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30. *Phase Angle of Waves Propagating on a Spherical Surface with Special Reference to the Polar Phase Shift and the Initial Phase.**

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Abstract

In order to make clear the phase relation of waves propagating on a spherical surface to various quantities, theoretical seismograms showing the azimuthal component of torsional disturbances are Fourier analysed. Each phase of G waves is employed in the analysis and the results are given in the figures, which show the polar phase shift clearly.

The possibility and method of deducing informations on the characteristics of waves from the Fourier analysis is proposed in this paper.

1. Introduction

The phase shift of waves propagating on a spherical surface when they pass the pole and antipode was found by Nafe, Brune and Alsop[1], and has been successfully applied to the analysis of dispersive surface waves.

Even in the plane boundary problem a similar phase shift to compensate the phase variation with which waves diverge from a point had been noticed and adopted in an attempt to reproduce the disturbance at the source[2].

The above phenomenon discovered by Nafe and others, which is today called the polar phase shift, is valid for each wave component with a definite value of frequency, while seismograms are usually a su-

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1) J. N. BRUNE, J. E. NAFE and L. E. ALSOP, "The Polar Phase Shift of Surface Waves on a Sphere," *Bull. Seism. Soc. Amer.*, **51** (1961) 247-257.

2) Y. SATÔ, "Synthesis of Dispersed Surface Waves by Means of Fourier Transform," *Bull. Seism. Soc. Amer.*, **50** (1960), 417-426.

perposition of waves with various frequencies. Consequently, it usually happens that the polar phase shift is not clearly observed on the seismograms of natural earthquakes.

In this paper, to visualize the phase variation of waves propagating on a spherical surface, the result of Fourier analysis on the theoretical seismograms is illustrated.

2. Notations and Formulas

Disturbances propagating on the surface of a radially heterogeneous sphere is expressed by the sum of free oscillations of various modes[3)]

$$D=\sum_{i,n,m} C_n^m \cdot R_n(a,\ {}_ip_n)\cdot f^*({}_ip_n)\cdot \Theta_n^m(\theta)\cdot {\cos \atop \sin} m\varphi \cdot \exp(j\,{}_ip_nt)\,, \tag{1}$$

where

a	:	Radius of the sphere.
C_n^m	:	Coefficient due to space distribution of the force applied, m and n being the degree and order of the free oscillation respectively.
$f^*({}_ip_n)$	:	Fourier conjugate of the time distribution of external force applied, $f(t)$.
j	:	Unit of imaginary number.
${}_ip_n$	:	Frequency of the free oscillation, i being the radial mode number and n the order of free oscillation.
$R_n(r,\ {}_ip_n)$	:	Function giving radial distribution of disturbance.
$\Theta_n^m(\theta)\cdot {\cos \atop \sin} m\varphi$	:	Function giving surface distribution of disturbance. $\Theta_n^m(\theta)$ implies associated Legendre function or its modified form. Actual expression of $\Theta_n^m(\theta)$ depends on the kind of oscillation, the quantity measured and the component of disturbances, and their actual forms were given in our previous paper[4)] for typical cases.

The spectrum of the disturbances D obtained at a point on the surface with an epicentral distance $\theta=\theta_0$ and an azimuth $\varphi=\varphi_0$ is

$${}_iS_n^m=C_n^m\cdot R_n(a,\ {}_ip_n)\cdot f^*({}_ip_n)\cdot \Theta_n^m(\theta_0)\cdot {\cos \atop \sin} m\varphi_0\,. \tag{2}$$

3) Y. SATÔ, T. USAMI and M. EWING, "Basic Study on the Oscillation of a Homogeneous Elastic Sphere. IV. Propagation of Disturbances on the Sphere," *Geophys. Mag.*, **31** (1962), 237-242.

4) Y. SATÔ and T. USAMI, "Method of Determining the Degree of Free Oscillation of a Radially Heterogeneous Elastic Sphere," *Bull. Earthq. Res. Inst.*, **41** (1963), 331-342.

When n is larger than m, and $\sin\theta$ is not very small, the following asymptotic expansion holds[5]

$$P_n^m(\cos\theta) \sim (-)^m n^m \sqrt{\frac{2}{n\pi\sin\theta}} \cos\left\{\left(n+\frac{1}{2}\right)\theta + \frac{m\pi}{2} - \frac{\pi}{4}\right\}$$
$$(0<\theta<\pi). \tag{3}$$

As $\Theta_n^m(\theta)$ is P_n^m itself or its modified function, the asymptotic expansion has a similar expression to the formula (3), namely

$$\Theta_n^m(\theta) \sim A_n^m(\theta) \cdot \cos\left\{\left(n+\frac{1}{2}\right)\theta + \frac{m\pi}{2} - \frac{\pi}{4} - \alpha\right\}, \tag{4}$$

in which $A_n^m(\theta)$ is a slowly varying function of θ, and

α is an additional phase angle when the associated Legendre function in Θ_n^m is differentiated with respect to θ. α, therefore, is expressed as $l\pi/2$, l being the number of differentiation. (A)

Combining the two relations (3) and (4), the spectrum is expressed as

$$_iS_n^m = |C_n^m \cdot R_n \cdot f^*({}_ip_n) \cdot A_n^m(\theta) \cdot {}^{\cos}_{\sin}\, m\varphi|$$
$$\cdot \cos\left\{\left(n+\frac{1}{2}\right)\theta + \left(\frac{m\pi}{2} - \frac{\pi}{4} - \alpha\right)(1+2s) + \varepsilon\right\}, \tag{5}$$

where

s is the number of passage over the pole and antipode. (B)

ε is the phase angle due to the sign of $\left(C_n^m \cdot R_n \cdot f^*({}_ip_n) \cdot A_n^m(\theta) \cdot {}^{\cos}_{\sin}\, m\varphi\right)$. When this function is negative $\varepsilon=\pi$, otherwise $\varepsilon=0$. (C)

In the formula (5), θ is not restricted between 0 and π.

3. Phase angle of waves propagating on a spherical surface

The phase angle of the complex spectrum, namely the Fourier transform of disturbance D, is from the above consideration

$$\mathfrak{L}_c = \left(n+\frac{1}{2}\right)\theta + \left(\frac{m\pi}{2} - \frac{\pi}{4} - \alpha\right)(1+2s) + \varepsilon, \tag{6}$$

5) E. W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics.* Cambridge Univ. Press, p. 303.

This is the formula when the real part of the time factor in equation (1) is adopted, while the phase angle should be

$$\mathfrak{L}_s=\left(n+\frac{1}{2}\right)\theta+\left(\frac{m\pi}{2}-\frac{\pi}{4}-\alpha\right)(1+2s)+\varepsilon+\frac{\pi}{2}\,, \tag{7}$$

if the imaginary part is employed.

In order to illustrate the phase relation of waves propagating on a spherical surface, theoretical seismograms are analysed by the method

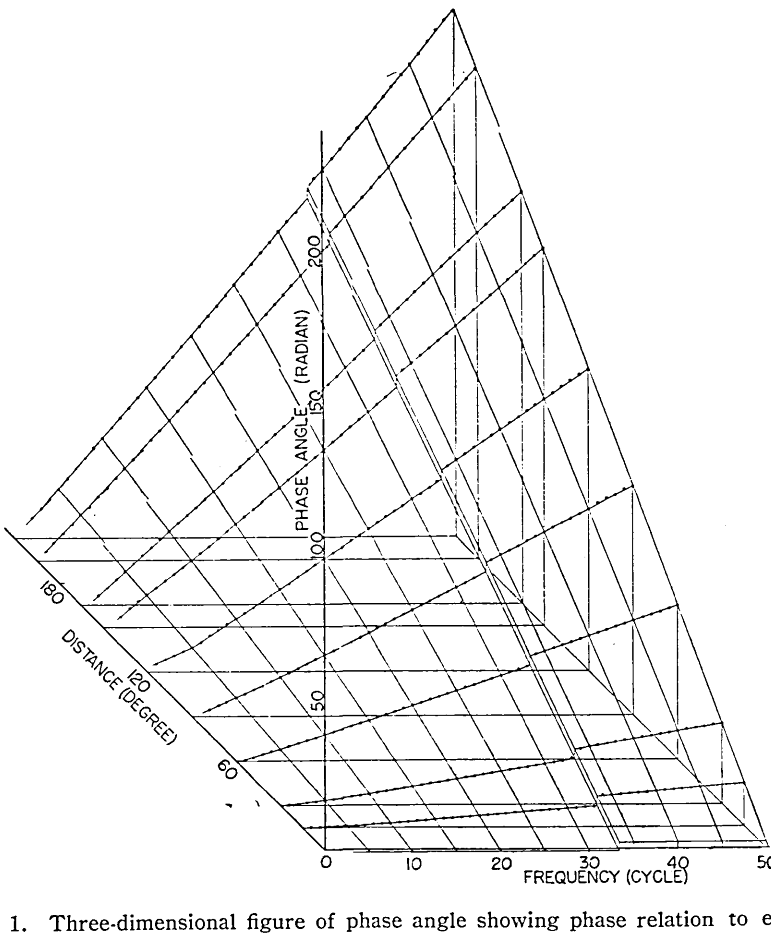

Fig. 1. Three-dimensional figure of phase angle showing phase relation to epicentral distance and frequency. Unit of time is (2π (radius of sphere)/S wave velocity).

Initial phase angle is $-\pi/4$ and $3\pi/4$, and the gap on the curves of constant frequency is $\pi/2$, which is the polar phase shift.

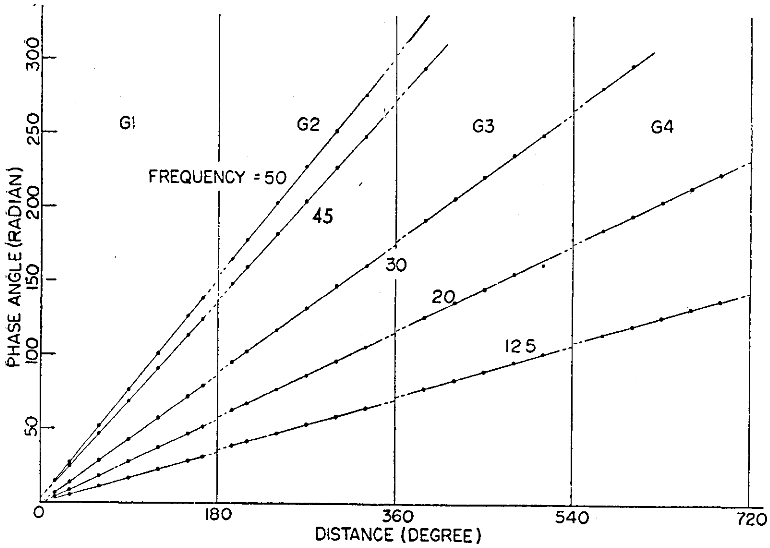

Fig. 2. Phase angle as a function of epicentral distance and frequency obtained by the method of Fourier transform from theoretical seismogram showing azimuthal displacement of torsional oscillation of degree 0 for a case of homogeneous isotropic elastic sphere. Unit of frequency is (Velocity of S wave on the surface)/(2π (radius of the sphere)). This figure shows polar phase shift of amout $\pi/2$ at each passage over the pole and antipode which shows good agreement with the theory. Initial phase for smaller frequency is $-\pi/4$, and is $3\pi/4$ for larger value of frequency.

of Fourier transform. These seismograms are the calculated disturbances of the torsional oscillation caused by a localised stress around the pole[6]. Each phase of G waves is analysed separately and the phase angle of the Fourier transform is shown in Figs. 1 and 2 as a function of epicentral distance and frequency. Since the theoretical seismogram employed is the azimuthal displacement of torsional oscillation mentioned above, we have

$$\left.\begin{aligned} &\Theta_n^m(\theta)=dP_n(\cos\theta)/d\theta\ , \\ \text{and}\qquad &m=0 \text{ and } \varepsilon=0\ . \end{aligned}\right\} \tag{8}$$

As the imaginary part of the time factor in the expression D is employed, the formula (7) is applied and we have $\alpha=\pi/2$ from (A)[7].

6) Y. SATÔ and others, *loc. cit.*, 3).

7) A similar consideration was given by one of the present authors more briefly. Y. SATÔ, "A Note on the Relation between the Initial Motion and the Azimuthal Characteristic of a Focus from the View-point of the Phase Shift near the Origin," *Bull. Earthq. Res. Inst.*, **40** (1962), 653-655.

As a whole, the phase angle at the origin is

$$\left(n+\frac{1}{2}\right)\times 0+\left(\frac{\pi}{2}\times 0-\frac{\pi}{4}-\frac{\pi}{2}\right)(1+2\times 0)+0+\frac{\pi}{2}=-\frac{\pi}{4} \tag{9}$$

which is clearly seen in Figs. 1 and 2[8]. The polar phase shift of amount

$$\frac{\pi}{2}\left[=2\pi+2\left(-\frac{\pi}{4}-\frac{\pi}{2}\right)\right], \tag{10}$$

is also seen in the same figures at an epicentral distance $\theta=n\pi$ (2π in formula (10) is an uncertainty appearing in the calculation of phase angle).

As is seen in the formulas (6) and (7), the phase angle of the spectrum is a function of m, n, θ and ε. The last factor ε takes π or 0 according to the time and space distribution of the force applied, the structure of the sphere, quantity measured and azimuth of observation station. (cf. (C))

4. Summary

The above consideration suggests that, if stations are favorably distributed, the following information can be deduced concerning the characteristics of waves from the complex spectrum of the disturbances at these stations.

i) Comparing the periods of spectral peaks obtained by Fourier transform of actual seismograms with those given by theory, we can determine whether the wave observed is spheroidal or torsional.

ii) Once the kind of oscillation, spheroidal or torsional, is identified, the actual form of function $\Theta_n^m(\theta)$ is determined except the value m since we know what quantity is measured.

iii) The degree (m) of waves is inferred from the following three methods. First, from the formulas (6) and (7), we can determine whether m is even or odd from the jumping amount of phase angle at $\theta=n\pi$. (See Figs. 1 and 2. Intercept phase angle, or initial phase.) Second, among the factors forming the expression of disturbance D in the equation (1), $\cos m\varphi$ (or $\sin m\varphi$) is the only function relating to the azimuth. Therefore, the number of sign changes of $\cos m\varphi$ (or $\sin m\varphi$) between

8) In another paper prepared by the present authors, $\pi/4$ was added and the initial phase angle for small p was put to 0. This is because the authors intended to show the value of $\arg(f^*(p))$ clearly. See Y. SATÔ and T. USAMI "Spectrum, Phase and Group Velocities of the Theoretical Seismograms and the Idea of the Equivalent Surface Source of Disturbances." (in print).

$\varphi=0°$ and $360°$ is same as the number of jumps of the phase angle by the amount π. This is closely connected with the push-pull mechanism of the source of disturbance. The third method is described in our previous paper[9].

iv) Frequency p has effects on the phase angle through ε (cf. (C)). The phase angle expressed as a function of p jumps by an amount π at frequencies where $C_n^m R_n(a, {}_i p_n) \cdot f^*({}_i p_n)$ changes its sign. Therefore, from these jumps of phase angle as a function of p, we are informed as to the nature of the above function.

5. A remark on the propagation of waves on a sphere

Although theoretical seismograms were computed using only the first kind associated Legendre function, they show similar features to those of natural earthquakes, and their Fourier analysis gives satisfactory coincidence with theory. This result indicates that the superposition of $P_n^m(\cos\theta)$ functions is enough for the complete description of the phenomenon including the propagating waves. This might be anticipated from the equations (4), the disturbances on a sphere being reduced to the sum of waves propagating in the positive and negative directions. These features are similar to the wave propagating on a string which is expressed as the sum of normal modes.

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9) Y. SATÔ and T. USAMI, *loc. cit.*, 4).

30. 球面上を伝わる弾性波の位相角 とくに極における位相の変りと震源における位相角に関連して

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地震波の解析にフーリエ変換の方法が使われるようになつてから久しく，この間に周期と位相角の関係，波の伝播に伴なう位相角の変化の有様は次第に明らかになつた．すなわち震源付近では位相角の変り方が特殊であることが，以前，筆者の一人によつて，平面を伝わる波についていわれた．同様なことが Nafe, Brune らによつて球面上を伝わる波についても発見され，Polar phase shift とよばれている．

一方，球面上を伝わる波は固有振動の和として表わされるが，polar phase shift は個々の振動周期について成立つものであるから，それらの重ね合せである実際の地震記録上では，一見してわかるほどはつきりと出ることは少ない．

この論文では半径方向に不均質な球面上を伝わるいろいろな弾性波動の位相角と，周波数，震央距離，デグリー，オーダーなどとの関係をしらべた．

この関係を明らかにするために，前に筆者らが計算した理論地震記録をフーリエ解析した結果を第1図，第2図に示した．この理論地震記録はデグリー m が 0 の場合の捩れ振動による方位角方向の変位を示している．両図の結果は理論とよく一致する．すなわち，震央での初期位相が $-\pi/4$ となること，スペクトルが符号を変ずる点で，位相に π の不連続を生ずること，極およびその対称点を通過する際に位相が $\pi/2$ とぶこと，しかしその他の部分では，位相と震央距離の関係は線形関係にあること，等である．

また，地震観測所が適当に分布しているとき実際の地震記録のフーリエ解析の結果から推定されるいろいろな波動の性質について言及した．