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1. *Soft Core Spectrum Splitting of the Torsional Oscillation of an Elastic Sphere and Related Problems.**

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Abstract

The torsional oscillation of an earth model consisting of a homogeneous mantle and a soft homogeneous core is studied and the following matters are discussed and proved. (1) When there is a soft core those modes are predominant which have nearly equal frequencies to that of a liquid core model. (2) These modes are very high radial higher modes, and a simple formula that gives the order number of them is presented. (3) Consecutive spectrum lines far stronger than the other ones sometimes have nearly equal amplitudes and frequencies, which fact proves to be a kind of spectrum splitting phenomenon. (4) Depending on the density and rigidity ratios of the core and mantle observable frequency is higher or lower than that of a liquid core model, and there is also a possibility of two spectrum peaks having nearly equal amplitudes. This phenomenon will be of use for the determination of the core rigidity by means of the observation of free oscillation periods.

1. Introduction

Ten years ago, when the possibility of exploring the nature of the earth core by the use of long period surface waves was suggested, the present author calculated the torsional oscillation period of a simple elastic earth model.[1] This model consisting of a homogeneous mantle and either a liquid or rigid core, was far too simple to explain observed data, but still gave some clue giving a free oscillation period of 42.6

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1) T. MATUMOTO and Y. SATÔ, "On the Vibration of an Elastic Globe with One Layer. The Vibration of the First Class," *Bull. Earthq. Res. Inst.*, **32** (1954), 247.

minutes for a liquid core and 33.8 minutes for a rigid core.

Later study,[2] however, revealed that the free torsional fundamental period for a model with a finite core rigidity did not come out between the two values cited above, but the period increased indefinitely when the rigidity of the core tended to zero. This apparently strange phenomenon, which one can hardly be expected to observe, was rather insufficiently explained in that study. In the present paper, however, in which a very soft core is assumed, the following matters are discussed and explained; 1) the periods which are to be observed, 2) the soft core splitting of spectrum lines, 3) comparison of the periods of a liquid core model and a soft core model (an idea of over, under and double frequencies) and 4) the radial mode number which is likely to give a large amplitude.

2. Characteristic Equation and the Solution

Following an ordinary way of solving the wave equation referred to polar coordinates solutions are obtained in the form:

$$\begin{aligned} u&=0\ , \\ v&=mR(r)\cdot P_n^m(\cos\theta)/\sin\theta\cdot\sin m\varphi\cdot\exp(ipt)\ , \qquad (2.1) \\ w&=\ R(r)dP_n^m(\cos\theta)/d\theta\cdot\cos m\varphi\cdot\exp(ipt)\ , \end{aligned}$$

where u, v and w are radial, colatitudinal and azimuthal displacement components respectively. The function $R(r)$ is expressed by spherical Bessel functions j_n and y_n. The boundary conditions are, at the surface $r=a$

$$\text{Displacement};\ R(a)=A\cdot j_n(k_0a)+B\cdot y_n(k_0a)=K(\text{constant})\ ,$$

$$\text{Stress};\ \mu_0\left(\frac{\partial R}{\partial r}-\frac{R}{r}\right)_a=\mu_0A\left\{a\frac{d}{da}\left(\frac{j_n(k_0a)}{a}\right)\right\}+\mu_0B\left\{a\frac{d}{da}\left(\frac{y_n(k_0a)}{a}\right)\right\}\ ,$$

$$=\delta'\ (\text{ordinarily zero})\ , \qquad (2.2)$$

In the same manner, at the interface $r=b$,

$$\text{Displacement};\ A\cdot j_n(k_0b)+B\cdot y_n(k_0b)=C\cdot j_n(k_ib)\ ,$$

$$\text{Stress};\ \mu_0A\left\{b\frac{d}{db}\left(\frac{j_n(k_0b)}{b}\right)\right\}+\mu_0B\left\{b\frac{d}{db}\left(\frac{y_n(k_0b)}{b}\right)\right\}=\mu_iC\left\{b\frac{d}{db}\left(\frac{j_n(k_ib)}{b}\right)\right\}\ , \qquad (2.3)$$

2) Y. SATÔ and T. MATUMOTO, "Vibration of an Elastic Globe with a Homogeneous Mantle over a Homogeneous Core. Vibration of the First Class," *J. Phys. Earth*, **9** (1961), 1.

in which $k_0^2=p^2\rho_0/\mu_0=(p/V_0)^2$ and $k_i^2=p^2\rho_i/\mu_i=(p/V_i)^2$, ρ, μ and V are the density, rigidity and shear velocity respectively, and o and i always refer to the mantle and core respectively.

From the above equations, after a little modification, the characteristic equation is obtained as follows:

$$E\left(n,\ \xi,\ \frac{b}{a},\ \frac{V_i}{V_0},\ \frac{\mu_i}{\mu_0},\ \delta\right)=\begin{vmatrix} FJ(\xi) & FY(\xi) & 0 & 1 \\ GJ(\xi) & GY(\xi) & 0 & \delta \\ GJ(\eta) & GY(\eta) & \mu_i/\mu_0\, GJ(\zeta) & 0 \\ FJ(\eta) & FY(\eta) & FJ(\zeta) & 0 \end{vmatrix}=0$$

where
$$FJ(\xi)=\xi\cdot j_n(\xi)\,,\quad GJ(\xi)=\xi^3\cdot\partial(j_n(\xi)/\xi)/\partial\xi\,,$$
$$FY(\xi)=\xi\cdot y_n(\xi)\,,\quad GY(\xi)=\xi^3\cdot\partial(y_n(\xi)/\xi)/\partial\xi \tag{2.4}$$

and

$$\xi=k_0 a\,,\quad \eta=(b/a)\xi\,,\quad \zeta=(V_0/V_i)\eta\,.$$

An ordinary form of the characteristic equation is

$$D_n(\xi,\ V_i/V_0)\equiv -E\left(n,\ \xi,\ \frac{b}{a},\ \frac{V_i}{V_0},\ \frac{\mu_i}{\mu_0},\ 0\right), \tag{2.5}$$
$$=FJ(\zeta)\cdot L_n(\xi,\ \eta)-(\mu_i/\mu_0)GJ(\zeta)\cdot R_n(\xi,\ \eta)=0\,,$$

where
$$L_n(\xi,\ \eta)=GJ(\xi)\cdot GY(\eta)-GY(\xi)\cdot GJ(\eta)\,, \tag{2.6a}$$
$$R_n(\xi,\ \eta)=GJ(\xi)\cdot FY(\eta)-GY(\xi)\cdot FJ(\eta)\,. \tag{2.6b}$$

The torsional frequency for a liquid or a rigid core model is obtained as a root of

$$L_n(\xi,\ \eta)=0\,, \tag{2.7a}$$

or

$$R_n(\xi,\ \eta)=0\,. \tag{2.7b}$$

When the core rigidity is finite and the density ratio being fixed, the free torsional frequency ξ is given as a function of (V_i/V_0), and the $(V_i/V_0)-\xi$ plane is divided into a mesh by the following two groups of straight lines:

(i) parallel lines determined by the equations (2.7a) and (2.7b),

and (ii) straight lines which pass the origin and are given by the equations

$$FJ_n(\zeta)=0\ , \tag{2.8a}$$

and

$$GJ_n(\zeta)=0\ , \tag{2.8b}$$

of which the first equation gives $\mu_i=0$ and the second gives $\mu_i=\infty$.

In Fig. 1 the shadow zones are the parts where no solution can exist, and the black circles are the points which the frequency curves must pass. In this way the curves are separated into independent branches. Fig. 2 shows the result of an exact calculation assuming the density ratio $\rho_i/\rho_0=2.0$. The lower-most curve is the fundamental mode and others are the radial higher modes.

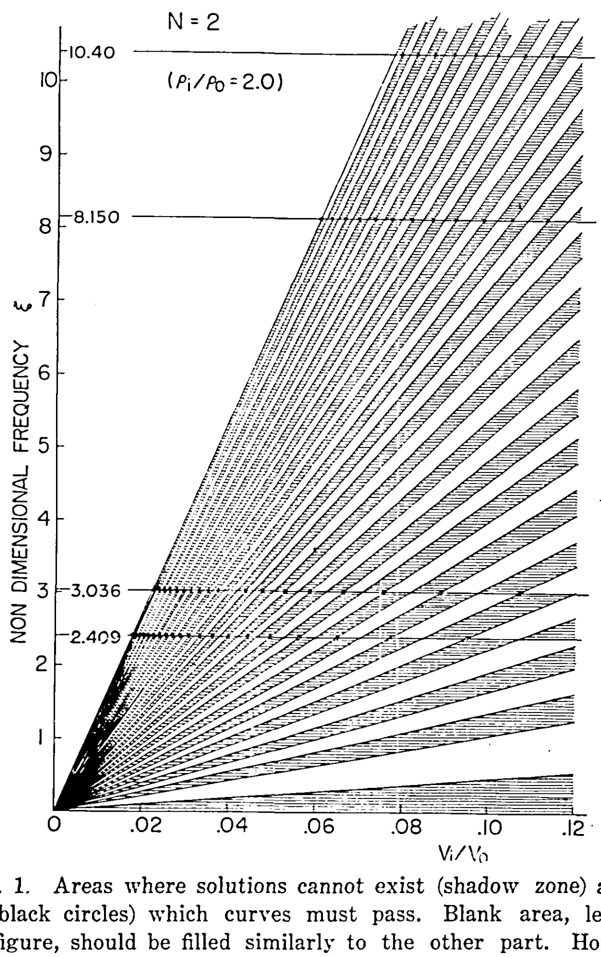

Fig. 1. Areas where solutions cannot exist (shadow zone) and the points (black circles) which curves must pass. Blank area, left half of the figure, should be filled similarly to the other part. Horizontal lines give the values of ξ for a liquid and a rigid core model.

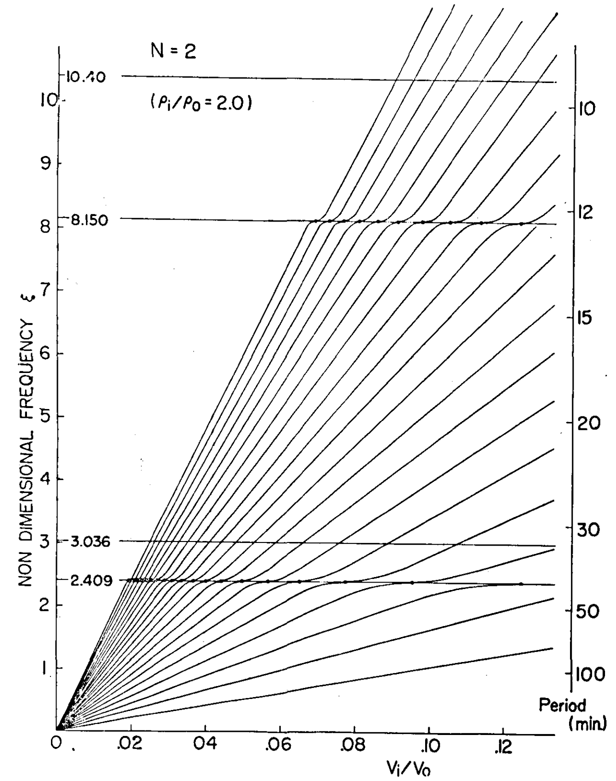

Fig. 2. Frequency curves as functions of V_i/V_0. In the blank area there is an infinitely large number of similar curves. Black circles on the straight lines indicate points where the frequency of each mode becomes equal to that of a liquid model. $\rho_i/\rho_0=2.0$ is assumed.

Most parts of the curves nearly coincide with the straight lines given by (2.8 a), which are the lines of $\mu_i=0$. This is a natural consequence of our assumption that the core rigidity is very small.

3. Soft Core Spectrum Splitting

Although there are many branches of radial higher modes, not all of them have appreciable amplitudes, but only those satisfying the following condition are actually excited and observed.

In our previous work[3] it was proved that the spectrum amplitudes

3) Y. SATÔ, T. USAMI, M. LANDISMAN and M. EWING, "Basic Study on the Oscillation of a Homogeneous Elastic Sphere (V)," *Geophys. J.*, **8** (1963), 44.

are proportional to

$$ {}_iA_n=\left(\frac{\partial\xi}{\partial(\partial R/\partial r-R/r)}\right)_{r=a}, \tag{3.1}$$

in which r is the radial distance and $R(r)$ is the displacement amplitude of the azimuthal component. The result of numerical computation of ${}_iA_n$ for various values of i is shown in Fig. 3 assuming $n=2$ and $\rho_i/\rho_0=$

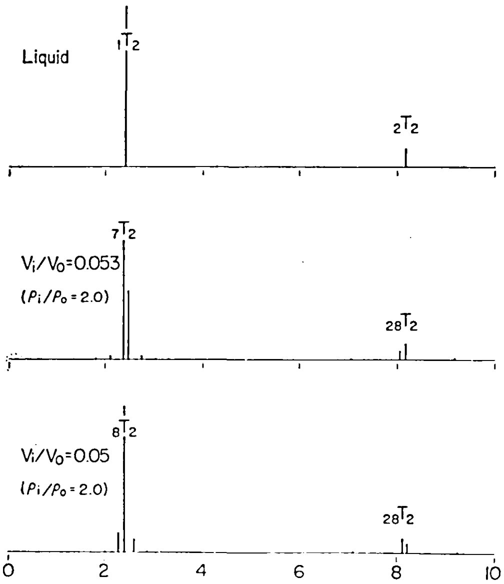

Fig. 3. Spectrum lines for the liquid core and soft core models. For soft core models there are many small spectrum lines between two large peaks. However, most of them are too small to be found in the figure. Abscissa is the non-dimensional frequency ξ. $(n=2)$

2.0. From this figure we can deduce a number of interesting conclusions.

1) When the core is liquid there are widely separated spectrum lines. The first and the largest is the fundamental mode.

2) When the core is a soft solid there are many spectrum lines corresponding to many curves given in Fig. 2. However, only those which have nearly equal frequency to that of the liquid core model are predominant.

3) There is a possibility that two or more spectrum lines have considerable amplitudes, which fact is a kind of spectrum splitting phenomenon caused by the existence of a soft core.

4) The intervals between consecutive spectrum lines are determined only by the distribution of material in the radial direction.

The phenomenon will be called the "soft core splitting" of spectrum lines hereafter.

4. Interpretation of the Soft Core Spectrum Splitting

If in the expression (2.5) we fix the two parameters V_i/V_0 and ρ_i/ρ_0, and change the variable ξ, then the value of D_n varies as illustrated in Fig. 4. The zero points of D_n are at first uniformly distributed. Near

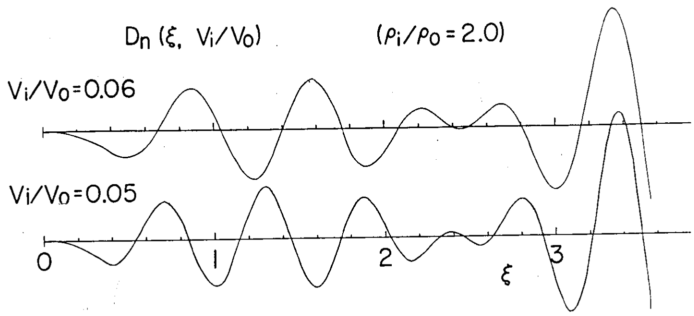

Fig. 4. $D_n(\xi, V_i/V_0)$ as a function of ξ. Near the point $\xi=\xi_{\text{liquid}}$, undulation, consequenly the slope of curves becomes small. ($n=2$)

the point $\xi=\xi_{\text{liquid}}$ ($=2.409$), however, behavior of the curve becomes different from the other part and extraordinary things happen. One is an irregular interval of zero points. The second is a phenomenally small amplitude of the undulations of the function D_n, and the last which naturally derives from the second, is the smallness of the inclination ($\partial D_n/\partial\xi$) of the curve D_n at these points.

The amplitude of a mode is, introducing the relations (2.4) and (2.5) into (3.1), calculated as follows:

$$
\begin{aligned}
{}_iA_n \propto 1 \Big/ \frac{\partial}{\partial\xi}\left\{\frac{\partial R}{\partial r}-\frac{R}{r}\right\}_{r=a} &= 1 \Big/ \frac{\partial}{\partial\xi}\left(\frac{\partial R}{\partial r}\right)_{r=a}, \\
&= 1 \Big/ \left(\frac{\partial\delta}{\partial\xi}\right)_{r=a,\,\delta=0}, \\
&= 1 \Big/ \left\{-\frac{\partial E}{\partial\xi}\Big/\frac{\partial E}{\partial\delta}\right\}, \\
&= \frac{\partial E}{\partial\delta}\Big/\frac{\partial D_n}{\partial\xi}.
\end{aligned}
$$

Since $(\partial D_n/\partial\xi)$ becomes small near the points $\xi=\xi_{\text{liquid}}$ the amplitude becomes large at these points.

5. Radial Mode Number

It was now proved that the radial higher modes which have large amplitudes and are likely to be observed have always nearly equal frequencies to those of the liquid core model. Consequently the radial mode number of these modes is easily calculated by the following procedure.

Black circles on the line $\xi=\xi_{\text{liquid}}$ are the roots of the equation

$$FJ(\zeta)=\zeta\cdot j_n(\zeta)=0\,. \tag{5.1}$$

From the asymptotic formula of Bessel functions

$$FJ(\zeta)\sim\cos\left[\zeta-\frac{\{2(n+1/2)+1\}}{4}\pi\right].$$

Equating this to zero

$$\zeta_N=\frac{b}{a}\frac{V_0}{V_i}\xi_{\text{liquid}}=\frac{\{2(n+1/2)+1\}}{4}\pi+\left(N-\frac{1}{2}\right)\pi\,.$$

Therefore

$$N=\frac{1}{\pi}\left(\frac{b}{a}\frac{V_0}{V_i}\xi_{\text{liquid}}\right)-\frac{n}{2}\,, \tag{5.2}$$

If $n=2$ and $V_i/V_0=0.05$, then inserting $\xi_{\text{liquid}}=2.409$,

$$N=7 \text{ or } 8 \tag{5.3}$$

which is the number found in Figs. 2 and 3.

6. Over Frequency, Under Frequency and Double Frequency

Let us look at Fig. 2 keeping in mind that the predominant modes always have frequencies nearly equal to those of the liquid core model. The strongest mode, which is the nearest to the line $\xi=\xi_{\text{liquid}}$, can have a larger value than that according to the value of (V_i/V_0), but might be smaller, and sometimes there may be two modes having nearly equal amplitudes, frequencies being located on both sides of $\xi=\xi_{\text{liquid}}$. In this way V_i/V_0 axis is divided into an infinite number of parts; over frequency, under frequency and double frequency areas. (Fig. 5)

This nature gives a clue for determining the shear velocity of the core using only the free torsional period. If we accurately observe free torsional periods for various values of n, we can determine whether they are larger or smaller than the liquid model frequency, or there are two peaks with comparable amplitudes. These correspond to over, under and double frequencies respectively, and referring to Fig. 6, we can choose values of V_i/V_0 that give the combination of over, under and double frequencies identical to the observed one. If a sufficiently large number of frequencies is available, there will be only one value of V_i/V_0 that satisfies the observed condition. Similar figures can be prepared for frequencies corresponding to higher modes of liquid core model.

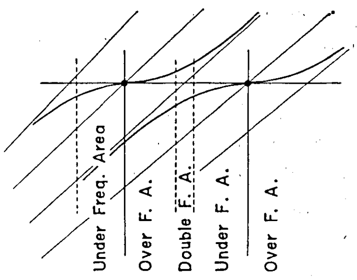

Fig. 5. V_0/V_i (or V_i/V_0) axis is divided into over and under frequency areas according as the strongest spectrum line has a frequency larger or smaller than that of a liquid core model. Between these areas there are double frequency areas where there are two peaks with comparable amplitudes. (Not to scale)

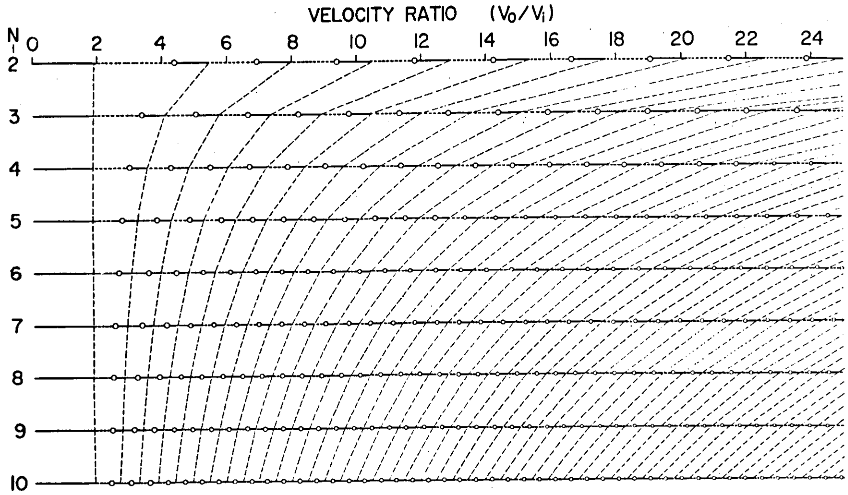

Fig. 6. (V_0/V_i)-axis is divided into over, under and double frequency areas. This figure is for the fundamental mode. Bold line: Over frequency area, Broken line: Under frequency area, Circle: Double frequency area.

7. Conclusion

In this paper a very simple earth model, a homogeneous mantle with a soft homogeneous core, is assumed. Consequently the numbers given here cannot be applied to a complicated heterogeneous earth model. Still, however, the same principle will hold even for the actual earth having a complicated structure with a very small core rigidity. Namely: (1) The observed period, which is a value for a high radial higher mode, will differ little from the liquid core frequency. (2) The soft core splitting of the torsional oscillation spectrum will occur. (3) According to the density and rigidity of the core the observed frequency will be larger or smaller than the liquid core frequency, or there may be peaks with comparable amplitude. (4) The order number of such a peak is given by a simple formula. A detailed discussion of these phenomenon will be given in the near future for heterogeneous earth models.

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1. 軟い核の存在による弾性球の捩れ振動スペクトルの分裂とこれに関連した問題

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一様なマントルと，一様で軟い核を持つ弾性球の捩れ振動に関して次のことが論ぜられた．

1) 核の剛性が小さい時には，基本振動周期は極めて長いが，実際に表はれるのは，核が流体の時とほとんど等しい周期をもつた振動である．

2) これらの振動は半径方向に多くの節を持つ高次のモードであり，その次数は簡単な公式によつて求めることができる．

3) 上に挙げたモードに対するスペクトル線は，他のものにくらべて著しく強い上に，極めて隣接して居るので，最近よく論ぜられる地球振動におけるスペクトルの分裂と同じ現象を呈することになる．

4) 核とマントルとの密度，剛性の比いかんによつて，最も強いスペクトル線に対する振動数は，流体のそれに対するよりも，ある時は大きく，ある時は小さく，また時にはほとんど強さの等しい二つのスペクトル線が表われることがある．この現象を利用すれば，自由振動の観測から，核の剛性を推測する手がかりが得られるであらう．