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46. *Behavior of a Seismic System under Vibrating Solid Friction of High Frequency (Report No. 1).*

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1. Introduction

Generally speaking, the restoring force provided in a seismic system of longer period is rather weak in comparison with its greater mass, and therefore any trial such as to support its greater mass by a member of producing restoring force (for instance, by a spring) would lead to an existence of appreciable amount of initial displacement, and also a trial to support the mass by a separate supporting device or by a bearing would produce a greater magnitude of solid friction at that place compared with its restoring force, which in turn would interfere with the smooth motion of seismic system. Thus, all these factors are making it very difficult to obtain any ideal seismic system of small size and of longer period.

If, however, such solid friction at supporting or bearing parts of a seismic system could be eliminated or at least reduced to a negligible amount, the production of a small size seismic system of longer period would be made possible.

In this regard, one alternative could be thought of in order to reduce such solid friction to a practically negligible amount, that is, the frictional parts are subjected to a vibration of higher frequency, by means of which the direction of frictional force changes its sign alternately. With such prediction, when it is put in practice, the mass concerned would of course be subjected to the fine vibration of high frequency, but up to this date no literature has yet been made available to clarify, on an average basis, the expected motion of mass under the condition mentioned qualitatively as well as quantitatively. Of course, a dither device is well adopted in the field of practice of automatic control, but the analytical study of its behavior has never been publicized by anyone up to date.

Thereupon, this paper (Report No. 1) covers the theoretical analysis of the proposed prediction and the experiment on the motion of mass pendulum under an existence of vibrating solid friction of high frequency, assuming that no external force other than the solid friction is acting upon the mass. And, succeeding to the current paper, in Report No. 2, the motion of pendulum mass under an existence of fluid friction in addition to those of vibrating solid friction of high frequency and then the motion of pendulum mass under an existence of any other external forces in addition to those of fluid friction and vibrating solid friction of high frequency will be taken up, discussed and analysed and the results obtained therefrom will be successively reported.

2. Theory

Let us consider a model vibration system as shown in Fig. 1, in which M is the mass, k the spring constant and F the solid friction between the mass and its supporting stand. The frictional force F, however, should be of a function of relative velocity of mass and stand when it is considered strictly, but for simplicity's sake, here we will assume it constant on an average basis and consideration. Further, the stand is subjected to the simple harmonic motion of $A \sin \omega t$.

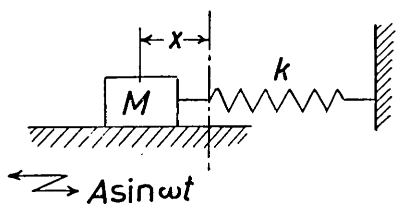


Fig. 1. Model of vibration system.

Then, the equation of motion of the mass under reference will be as follows:

$$M\ddot{x}+kx\pm F=0. \tag{1}$$

The positive or negative sign of F is dependent upon the relative velocity and thus,

$$\left.\begin{array}{lll}\text{when} & \dot{x}-(A\sin\omega t)'>0, & +, \\ \text{and when} & \dot{x}-(A\sin\omega t)'<0, & -. \end{array}\right\} \tag{2}$$

Namely, if the change-over times at which the sign of F changes by turns are denoted by $t_1, t_2, t_3, \cdots, t_n, \cdots$, then,

$$\left.\begin{array}{ll} t_{2n}\leq t\leq t_{2n+1}; & M\ddot{x}+kx-F=0, \\ t_{2n-1}\leq t\leq t_{2n}; & M\ddot{x}+kx+F=0. \end{array}\right\} \tag{1$'$}$$

These equations can be solved successively assuming that t_1, t_2, $\cdots$, t_n, $\cdots$ are known and under the initial condition that $x=x_0$, $\dot{x}=0$, when $t=0$. Rewriting (1), we have

$$\ddot{x}+\omega_0^2(x\mp x_f)=0\,, \qquad (1)''$$

where $\qquad \omega_0=\sqrt{k/M}\,, \quad x_f=F/k\,.$

For the interval $0\leq t\leq t_1$:
Using the initial condition that $x=x_0$, $\dot{x}=0$ when $t=0$, we have

$$x=x_f-(x_f-x_0)\cos\omega_0 t\,. \qquad (3)$$

For the interval $t_1\leq t\leq t_2$:
Using the condition that the displacement and velocity at $t=t_1$ should be continuous with those indicated by (3), we have

$$x_{t=t_1}=x_f-(x_f-x_0)\cos\omega_0 t_1\,,$$
$$\dot{x}_{t=t_1}=\omega_0(x_f-x_0)\sin\omega_0 t_1\,.$$

And when we solve equation (1)″ for $+x_f$, we have

$$x=-x_f-(x_f-x_0)\cos\omega_0 t+2x_f\cos\omega_0(t-t_1)\,. \qquad (4)$$

For the interval $t_2\leq t\leq t_3$:
Using the condition, in the same way, that the displacement and velocity at $t=t_2$ should be continuous with those for (4), solving equation (1)″ for $-x_f$, we have

$$x=x_f-(x_f-x_0)\cos\omega_0 t+2x_f\{\cos\omega_0(t-t_1)-\cos\omega_0(t-t_2)\}\,. \qquad (5)$$

For the interval $t_3\leq t\leq t_4$:
We have in the same way,

$$x=-x_f-(x_f-x_0)\cos\omega_0 t+2x_f\{\cos\omega_0(t-t_1)-\cos\omega_0(t-t_2)+\cos\omega_0(t-t_3)\}\,. \qquad (6)$$

Thus, we can obtain the solution in a general form as under:

$$t_{n-1}\leq t\leq t_n:\quad x=(-)^{n-1}x_f-(x_f-x_0)\cos\omega_0 t+2x_f\sum_{i=1}^{n-1}(-)^{i-1}\cos\omega_0(t-t_i)\,. \qquad (7)$$

Thereby, the motion of mass can be solved, the time t_i, however, is an unknown quantity, thus it must be obtained first of all. The change-over time of sign of relative velocity is the time at which $\dot{x}$ becomes equal to $(A\sin\omega t)'$. That is,

$$\dot{x}=\omega_0(x_f-x_0)\sin\omega_0 t_n+2\omega_0 x_f\sum_{i=1}^{n-1}(-)^i\sin\omega_0(t_n-t_i)=A\omega\cos\omega t_n\,. \qquad (8)$$

Since the accurate and strict solution of t_n is not obtainable, an approximate solution will hereby be sought for through the analytical method and also through the calculation by a digital computer.

3. Analysis of Approximate Solution

As mentioned in the aforesaid paragraph, although the exact value of t_i cannot be calculated from equation (8), an approximate calculation can be made possible when $\omega \gg \omega_0$, $A\omega \gg x_f\omega_0$ and $A\omega \gg x_0\omega_0$ are in effect. Under such condition, since $A\omega \gg x_f\omega_0$ and $A\omega \gg x_0\omega_0$, t_n can be considered very close to $\frac{2n-1}{2\omega}\pi$, thus putting

$$t_n = \frac{2n-1}{2\omega}\pi + \Delta t_n, \tag{9}$$

and Δt_n is to be obtained instead, hereby Δt_n is $\Delta t_n \ll \frac{\pi}{2\omega}$, the right hand side of equation (8) becomes

$$A\omega \cos \omega t_n = A\omega \sin\left(\frac{\pi}{2} - \frac{2n-1}{2}\pi - \omega\Delta t_n\right) \doteqdot A\omega(-)^n \omega\Delta t_n, \tag{10}$$

in which, however, the third order and higher terms of $\omega\Delta t_n$ were neglected. The same will be applied in the calculation to be made hereafter on the third order and higher terms of $\omega\Delta t_n$ or ω_0/ω. In the left hand side of equation (8), since,

$$\left.\begin{aligned} \sin \omega_0 t_n &\doteqdot \sin\left(\frac{\omega_0}{\omega}\right)\left(n-\frac{1}{2}\right)\pi + \cos\left(\frac{\omega_0}{\omega}\right)\left(n-\frac{1}{2}\right)\pi \times \left(\frac{\omega_0}{\omega}\right)\cdot \omega\Delta t_n, \\ \sin \omega_0(t_n - t_i) &\doteqdot \sin\left(\frac{\omega_0}{\omega}\right)(n-i)\pi + \cos\left(\frac{\omega_0}{\omega}\right)(n-i)\pi \times \left(\frac{\omega_0}{\omega}\right)(\omega\Delta t_n - \omega\Delta t_i), \end{aligned}\right\} \tag{10}'$$

equation (8) can be converted into the following form:

$$\begin{aligned} A\omega(-)^n \omega\Delta t_n \doteqdot \omega_0(x_f - x_0)&\left\{\sin\left(\frac{\omega_0}{\omega}\right)\left(n-\frac{1}{2}\right)\pi + \cos\left(\frac{\omega_0}{\omega}\right)\left(n-\frac{1}{2}\right)\pi \right. \\ &\left. \times \left(\frac{\omega_0}{\omega}\right)\cdot \omega\Delta t_n\right. + 2\omega_0 x_f \sum_{i=1}^{n-1} (-)^i \left\{\sin\left(\frac{\omega_0}{\omega}\right)(n-i)\pi \right. \\ &\left.\left. + \cos\left(\frac{\omega_0}{\omega}\right)(n-i)\pi \times \left(\frac{\omega_0}{\omega}\right)(\omega\Delta t_n - \omega\Delta t_i)\right\}\right., \end{aligned} \tag{11}$$

or this can be rewritten into:

$$\begin{aligned} (-)^n \omega\Delta t_n \doteqdot \frac{x_f - x_0}{A}\left(\frac{\omega_0}{\omega}\right)&\left\{\sin\left(\frac{\omega_0}{\omega}\right)\left(n-\frac{1}{2}\right)\pi + \cos\left(\frac{\omega_0}{\omega}\right)\left(n-\frac{1}{2}\right)\pi \times \left(\frac{\omega_0}{\omega}\right)\omega\Delta t_n \right. \\ &\left. + \frac{2x_f}{A}\left(\frac{\omega_0}{\omega}\right) \sum_{i=1}^{n-1} (-)^i \left\{\sin\left(\frac{\omega_0}{\omega}\right)(n-i)\pi + \cos\left(\frac{\omega_0}{\omega}\right)\pi \times \left(\frac{\omega_0}{\omega}\right)(\omega\Delta t_n - \omega\Delta t_i)\right\}\right.. \end{aligned} \tag{11}'$$

As can be easily seen from this equation, $\omega\Delta t_n$ is of the first order with regard to ω_0/ω, thus, since those terms on the right hand side which include $\omega\Delta t_n$ and $\omega\Delta t_i$, become of the third order with regard to ω_0/ω, they can be well neglected and it becomes

$$(-)^n\omega\Delta t_n \doteqdot \frac{x_f-x_0}{A}\left(\frac{\omega_0}{\omega}\right)\sin\left(\frac{\omega_0}{\omega}\right)\left(n-\frac{1}{2}\right)\pi + \frac{2x_f}{A}\left(\frac{\omega_0}{\omega}\right)\sum_{i=1}^{n-1}(-)^i\sin\left(\frac{\omega_0}{\omega}\right)(n-i)\pi\,. \tag{11''}$$

The second term of this right side can be converted into the following form for actual calculation:

$$\begin{aligned}
&\sum_{i=1}^{n=1}(-)^i\sin\left(\frac{\omega_0}{\omega}\right)(n-i)\pi=\sum_{i=1}^{n-1}(-)^i\cdot\frac{e^{j(\omega_0/\omega)(n-i)\pi}-e^{-j(\omega_0/\omega)(n-i)\pi}}{2j}\\
&=\frac{1}{2j}\left\{\sum_{i=1}^{n-1}(-)^i e^{j(\omega_0/\omega)(n-i)\pi}-\sum_{i=1}^{n-1}(-)^i e^{j(\omega_0/\omega)(i-n)\pi}\right\}\\
&=\frac{1}{2j}\left[\frac{e^{j(\omega_0/\omega)(n-1)\pi}\{1-(-)^{n-1}e^{-j(\omega_0/\omega)(n-1)\pi}\}}{1+e^{-j(\omega_0/\omega)\pi}}-\frac{e^{-j(\omega_0/\omega)(n-1)\pi}\{1-(-)^{n-1}e^{j(\omega_0/\omega)(n-1)\pi}\}}{1+e^{j(\omega_0/\omega)\pi}}\right]\\
&=-\frac{1}{2\cos\left(\frac{\omega_0}{\omega}\right)\frac{\pi}{2}}\left\{\sin\left(\frac{\omega_0}{\omega}\right)\left(n-\frac{1}{2}\right)\pi-(-)^{n-1}\sin\left(\frac{\omega_0}{\omega}\right)\frac{\pi}{2}\right\}.
\end{aligned}$$

Thus, equation (11)″ becomes

$$\begin{aligned}
\omega\Delta t_n &\doteqdot (-)^n\frac{x_f-x_0}{A}\left(\frac{\omega_0}{\omega}\right)\sin\left(\frac{\omega_0}{\omega}\right)\left(n-\frac{1}{2}\right)\pi\\
&\quad-(-)^n\frac{x_f}{A}\left(\frac{\omega_0}{\omega}\right)\frac{1}{\cos\left(\frac{\omega_0}{\omega}\right)\frac{\pi}{2}}\sin\left(\frac{\omega_0}{\omega}\right)\left(n-\frac{1}{2}\right)\pi\\
&\quad-\frac{x_f}{A}\left(\frac{\omega_0}{\omega}\right)\tan\left(\frac{\omega_0}{\omega}\right)\frac{\pi}{2}\\
&\doteqdot(-)^{n-1}\frac{x_0}{A}\left(\frac{\omega_0}{\omega}\right)\sin\left(\frac{\omega_0}{\omega}\right)\left(n-\frac{1}{2}\right)\pi-\frac{x_f}{A}\left(\frac{\omega_0}{\omega}\right)^2\frac{\pi}{2}\,.
\end{aligned}\tag{12}$$

This equation indicates the value of $\omega\Delta t_n$ as an approximation. With the approximate value of $\omega\Delta t_n$ thus obtained, equation (7) can be analytically solved. However, let us convert equation (7) into a different form of direct and easier understanding through further approximation as below:

In equation (7), i.e.,

$$x=(-)^{n-1}x_f-(x_f-x_0)\cos\omega_0 t+2x_f\sum_{i=1}^{n-1}(-)^{i-1}\cos\omega_0(t-t_i)\,,$$

the portion of summation in the third term is firstly rewritten, that is,

$$\begin{aligned}
&2\sum_{i=1}^{n-1}(-)^{i-1}\cos\omega_0(t-t_i)=\sum_{i=1}^{n-1}(-)^{i-1}\{e^{j\omega_0(t-t_i)}+e^{-j\omega_0(t-t_i)}\}\\
&=\sum_{i=1}^{n-1}(-)^{i-1}[e^{j\omega_0\{t-(i-1/2)\pi/\omega-\varDelta t_i\}}+e^{-j\omega_0\{t-(i-1/2)\pi/\omega-\varDelta t_i\}}]\\
&\doteqdot\sum_{i=1}^{n-1}(-)^{i-1}[e^{j\omega_0\{t-(i-1/2)\pi/\omega\}}+e^{-j\omega_0\{t-(i-1/2)\pi/\omega\}}\\
&\qquad+\{e^{-j\omega_0\{t-(i-1/2)\pi/\omega\}}-e^{j\omega_0\{t-(i-1/2)\pi/\omega\}}\}\left(\frac{j\omega_0}{\omega}\right)\omega\varDelta t_i]\\
&=\sum_{i=1}^{n-1}(-)^{i-1}[e^{j\omega_0\{t-(i-1/2)\pi/\omega\}}+e^{-j\omega_0\{t-(i-1/2)\pi/\omega\}}]\\
&\quad+\frac{x_0}{2A}\left(\frac{\omega_0}{\omega}\right)^2\sum_{i=1}^{n-1}[e^{-j\omega_0\{t-(2i-1)\pi/\omega\}}+e^{j\omega_0\{t-(2i-1)\pi/\omega\}}-e^{j\omega_0 t}-e^{-j\omega_0 t}]\\
&\quad-\frac{x_f}{A}\left(\frac{\omega_0}{\omega}\right)^2\left(\frac{j\omega_0}{\omega}\right)\frac{\pi}{2}\sum_{i=1}^{n-1}(-)^{i-1}[e^{-j\omega_0\{t-(i-1/2)\pi/\omega}-e^{j\omega_0\{t-(i-1/2)\pi/\omega\}}]\,.
\end{aligned}\tag{13}$$

Each portion of summation, however, can be indicated by the following equations:

$$\begin{aligned}
&\sum_{i=1}^{n-1}(-)^{i-1}[e^{j\omega_0\{t-(i-1/2)\pi/\omega\}}+e^{-j\omega_0\{t-(i-1/2)\pi/\omega\}}]\\
&=\frac{\cos\omega_0 t-(-)^{n-1}\cos\omega_0\{t-(n-1)\pi/\omega\}}{\cos(\omega_0/\omega)(\pi/2)}\,,
\end{aligned}\tag{14}$$

$$\begin{aligned}
&\frac{x_0}{2A}\left(\frac{\omega_0}{\omega}\right)^2\sum_{i=1}^{n-1}[e^{j\omega_0\{t-(2i-1)\pi/\omega\}}+e^{-j\omega_0\{t-(2i-1)\pi/\omega\}}-e^{j\omega_0 t}-e^{-j\omega_0 t}]\\
&=\frac{x_0}{2A}\left(\frac{\omega_0}{\omega}\right)^2\cdot\frac{\sin\omega_0 t-\sin\omega_0\{t-2(n-1)\pi/\omega\}}{\sin(\omega_0/\omega)\pi}-\frac{x_0}{A}\left(\frac{\omega_0}{\omega}\right)^2(n-1)\cos\omega_0 t\,,
\end{aligned}\tag{15}$$

$$\begin{aligned}
&-\frac{x_f}{A}\left(\frac{\omega_0}{\omega}\right)^2\left(\frac{j\omega_0}{\omega}\right)\frac{\pi}{2}\sum_{i=1}^{n-1}(-)^{i-1}[e^{-j\omega_0\{t-(i-1/2)\pi/\omega\}}-e^{j\omega_0\{t-(i-1/2)\pi/\omega\}}]\\
&=-\frac{x_f}{A}\left(\frac{\omega_0}{\omega}\right)^3\frac{\pi}{2}\cdot\frac{\sin\omega_0 t-(-)^{n-1}\sin\omega_0\{t-(n-1)\pi/\omega\}}{\cos(\omega_0/\omega)\pi/2}\,.
\end{aligned}\tag{16}$$

Next, substituting these expressions (14), (15) and (16) into (13) and again putting (13) in equation (7), we finally have after considering the order as far as $(\omega_0/\omega)^2$,

$$x=(-)^{n-1}x_f-(x_f-x_0)\cos\omega_0 t+x_f\frac{\cos\omega_0 t-(-)^{n-1}\cos\omega_0\{t-(n-1)\pi/\omega\}}{\cos(\omega_0/\omega)(\pi/2)}$$
$$+x_0\left[\frac{x_f}{2\pi A}\left(\frac{\omega_0}{\omega}\right)\left\{\sin\omega_0 t-\sin\omega_0\{t-2(n-1)\pi/\omega\}\right\}-\frac{x_f}{A}\left(\frac{\omega_0}{\omega}\right)\left(\frac{\omega_0}{\omega}\right)(n-1)\cos\omega_0 t\right].$$

Or re-writing, we have,

$$x=x_0\left[\left\{1-\frac{x_f}{A}\left(\frac{\omega_0}{\omega}\right)^2(n-1)\right\}\cos\omega_0 t\right.$$
$$\left.+\frac{x_f}{\pi A}\left(\frac{\omega_0}{\omega}\right)\cos\left\{\left(\frac{\omega_0}{\omega}\right)\left(\omega t-(n-1)\pi\right)\right\}\cdot\sin\left(\frac{\omega_0}{\omega}\right)(n-1)\pi\right]$$
$$+x_f\left[\frac{\pi^2}{8}\left(\frac{\omega_0}{\omega}\right)^2\cos\omega_0 t+(-)^{n-1}\left\{1-\cos\left\{\left(\frac{\omega_0}{\omega}\right)(\omega t-(n-1)\pi)\right\}\right\}\right]. \tag{17}$$

The term $(n-1)$ in the above equation produces an unrealistic result when n is made $n\to\infty$, and this means that an effective domain of equation (17) is corresponding to the approximation degree of $(\omega_0/\omega)^2$-order, or in other words, equation (17) is an approximate relationship which can be made in effect for those values of n which are not too great. $\dot{x}$ can be obtained through a similar process of calculation by differentiating equation (7) or by direct differentiation of equation (17). Neglecting all intervening procedures, we write below the result of calculation:

$$\dot{x}=-\omega_0 x_0\left[\left\{1-\frac{x_f}{A}\left(\frac{\omega_0}{\omega}\right)(n-1)\right\}\sin\omega_0 t\right.$$
$$\left.-\frac{x_f}{\pi A}\left(\frac{\omega_0}{\omega}\right)\sin\left\{\left(\frac{\omega_0}{\omega}\right)(\omega t-(n-1)\pi)\right\}\cdot\sin\left(\frac{\omega_0}{\omega}\right)(n-1)\pi\right]$$
$$+\omega_0 x_f\left[-\frac{\pi^2}{8}\left(\frac{\omega_0}{\omega}\right)^2\sin\omega_0 t+(-)^{n-1}\sin\left\{\left(\frac{\omega_0}{\omega}\right)(\omega t-(n-1)\pi)\right\}\right]. \tag{18}$$

For both equations (17) and (18), the first term represents the average value of x and a very fine vibration depending on the initial condition. Let us now consider the magnitude of displacement and velocity after the duration of one period $2\pi/\omega_0$. For simplicity's sake, let us put

$$\frac{\omega_0}{\omega}=\frac{1}{N}\,, \tag{19}$$

and t and t_n are indicated by the following equations:

$$\mathrm{t}=\frac{2\pi}{\omega_0}=\frac{2N\pi}{\omega}\,, \tag{20}$$

$$t_n = t + \frac{\pi}{2\omega} + \Delta t_n = \frac{4N+1}{2\omega}\pi + \Delta t_n = \frac{2n-1}{2\omega}\pi + \Delta t_n ,$$

$$\therefore \quad n = 2N+1 . \tag{21}$$

And substituting these into (17) and (18), we have,

$$\left.\begin{aligned} &\dot{x} = 0 , \\ &x = x_0\left\{1 - \frac{2x_f}{A}\left(\frac{\omega_0}{\omega}\right)\right\} , \end{aligned}\right\} \tag{22}$$

which means that the motion brought about under the initial condition $t=0$: $x=x_0$ and $\dot{x}=0$, reduces its amplitude to $x_0\left(1-\frac{2x_f\omega_0}{A\omega}\right)$ after the duration of one period of time.

When $(2x_f/A)(\omega_0/\omega) \ll 1$, we can write

$$1 - \frac{2x_f\omega_0}{A\omega} \doteqdot e^{-\frac{2x_f\omega_0}{A\omega}} = e^{-2\pi h} , \tag{23}$$

where
$$h = \frac{x_f\omega_0}{\pi A\omega} .$$

The average displacement and velocity in (17) and (18) can be re-written with an introduction of h as below:

$$x \doteqdot x_0(\cos\omega_0 t + h\sin\omega_0 t)e^{-h\omega_0 t} + \Delta x(\omega t) , \tag{17$'$}$$

$$\dot{x} = \omega_0(1-h^2)x_0 \sin\omega_0 t e^{-h\omega_0 t} + \Delta\dot{x}(\omega t) , \tag{18$'$}$$

and the vibration under reference becomes equivalent to the free vibration with the damping decrement ε which is expressed by

$$\varepsilon = \frac{x_f\omega_0^2}{\pi A\omega} .$$

The frequency of such free vibration should deviate from the value of ω_0. Such deviation, however, is of a higher order than (ω_0/ω), and thus it does not make its appearance in the current approximate calculation.

The above calculation relates to the case where the motion starts under the initial condition $t=0$; $x=x_0$ and $\dot{x}=0$, but there does not exist any restriction as to the initial condition itself, since the result would turn out the same if the origin of time $t=0$ should be taken at such time when $\dot{x}$ becomes $\dot{x}=0$. Further, the vibrating motion given to the supporting stand was assumed as $A\sin\omega t$ in the current analysis, but such motion as indicated by $A\sin(\omega t-\phi)$ in general would still lead to the same result, if the change-over time of relative velocity is shifted from $t_n=\frac{2n-1}{2\omega}\pi+\Delta t_n$ to $t_n=\frac{2n-1}{2\omega}\pi+\frac{\phi}{\omega}+\Delta t_n$.

Summarizing the above, we obtained the following analytical results: Under the conditions that when t is $t=0$; $x=x_0$ and $\dot{x}=0$; and that $(\omega_0/\omega)\ll 1$, the equation of motion

$$\ddot{x}+\omega_0(x\mp x_f)=0 \tag{1$''$}$$

can be solved as below:

For any interval of t such as $\frac{2n-3}{2\omega}\pi+\Delta t_{n-1}\leq t\leq\frac{2n-1}{2\omega}\pi+\Delta t_n$,

$$\begin{aligned}x=x_0&\left[\left\{1-\frac{x_f}{A}\left(\frac{\omega_0}{\omega}\right)(n-1)\right\}\cos\omega_0 t\right.\\&\left.+\frac{x_f\omega_0}{\pi A\omega}\cos\left\{\left(\frac{\omega_0}{\omega}\right)(\omega t-(n-1)\pi)\right\}\sin\left(\frac{\omega_0}{\omega}\right)(n-1)\pi\right]\\&+x_f\left[\frac{\pi^2}{8}\left(\frac{\omega_0}{\omega}\right)^2\cos\omega_0 t+(-)^{n-1}\left\{1-\cos\left\{\left(\frac{\omega_0}{\omega}\right)(\omega t-(n-1)\pi)\right\}\right\}\right],\end{aligned} \tag{17}$$

where $$\omega\Delta t_n=(-)^n\frac{x_0\omega_0}{A\omega}\sin\left(\frac{\omega_0}{\omega}\right)\left(n-\frac{1}{2}\right)\pi-\frac{x_f}{A}\left(\frac{\omega_0}{\omega}\right)^2\frac{\pi}{2}. \tag{12}$$

And the first term is composed of a damped oscillation and very fine vibration of frequency ω depending upon the initial condition. When we consider the term $\omega\Delta t_n$ to its third order, the term $(-)^n\frac{x_0}{A}\cdot\frac{x_f}{A}\left(\frac{\omega_0}{\omega}\right)^3\times(n-1)\sin\left(\frac{\omega_0}{\omega}\right)\left(n-\frac{1}{2}\right)\pi$ appears, with a result that the absolute value of $\omega\Delta t_n$ reveals itself to be on a gradual decrease. The calculation containing the third order term of $\omega\Delta t_n$, however, will be currently suspended herewith.

4. Comparison of the Result of Exact Calculation by Computer and that of Approximate Calculation

An exact calculation by using digital computer was made on equation (7) with degree of accuracy of the order of 10^{-5} and its result will be compared with those of approximate calculation expressed by (17).

(i) $x_0=1$, $x_f=0.5$, $A=0.5$, $\omega_0=1$, $\omega=10$, $(h=0.1/\pi)$:

Fig. 2 shows the calculated result by computer and the left hand side of Table 1 shows the values of maximum and minimum of the curve in Fig. 2 and the right hand side shows those for $x=x_0\cdot(\cos\omega_0 t+h\sin\omega_0 t)\cdot e^{-h\omega_0 t}$ which becomes the average values of approximate solution of x.

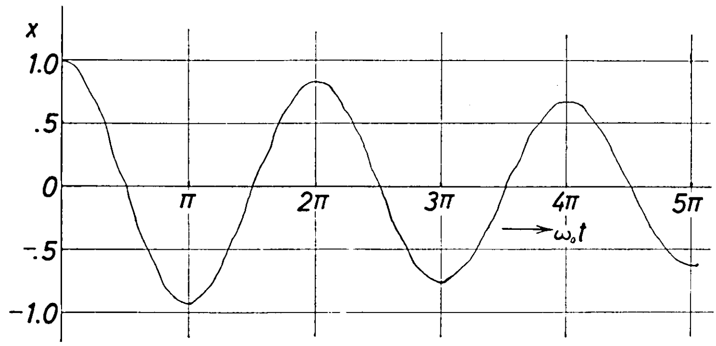


Fig. 2. Variation of x when $x_0=1$, $x_f=0.5$, $A=0.5$, $\omega_0=1$, $\omega=10$, $h=\frac{0.1}{\pi}$.

Table 1.

Exact Solution		Approximate Solution	
Maximum Value	Minimum Value	Maximum Value	Minimum Value
1.0000	−0.9162	1.0000	−0.9048
0.8168	−0.7506	0.8189	−0.7408
0.6671	−0.6153	0.6704	−0.6068

The difference observed between these two results is considered to be due to an existence of very fine vibration and any possible errors that were introduced in the process of approximate calculation.

Fig. 3 shows the values of exact solution subtracted by $x_0\cos\omega_0 te^{-h\omega_0 t}$, that is, sum of the very fine vibration and the value of $h\sin\omega_0 te^{-h\omega_0 t}$ corresponding to the shift of phase which is in fair coincidence with the approximate solution.

Fig. 4 shows the exact value of $\omega\Delta t_n$ which, although the value of $\omega\Delta t_n$ is discontinuous, is represented by a smooth and continuous curve for easiness of understanding, and it is also in fair coincidence with the approximate solution calculated to the order of $(\omega_0/\omega)^3$.

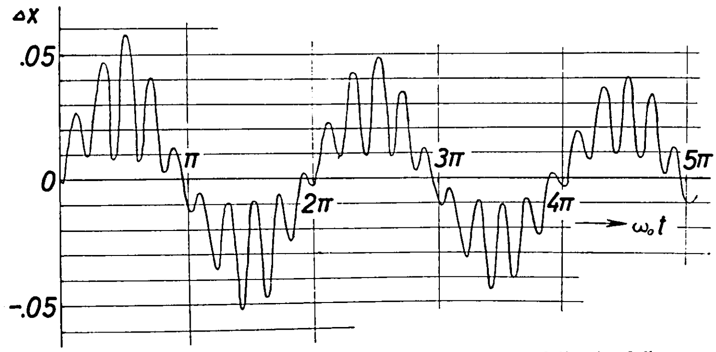


Fig. 3. $\Delta x = x - x_0 \cos \omega_0 t e^{-h\omega_0 t}$ when $x_0=1$, $x_f=0.5$, $A=0.5$, $\omega_0=1$, $\omega=10$, $h=\frac{0.1}{\pi}$.

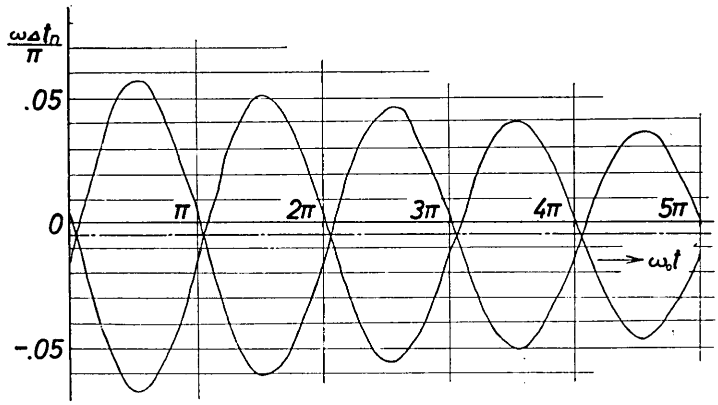


Fig. 4. Variation of $\omega \Delta t_n$ when $x_0=1$, $x_f=0.5$, $A=0.5$, $\omega_0=1$, $\omega=10$, $h=\frac{0.1}{\pi}$.

(ii) $x_0=1$, $x_f=0.5$, $A=1$, $\omega_0=1$, $\omega=3$, $h=1/6\pi$:

Fig. 5 shows the result of exact calculation. The values of maximum and minimum for the exact solution and those for the approximate solution $x=x_0(\cos \omega_0 t + h \sin \omega_0 t)e^{-h\omega_0 t}$ are shown in Table 2.

Table 2.

Exact Solution		Approximate Solution	
Maximum Value	Minimum Value	Maximum Value	Minimum Value
1.0000	−0.8315	1.0000	−0.8464
0.6900	−0.5709	0.7164	−0.6064
0.4703		0.5136	

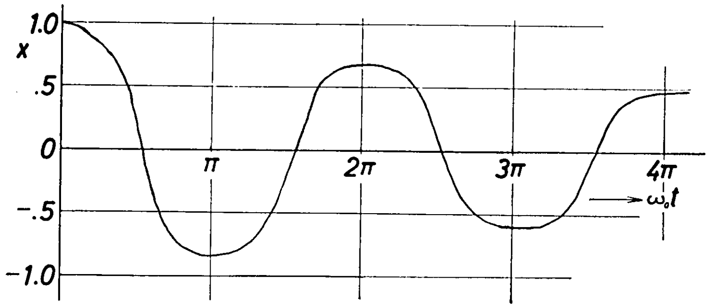


Fig. 5. Variation of x when $x_0=1$, $x_f=0.5$, $A=1$, $\omega_0=1$, $\omega=3$, $h=\frac{1}{6\pi}$.

The difference between the two results, although it could be referred to the same causes mentioned in (i), is greater than those for (i). Since (ω_0/ω) is $\omega_0/\omega=1/3$, which cannot be said to be sufficiently small enough compared with 1, thus it is considered that the approximate equation thus derived may need further correction of a greater extent.

Fig. 6 shows the value of exact solution subtracted by $x_0 \cos \omega_0 t e^{-h\omega_0 t}$,that is, a curve indicating the sum of $h \sin \omega_0 t e^{-h\omega_0 t}$ and the very fine vibration. It cannot be said that the approximate solution is quantitatively in coincidence with this curve.

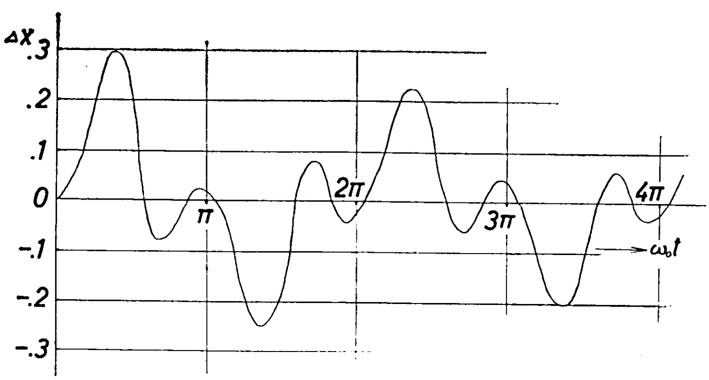


Fig. 6. $\Delta x = x - x_0 \cos \omega_0 t e^{-h\omega_0 t}$ when $x_0=1$, $x_f=0.5$, $A=1$, $\omega_0=1$, $\omega=3$, $h=\frac{1}{6\pi}$.

Fig. 7 shows the value of $\omega \Delta t_n$, which is also in fair coincidence with the result of approximate calculation considered to the order of $(\omega_0/\omega)^3$.

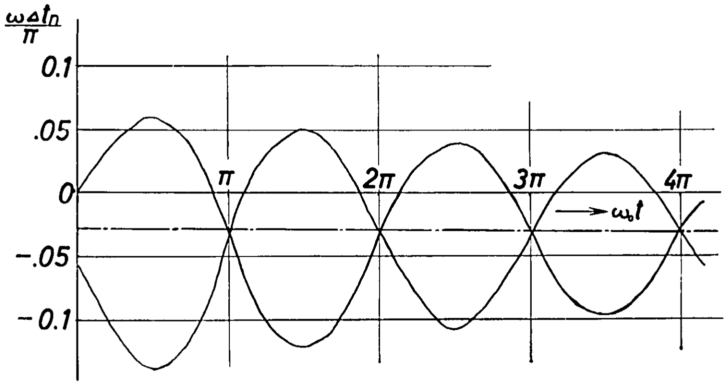


Fig. 7. Variation of $\omega\Delta t_n$ when $x_0=1$, $x_f=0.5$, $A=1$, $\omega_0=1$, $\omega=3$, $h=\dfrac{1}{6\pi}$.

5. Experiment

We have found in the aforesaid paragraph that, when the vibration of high frequency is given between supporting stand and vibrating body under the condition $(\omega_0/\omega)\ll 1$ and $(x_0\omega_0)/(A\omega)$ or $(x_f\omega_0)/(A\omega)\ll 1$, the vibration becomes a damped vibration and it is equivalent to the free vibration having a damping constant h which is expressed by $h=(x_f\omega_0)/(\pi A\omega)$. In this regard, the experiment was conducted under a few conditions to ascertain and endorse the facts derived analytically.

5.1. Apparatus of Experiment.

The apparatus for the current experiment is shown in Fig. 8 schematically and the actual illustration of it in Fig. 9. In Figs. 8 and 9, ① is a gravity pendulum which is able to rotate around an axle ③ through a plain bearing which is fixed to the pendulum. Therefore, the solid frictional force may act on the pendulum through the relative motions between pendulum ① and axle ③. A weight ② can be moved up and down so as to adjust or vary the period of the gravity pendulum and the value of x_f which was shown in the aforementioned equations. For obtaining the vibrating solid friction of high frequency, the rotational vibrations of axle ③ are excited by the action of electromagnet ⑥: The frequency of these vibrations is made equal to the natural frequency of the mechanical vibration system composed of plate spring ⑦, levers ④, ⑤, axle ③ and pick-up ⑩ for exciting the rotational vibrations of greater resonance amplitude of axle ③. Of course, the frequency of

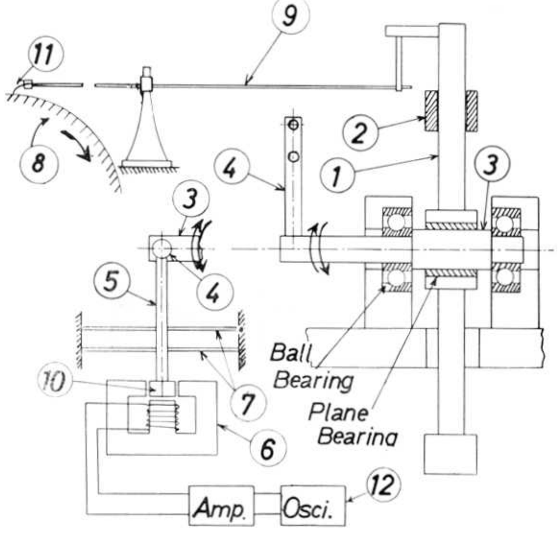


Fig. 8. Apparatus of Experiment.

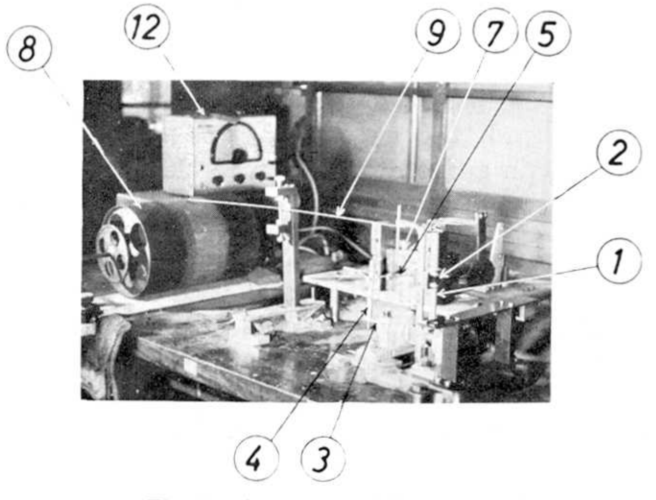


Fig. 9. Apparatus of Experiment.

electric current in the coil of magnet ⑥ is adjusted to become equal to the natural one of the mechanical vibration system by controlling the electric oscillator ⑫, and also the current intensity is adjusted by the electric amplifier, shown in Fig. 8. Thus, the rotational vibrations of pendulum ① are recorded, through a suitable device, stem ⑨ and recording needle ⑪, on the soot coated paper tied round the surface of drum ⑧ which is made to rotate by a synchronous motor with reduction gears.

5.2. Analysis of Motion of Gravity Pendulum.

The equation of motion of gravity pendulum ① is as belows:

$$I\ddot{x}+kx\mp F=0\,, \tag{24}$$

where $I=I_0+mr^2$, $k=(MR-mr)/g$, $F=r'(M+m)$, and

x: revolution angle,
I_0: moment of inertia,
I: moment of inertia of revolving parts excluding weight ②,
M: mass of revolving parts excluding weight ②,
R: distance between the center of revolving axle ③ and the center of gravity of parts excluding weight ②,
r: distance between the center of revolving axle ③ and the center of gravity of weight ②,
m: mass of weight ②,
r': radius of revolving axle ③,
μ: coefficient of friction between axle and bearing,
g: acceleration due to gravity.

Rewriting equation (24), we have

$$\ddot{x}+\omega_0^2(x\mp x_f)=0\,, \tag{24$'$}$$

where

$$\omega_0=\sqrt{\frac{(MR-mr)g}{I_0+mr^2}}\,, \tag{25}$$

$$x_f=\frac{\mu r'(M+m)}{MR-mr}\,, \tag{26}$$

and, in which ω_0 and x_f can be varied by changing the value of r. According to equation (17), an average curve, that is the curve to be recorded on the recording paper should be of the form of damped vibration having the damping constant $h=(x_f\omega_0)/(\pi A\omega)$.

5.3. Results of Experiment and Consideration.

The experiment was conducted by varying r (consequently ω_0 and x_f) and amplitude A of high frequency vibration.

Figs. 10 and 11 show the change of $T_0=\dfrac{2\pi}{\omega_0}$ and x_f with that of r, in which, however, x_f is the displacement in the recording paper. The

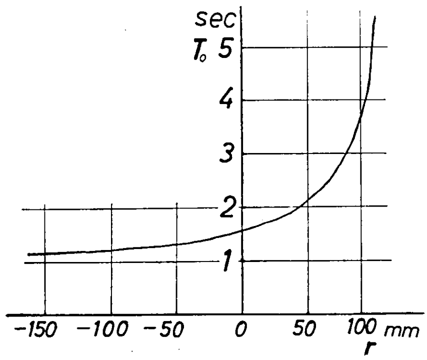


Fig. 10. Relationship between r and $T_0\left(=\frac{2\pi}{\omega_0}\right)$.

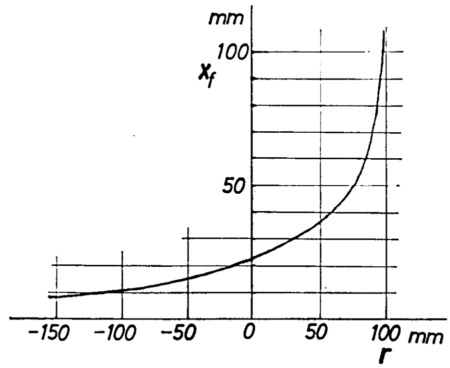


Fig. 11. Relationship between r and x_f.

conversion ratio of revolution angle on the recording paper is 211 mm/rad. With a greater value of r exceeding r=80 mm, x_f was made too great to be measured; when r is made even greater, equation (26) is made to forfeit its physical significance and the pendulum would be made to stop at any position. This means that since the current experiment is conducted under such magnitude of amplitude which is not too great, x_f could be considered to be F/K and to become infinity with $k \rightarrow 0$.

Fig. 12 shows the relationship between the input current for exciting vibration and converted value of vibration amplitude on the recording paper.

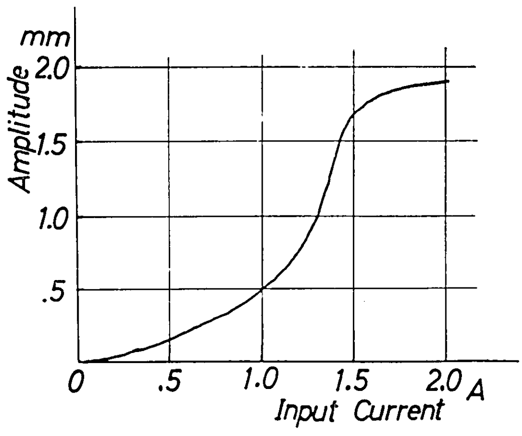


Fig. 12. Relationship between input exciting current and amplitude of vibration.

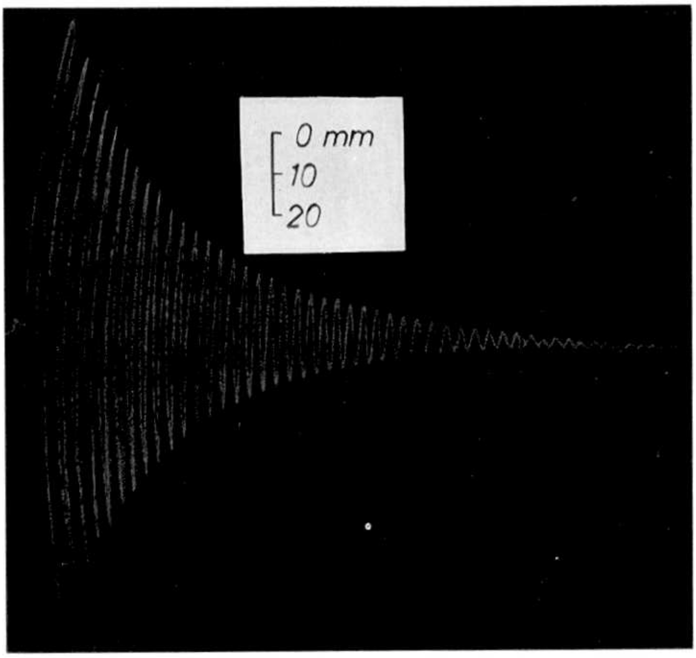


Fig. 13. Record of pendulum motion when $r=-149.5\,mm$, $T_0=1.17\,s$, $x_f=9.4\,mm$, $A=1.5\,mm$, $\frac{\omega}{2\pi}=100\,c/s$.

Figs. 13, 14 and 15 show some examples of pendulum vibration.

Figs. 16 (a)$\sim$16 (g) were so obtained by plotting the difference of adjacent width of amplitude Δw with regard to the concerned width of amplitude w of pendulum vibration subjected to the three kinds of vibration of high frequency by varying the value of r from $r\doteqdot 100$ to $-150\,mm$. The inclination or tangent of this straight line with regard to w axis represents the magnitude $(e^{\pi h}-1)$ and the distance between the origin and intersecting point of this line with regard to Δw axis indicates those of $2\rho(1+e^{\pi h})$ in which ρ represents the solid friction. Tables 3 (a)$\sim$3 (d) show the values of h_e thus calculated from these graphs and those of $h_t\left(h_t=\dfrac{x_f\omega_0}{\pi A\omega}\right)$ obtained theoretically.

In Table 3 (b), however, the value of x_f corresponding to $T_0=5.2$ sec is not that directly obtained by an actual measurement, but it is the calculated value from the equation $x_f\omega_0^2=F/I=\text{const}$. Moreover, when T_0 is large and A becomes small, it is very difficult to obtain the real value of damping constant h_e in the current experiment.

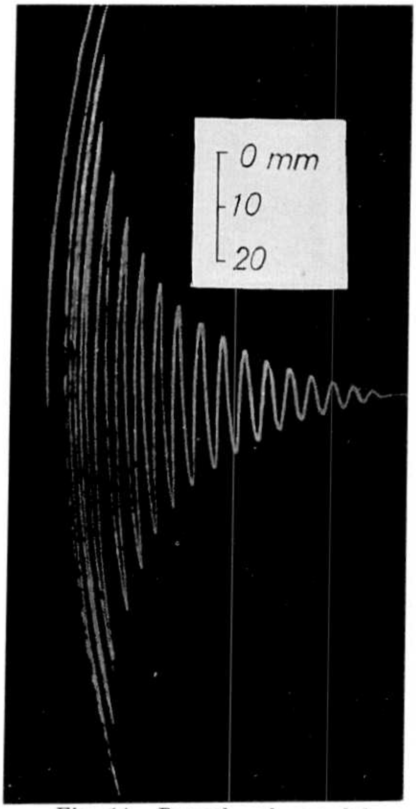


Fig. 14. Record of pendulum motion when $r=-30.2\,mm$, $T_0=1.40\,s$, $x_f=9.4\,mm$, $A=1.5\,mm$, $\frac{\omega}{2\pi}=100\,c/s$.

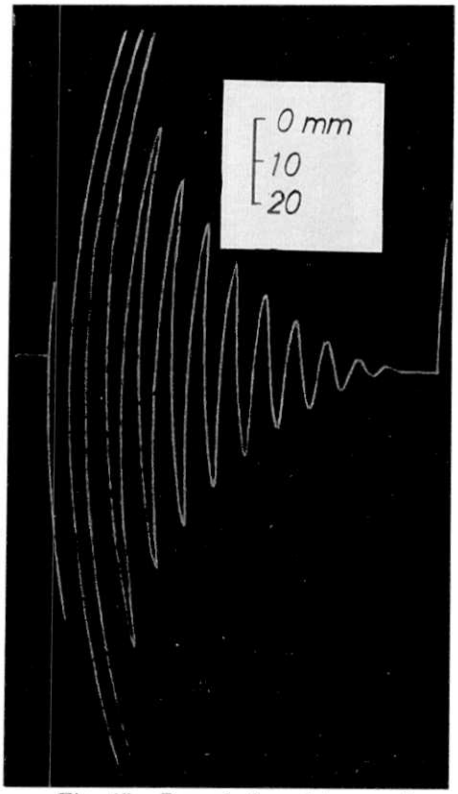
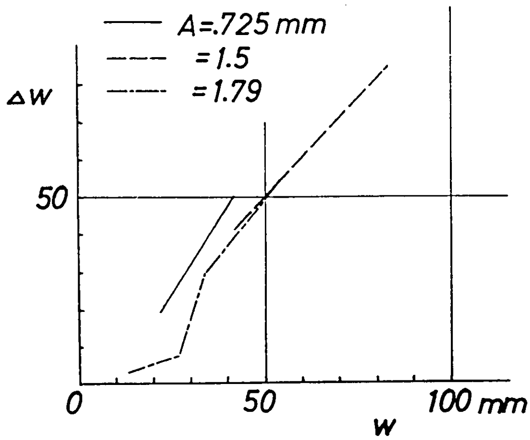


Fig. 15. Record of pendulum motion when $r=65.1\,mm$, $T_0=2.48\,s$, $x_f=47\,mm$, $A=1.5\,mm$, $\frac{\omega}{2\pi}=100\,c/s$.

From Tables 3 (a)$\sim$3 (d), it is clear that the damping constant h_e obtained by the experiment is somewhat greater in its magnitude than h_t calculated theoretically and Figs. 17 (a)$\sim$17 (d) show the relationship between the values of h experimental and those by theoretical method.

The fact that the experimental value h_e differs from the theoretical one h_t and the difference between them is dependent on the magnitude of amplitude, is not as yet given a full and complete explanation. For clarification at this stage, however, various factors could be considered; for instances, in controlling the amplitude of vibration, an indirect method had to be taken by reading and controlling the input electric current which excites the vibration after previously converting the magnitude of amplitude into the corresponding amount of current, and



(a). $r=100.1\,mm$, $T_0=5.2\,s\,(x_f \fallingdotseq 205)$.

Fig. 16. Relationship between the amplitude w and the difference of the adjacent amplitudes Δw.

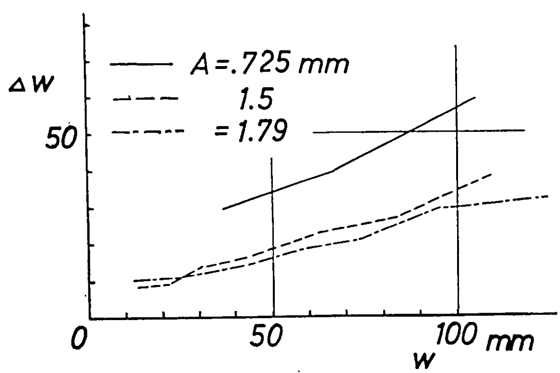


Fig. 16 (b). $r=85.0\,mm$, $T_0=3.35\,s$, $x_f=85.0\,mm$.

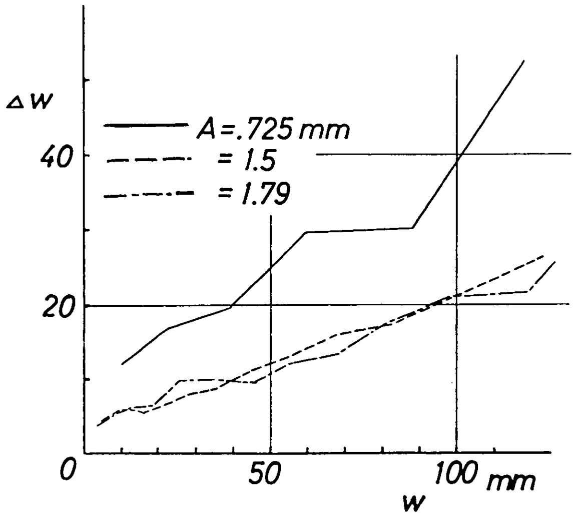


Fig. 16 (c). $r=65.1\,mm$, $T_0=2.48\,s$, $x_f=47\,mm$.

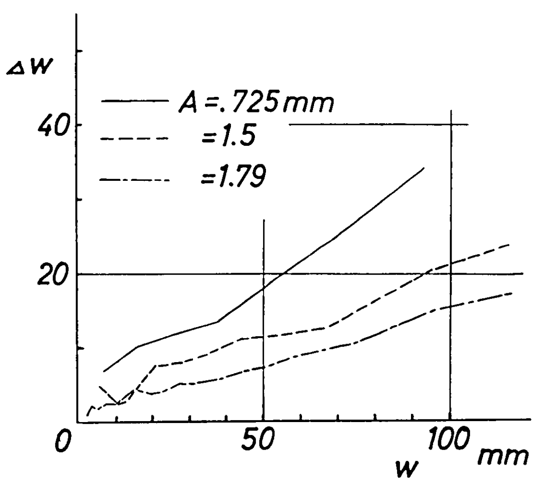


Fig. 16 (d). $r=40.0\,mm$, $T_0=1.94\,s$, $x_f=32\,mm$.

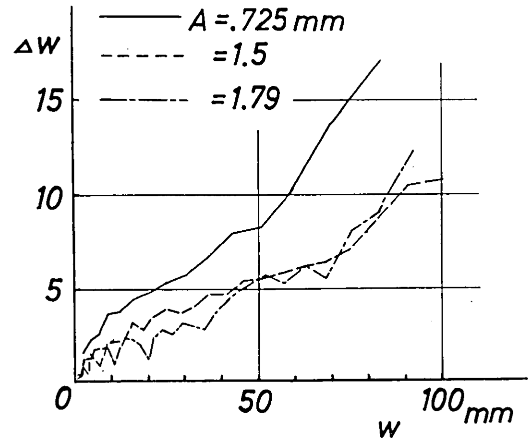


Fig. 16 (e). $r=-30.2\,mm$, $T_0=1.40\,s$, $x_f=17\,mm$.

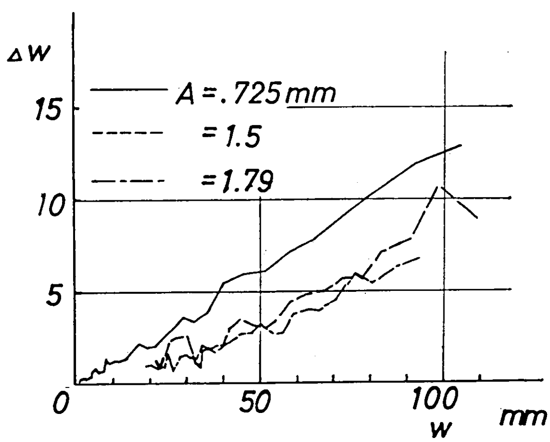


Fig. 16 (f). $r=-85.15\,mm$, $T_0=1.23\,s$, $x_f=12\,mm$.

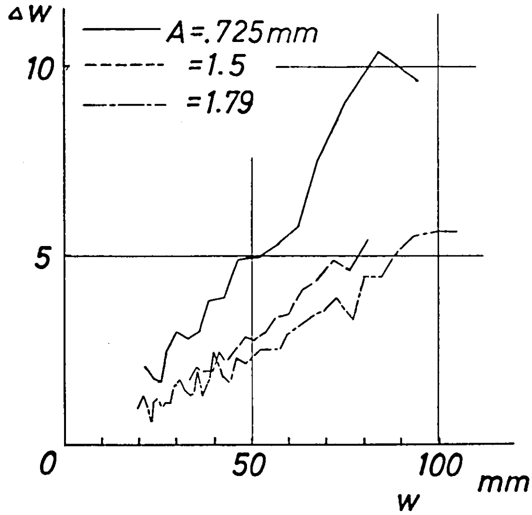


Fig. 16 (g). $r=-149.5\,mm$, $T_0=1.17\,s$, $x_f=9.4\,mm$.

the variation of the current greatly affects the value of A; or due to some unknown reasons, the conversion rate of A and current does not remain invariable according to the conditions of experiment; or further, the frictional force that would appear between the axle and the plain bearing would vary according to the condition of experiment and the magnitude of relative velocity, and so forth.

Next, referring to Figs. 16 (a)~16 (g), there can be considered such a case where the solid friction continues to keep its existence even though it is exposed to the axle vibration under the large value of x_f. In the actual experiment, however, when the pendulum comes to a

Table 3 (a). $\frac{\omega}{2\pi}=72\,c/s$.

T_0 (sec)	x_f (mm)	h_e (by experiment)				h_t (theoretical)			
		$A=0.78$	1.10	1.30	1.65	0.78	1.10	1.30	1.65 ^{mm}
1.17	8.9	0.0565	0.0334	0.0202	—	0.0465	0.0228	0.0204	—
1.37	13.3	0.140	0.050	0.0567	—	0.0615	0.0296	0.0245	—
1.90	28.2	—	0.0695	0.0759	0.0688	—	0.0461	0.0398	0.0338
2.40	43.2	—	0.0995	0.120	0.0541	—	0.0551	0.0468	0.0410
3.10	71.3	—	0.127	0.131	0.0786	—	0.0698	0.0580	0.0506
4.20	(130)	—	0.245	0.227	0.116	—	0.0962	0.0825	0.0708

Table 3 (b). $\frac{\omega}{2\pi}=100\,c/s.$

T_0 (sec)	x_f (mm)	h_e (by experimental)			h_t (theoretical)		
		$A=0.725$	1.5	1.8	0.725	1.5	1.8 ^{mm}
1.17	9.4	0.0376	0.0204	0.0185	0.0353	0.0172	0.0159
1.23	12.0	0.0392	0.0324	0.0258	0.0427	0.0207	0.0182
1.40	18.5	0.055	0.0328	0.0363	0.0580	0.0281	0.0264
1.94	34	0.097	0.0583	0.0459	0.0770	0.0375	0.0346
2.48	47	0.097	0.0590	0.0510	0.0835	0.0403	0.0374
3.35	85	0.137	0.086	0.0636	0.112	0.0541	0.0452
(5.2)	(205)	(1.0)	(1.0)	0.675	—	0.835	0.697

Table 3 (c). $\frac{\omega}{2\pi}=180\,c/s.$

T_0 (sec)	x_f (mm)	h_e (by experimental)					h_t (theoretical)				
		$A=0.28$	0.52	0.74	1.05	1.25	0.28	0.52	0.74	1.05	1.25 ^{mm}
1.17	8.1	0.503	0.0300	0.0156	0.0132	0.0118	0.0506	0.0258	0.0173	0.0131	0.0106
1.37	15.3	0.846	0.0698	0.0255	0.0204	0.0134	0.0786	0.0407	0.0272	0.0211	0.0171
1.90	32.5	(0.424)	(0.0309)	0.0255	0.0284	0.0318	0.138	0.0678	0.0426	0.0324	0.0272
2.40	48.0	—	(0.0475)	0.0455	0.0430	0.0370	—	0.0802	0.0500	0.0382	0.0317
3.10	(91.6)	—	(0.0430)	0.0700	0.0620	0.0459	—	0.115	0.0752	0.0564	0.477

Table 3 (d). $\frac{\omega}{2\pi}=316\,c/s.$

T_0 (sec)	x_f (mm)	h_e (by experimental)			h_t (theoretical)		
		$A=0.23$	0.29	0.34	0.23	0.29	0.34 ^{mm}
1.19	9.4	0.0252	0.0245	0.0185	0.0370	0.0294	0.0250
1.40	17.9	0.0425	0.0325	0.0392	0.0583	0.0462	0.0386
1.98	31.6	0.0473	0.0538	0.0410	0.0748	0.0595	0.0507
2.50	47.2	0.0637	0.0619	0.0530	0.0885	0.0770	0.0600
3.30	73.1	0.0955	0.0835	0.0637	0.104	0.0825	0.0705
4.1	(113)	0.137	0.115	0.087	0.129	0.103	0.087

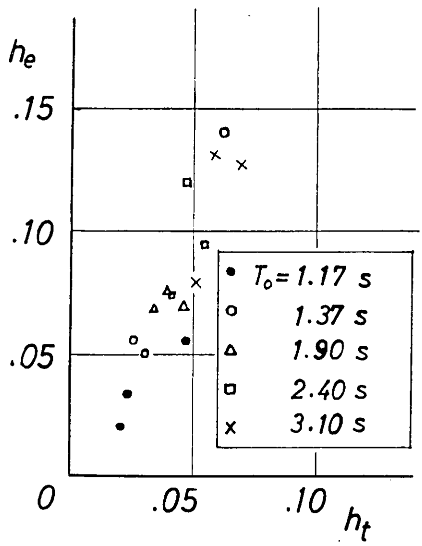


Fig. 17 (a). Relationship between h_e and h_t when f=72 c/s.

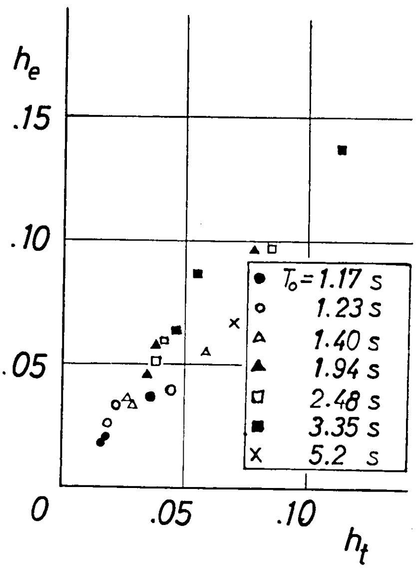


Fig. 17 (b). Relationship between h_e and h_t when f=100 c/s.

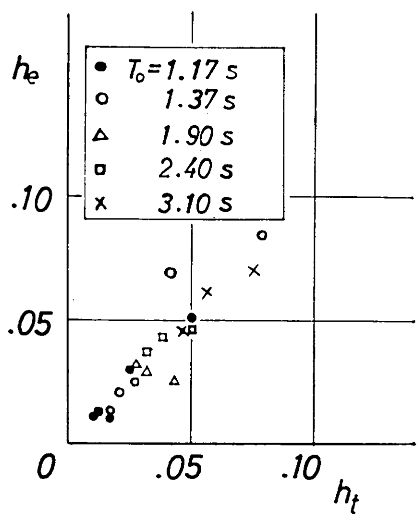


Fig. 17 (c). Relationship between h_e and h_t when f=180 c/s.

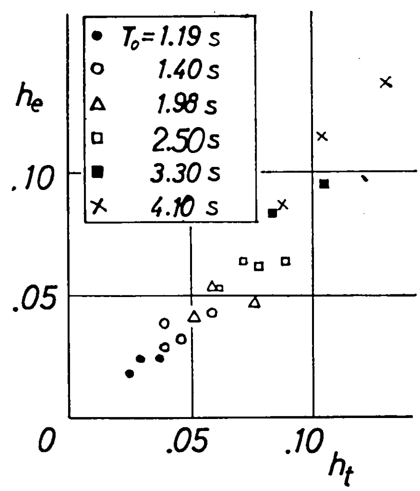


Fig. 17 (d). Relationship between h_e and h_t when f=316 c/s.

complete stop, it has returned to and assumes its zero point, which indicates that the so-called non-sensitive zone due to the solid friction does not make its appearance in the current experiment. Therefore, the aforementioned phenomenon could presumably be attributed to any possible difference in the way of action of frictional force when the ampliude of vibration of pendulum is large and when it becomes small.

6. Conclusion

The current experiment thus conducted would still involve problems that would after all remain unsolved and in need of further study for their clarification in future. Nevertheless the following conclusions can be derived from the analytical study developed thus far and within the scope of results obtained by the current experiment.

1. The solid friction can be eliminated from the solid frictional surfaces when they are subjected to the vibration of high frequency.

2. When the pendulum is subjected to the free vibration under an existence of vibrating solid friction of high frequency, its motion becomes the damped vibration, and its damping constant h is, when it is calculated theoretically, $h=\frac{x_f\omega_0}{\pi A\omega}$, whereas it becomes a little greater than the theoretically calculated value when it is obtained experimentally.

3. From a purely theoretical view-point, any small size seismic system is so made as to be possible to provide the period of infinite magnitude.

In closing, the authors would like to express their appreciations to Professor Yasuo Jimbo of the Department of Precision Mechanics, Faculty of Engineering, Univ. of Tokyo for his cordial discussions regarding their researches.

46. 振動数の高い振動固体摩擦の作用する 1自由度振動系の振動挙動（第一報）

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普通長周期の振動系は大きな質量に対して相対的に弱い復元力をもち，その大きな質量を復元力を構成するもの例えばバネで支持しようとすると初期変位が大きくなり，また別の台や軸受などで支持しようとすれば，その部分の固体摩擦が復元力と比較して大きくなり，理想的な自由な運動ができなくなる．かくして長周期の振動系を小型にすることが困難になる．もしもその支持台や軸受の固体摩擦を全くなくするか，あるいは非常に小さくすることができれば小型で長周期の振動系の出現が可能になる．そこで考えられることはその摩擦部分を高振動数で振動さすと摩擦の作用する方向は交互に正負となり，事実上固体摩擦がなくなるであろうということである．このような予想をたてて，このような場合，平均して系の質量はどのような運動をするであろうかということを取扱つてみた．

もちろん自動制御機構には dither 装置が利用されているが，この特性についての詳しい解析もないので著者らは，本報では，質量に固体摩擦以外には何の外力も作用しない場合，この振動系の挙動を理論的に明らかにし，これを実験によつて明確にし，次の結論を得た．

1. 固体摩擦面に高振動数の振動を与えると固体摩擦は有効でなくなる．
2. 高振動数で振動する固体摩擦のある場合，振子を自由振動させると，それは減衰振動の形態をとり，その減衰定数は理論的には $h=(x_f\cdot\omega_0)/(\pi A\omega)$ となり，実験結果もこの特性のあることがわかつた．
3. 理論的には如何なる小型の振動系でも無限の周期ををもつようにすることが可能である．