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17. *Maximum Likelihood Estimate of b in the Formula* $\log N = a - bM$ *and its Confidence Limits.*

By Keiiti AKI,

Earthquake Research Institute.

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Usually, the b value in the Gutenberg-Richter's[1] formula, $\log N = a - bM$, expressing the magnitude frequency relation is determined by the least squares method. Recently, Utsu[2] proposed a different method, in which b is given by $\log_{10} e / (\overline{M} - M_{\min})$, where $\overline{M}$ is the average magnitude and $M_{\min}$ is the minimum magnitude in a given sample. The purpose of the present paper is to show that Utsu's estimate of b is the maximum likelihood estimate and also to give the confidence limits of the estimate from a given sample.

We shall deal with earthquakes with magnitude greater than M_0, and assume that the probability density function $f(M, b')$ is expressed by

$$f(M,\ b') = b' e^{-b'(M-M_0)}\ , \qquad M_0 \leq M$$

where $b' = b/\log_{10} e$. Suppose we have a sample of n earthquakes with magnitudes $M_1, M_2, M_3, \ldots, M_n$. Then, we define y_i and Y by the following formulas.

$$y_i = \frac{\partial}{\partial b'} \log f(M_i,\ b')\ ,$$

$$Y = \sum_{i=1}^{n} y_i\ .$$

From the central limit theorem, the distribution function of Y will be Gaussian if n is sufficiently large and if y_i has finite mean and variance. The mean of y_i is zero, because

$$y = \frac{1}{b'} + M_0 - M$$

$$\mathrm{E}(y) = \int_{M_0}^{\infty} y f(M,\ b') dM = b' \int_{M_0}^{\infty} \left(\frac{1}{b'} + M_0 - M \right) e^{-b'(M-M_0)} dM$$
$$= 0\ .$$

1) B. GUTENBERG and C. F. RICHTER, *Seismicity of the Earth* (Princeton, 1954).
2) T. UTSU, read at the meeting of the *Seismological Society of Japan*, Oct., 1964.

The variance of y_i may be obtained by

$$\mathrm{E}(y^2)=\int_{M_0}^{\infty} y^2 f(M,\ b')dM=b'\int_{M_0}^{\infty}\left(\frac{1}{b'}+M_0-M\right)^2 e^{-b'(M-M_0)}dM$$
$$=\frac{1}{b'^2}\ .$$

Therefore, the mean of Y will be zero, and the variance of Y will be n/b'^2. It follows that the probability ε of $b'Y/\sqrt{n}$ falling in the range from $-d_\varepsilon$ to $+d_\varepsilon$ is given by $\frac{1}{\sqrt{2\pi}}\int_{-d_\varepsilon}^{d_\varepsilon} e^{-x^2/2}dx$. For $\varepsilon=95\%$, $d_\varepsilon=1.96$ and the confidence limits are given by $-1.96\leq b'Y/\sqrt{n}\leq 1.96$, that is, $-1.96\leq\frac{b'}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{1}{b'}+M_0-M_i\right)\leq 1.96$. This can be rewritten in terms of b' as

$$\frac{(1-d_\varepsilon/\sqrt{n})}{\sum_1^n M_i/n-M_0}\leq b'\leq\frac{(1+d_\varepsilon/\sqrt{n})}{\sum_1^n M_i/n-M_0}\ .$$

The central value of b' $\left(=\frac{1}{\sum M_i/n-M_0}\right)$ in the above range is the solution of $Y=0$. This is the b' value which maximizes the likelihood function $\Pi_i f(M_i,\ b')$. The corresponding b value is given by the same formula as Utsu's.

The following table shows the values of $d_\varepsilon/\sqrt{n}$ for various ε's. They show one half of the interval between the confidence limits in fraction of the b value estimated.

Table 1. values of $d_\varepsilon/\sqrt{n}$ for various ε's.

n \ ε	0.50	0.80	0.90	0.95	0.98
50	.090	.181	.233	.277	.329
100	.067	.128	.165	.196	.233
200	.048	.091	.116	.139	.165
500	.030	.057	.075	.088	.104
1000	.021	.041	.052	.062	.074

17. $\log N=a-bM$ の関係式における b の値の最尤度法による推定と信頼限界

地震研究所 安 芸 敬 一

普通，地震のマグニチュードと頻度の関係式 $\log N-a=bM$ の b の値は，最小自乗法できめられている．最近，宇津は，異なつた方法によるこの推定法を発表した．この論文では，宇津の方法が，統計学でいう最尤度法に等しいことを示し，併せて b の値の信頼限界を求めた．
