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23. *Studies of the Thermal State of the Earth. The 14th Paper: A Theory of Non-Steady Mantle Convection.*

By Tsuneji RIKITAKE,

Earthquake Research Institute.

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Summary

Rotational motion of a rigid sphere placed in gravity and temperature fields is studied in the hope of getting some clue as to time-dependent mantle convection. The estimate of mechanical couple exerted to the sphere due to thermal expansion leads to an equation of motion which is given in a form of non-linear integral equation provided proper account is taken of heat conduction. Solving numerically the integral equation thus obtained, it is found that the sphere overturns in a way studied by Griggs. For cells having radii of 100 and 1000 *km*, the rapidly accelerating period is estimated to last some 10^8 and 10^6 *yrs* respectively. The effect of heat conduction is serious for a small cell of which the radius is of the order of 100 *km*, while it can be ignored for a cell having a radius of 1000 *km* or more.

1. Introduction

The theory of non-steady convection in the earth's mantle developed by D. Griggs[1] and F. A. Vening Meinesz[2] has been attracting much attention of geologists as well as geophysicists because the theory seems to account for a number of geological and geophysical events, for example mountain-building, formation of basin and such like, which otherwise cannot be explained very successfully.

The theory is based on dynamical instability arising in the mantle due to the cooling of the earth. If we think of a mass in the mantle of which the temperatue is low at the top and high at the bottom, it is obvious that the mass is under an unstable equilibrium because, owing to thermal expansion, the top of the mass is heavier than the bottom. In the presence of a small force such as that due to a horizontal

1) D. GRIGGS, *Amer. Journ. Soc.*, **237** (1939), 612.
2) F. A. VENING MEINESZ, *Quart. Jour. Geol. Soc. London*, **103** (1948), 191.

temperature inequality as suggested by Griggs and Vening Meinesz, the whole mass tends to overturn on condition that the surroudings have some fluidity. Such a rotation comes to an end when the lighter material occupies the top. After that, the mass would slowly recover the temperature gradient given in the beginning, the condition necessary for the next cycle being thus provided. During the overturn the bottom of the overlying crust would be dragged by the subcrustal material in motion forming mountains at the earth's surface.

It has been considered that the influence of heat conduction on the cooling and heating of the moving medium is very small because even a motion having a velocity of the order of $1\,cm/yr$ as generally accepted for mantle convection current is so fast that the heat carried away by the motion is overwhelming. Griggs estimated the lengths of period required for the various stages of convection cycle though it does not seem to the writer that his estimate was based on exact mathematics. According to Griggs who dealt with a convection cell extending from the surface to the core of the earth, the overturning would start very slowly, so that the phase corresponding to the slowly accelerating currents covers some 25 million years. If the overturning has once started, the currents are greatly accelerated because of the dynamical instability. This period of rapid currents lasts from 5 to 10 million years. Then comes the decelerating stage because the instability becomes less as the hot material goes up and the cool material covers bottom of the cell. The period during which we have decelerating currents is also estimated as 25 million years. Finally, the stage of dynamical equilibrium is to be attained though it would take some 500 million years to recover the initial temperature distribution by conduction, so that the part of the mantle occupied by the cell would keep quiescence for at least 500 million years. Another cycle of convection may take place after this period.

Vening Meinesz has applied a similar idea to tectonophysics on a smaller scale. Being classified by him as "episodic" convection current, the hypothesis of mantle convection seems to have much advantage in the explanation of a number of geological and geophysical phenomena. In recent years, the existence of convection currents in the mantle becomes more plausible than it was some time before through measurements of heat-flow anomaly on deep sea-floor.[3),4)]

In view of the importance of mantle convection hypothesis, the

3) R. VON HERZEN, *Nature*, **183** (1959), 882.
4) R. VON HERZEN and S. UYEDA, To be published.

writer would here like to re-examine the Griggs theory on the basis of dynamics coupled with the theory of heat conduction. Since it is of great difficulty to advance a theory of non-steady convection in a viscous fluid, a rotational motion of a rigid sphere placed in a viscous fluid is studied in the presence of gravity force and temperature gradient. Although no accurate results can be obtained by examining such a simple model, it is hoped that we may have some clue as to the time-dependent behaviour of convection currents in the mantle.

2. Theory of heat conduction related to a slowly rotating sphere embedded in a medium of infinite extent

Let us suppose a sphere of radius a buried in a medium of infinite extent. The thermal conductivity and diffusivity are assumed to be the same both for the medium and the sphere. It is assumed that the sphere is slowly rotating, not necessarily with a constant angular velocity, about the z-axis, while the medium outside the sphere is stationary. Let us then take a co-ordinate system (x', y', z) fixed to the sphere which is rotating about the z-axis as can be seen in Fig. 1 in which a co-ordinate system (x, y, z) fixed to the space is also shown.

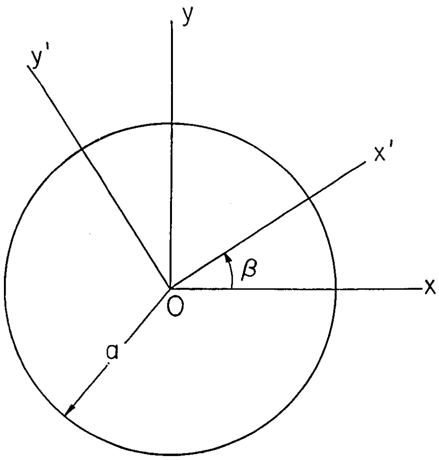

Fig. 1. Co-ordinate systems.

We are now in a position to study heat conduction referring to the rotating co-ordinate. As for the initial distribution of temperature, which is applicable to the later problem, we may assume a linear one parallel to the x'-axis in the sphere and zero temperature outside the sphere. Only the deviation from the equilibrium state is dealt with. The equations of heat conduction, that are not affected by a slow motion, are then given in the following for the outside and inside of the sphere respectively.

$$pu_1 = \kappa \nabla^2 u_1 \ , \tag{1}$$

$$pu_2 = \kappa \nabla^2 u_2 - pU_0 r \sin\theta \sin\phi' \ , \tag{2}$$

where u_1 and u_2 are the temperatures in the respective regions, while κ and p denote the thermal diffusivity and time-operator $\partial/\partial t$. $-U_0$ is the temperatue gradient initially assumed in the sphere as well as in the surrouning, (r, θ, ϕ) and (r, θ, ϕ') are the polar co-ordinates relevant to the stationary and rotating systems.

If we put

$$k=\kappa^{-1/2}p^{1/2} , \tag{3}$$

the solutions of (1) and (2), which do not become infinitely large at $r\rightarrow\infty$ and $r\rightarrow 0$ respectively, can be given as

$$u_1=A(kr)^{-1/2}K_{3/2}(kr)\sin\theta\sin\phi' , \tag{4}$$

$$u_2=-U_0 r\sin\theta\sin\phi'+B(kr)^{-1/2}I_{3/2}(kr)\sin\theta\sin\phi' , \tag{5}$$

in which A and B are functions of p, while $I_{3/2}$ and $K_{3/2}$ are modified Bessel functions of degree 3/2.

The conditions to be satisfied at the boundary of the sphere or at $r=a$ are the continuity of heat flow and temperature provided the interface drop is ignored. They are written as

$$u_1=u_2 , \tag{6}$$

$$\frac{\partial u_1}{\partial r}=\frac{\partial u_2}{\partial r} , \tag{7}$$

(6) and (7) enable us to determine A and B involved in (4) and (5). We thus obtain

$$A=-U_0 a(ka)^{3/2}I_{5/2}(ka) , \tag{8}$$

$$B=U_0 a(ka)^{3/2}K_{5/2}(ka) . \tag{9}$$

Putting (9) into (5), the temperatue inside the sphere is then determined as

$$u_2=-U_0 r\sin\theta\sin\phi'+U_0 a(ka)^{3/2}(kr)^{-1/2}K_{5/2}(ka)I_{3/2}(kr)\sin\theta\sin\phi' . \tag{10}$$

3. Mechanical couple exerted to the sphere

When we refer to the co-ordinate system fixed to the space, the temperature within the sphere as given by (10) can be transformed to

$$u_2=-U_0 r\sin\theta\sin(\phi-\beta)+U_0 a(ka)^{3/2}(kr)^{-1/2}K_{5/2}(ka)I_{3/2}(kr)\sin\theta\sin(\phi-\beta) , \tag{11}$$

in which β is the angle xOx' as shown in Fig. 1.

In the presence of a gravity field of which the acceleration is denoted by g, a volume element of the sphere undergoes a force which is given as

$$\rho\alpha g u_2 r^2 \sin\theta \, dr d\theta d\phi \ , \tag{12}$$

where ρ and α denote the density and the volume coefficient of expansion. The mechanical couple exerted to the sphere is therefore given by

$$\Gamma = \rho\alpha g \int_0^a \int_0^\pi \int_0^{2\pi} u_2 \cos\phi \, r^4 \sin^2\theta \, dr d\theta d\phi \ , \tag{13}$$

while the gravity field is assumed to be perpendicular to the x-axis.

On performing the integrations involved, (13) reduces to

$$\Gamma = \frac{4\pi}{15} a^5 \rho\alpha g U_0 [1 - 5 I_{5/2}(ka) K_{5/2}(ka)] \sin\beta \ , \tag{14}$$

which may be regarded as the operational form of the couple, so that the couple as a function of time can be written as

$$\Gamma(t) = \frac{4\pi}{15} a^5 \rho\alpha g U_0 \left[\sin\beta - 5 \frac{d}{dt} \int_0^t \varphi(t-\tau) \sin\beta d\tau \right] \quad \text{for } t \geq 0 \ , \tag{15}$$

on the assumption that β is zero for $t<0$. $\varphi(t)$ is given by

$$\varphi(t) = \frac{1}{2} \int_0^t t^{-1} \exp\left(-\frac{a^2}{4\kappa t}\right) I_{5/2}\left(\frac{a^2}{4\kappa t}\right) dt \ . \tag{16}$$

4. Equation of motion

Supposing that the sphere has been placed in a temperature field having a gradient $-U_0$ in the direction of the y-axis over such a long period that the sphere reaches a state of thermal equilibrium, the mechanical couple obtained in the above can well be applied to forming an equation of motion of the sphere. Denoting the moment of inertia and the coefficient of resistance respectively by C and R, the equation of motion becomes

$$C \frac{d^2\beta}{dt^2} + R \frac{d\beta}{dt} = G \left[\sin\beta - 5 \frac{d}{dt} \int_0^t \varphi(t-\tau) \sin\beta d\tau \right] + f \ , \tag{17}$$

where

$$G = \frac{4\pi}{15} a^5 \rho\alpha g U_0 \ . \tag{18}$$

A small couple f is introduced in the equation. f plays an important role in starting the motion though the dynamical instability of the sphere controls in the main the course of overturning after it has started.

According to the hydrodynamics of viscous liquid,[5] it is known that

$$R=8\pi\mu a^3 , \tag{19}$$

for a slow rotational motion where μ denotes the viscosity.

The first term on the left-handside of (17) may well be ignored because the motion concerned is extremely slow. In that case, measuring the time in units of R/G, (17) can be rewritten in a form

$$\frac{d\beta}{dt}=\sin\beta-5\frac{d}{dt}\int_0^t\varphi(t-\tau)\sin\beta d\tau+c , \tag{20}$$

in which all the quantities are non-dimensional and c is defined by

$$c=f/G . \tag{21}$$

Meanwhile, $\varphi(t)$ becomes

$$\varphi(t)=\frac{1}{2}\int_0^t\frac{e^{-\nu/t}}{t}I_{5/2}(\nu/t)dt , \tag{22}$$

where

$$\nu=\frac{a^2}{4\kappa}\frac{G}{R} . \tag{23}$$

(20) can be integrated as

$$\beta=ct+\int_0^t[1-5\varphi(t-\tau)]\sin\beta d\tau , \tag{24}$$

because $\beta=0$ at $t=0$ as already mentioned. (24) is a Volterra's integral equation of the second kind. Since the equation is non-linear, no analytical solution of (24) can easily be obtained.

5. A solution of the equation of motion

We may assume that

$$\left.\begin{array}{ll} a=100\,km , & \rho=3.5\,gm/cm^3 , \\ \alpha=2\times10^{-5}/^\circ C , & g=10^3\,cm/sec^2 , \\ U_0=1^\circ C/km , & \end{array}\right\} \tag{25}$$

5) H. LAMB, *Hydrodynamics*, 6th ed. Cambridge (1932), 589.

which can possibly be taken as likely values for estimating a time-dependent mantle convection.

$$\mu=3\times10^{21}\,c.\,g.\,s. \tag{26}$$

is also assumed after N. W. Haskell.[6] In that case, we obtain

$$\left.\begin{aligned} G&=5.86\times10^{23}\,cm^2\,gm/sec^2\ ,\\ R&=7.54\times10^{43}\,cm^2\,gm/sec\ ,\end{aligned}\right\} \tag{27}$$

so that

$$R/G=1.29\times10^{15}\,sec\ \ (=4.10\times10^{7}\,yr)\ , \tag{28}$$

Nothing is known about f. We may assume, however, that the initial velocity of the sphere, which is governed solely by f, takes a very small value, say, $10^{-2}\,cm/yr$ at the surface of the sphere. Such a velocity is much smaller than the one generally accepted in theories of mantle convection. We thus assume that

$$a\left(\frac{d\beta}{dt}\right)_0=10^{-2}cm/yr\ , \tag{29}$$

which leads to

$$\left(\frac{d\beta}{dt}\right)_0=3.18\times10^{-17}\,sec^{-1}\ . \tag{30}$$

On the other hand, we have

$$R\left(\frac{d\beta}{dt}\right)_{t=0}=f\ , \tag{31}$$

which enables us to estimate f as

$$f=2.40\times10^{27}\,cm^2\,gm/sec^2\ . \tag{32}$$

c is therefore estimated from (21) as

$$c=4.1\times10^{-2}\ . \tag{33}$$

We next have to calculate ν from (23). On assuming $\kappa=0.01\,c.\,g.\,s.$, which is a likely value of the thermal diffusivity in the mantle, (23) leads to

$$\nu=1.94\ . \tag{34}$$

6) N. W. HASKELL, *Physics*, **6** (1935), 265.

Since we could, in the above, estimate all the numerical values of the constants involved in (24), the next task is to solve non-linear integral equation (24).

K. Hidaka[7] advanced a numerical method for solving a Volterra's integral equation of the second kind. Although his example was linear, his method can be applied to a non-linear one such as

$$u(x)=f(x)-\int_0^x K(x, \xi)F[u(\xi)]\,d\xi \,. \tag{35}$$

For $x=0$, it is obvious that

$$u(0)=f(0) \,. \tag{36}$$

If we choose a sufficiently small h, we approximately obtain

$$u(h)=f(h)-\left\{\frac{1}{2}K(h, 0)F[u(0)]+\frac{1}{2}K(h, h)F[u(h)]\right\}h \,, \tag{37}$$

so that we may get $u(h)$ that satisfies (37) by making use of a trial-and-error method, all other quantities involved being known. We further have

$$u(2h)=f(2h)-\left\{\frac{1}{6}K(2h, 0)F[u(0)]+\frac{4}{6}K(2h, h)F[u(h)]\right.$$
$$\left.+\frac{1}{6}K(2h, 2h)F[u(2h)]\right\}2h \,, \tag{38}$$

$$u(3h)=f(3h)-\left\{\frac{1}{8}K(3h, 0)F[u(0)]+\frac{3}{8}K(3h, h)F[u(h)]\right.$$
$$\left.+\frac{3}{8}K(3h, 2h)F[u(2h)]+\frac{1}{8}K(3h, 3h)F[u(3h)]\right\}3h \,, \tag{39}$$

$$u(4h)=f(4h)-\left\{\frac{1}{6}K(4h, 0)F[u(0)]+\frac{4}{6}K(4h, h)F[u(h)]\right.$$
$$+\frac{2}{6}K(4h, 2h)F[u(2h)]+\frac{4}{6}K(4h, 3h)F[u(3h)]$$
$$\left.+\frac{1}{6}K(4h, 4h)F[u(4h)]\right\}4h \,, \tag{40}$$

which can be successively solved with regard to $u(2h)$, $u(3h)$, $u(4h)$ and so on.

7) K. HIDAKA, *Sekibun Hoteishiki Ron*, Kawade Shobo (1941), (in Japanese).

In applying the method to (24), there are a few practical difficulties. It is not practicable, for instance, to continue the procedure over a wide range of t, so that some larger h should be taken for solutions for large values of t. This causes errors in β to be solved. The part of the solution for large values of t as is illustrated in Fig. 2 is thus not very accurate. $\varphi(t)$, which has been already estimated by the present writer[8] in a theory of electromagnetic couple related to rotation of a conducting sphere placed in a magnetic field, is obtained also by numerical integration which necessarily causes some error.

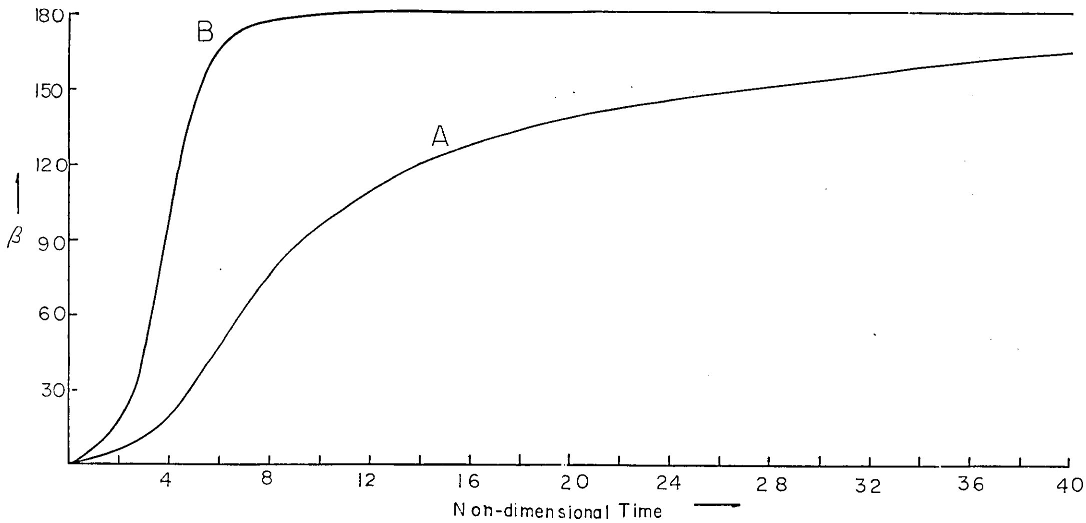

Fig. 2. Changes in β for a sphere of $100\,km$ in radius. The influence of heat conduction is taken into account for Curve A, while it is completely neglected for Curve B.

In spite of difficulties, β is solved as shown in Fig. 2 (curve A) in which the non-dimensional time is taken as the abscissa. As the curve for β is smooth, we may think that there would not be serious errors.

6. The case in which the heat conduction is completely neglected

Going back to (20), the equation of motion for a case in which the influence of heat conduction is ignored becomes

$$\frac{d\beta}{dt}=\sin\beta+c\ , \tag{41}$$

8) T. RIKITAKE, *Journ. Geomagn. Geoelectr.*, **14** (1962), 66.

which can be solved as

$$\frac{1+\sqrt{1-c^2}}{c}\exp(-\sqrt{1-c^2}t)=\frac{\sqrt{1-c^2}\cos\beta+c\sin\beta+1}{\sin\beta+c}, \tag{42}$$

on condition that $\beta=0$ at $t=0$.

The course of overturning for such a case is also shown in Fig. 2 (curve B) while the same c as before is taken. It is apparent that β asymptotically reaches a value $\sin^{-1}(-c)$ which is slightly larger than π.

7. Discussion

Curve B of Fig. 2 shows that the rotational motion of the sphere agrees qualitatively with that suggested by Griggs.[1] The overturn starts rather slowly being followed by a rapid rotation. The angular velocity then decreases asymptotically. We see, however, that the influence of heat conduction is so large that Curve A is significantly different from Curve B. The speed of overturning becomes much less and no marked contrast between the second and third stages as defined by Griggs can be observed.

For a sphere of 100 km in radius here concerned, the periods covered by each stage are estimated as shown in Table 1.

These periods are largely affected by μ because they are proportional to μ. If we assume a μ ten times as small as the one given in (26), the periods in Table 1 become one-tenth. As has been pointed out by Griggs,[1] however, there is a possibility of a much smaller viscosity in the mantle. According to the results of experiment, rocks do not behave as a viscous fluid, but as a "pseudo-viscous" fluid as was called by Griggs. In a pseudo-viscous flow, the logarithm of velocity is proportional to the stress, so that the pseudo-viscosity becomes small when the stress increases. For lack of experimental data for the mantle, however, it would not be practical to pursue discussion in further detail.

Table 1. The periods of various stages of convection cycle for a sphere of 100 km in radius.

Stage of convection cycle	A	B
Slowly accelerating stage	$1.2\times10^8\,yr$	$8\times10^7\,yr$
Rapidly accelerating stage	1.2×10^8	8×10^7
Decelerating stage	1.5×10^9	1.6×10^8

As long as we take a viscosity value generally accepted, it is estimated from Curve A that the maximum rotation speed is 0.06 cm/yr

at the surface of the sphere. Even from Curve B, it is estimated to be as small as $0.2\,cm/yr$. It is therefore unlikely that a speed of convection current of the order of $1\,cm/yr$ is attained by a mechanism considered in this paper provided an eddy having a $100\,km$ radius is assumed.

Let us next study convection currents of a larger extent. Assuming $a=1000\,km$, we obtain

$$R/G=1.29\times 10^{13}\,sec\ (=4.10\times 10^{5}\,yr)\ , \tag{43}$$

while we get

$$\nu=1.94\times 10^{4}\ , \tag{44}$$

so that we see that φ involved in (24) has an appreciable influence on the solution (24) only when the non-dimensional t reaches a value of the order of 10^4, that is of the order of $10^9\,yr$ as we can readily see

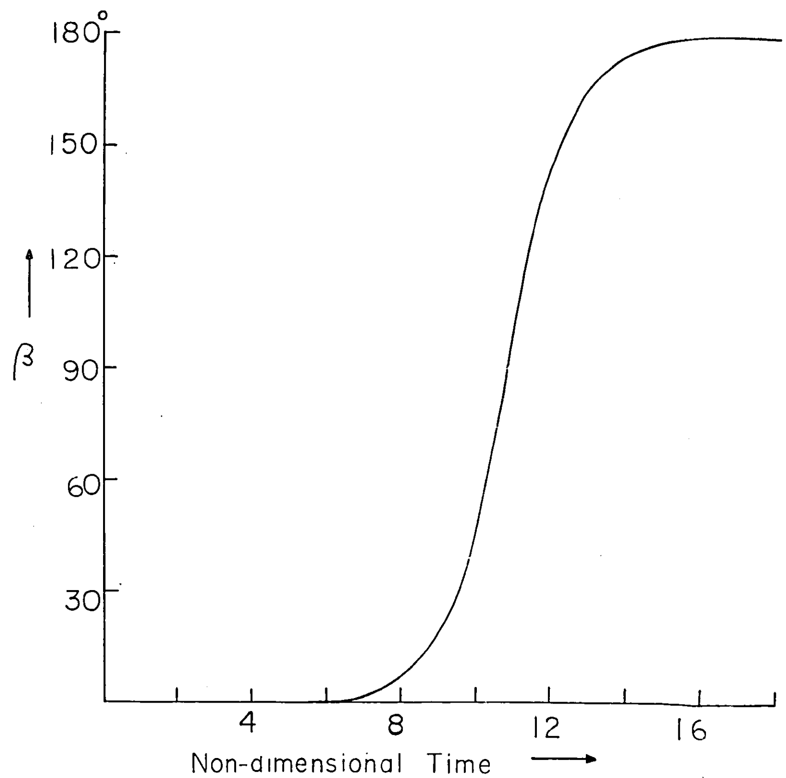

Fig. 3. Changes in β for a sphere of $1000\,km$ in radius.

by taking (43) into account. For values of t that cover the periods of overturning, φ takes a very small value as will be seen in the following. It is therefore seen that the influence of heat conduction can be ignored for the equation of motion of a sphere having a radius of 1000 km or larger as long as a convection cycle having a period of 10^6–10^7 yr is concerned.

Ignoring the heat conduction, the equation of motion becomes the same as (41). If we assume a value for c which initially gives a velocity of 0.01 cm/yr at the surface of the sphere, we have to take $c=4.1\times10^{-5}$ in this case. The changes in β with time are then calculated on the basis of (42) as shown in Fig. 3. In this case, the lengths of various periods are estimated as in Table 2.

Table 2. The periods of various stages of convection cycle for a sphere of 1000 km in radius

Stage of convection cycle	Period
Slowly accelerating stage	3.6×10^6 yr
Rapidly accelerating stage	1.2×10^6
Decelerating stage	2.0×10^6

Comparing these periods with those estimated by Griggs,[1] who studied a convection cell extending from the top to the bottom of the mantle, we observe that his estimate gave periods much longer than those of the present estimate. The present writer is of the opinion that Griggs's estimate should be subjected to a modification though his result is qualitatively correct. Attention should be paid to the fact that the length of slowly accelerating period is largely controled by the initial velocity. Since the initial velocity has been taken rather arbitrarily, the lengths of this period either for $a=100$ km or 1000 km are by no means realistic.

The maximum linear velocity of the convection cell of 1000 km in radius can also be estimated from the curve in Fig. 3. It amounts to as much as 2 m/yr which is so large that an objection from the geological standpoint may be made against such a rapid motion.

It is of considerable interest that the behaviours of convection cycle for small ($a=100$ km) and large scales are distinctly different from one another. The couple due to dynamical instability is proportional to the fifth power of the radius while the viscous force to the third power, so that the ratio of the former to the latter increases enormously as the size of convection cell becomes large. Meanwhile it becomes clear that the effect of heat conduction cannot be neglected for a small cell having a raduius of 100 km or so though there is practically no cooling in the course of a convection cycle in the case of a relatively large cell having a radius of the order of 1000 *km*.

No account has been taken of the threshold stress below which no continuing flow takes place. Even if account is taken of the threshold strength of the mantle material, it seems unlikely that the behaviour of overturning would become very much different from the ones studied in the above because the dynamical instability controls the motion if it once starts. Some modification would however be necessary for the initial stage if the effect of threshold strength is to be taken into account.

8. Concluding remarks

A theory of non-steady convection cycle, based on a spherical model, in which proper account of the dynamical instability due to tempeature distribution is taken together with the effect of heat conduction, leads to the following conclusions.

A period of the order of 10^8 *yrs* is required for a cell of 100 *km* in radius to complete the rapidly acceleration stage of overturn. The influence of heat conduction for such a small cell cannot be neglected. Owing to the gradual cooling of the cell during the overturning, the rotation speed becomes markedly smaller than that for a model in which the effect of heat conduction is entirely ignored. Meanwhile, the rapidly accelerating period is estimated as 10^6 *yrs* or so for a cell having a 1000 *km* radius. In this case, very little cooling occurs during a cycle. General aspects of the convection cycle here studied agree with what Griggs suggested many years ago.

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23. 地球熱学 第14報 非定常マントル対流論

地震研究所 力 武 常 次

温度勾配のある重力場中に置かれた剛体球の力学的不安定を，熱伝導を考慮した上で，吟味した．このモデルはマントル熱対流に関する Griggs の理論を数式によつて定量化したものである．

球が粘性流体にかこまれているとし，各種物理量をマントルについてふつう考えられている程度にとれば，半径 100 *km* の球では，ゆつくりと回転がはじまり，不安定が増加するとともに急激に回転し，ついにはゆつくりと安定状態に近づくが，各 stage はそれぞれ 1.2×10^8，1.2×10^8 および 1.5×10^9 年程度となる．熱伝導の影響はいちじるしく，熱伝導を全く省略した場合には，安定状態に近づく時間が数十パーセント短くなる．半径 1000 *km* の球では，熱伝導の影響はほとんど無視することができる．また回転の速度は前の場合にくらべて100倍程度大きくなる．これらの結果は Griggs の見積りと定性的には一致している．
