BULLETIN OF THE EARTHQUAKE
RESEARCH INSTITUTE
Vol. 44 (1966), pp. 1397-1418

# 66. Reflection and Refraction of Elastic Waves at a Corrugated Boundary Surface. Part III. The Case of Incidence of an SV Wave and the Phase Change.

By Shuzo ASANO,

Earthquake Research Institute.

(Read June 28, 1966.—Received Sept. 30, 1966.)

## Abstract

To ascertain and supplement previous results obtained in the case of a $P$ wave incident upon a corrugated interface, the case of incidence of an $SV$ wave and the phase change upon reflection and refraction were considered. In the case of normal incidence of an $SV$ wave, computation for three models and in the case of oblique incidence of an $SV$ wave that for angles of incidence smaller than 16° for one model were made. Although these results support those for the case of an incident $P$ wave, there are several new features. For example, it is not always valid for large angles of incidence that the effect of corrugation is larger on reflection than on refraction, since the amplitude of regularly reflected $S$ waves is affected by the angle of incidence.

Phase changes upon reflection and refraction are not large for regular waves. The velocity contrast between media concerned seems to play an important role for phase changes. For irregular waves there are large phase changes, but it is considered that the contribution to the amplitude of waves is not large since irregular waves travel in different directions to each other.

## 1. Introduction

In order to see the effect of corrugated interface on the propagation of elastic waves the author presented the results for the case of normal incidence of $SH$, $P$, and $SV$ waves and also for the case of oblique incidence of a $P$ wave by applying Rayleigh's method.[1),2)] The formulas were derived for an arbitrary periodic interface by assuming the amplitude

1) S. ASANO, *Bull. Earthq. Res. Inst.*, **38** (1960), 177; **39** (1961), 367.
2) S. ASANO, *Bull. Seism. Soc. Amer.*, **56** (1966), 201.

and the slope of a corrugated boundary surface to be small. In a case of normal incidence it is found that the effect of corrugated interface on reflection is larger than on refraction and the velocity contrast being important in this problem. In a case of oblique incidence of a $P$ wave for one model, although with small angles of incidence, it was concluded that almost all results for the case of normal incidence are supported, the effect of the angle of incidence on the reflected $S$ (both regular and irregular) waves being the largest of all though not large in general, and that large differences in the amplitude of irregular waves at different angles of incidence may be expected, and so on.

In this paper, first the results of incidence of an $SV$ wave will be presented and compared with previous results and, second, phase changes for all kinds of waves upon reflection and refraction at a corrugated interface will be considered.

## 2. Formulas used[3)]

The coordinate system is shown in Fig. 1. The equations of motion are

$$(\nabla^2+h_i^2)\cdot\phi_i=0\ ,\qquad (\nabla^2+\sigma_i h_i^2)\cdot\psi_i=0$$

where $\nabla^2$—Laplacian, $\lambda_i$, $\mu_i$=Lamé's constants,

$\phi_i$, $\psi_i$=displacement potentials, $h_i^2=\rho_i\omega^2/(\lambda_i+2\mu_i)$

$\rho_i$=density, $\sigma_i=(\lambda_i+2\mu_i)/\mu_i$, $i=1$ or 2.

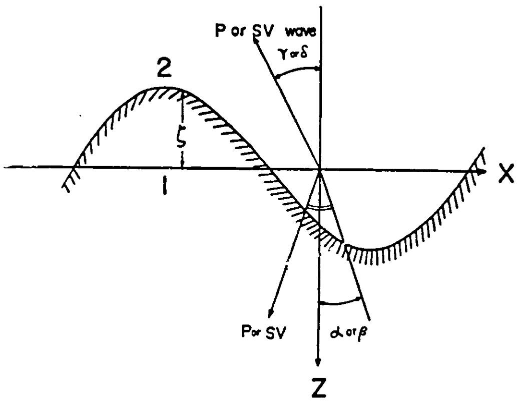


Fig. 1. Coordinate system.

3) Presented in more detail in 1) or 2).

The periodic interface is represented as

$$\zeta=\sum_{n=1}^{\infty}\{\zeta_n\cdot\exp(inpx)+\zeta_{-n}\cdot\exp(-inpx)\}$$

where $\zeta_1=\zeta_{-1}=c_1/2$, $\zeta_{\pm n}=(c_n\pm is_n)/2$, $c_n$ and $s_n$ are the cosine and sine Fourier coefficients, respectively.

The following displacement potentials are adopted:
for the incident $SV$ wave

$$\psi=\exp[i\sqrt{\sigma_1}\,h_1(z\cos\beta+x\sin\beta)]\,,$$

for regular waves
reflected waves:

$$\phi=A_0\exp[ih_1(-z\cos\alpha+x\sin\alpha)]\,,$$
$$\psi=B_0\exp[i\sqrt{\sigma_1}\,h_1(-z\cos\beta+x\sin\beta)]\,;$$

refracted waves:

$$\phi=C_0\exp[ih_2(z\cos\gamma+x\sin\gamma)]\,,$$
$$\psi=D_0\exp[i\sqrt{\sigma_2}\,h_2(z\cos\delta+x\sin\delta)]\,;$$

for irregular waves of the $n$th order
reflected waves:

$$\phi=A_n\exp[ih_1(-z\cos\alpha_n+x\sin\alpha_n)]+A'_n\exp[ih_1(-z\cos\alpha'_n+x\sin\alpha'_n)]\,,$$
$$\psi=B_n\exp[i\sqrt{\sigma_1}\,h_1(-z\cos\beta_n+x\sin\beta_n)]$$
$$+B'_n\exp[i\sqrt{\sigma_1}\,h_1(-z\cos\beta'_n+x\sin\beta'_n)]\,;$$

refracted waves:

$$\phi=C_n\exp[ih_2(z\cos\gamma_n+x\sin\gamma_n)]+C'_n\exp[ih_2(z\cos\gamma'_n+x\sin\gamma'_n)]\,,$$
$$\psi=D_n\exp[i\sqrt{\sigma_2}\,h_2(z\cos\delta_n+x\sin\delta_n)]$$
$$+D'_n\exp[i\sqrt{\sigma_2}\,h_2(z\cos\delta'_n+x\sin\delta'_n)]\,.$$

In the above formulas, the common time factor $\exp(i\omega t)$ is omitted. There are the following relations between $\alpha_n$, $\alpha'_n$, $\beta$, etc:

$$\sin\alpha_n-\sin\alpha=np/h_1\,,\qquad \sin\alpha'_n-\sin\alpha=-np/h_1\,,$$
$$\sin\beta_n-\sin\beta=np/(\sqrt{\sigma_1}\,h_1)\,,\qquad \sin\beta'_n-\sin\beta=-np/(\sqrt{\sigma_1}\,h_1)\,,$$
$$h_1\sin\alpha_n=\sqrt{\sigma_1}\,h_1\sin\beta_n=h_2\sin\gamma_n=\sqrt{\sigma_2}\,h_2\sin\delta_n\,.$$

The above potentials are substituted into the boundary conditions, the continuity of stresses and displacements across the interface $z=\zeta$. Then by assuming $\zeta_{\pm n}$ and $\zeta'_{\pm n}$ to be small and taking terms of the order as high as $\zeta^2_{\pm n}$ into account, the formulas of first approximation for $A_1$, $B_1$, $A'_1$, etc. and then those of second approximation for $A_0$, $B_0$, etc. are derived successively. In computation, $\lambda_i=\mu_i$ and $\zeta=c\cos px$ were assumed.

## 3. The case of incidence of an $SV$ wave

First, in order to see mainly the effect of velocity contrast on this problem, the results of normal incidence of an $SV$ wave for three models in Table 1 are presented in Figs. 2 through 5. These figures show that these results support those in a case of normal incidence of a $P$ wave, that is, the larger the corrugation amplitude, the larger the variation of amplitude with the wavelength of corrugation; the larger the velocity contrast, the larger both the variation of amplitude of elastic waves with the wavelength of corrugation for a given $c/L_{S1}$, where $c$ is the corrugation amplitude and $L_{S1}$, the wavelength of an incident $SV$ wave, and the amount of decrease or increase in the amplitude of elastic waves for a given $L/L_{S1}$ where $L$ is the wavelength of corrugation, there existing a possibility of there being a significant difference in the amplitude of elastic waves for a given interface with a difference in elastic constants (Figs. 2 and 4). In Fig. 3 results of regular waves for the three models are superimposed so that the points for each model in the case of a plane interface, $c=0$, correspond to the same point in the figure. This figure shows results quite similar to those of an incident $P$ wave such that the amplitude of the reflected $SV$ wave decreases while that of the refracted $SV$ wave increases as the corrugation amplitude increases and the amount of decrease in the amplitude of the reflected $SV$ wave is, in general, greater than that of the increase in the amplitude

Table 1. Models used

| $P$ wave velocity in the incident medium km/sec | $P$ wave velocity in the refracted medium km/sec | Rigidity ratio $\mu_2/\mu_1$ |
|---|---|---|
| 2.5 | 5.0 | 4.8* |
| 6.0 | 8.0 | 2.0 |
| 5.5 | 6.0 | 1.2 |

* This is called "the model (2.5/5.0)" in this paper.

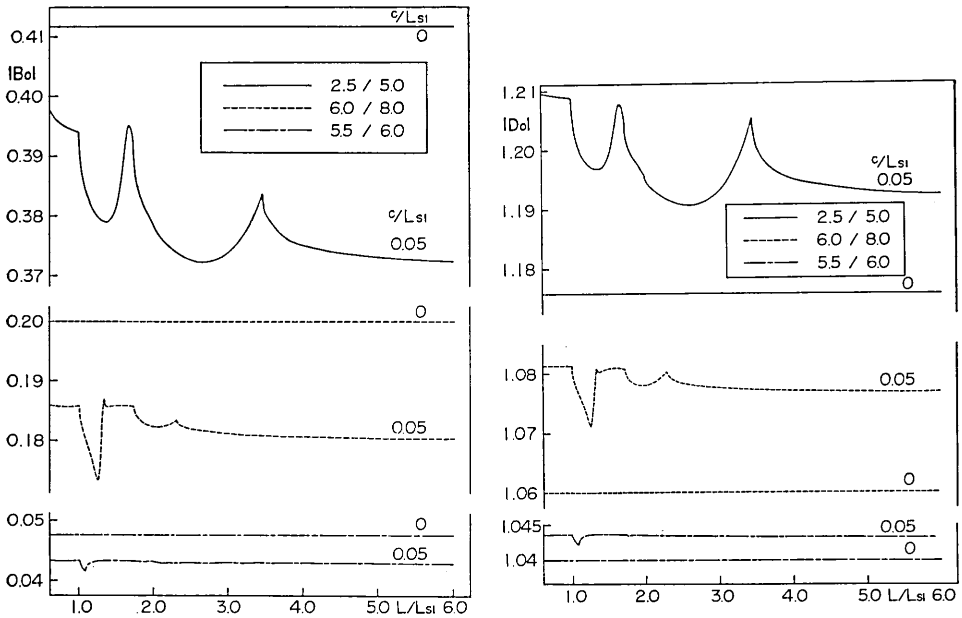


Fig. 2. Reflection ($B_0$) and refraction ($D_0$) coefficients of regular waves for three models in the case of normal incidence of *SV* waves. The abscissa is the wavelength of corrugation in units of the wavelength of the incident *SV* wave.

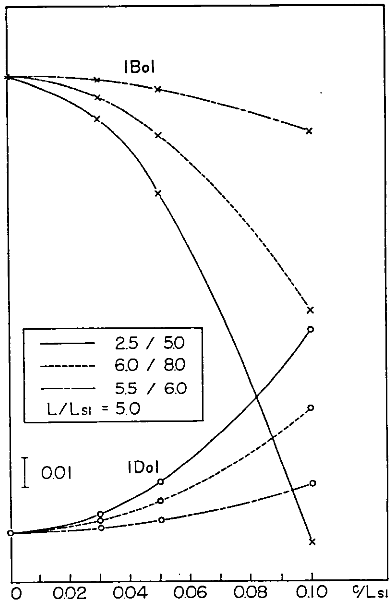


Fig. 3. Reflection and refraction coefficients of regular waves for three models in the case of normal incidence of *SV* waves.

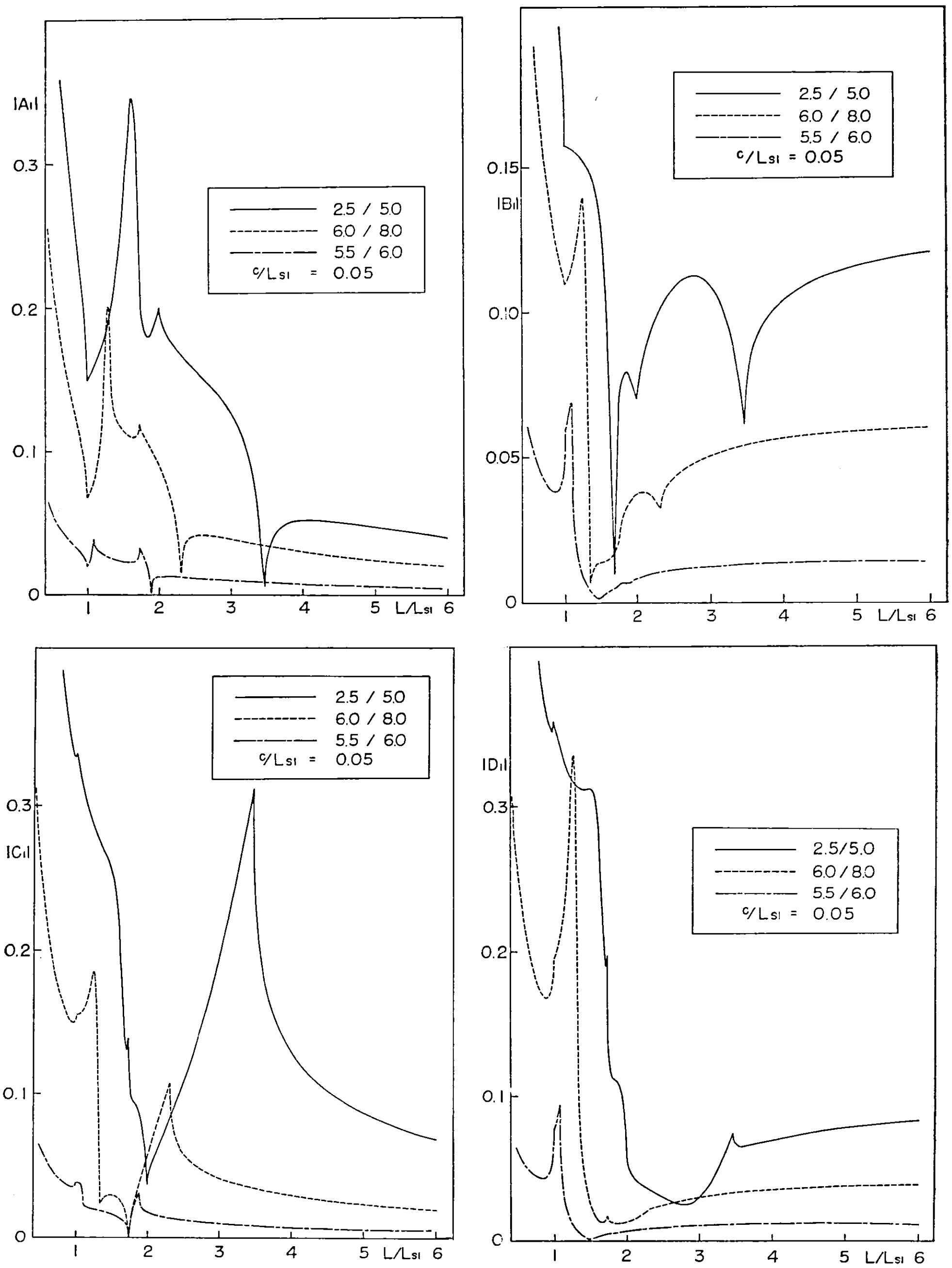


Fig. 4. Reflection and refraction coefficients of irregular waves for three models in the case of normal incidence of *SV* waves.

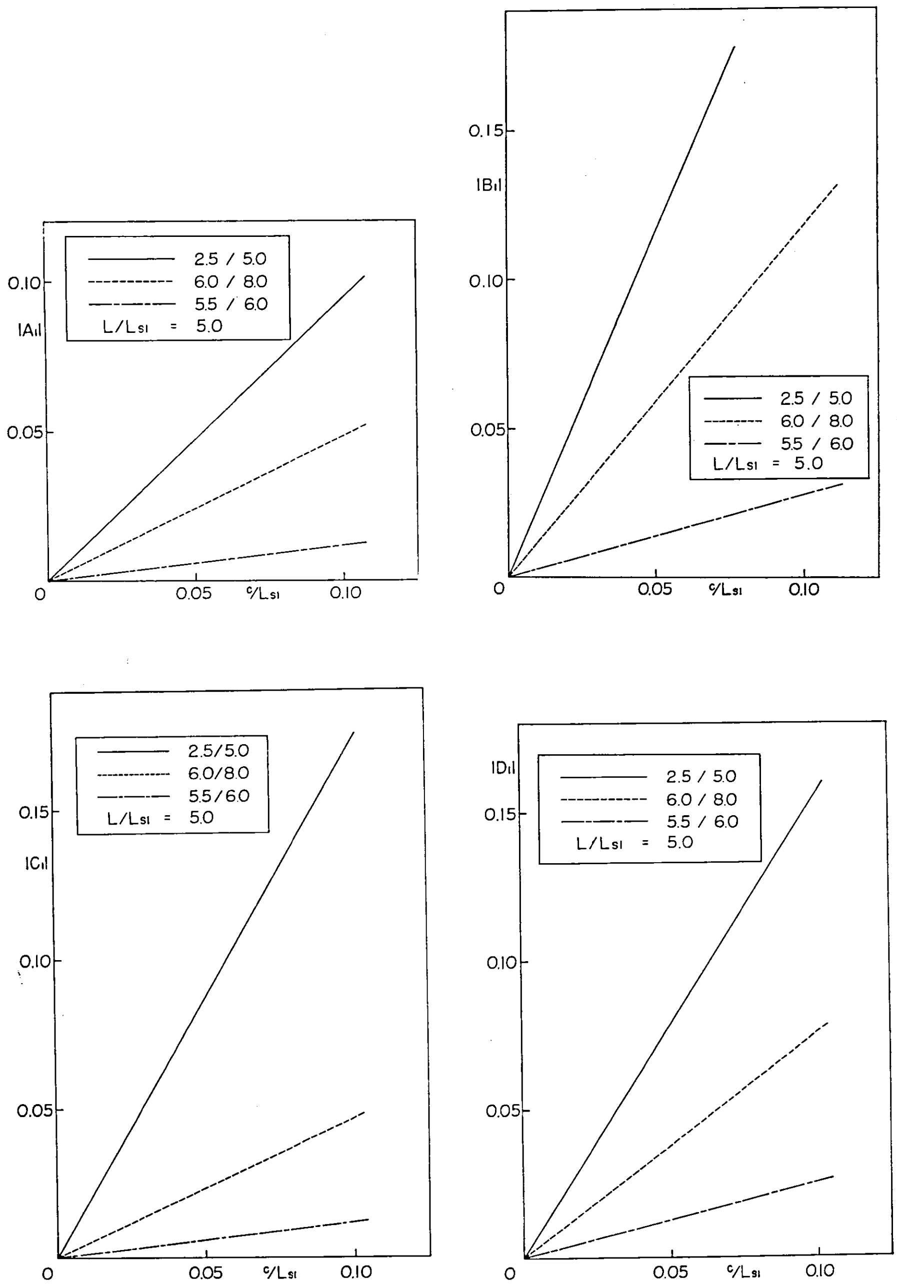


Fig. 5. Reflection and refraction coefficients of irregular waves for three models in the case of normal incidence of *SV* waves.

of the refracted $SV$ wave.

In Fig. 4, it is clear that the amplitude of irregular $P$ waves, especially the reflected wave (represented by $A_1$), and that of the irregularly refracted $SV$ wave (represented by $D_1$) increases with the decrease of wavelength of corrugation in units of that of the incident wave, $L/L_{S1}$. This result corresponds qualitatively to the increase in amplitude of irregular $S$ waves with the decrease of wavelength of corrugation in units of that of the incident wave, $L/L_{P1}$ in the case of a normal incidence of a $P$ wave.

Next, the results of oblique incidence of an $SV$ wave are given mostly for 4° and 16° of angles of incidence in Figs. 6 through 10. The computation was carried out for the model (6.0/8.0) and for angles of incidence smaller than 16°.

Fig. 6 gives the results of regular waves for $c/L_{S1}=0.05$ and shows the following. Most of them are quite similar to those for an incident $P$ wave, but some are different from such.

(1) Regardless of the angle of incidence, the variation of amplitude with the wavelengh of corrugation is not large but becomes larger as the wavelength of corrugation decreases.

(2) Reflected and refracted $P$ waves have a tendency to increase in amplitude as the wavelength of corrugation decreases. This tendency becomes clearer as the angle of incidence increases. For an incident $P$ wave, regular $P$ waves, especially a refracted wave, has a similar tendency. Reflected and refracted $S$ waves have the same tendency for small angles of incidence, though not so clear as $P$ waves, but for the angle of incidence, 16°, there appears a tendency to decrease as the wavelength of corrugation becomes smaller.

(3) The amplitude of $SV$ waves, the same kinds of waves with an incident one, varies more than that of $P$ waves, especially for large angles of incidence although in the case of incidence of a $P$ wave the variation of amplitude of $SV$ waves is large.

(4) The dependency of the amplitude of reflected $P$ wave on the corrugation amplitude in units of wavelength of an incident wave, $c/L_{S1}$, is reversed at a fixed value of wavelength of corrugation, $L/L_{S1}$, that is, in the range of $L/L_{S1}$ larger than this fixed value the amplitude of reflected $P$ wave decreases with the corrugation amplitude increasing, while for another range, it has an opposite tendency. In the case of an incident $P$ wave, only the dependency of refracted $S$ wave on $c/L_{P1}$ is reversed in some ranges of $L/L_{P1}$.

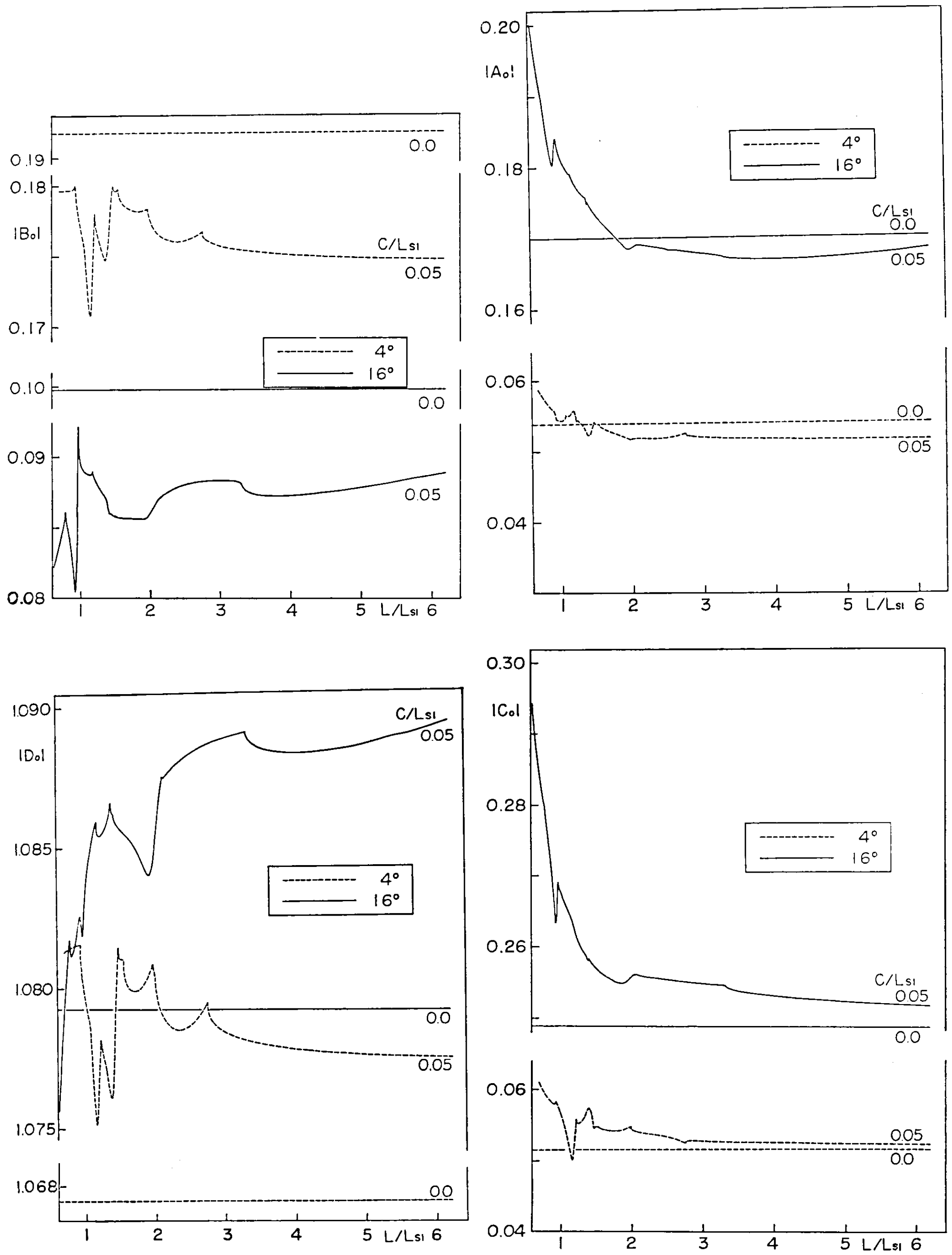


Fig. 6. Reflection and refraction coefficients of regular waves in the case of oblique incidence of *SV* waves. $A_0$ is the reflection coefficient for *P* waves; $D_0$ is the refraction coefficient for *S* waves; and so on. The abscissa is the wavelength of corrugation in units of the wavelength of the incident *SV* wave: the parameter is the amplitude of corrugation in units of the wavelength of the incident *SV* wave.

This feature is quite interesting from the macroseismic view-point, since anomalous distribution of seismic intensity may be explained by this feature. In this calculation, small amplitude of corrugation is assumed, the cases for small angles of incidence only being considered. Yet, the above result for limited cases suggests that for large angles of incidence and for large amplitude of corrugation, the dependency of amplitude of seismic waves on the corrugation amplitude may vary quite significantly so that this may cause the anomalous distribution of seismic intensity. Also there is a possibility of explaining this phenomenon by certain irregular waves as seen from figures for such waves.

In Fig. 7, to see the dependency of amplitude of regular waves on both corrugation amplitude and angles of incidence, all curves for different angles of incidence are superimposed so that the points for $c/L_{S1}=0$ coincide with one another. Therefore, the absolute value of ordinate is arbitrary. The wavelength of corrugation in units of that of incident wave, $L/L_{S1}$, is fixed as 5.8. Fig. 7 shows the following:

(1) The amplitude of reflected $SV$ wave depends fairly largely on

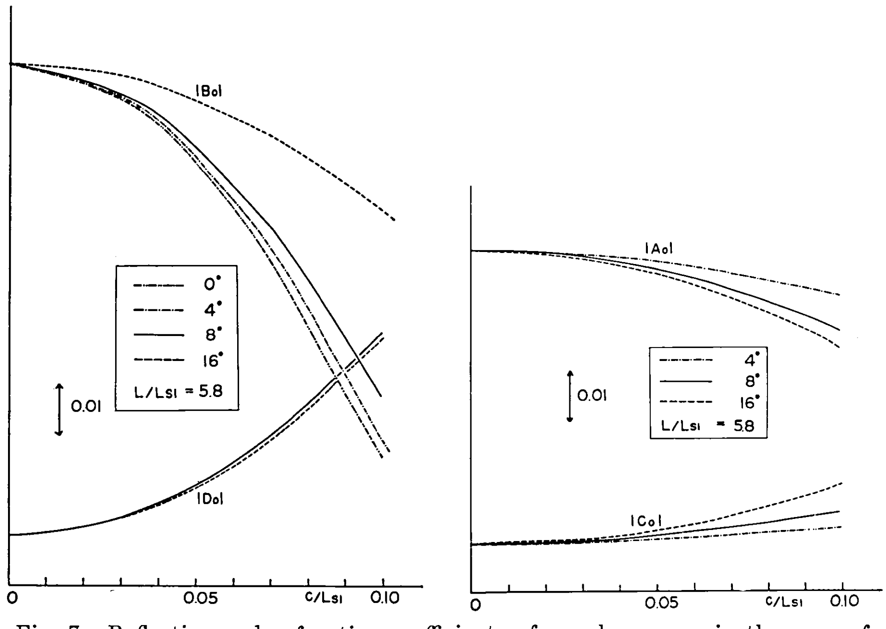


Fig. 7. Reflection and refraction coefficients of regular waves in the case of the oblique incidence of $SV$ waves. $A_0$ is the reflection coefficient for $P$ waves; $D_0$, the refraction coefficient for $S$ waves and so on. The abscissa is the amplitude of corrugation in units of the wavelength of the incident $SV$ wave.

angles of incidence, while that of a refracted $S$ wave has a rather small dependence. The dependency of amplitude of reflected and refracted $P$ waves is between that of $S$ waves. However, the extent of dependency of $S$ waves on the corrugation amplitude, $c/L_{S1}$, larger than that of $P$ waves irrespective of angles of incidence. The amplitude of reflected $P$ as well as $SV$ waves decreases and that of refracted both $P$ and $SV$ waves increases as the corrugation amplitude becomes larger. It is interesting to compare this result with the result for an incident $P$ wave. Irrespective of incident waves, the reflected $SV$ wave varies fairly largely in amplitude. However, as the angles of incidence increase, the amount of decrease in the amplitude of a reflected $S$ wave increases in the case of an incident $P$ wave and decreases in that of an incident $SV$ wave. Since the dependency of amplitude of a regularly reflected $S$ wave on the angle of incidence is large in the case of an incident $SV$ wave, it is not always valid for large angles of incidence that the effect of corrugation is larger on reflection than on refraction.

With regard to irregular (or scattered) waves, two kinds of waves, forward and backward, have to be taken into account for the oblique incidence. In Fig. 8 the results for forward irregular waves and in Fig. 9 those for backward ones are given. The following points can be observed from these figures.

(1) The general pattern of variation of amplitude does not change much with the angles of incidence.

(2) The pattern of variation of amplitude with the wavelength of corrugation is quite similar for both forward and backward irregular waves.

(3) There is a possibility of there being a significant difference in amplitude between different angles of incidence at a given $L/L_{S1}$.

(4) There is a tendency for an increase in amplitude as the wavelength of corrugation decreases.

(5) The amplitude of refracted $S$ waves does not change much with the angles of incidence.

The above features are quite similar to those for an incident $P$ wave.

(6) Among forward irregular waves, the amplitude of $P$ wave components is in general comparable with or larger than that of $S$ wave components. In the case of an incident $P$ wave, the amplitude of $P$ wave components, the same kinds of waves with the incident wave is usually larger than that of $S$ wave components.

(7) Some of the irregular waves, especially backward ones of both

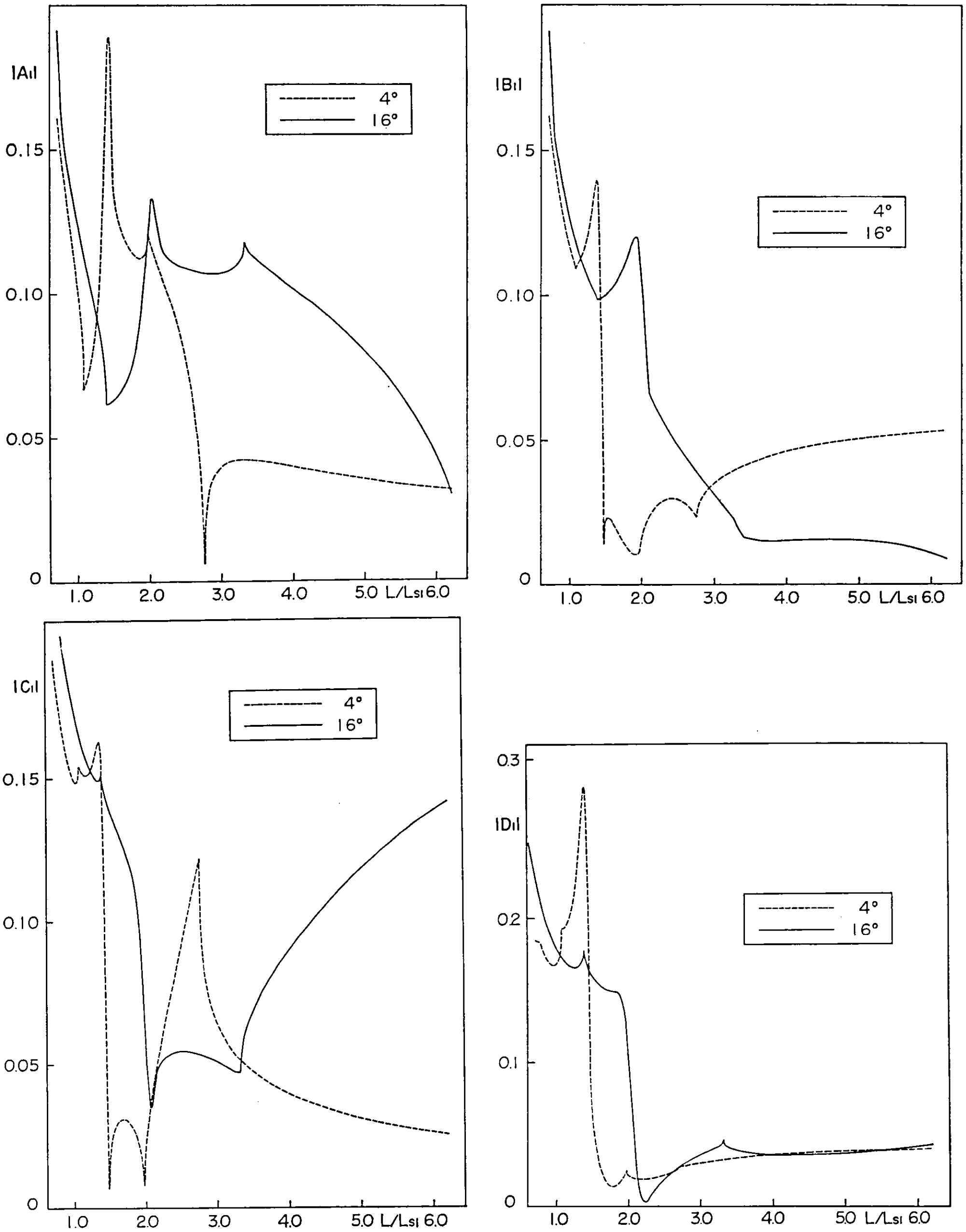


Fig. 8. Reflection and refraction coefficients of forward irregular waves in the case of the oblique incidence of $SV$ waves. $A_1$ is the reflection coefficient for forward irregular $P$ waves; $D_1$, the refraction coefficient for forward irregular $S$ waves; and so on.

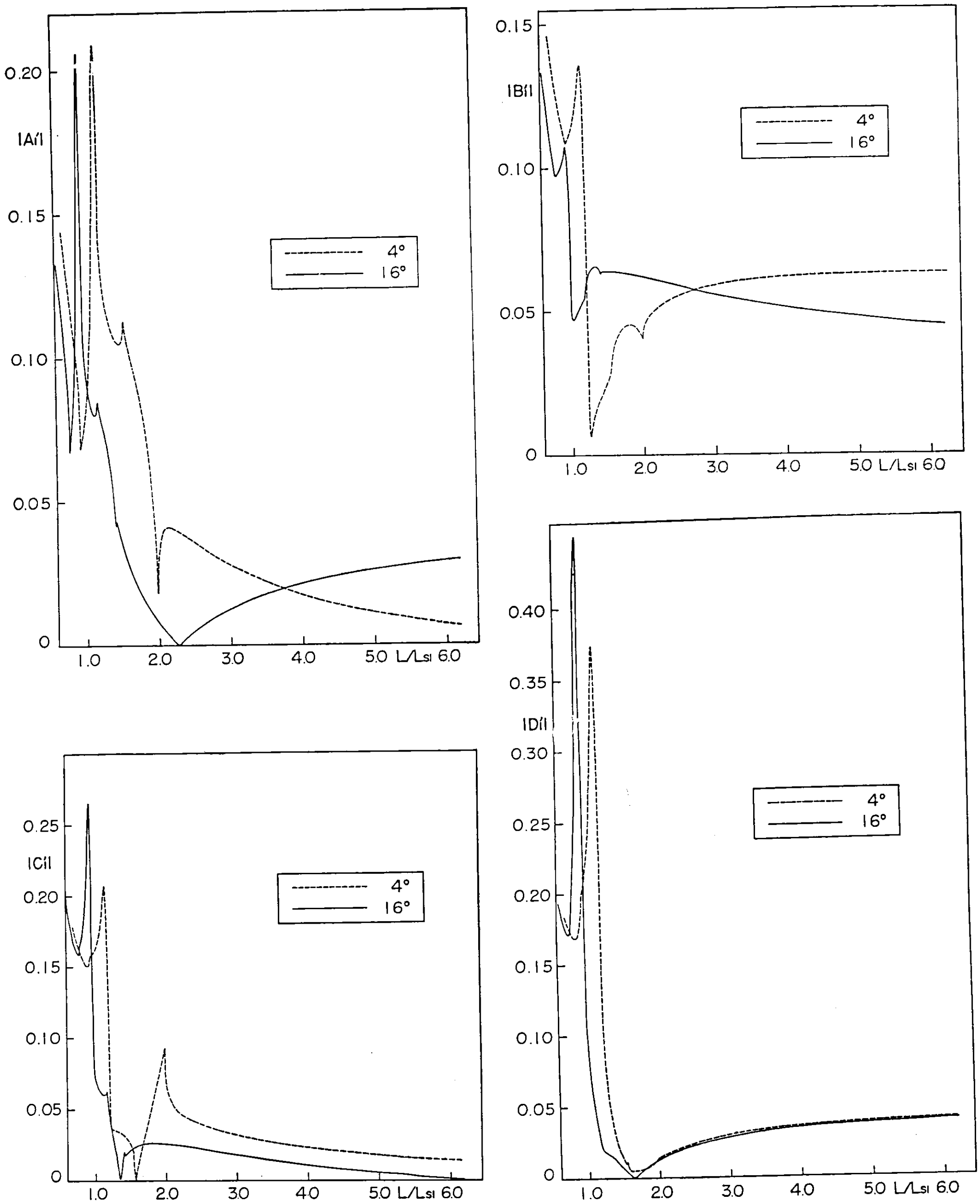


Fig. 9. Reflection and refraction coefficients of backward irregular waves in the case of the oblique incidence of $SV$ waves. $A_1'$ is the reflection coefficient for backward irregular $P$ waves; $D_1'$, the refraction coefficient for backward irregular $S$ waves; and so on.

reflected $P$ and refracted $SV$ waves, have a peak for small $L/L_{S1}$. This feature seems to have a slight dependency on angles of incidence. There is no steep decrease as in the case of an incident $P$ wave.

Fig. 10 shows the dependency of amplitude of irregular waves on the corrugation amplitude. The abscissa of this figure is the corrugation amplitude in units of the wavelength of incident wave, $c/L_{S1}$. From this figure the following can be pointed out.

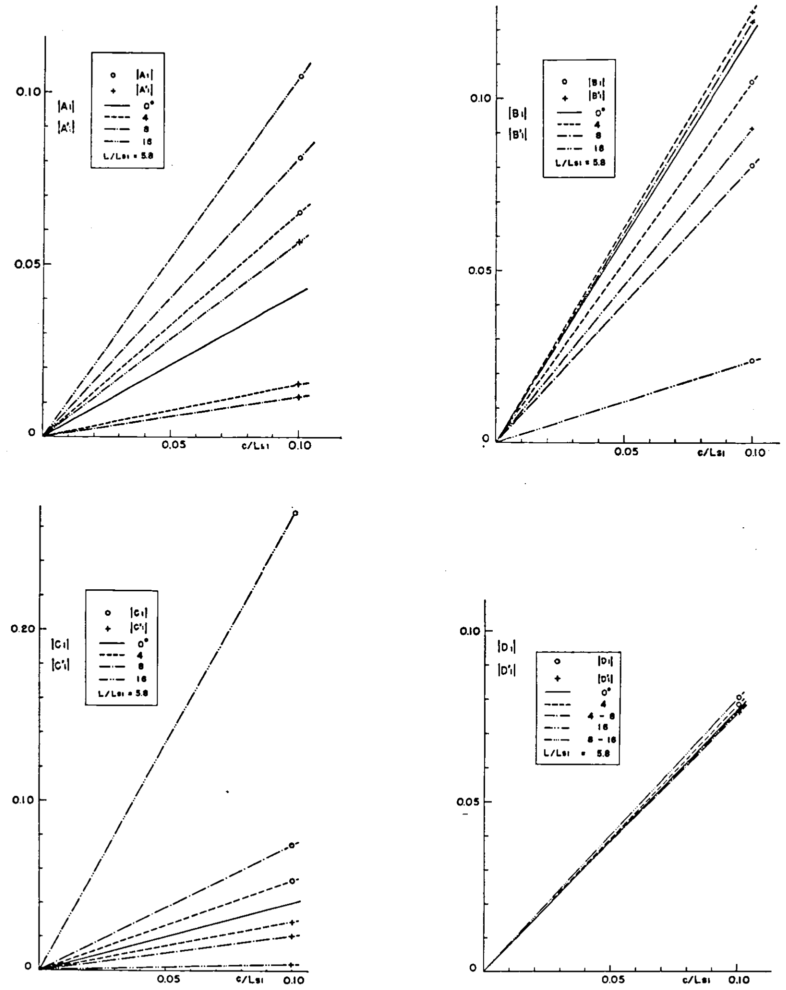


Fig. 10. Reflection and refraction coefficients of irregular waves in the case of the oblique incidence of $SV$ waves. The abscissa is the corrugation amplitude in units of the wavelength of the incident $SV$ wave.

(1) The effect of the corrugation amplitude on refracted $S$ waves is small.

(2) Besides reflected $S$ wave, reflected and refracted $P$ waves are affected fairly largely.

(3) The effect on backward irregularly reflected waves is extraordinary for 16°. This is probably because one of the irregular waves is a boundary wave when $L/L_{S1}=5.8$.

## 4. Phase changes upon reflection and refraction at a corrugated boundary surface

So far attention has been paid to the effect of corrugation on the absolute values of reflection or refraction coefficients, that is, the amplitude of elastic waves, and the consideration of the phase changes of each wave upon reflection and refraction has been omitted. As mentioned previously,[4] once one of the irregular waves becomes a boundary wave, all coefficients become complex numbers. This means that there are phase changes for each wave. For example, since for the regularly reflected $P$ wave a complex coefficient $A_0$ is written as $A_0=|A_0|\cdot\exp\{i\varphi(A_0)\}$, the displacement potential can be represented as follows:

$$\phi=|A_0|\cdot\exp[i\{h_1(x\cdot\sin\alpha-z\cdot\cos\alpha)+\varphi(A_0)\}]\ .$$

For other waves, the formulas for displacement potentials are quite similar. It is interesting to show that even for a boundary wave there is phase change, for example, the displacement potential of forward irregularly reflected $P$ waves is represented as

$$\phi=|A_1|\cdot\exp(-h_1z\cdot|\cos\alpha_1|)\cdot\text{ext}\,[i\{h_1x\sin\alpha_1+\varphi(A_1)\}]$$

which has phase change $\varphi(A_1)$.

If we are concerned with the change of wave form, attention has to be paid only to the case in which the waves concerned travel almost in the same direction with the same phase velocity, since the distortion of wave form is expected by superposition of waves with different phases. Although there are significant phase changes for irregular waves, in general irregular (or scattered) waves propagate with each other in different directions and the contribution from irregular waves can be ignored. On the other hand, if the difference of phase velocity along different wave paths is the subject to be studied, phase changes sometimes become

4) S. ASANO, *loc. cit.*, 1) or 2).

important.

Now, the results mainly for an incident $P$ wave are given in Figs. 11 through 15, the general features being summarized briefly in the following.

(1) In general, the phase changes of regular waves are not large and do not depend much on the wavelength of corrugation (Figs. 11 and 13).

(2) In a case of normal incidence of a $P$ wave, the larger the velocity contrast, the larger the phase changes of regular waves (Fig. 11).

(3) The phase changes are relatively large around the wavelength of corrugation in units of that of incident wave, $L/L_{P1}$, where the variation of amplitude is fairly large irrespective of angles of incidence.

(4) It is interesting to note that the phase change of regular $S$ waves is larger than that of regular $P$ waves for an incident $P$ wave. Since also the amplitude of a regularly reflected $S$ wave is affected more than other regular waves by angles of incidence, it may be said that $S$ wave components are sensitive to the corrugation.

(5) The phase changes of irregular waves are quite large. There are even discontinuous phase changes depending upon the wavelength of

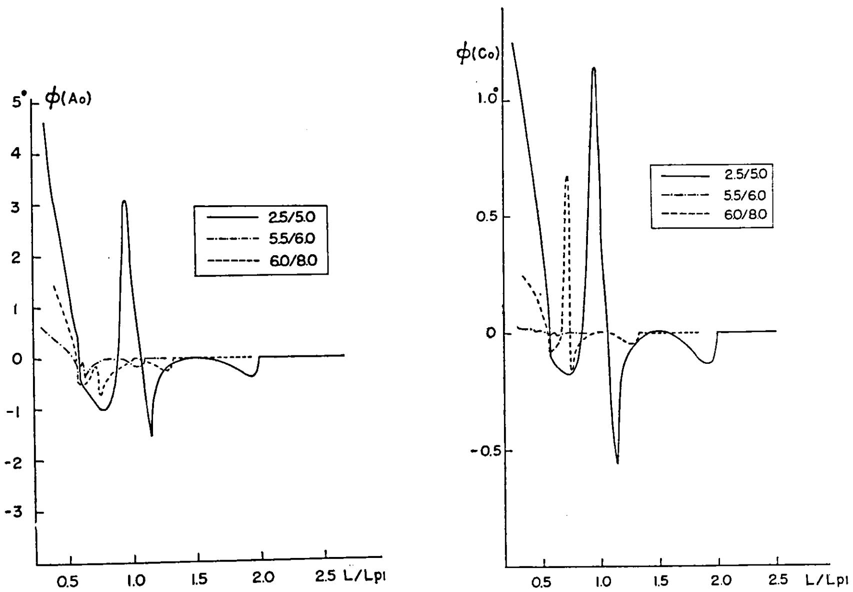


Fig. 11. Phase changes of regular waves upon reflection and refraction for three models in the case of normal incidence of $P$ waves. $\phi(A_0)$ stands for the phase change of regularly reflected $P$ waves; $\phi(C_0)$ is the phase change of regularly refracted $P$ waves. $c/L_{P1}$, the corrugation amplitude to the wavelength of incident $P$ wave, is put equal to be 0.05.

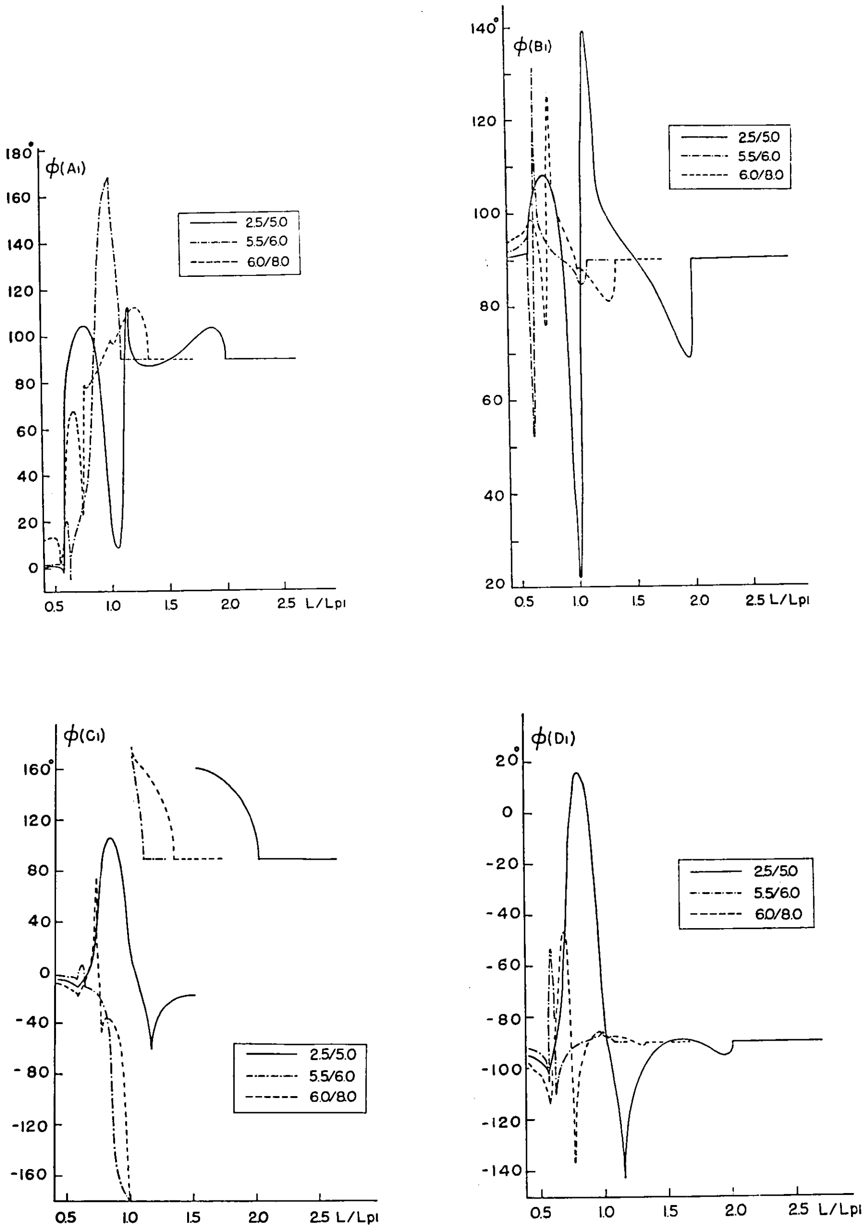


Fig. 12. Phase changes of irregular waves for three models in the case of normal incidence of $P$ waves. $\phi(A_1)$ is the phase change of irregularly reflected $P$ wave; $\phi(D_1)$, that of irregularly refracted $S$ wave and so on.

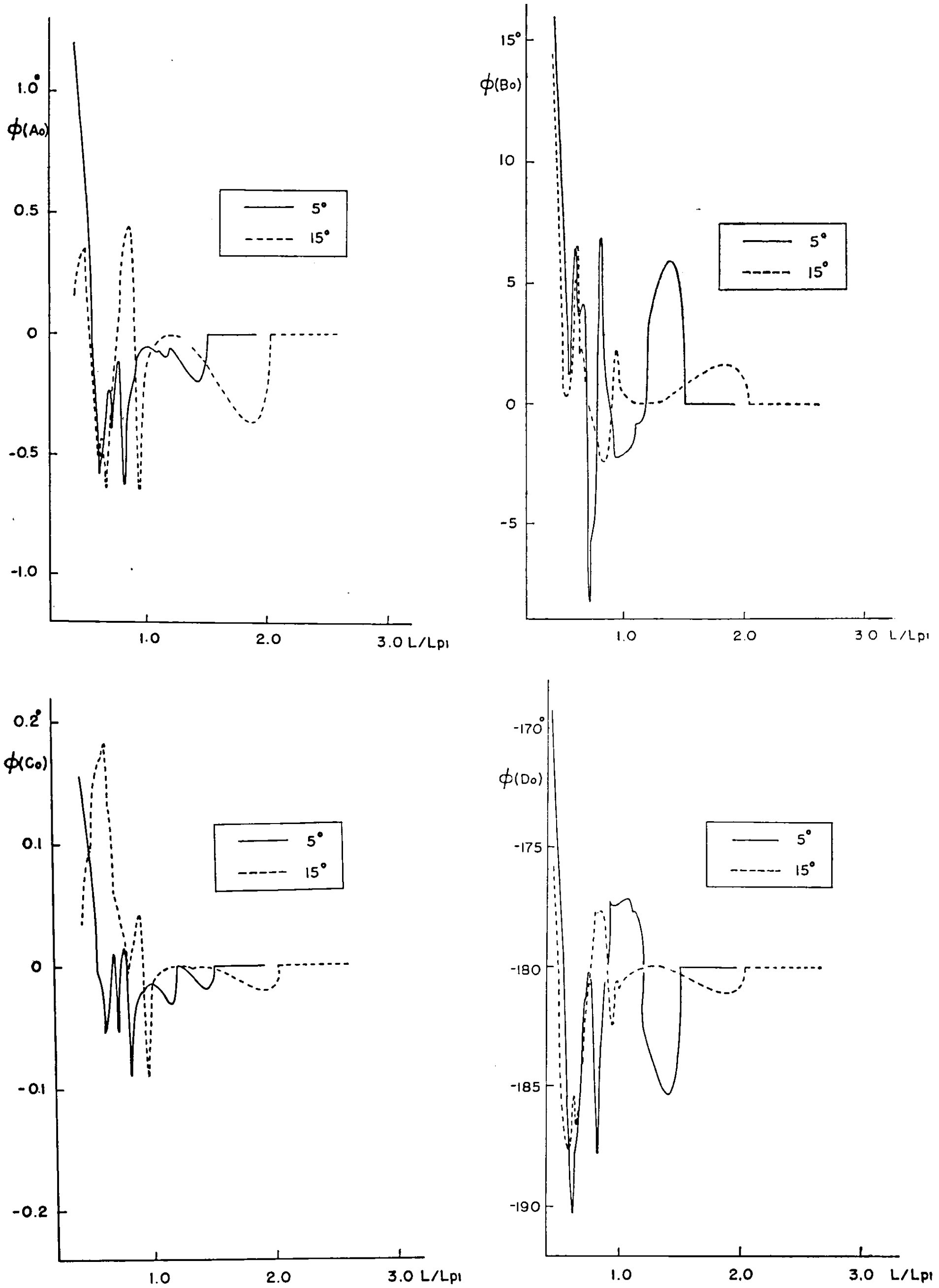


Fig. 13. Phase changes of regular waves in the case of oblique incidence of $P$ waves. For example $\phi(A_0)$ is the phase change of regularly reflected $P$ waves, $\phi(D_0)$, that of regularly refracted $S$ waves. $c/L_{P1}$, the corrugation amplitude to the wavelength of incident $P$ wave, is put equal to 0.05.

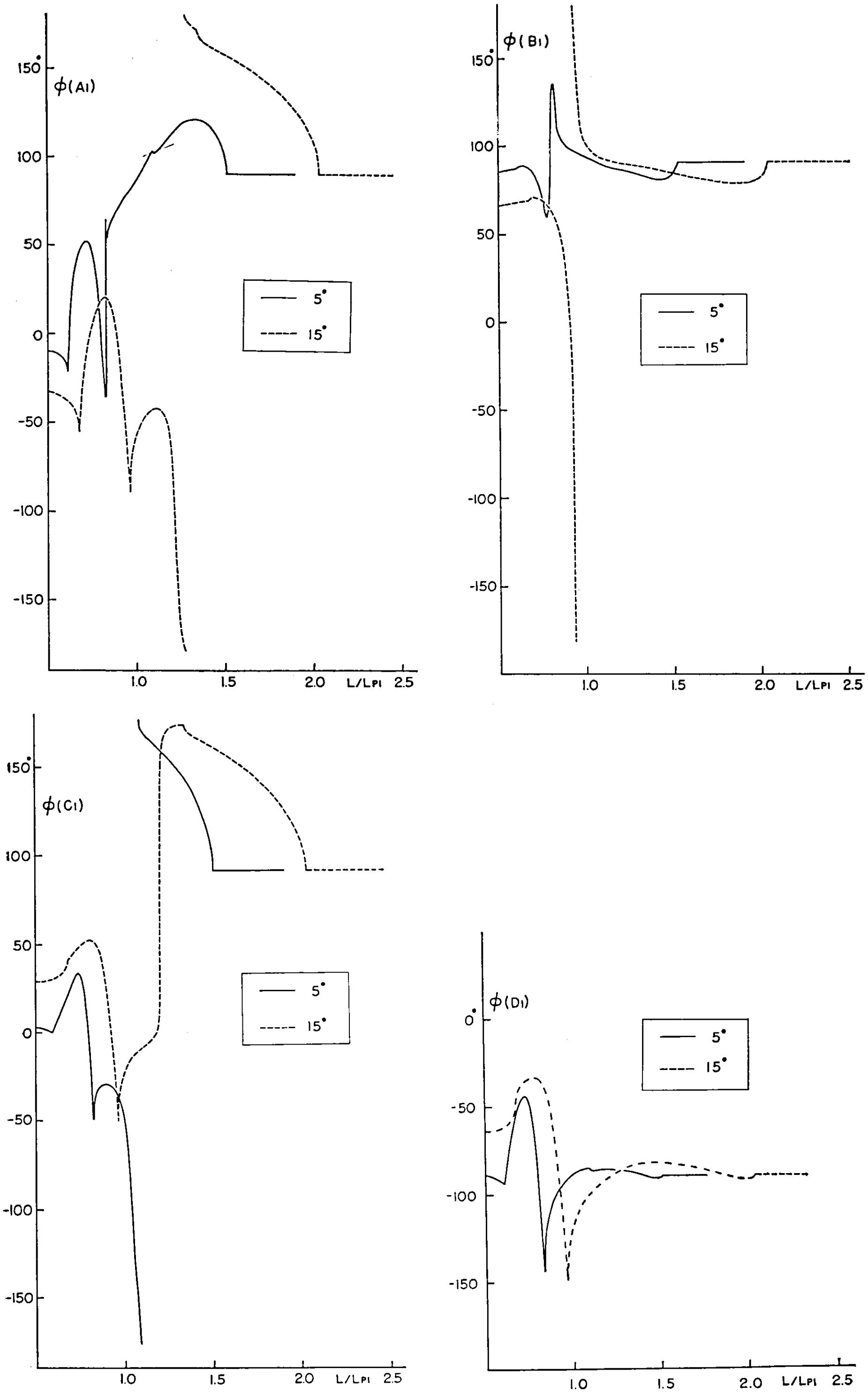


Fig. 14. Phase changes of forward irregular waves in the case of oblique incidence of $P$ waves. For example, $\phi(A_1)$ is the phase change of forward irregularly reflected $P$ waves; $\phi(D_1)$, that of forward irregularly refracted $S$ waves.

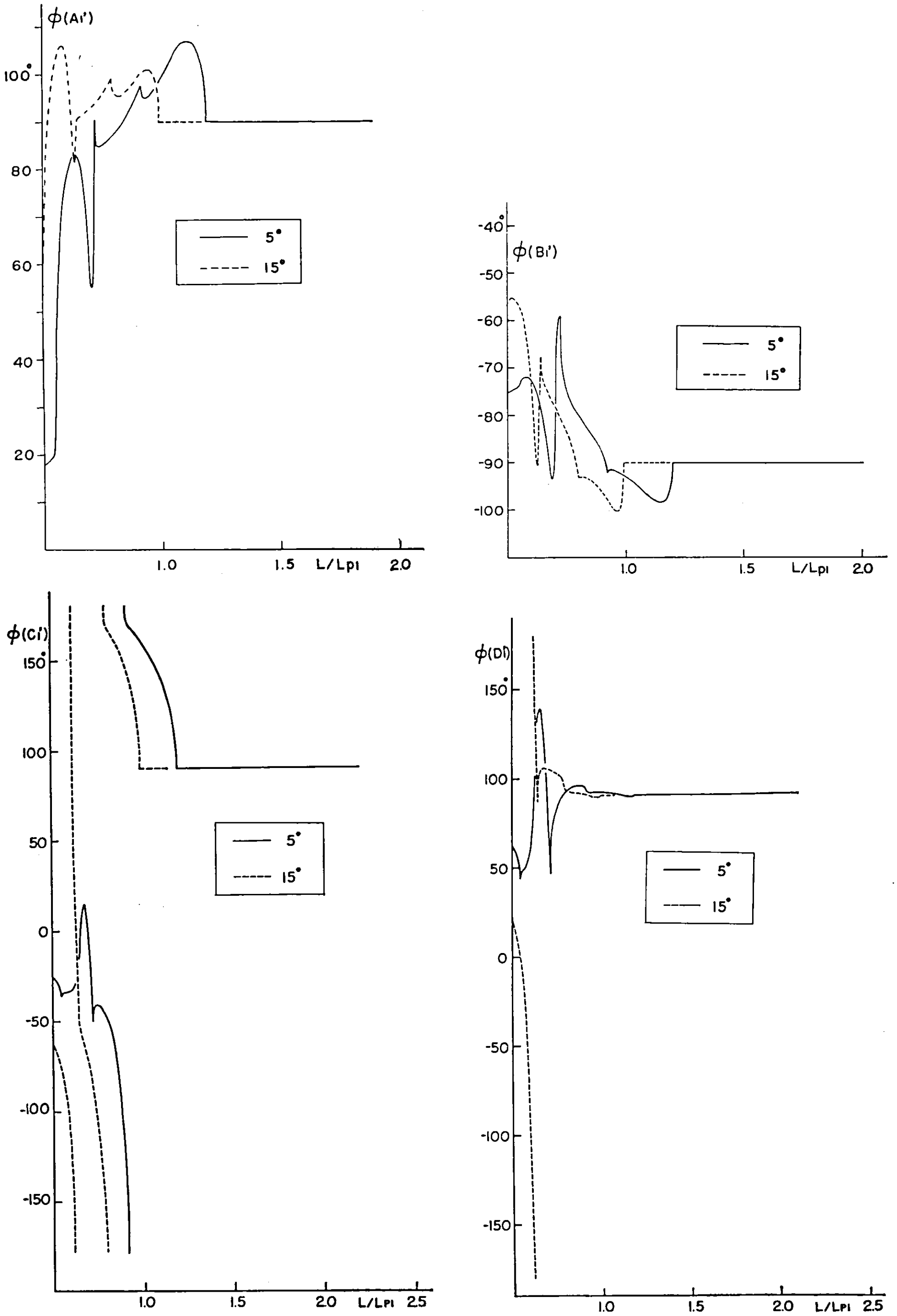


Fig. 15. Phase changes of backward irregular waves in the case of oblique incidence of $P$ waves. For example, $\phi(A_1')$ is the phase change of backward irregularly reflected $P$ waves; $\phi(D_1')$, that of backward irregularly refracted $S$ waves.

corrugation as well as angles of incidence.

(6) The phase changes of irregular waves increase rapidly as the angle of incidence increases. Therefore, the dependency of phase changes on the angles of incidence seems to be fairly large.

## 5. Concluding remarks

In order to ascertain and supplement the results obtained in the case of an incident $P$ wave, the case of the incidence of an $SV$ wave and the phase changes of waves upon reflection and refraction were considered in this paper by using Rayleigh's method as previously. Most of the results for the case of incidence of an $SV$ wave support previous results for an incident $P$ wave. That is, (1) the larger the velocity contrast, the larger the effect of corrugation although computation was done only for three models in the case of normal incidence; (2) the variation of $S$ waves with both wavelength and amplitude of corrugation is large; (3) the amplitude of regularly reflected waves decreases and that of regularly refracted waves increases and so on.

On the other hand, there are a few results particular to the case of an incident $SV$ wave. For example, (1) for large angles of incidence, the amplitude of regularly reflected as well as refracted $S$ waves has a tendency to decrease as the wavelength of corrugation decreases; (2) it is not always valid for large angles of incidence that the effect of corrugation is larger on reflection than on refraction, since regularly reflected $S$ waves are affected by the angle of incidence.

With regard to phase changes, mainly the results for the case of an incident $P$ wave are presented since those for the case of an incident $SV$ wave have similar features. Generally speaking, the phase does not change much for regular waves with the wavelengh of corrugation. Even for phase changes, it seems that the velocity contrast between media concerned play an important role. Also phase changes depend on both the angle of incidence and the wavelength of corrugation. Large phase changes are obtained for irregular waves especially in the range of small wavelength of corrugation. On occasions, this becomes important. However, since the irregular waves travel generally in different directions to each other and some of them become boundary waves, it seems that the contribution to the amplitude of waves from irregular waves is not large. These results are expected to be examined from different approaches.

In conclusion, the author would like to express his sincere gratitude to Mrs. J. Yanagisawa, Mr. Y. Ichinose and Miss E. Iwata for their assistance in preparing figures in the text.

A part of the computation was carried out at the NASA Institute for Space Studies, New York while the author was at Lamont Geological Observatory, and also by IBM 7090 through the project UNICON, IBM, Japan, for which his sincere thanks are due.

---

## 66. 波型の境界面における弾性波の反射，屈折

### 第3報 *SV* 波入射の場合および位相変化

地震研究所 浅野周三

波型の境界面へ *SV* 波が入射する場合を，前報と同じく Rayleigh の方法によつて解き，垂直入射の場合には *P* 波入射の時と同じ3種類のモデルについて，斜め入射の場合には1つのモデルについて入射角 16° 以下について，計算を行なつて従来の結果と比較してみた．大部分は *P* 波入射の場合の結果と矛盾しないが，入射波が異なるために若干の細かい点について異つた傾向を得た．例えば正常 *S* 波について入射角が大きくなり境界面の波長が短かくなると減少の傾向が見られる．また，正常反射 *S* 波が入射角によつて振幅がかなり変わるので，大きい入射角については凹凸の効果が屈折波よりも反射波に大きいとは必ずしもいえない．

反射，屈折の際の波の位相変化については，*SV* 波入射の時も主な傾向は似ているので，*P* 波入射の場合のみを示したが，正常波については位相変化そのものも小さいが，凹凸の波長によつてあまり変わらない．垂直入射の場合に速度比の異なる3種類のモデルに対する結果から，位相変化にも速度比が重要と考えられる．異常波については，入射角，凹凸波長によつて大きく位相が変化する．ただ実際には伝播方向がお互いに異なり，また，境界波となるものもあるので影響は少ないと考えられるが，他の方法による検討が望まれる．