eta' \xi_0 \sin 2\sigma) , \tag{4.3}$$

where

$$\xi_0=\frac{\gamma x_0}{2} , \qquad \eta'=\frac{\varepsilon}{2} ,$$

$$\tan \psi=\frac{\tanh (\eta' \xi_0 \sin 2\sigma)(\cos 2\sigma-2\delta-2\delta^2 \cos 2\sigma)}{\sin 2\sigma} ,$$

$$\delta=\frac{\varepsilon}{8}+\varepsilon^2\left(\frac{1}{6}-\frac{\cos 2\sigma}{32}\right) ,$$

$$\frac{2\omega}{\gamma c_0}=1+\frac{\varepsilon}{2} \cos 2\sigma-\frac{7}{16}\varepsilon^2 .$$

From (4.3), an apparent attenuation occurs near the frequency $\omega=c_0\pi/L$, where L is the structural wave length of heterogeneity, and the nature of the attenuation is determined by

$$\left.\begin{aligned} \operatorname{sech} (\eta' \xi_0) &= \operatorname{sech}\left(\varepsilon \frac{c_0\pi}{\omega L} \frac{\omega x_0}{2c_0}\right) , \\ &= \operatorname{sech}\left(\varepsilon \frac{\omega x_0}{2c_0}\right) . \end{aligned}\right\} \tag{4.4}$$

Although the behaviour of the attenuation is rigorously represented by a secant hyperbolic function, the secant hyperbolic function can be expanded as

$$\begin{aligned} \operatorname{sech} \xi &= 2e^{-\xi}\{1+O(e^{-2\xi})\} && \text{for large } \xi , \\ &= e^{-\xi}\{1+O(e^{-\xi} \sinh \xi)\} && \text{for small } \xi . \end{aligned}$$

The term of the wave attenuation can be written as

$$U_0 \exp\left(-\frac{1}{Q} \frac{\omega x}{2c}\right) ,$$

where U_0 is the amplitude at $x=0$. Consequently, the factor Q^{-1} is derived as

$$\left.\begin{aligned} Q^{-1}&=\varepsilon && \text{near the frequency } \omega=\gamma c_0/2\ ,\\ &=0 && \text{for other frequencies.}\end{aligned}\right\} \tag{4.5}$$

If a velocity distribution is given by

$$c(x)=c_0(1+\sum_{r=1}^{r_0}\varepsilon_r \cos r\gamma x)\ ,$$

then from the previous study (Onda 1964) the factor Q^{-1} is

$$\left.\begin{aligned} Q^{-1}&=\varepsilon_n && \text{near } \omega=n\cdot\gamma c_0/2\ ,\ n \text{ being an integer,}\\ &=0 && \text{for other frequencies.}\end{aligned}\right\} \tag{4.6}$$

As an example, let us assume that the crust consists of an alternation of two layers with velocities 3.40 and 3.32 km/sec and thicknesses d_1 and d_2, respectively, whereas the density is uniform throughout the whole medium. The structural wave length is assumed to be 12 km; *i.e.*, $d_1+d_2=12$ km. These velocities were found in the explosion seismic observations near Kamaisi Mine (Asano, Den, Mikumo, Shima and Usami, 1959). The velocity of S waves derived from 16 data points situated 10 to 25 km from the shot epicentre was 3.36 ± 0.04 km/sec. It is assumed that the origin of the ± 0.04 km/sec arises from a horizontally heterogeneous medium. In this medium, the velocity distribution is expressed by means of a Fourier series as

$$\begin{aligned} c&=3.32+0.08Z\sum_{n=0}\sin(n\pi Z)\cos(n\gamma x)/(n\pi Z)\\ &=(3.32+0.08Z)\left[1+\frac{1}{(3.32+0.08Z)}\times\sum_{n=1}\frac{\sin(n\pi Z)\cos(n\gamma x)}{n\pi Z}\right],\quad \text{(km/sec)}\end{aligned}$$

where

$$\gamma=2\pi/12=0.524\,(1/\text{km})\ ,\qquad Z=d_1/(d_1+d_2)\ .$$

The summation is carried out by using all terms with values greater than about 0.007 km/sec. Each coefficient for any Z is shown in Table 2. From (4.5), progressive waves in this medium are attenuated near the periods 7.1, 3.6, 2.4, $\cdots$ seconds and corresponding factors of attenuation, Q, are between 100 and 400.

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Table 2. Fourier coefficients of each term of the velocity distribution for some Z; upper and lower values are c_n and ε_n respectively:

$$c(x)=\sum_{n=1} c_n \cos n\gamma x = c_0(1+\sum_{n=0} \varepsilon_n \cos n\gamma x) \ .$$

Z \ n	0	1	2	3	4
0.1	3.328	0.008 0.003	0.007 0.003	0.007 0.003	0.006 0.002
0.2	3.336	0.014 0.005	0.012 0.004	0.008 0.003	
0.3	3.344	0.020 0.008	0.010 0.004		
0.4	3.352	0.024 0.009	0.006 0.002		
0.5	3.360	0.025 0.009	0.000	−0.008 −0.003	
0.6	3.368	0.024 0.009	−0.006 −0.002		
0.7	3.376	0.020 0.008	−0.010 −0.004		
0.8	3.384	0.014 0.005	−0.012 −0.004	0.008 0.003	
0.9	3.392	0.008 0.003	−0.007 −0.003	0.007 0.003	−0.006 −0.002

may result from block structures of the crust (*e.g.*, Tatel, Adams and Tuve, 1953). As shown in this example, the apparent attenuation resulting from a periodic structure will not be negligible.

Summary and Conclusions

In most of the current investigations, wave propagation has been discussed for a structure with large scale heterogeneity. There are a few observations in which small scale heterogeneity is verified, and theoretical investigations of the latter subject have seldom been performed. This paper analyzes theoretically elastic waves propagated through small scale heterogeneity. The results are as follows:

(1) Substitution of an independent variable into the corresponding travel time for the spatial coordinate leads to the separation of the frequency from the other variables in the wave equation. It is possible to discuss directly the frequency characteristics of solutions of the equation. As the nature of the medium varies periodically, the wave equation treated is Hill's equation. The solution is given by means of

the modified Whittaker's sigma method. Heterogeneity is considered in two cases; in one case, the velocity varies periodically, and in the other, elasticity varies similarly. The independent variable is expressed in terms of a trigonometric function for the velocity variation, while a Jacobi's elliptic function is used for the elasticity variation. Both solutions have strong similarity to each other; the most important solution in wave propagation is associated with the lowest, or first, unstable region. These results hold for rather complicated variations in velocity, too, if the degree of approximation is properly considered. This unstable region occurs when the wave length Λ of waves propagated is nearly equal to twice the wave length L of the structure, that is, $\Lambda=2L$. The solution is given by means of the superposition of higher mode scattered waves and can be regarded as a standing wave. According to Floquet's theorem, if a force or stress is applied at some location in an area widely covered by a periodic structure, a regular oscillation lasting for a relatively long duration will develop into microseisms, so-called wave guide phenomena, and so on.

(2) Stability of a progressive wave is discussed by calculating the transmission coefficient of a wave through this medium. From the result calculated over the whole frequency range, it follows that the first unstable region plays an important role; the larger the structural undulation, the greater becomes the apparent attenuation and the wider is the associated frequency band; and the thicker the heterogeneous medium, the greater becomes the attenuation and the narrower is the frequency band. The attenuation factor Q, obtained from this hypothesis, may be as large as 10^2 to 10^3, provided that the scattering of travel times observed in explosion seismology originates in a certain layering of media with different velocities.

(3) From the integration of the transmission coefficient, the apparent above-mentioned attenuation is interpreted as a portion of wave energy with that frequency component reflected a number of times in the periodic structure, *i.e.*, the unstable solution is regarded as resulting from a resonance in the periodic structure.

(4) Wave propagation along the surface of a homogeneous medium bounded by a sinusoidal surface is solved, assuming that the ratio of elevation to wave length of the undulation is of the order of 10^{-2} or less. The stress induced from SH wave propagation along a flat surface vanishes throughout the whole medium. A suitable conformal mapping is carried out, and an equation of wave motion with a sinu-

soidal character is introduced. The resulting wave equation is in formal agreement with that in a periodic structure. This procedure may not be rigorous, but it seems that the main feature of wave propagation is expressed sufficiently by the solution obtained.

It becomes evident that the effect of small scale heterogeneity on wave propagation is not negligible. It is hoped that the degree of small scale heterogeneity is confirmed from the observational point of view.

In the future, further investigations of problems of a periodic structure with a more complicated variation and of surface waves passing along an uneven surface will be quantitatively performed.

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References

ABUBAKAR, I., 1962, "Reflection and Refraction of Plane SH Waves at Irregular Interfaces," *J. Phys. Earth,* **10**, 1-14, 15-20.

ANDERSON, D. L., and ARCHAMBEAU, C. B., 1964, "The Anelasticity of the Earth," *J. Geophys. Res.*, **69**, 2071-2084.

ASANO, S., 1960, 1961, "Reflection and Refraction of Elastic Waves at a Corrugated Boundary Surface, Part I and II", *Bull. Earthq. Res. Inst.*, **38**, 177-197; **39**, 367-466.

ASANO, S., DEN, N., MIKUMO, T., SHIMA, E., and USAMI, T., 1959, "On the Travel-Times of *S*-Waves, Derived from the Explosion Seismic Observations," *Bull. Earthq. Res. Inst.*, **37**, 279-306.

BOSE, S. K., 1966, "Effect of Undulating Bed on Singing," *Bull. Seism. Soc. Amer.*, **56**, 409-419.

BREKHOVSKIKH, L. M., 1959, "On the Attenuation of Rayleigh Waves during Propagation along an Uneven Surface," *Soviet Physics, Doklady*, **4**, 150-153, (English Translation).

BURG, K. E., EWING, M., PRESS, F., and STULKEN, E. J., 1951, "A Seismic Wave Guide Phenomenon," *Geophysics*, **16**, 594-612.

CHUJO, J., and KIMURA, S., 1959, "A Possible Case of the Wave Guide Phenomenon," *Geophysical Exploration (Butsuri-Tankō)*, **12**, 190-193, (in Japanese).

DENOYER, J., 1961. "The Effect of Variation in Layer Thickness on Love Waves," *Bull. Seism. Soc. Amer.*, **51**, 221-236.

EMURA, K., 1965, "Effects of Transition Zone and Low Velocity Layer on Frequency Spectra of Plane SH Waves," *Sci. Rep. Tôhoku Univ., Ser.* 5, **16**, 164-172.

EWING, M., JARDETZKY, W. S., and PRESS, F., 1957, *Elastic Waves in Layered Media*, McGraw-Hill, 380pp.

FÖRTSCH, O., 1956, "Die Ursachen der Absorption elastischer Wellen," *Ann. Geofisica*, **9**, 469-524.

GILBERT, F., and KNOPOFF, L., 1960, "Seismic Scattering from Topographic Irregularities," *J. Geophys. Res.*, **65**, 3437-3444.

GUTENBERG, B., 1958, "Microseisms," *Advances in Geophysics*, **5**, 51-92.

HOMMA, S., 1941, 1942, "On the Effect of Topography on the Surface Oscillation," *Quart. J. Seism.*, **11**, 349-364; **12**, 17-23, 24-36, (in Japanese).

HOSALI, N. M., 1923, "On Seismic Waves in a Visco-Elastic Earth," *Proc. Roy. Soc. London A*, **104**, 271-278.

KATAOKA, A., and OGURI, M., 1959, "Some Dynamic Properties of Rocks at Room Temperature," *Zisin*, [ii], **12**, 91-100, (in Japanese).

KNOPOFF, L., 1964, "Q", *Reviews of Geophysics*, **2**, 625-660.

KNOPOFF, L., and HUDSON, J. A., 1964, "Scattering of Elastic Waves by Small Inhomogeneities," *J. Acoust. Soc. Amer.*, **36**, 338-343.

KNOPOFF, L., and MACDONALD, G. J. F., 1958, "Attenuation of Small Amplitude Stress Waves in Solids," *Reviews of Modern Physics*, **30**, 1178-1192.

KNOPOFF, L., and PORTER, L. D., 1963, "Attenuation of Surface Waves in a Granular Material," *J. Geophys. Res.*, **68**, 6317-6321.

KUO, J. T., and NAFE, J., 1962, "Periodic Equation of Rayleigh Waves in a Layer overlying a Half-space with a Sinusoidal Interface," *Bull. Seism. Soc. Amer.*, **52**, 807-822.

KURODA, H., and KIMURA, S., 1962, "Field Experiment of Wave Guide on Ariake Sea," *Geophysical Exploration (Butsuri Tankō)*, **15**, 138-145, (in Japanese).

LOMNITZ, C., 1957, "Linear Dissipation of Solids," *J. Appl. Phys.*, **28**, 201-205.

MACDONALD, J. R., 1961, "Theory and Application of Superposition Model of Internal Friction and Creep," *J. Appl. Phys.*, **32**, 2385-2398.

MILLER, M. K., 1964, "Acoustic Wave Motion along a Periodic Surface," *J. Acoust. Soc. Amer.*, **36**, 2143-2148.

NAGAOKA, H., 1906, "On Damped Progressive Waves and the Formation of Tail in Distant Earthquakes," *Tôkyô Sûgaku Buturigakkai Kizi-Gaiyô*, **3**, 17-26.

OBUKHOV, G. G., 1963, "The Effect of Periodic Irregularities in a Relief on the Dispersion Curves of Seismic Surface Waves," *Izvestya, Geophys. Ser.*, 340-342, (English Translation).

ONDA, I., 1964, "Propagation and Apparent Attenuation of Elastic Waves in a Heterogeneous Medium with Certain Periodic Structures," *Bull. Earthq. Res. Inst.*, **42**, 427-447.

ONDA, I., 1965, "Effects of Intervenient Boundaries on the Elastic Propagation," *Bull. Earthq. Res. Inst.*, **43**, 1-14.

ONDA, I., 1966*a*, "Plane Pulse Propagation through a Heterogeneous Medium with a Periodic Structure," *Bull. Earthq. Res. Inst.*, **44**, 1-12.

ONDA, I., 1966*b*, "Solution of Elastic Waves through a Heterogeneous Medeum with Regular Variation in Elasticity," *Bull. Earthq. Res. Inst.*, **44**, 765-777.

ONDA, I., 1967, "Notes on Propagation of Elastic Waves through a Heterogeneous Medium with Periodic Structures," *Bull. Earthq. Res. Inst.*, **45**, 559-566.

PESELNICK, L., and OUTERBRIDGE, W. F., 1961, "Internal Friction in Shear and Shear Modulus of Solenhofen Limestone over a Frequency Range of 10^7 cycles per second," *J. Geophys. Res.*, **66**, 581-588.

RAYLEIGH, LORD, 1907, "On the Dynamical Theory of Gratings," *Proc. Roy. Soc. London*

A, **79**, 399-417.

SATO, R., 1955*a*, "The Reflection of Elastic Waves on Corrugated Surface," *Zisin*, [ii], **8**, 8-22, (in Japanese).

SATO, R., 1955*b*, "On Rayleigh Waves Generated at Rough Surfaces (I)," *Zisin*, [ii], **8**, 121-137, (in Japanese).

SATÔ, Y., 1952, "Study of Surface Waves VI. Generation of Love- and Other Type of SH-Waves," *Bull. Earthq. Res. Inst.*, **30**, 101-120.

SAVAGE, J. C., 1965, "Attenuation of Elastic Waves in Granular Mediums," *J. Geophys. Res.*, **70**, 3935-3942.

SEZAWA, K., 1927, "On the Decay of Waves in Visco-Elastic Solid Bodies," *Bull. Earthq. Res. Inst.*, **3**, 43-53.

SHIMA, E., 1962, "Modifications of Seismic Waves in Superficial Soil Layers as Verified by Comparative Observations on and beneath the Surface," *Bull. Earthq. Res. Inst.*, **40**, 187-260.

TAKAHASI, K., 1937, "Seismic Waves and Block Structure of the Earth's Crust," *Geophys. Mag.*, **11**, 117-138.

TATEL, H. E., ADAMS, L. H., and TUVE, M. A., 1953, "Studies of the Earth's Crust using Waves from Explosions," *Proc. Amer. Phil. Soc.*, **97**, 658-669.

TAZIME, K., 1957, "Minimum Group Velocity, Maximum Amplitude and Quarter Wave-Length Law," *J. Phys. Earth*, **5**, 43-50.

WESLEY, J. P., 1965, "Diffusion of Seismic Energy in the Near Range," *J. Geophys. Res.*, **70**, 5099-5106.

YAMAKAWA, N., 1962, "Scattering and Attenuation of Elastic Waves, I and II," *Geophys. Mag.*, **31**, 63-95, 97-103.

YAMAKAWA, N., and SATÔ, Y., 1964, "Q of Surface Waves," *J. Phys. Earth*, **12**, 5-18.

YOSHIYAMA, R., 1941, "Waves through a Heterogeneous Medium," *Zisin*, **13**, 363-366, (in Japanese).

YOSHIYAMA, R., 1960, "Stability of Waves through a Heterogeneous Medium and Apparent Internal Friction," *Bull. Earthq. Res. Inst.*, **38**, 467-478.

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31. 周期構造を有する不均質媒質を伝わる弾性波 —地震学への応用—

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地球表面の構造や地形の複雑さが，地震波に及ぼす影響を研究するため，これまで単純化された基本的な問題を取扱つて来た．すなわち地震波伝播の媒質として，地殻構造の複雑さを，その中の地震波の伝播速度あるいは弾性定数がそれぞれある値を平均値として，場所的に周期的に変化している状態におきかえた．要するに周期構造を伝播する一次元の波動の特性を詳しく調べた．この問題は数学的にはヒルの方程式の取扱いに帰着される．空間座標として所謂"走時"を用いることにより，地震波の問題について，定量的取扱いに便利な形に波動方程式を書き改めた．そして解はその性質を把握するに便利な点で，"ホイッタカーの σ 法"によつて解かれた．

構造上の波長の約2倍に相当する長さをもつ波動方程式の解は不安定となつており，進行波が受ける周期構造の影響は，この特定の波長を中心として見掛けの減衰を示すことがわかつている．なお，この見掛けの減衰は周期構造の中で生じた共鳴に因るものである．

起伏のある表面に沿つて伝わる波動について最初に考える．平らな表面に沿つて伝わる SH 波によつて誘発される，境界条件を満す応力成分は媒質内の至るところで零である．起伏の続く表面に対して適当な等角写像をとつて，この性質をもつ波動方程式を作ると，その波動方程式の形は周期構造におけるそれに近似される．起伏の続く表面を伝わる表面波の主な性質は第一近似として，この解によつて十分に説明され得るものと思われる．

次に，脈動やウェーブガイド等の現象は規則的な波が可成り長時間続くことが大きな特徴であり，特に周期1秒以下の脈動は地表近くの層内における，またウェーブガイドは海水中における垂直方向の多重反射によつて解釈されている．もしもエネルギー源から観測点まで周期構造が続いているならば，得られるべき振動記録には規則的な波が現われることが予想される．従つて上記の現象についても水平方向の不均質を考慮に入れる必要があると思われる．

周期構造を透過する波が，特定の周波数のところで見掛けの吸収が現われることを示した．透過係数のへこみ量は厳密にいえば，双曲線正割関数で与えられるけれども，これを指数函数で近似して Q 値に相当する量を求めた．浅野博士その他によつて研究された釜石爆破の S 波が一例として考えられた．S 波速度の観測誤差が地殻のブロック構造に起因するものと仮定すれば上記の見掛けの Q 値は2から7秒の範囲で100から400なる値が得られた．この大きさは多くの研究から得られている Q 値と同じ大きさである．

以上のように波の伝播に及ぼす周期構造の影響は無視できないことは明らかであり，今後観測から周期構造の程度を確めるようデータが整理されることが望まれる．