19. *Stability of the Earth's Dynamo.*

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(Read Sept. 25, 1956.—Received Sept. 25, 1956.)

Summary

Magneto-hydrodynamic oscillations of a simple mode are studied in relation to the stability condition of the earth's dynamo. The influence of the Coriolis force is found to be considerable. The previous result, that the toroidal magnetic field would not be large in order to have a stable dynamo, should be abandoned when we take the earth's rotation into account. It seems acceptable that strong magnetic fields which have been suggested by E. C. Bullard exist in the earth's core.

1. Introduction

The writer[1),2),3),4)] has been studying magneto-hydrodynamic oscillations in the earth's core. One of the interesting results of these studies is that the earth's dynamo which has been investigated by E. C. Bullard and H. Gellman[5)] is not stable for small disturbances. A small magneto-hydrodynamic disturbance given to the dynamo system which consists of steady magnetic fields and fluid motions of certain types should not grow indefinitely if the dynamo is stable. But the above study showed that such a disturbance which gives a magnetic field of the S_1^0 type does grow if we assume strong magnetic fields of toroidal type in the earth's core as has been suggested by Bullard.

Bullard[6)] has pointed out, however, that the stability condition would be seriously affected by the effect of the Coriolis force due to the earth's rotation, because the above study was made for the non-rotating earth. Since the influence of the Coriolis force on a rotating fluid mass of large dimension is usually appreciable, it has been hoped to carry out an investigation on magneto-hydrodynamic behaviour of the rotating

1) T. RIKITAKE, *Bull. Earthq. Res. Inst.*, **33** (1955), 1.
2) T. RIKITAKE, *Bull. Earthq. Res. Inst.*, **33** (1955), 175.
3) T. RIKITAKE, *Bull. Earthq. Res. Inst.*, **33** (1955), 583.
4) T. RIKITAKE, *Bull. Earthq. Res. Inst.*, **34** (1956), 139.
5) E. C. BULLARD and H. GELLMAN, *Phil. Trans. Roy. Soc. London A*, **247** (1954), 213.
6) E. C. BULLARD, Personal communication.

earth's core. The writer[1] studied magneto-hydrodynamic oscillations of a conducting fluid sphere under the influence of the Coriolis force, a uniform external magnetic field being assumed in this case. The effect of the Coriolis force is so strong that the free oscillations are possible for a sphere of the earth's size when the external magnetic field is of the order of 10^5 *gauss* or more. The result would suggest that there may be a strong magnetic field in the earth's core where a stable dynamo system is existing. Since the steady magnetic field includes the ones of the toroidal types in the dynamo considered, however, a simple calculation such as was made in the previous study[1] can not be applied to the stability problem of the dynamo. The present study is made in the hope of obtaining the stability condition of the dynamo by taking into account the effect of the Coriolis force due to the earth's rotation.

2. Theory

The mathematical treatment in this paper is almost the same as that in the previous paper[1]. Maxwell's equations are reduced to the form such as

$$\{D-(4\pi\sigma)^{-1}\nabla^2\}\vec{h}=\mathrm{curl}\,(\vec{V}_0\wedge\vec{h})+\mathrm{curl}\,(\vec{v}\wedge\vec{H}_0)\,, \qquad (1)$$

where the velocity of the steady fluid motions and magnetic fields which consist of the dynamo are denoted by $\vec{V}_0$ and $\vec{H}_0$ respectively. $\vec{h}$ and $\vec{v}$ denote respectively small disturbances of magnetic field and fluid velocity which are superposing on the steady ones, their second order terms being ignored. D is written in place of $\partial/\partial t$ which is regarded as an operator hereafter. σ denotes the electrical conductivity, while the magnetic permeability is assumed to be unity in the electromagnetic unit. (1) is exactly the same as the expression (13) in the previous paper[1].

The equation of motion of the fluid is written as

$$\rho D\vec{v}+2\rho(\vec{\omega}_0\wedge\vec{v})=(4\pi)^{-1}(\mathrm{curl}\,\vec{h}\wedge\vec{H}_0+\mathrm{curl}\,\vec{H}_0\wedge\vec{h})-\mathrm{grad}\,p\,, \qquad (2)$$

where the second term in the left-hand side is the effect of the rotation, the introduction of this term being the only difference between expression (14) of the previous paper[1] and (2). In (2), $\vec{\omega}_0$, ρ and p denote respectively the rotation vector, density and pressure.

If we make div of (2), we obtain

$$\nabla^2 p = -(4\pi)^{-1}(\vec{H}_0 \cdot \nabla^2 \vec{h} + \vec{h} \cdot \nabla^2 \vec{H}_0 + 2 \operatorname{curl} \vec{h} \cdot \operatorname{curl} \vec{H}_0) + 2\rho\vec{\omega}_0 \cdot \operatorname{curl} \vec{v}, \quad (3)$$

because the incompressibility of the fluid gives div $\vec{v}=0$ and the uniform rotation curl $\vec{\omega}_0=0$.

By solving p from (3) and putting it into (2) as has been done in the previous paper[1], we can obtain the relation between $\vec{v}$ and $\vec{h}$. The relation thus obtained may be regarded as one of the simultaneous equations together with (1).

Under the assumption that we take the steady magnetic fields of the S_1^0, T_2^0, T_2^{2c} and T_2^{2s} types as well as the steady fluid motions of the T_1^0 and S_2^{2c} types as the elements of the dynamo and also that only the small magnetic disturbance of the S_1^0 type may be taken into account, the relation between the radial parts of the magnetic field and of the velocities is to be obtained. Since all the interactions which do not give rise to a magnetic field of the S_1^0 type are neglected, it is sufficient to take into consideration only the velocities of the S_2^0, S_2^{2c} and S_2^{2s} types.

When we express the S_1^0-type magnetic field as

$$\vec{h} = \begin{cases} -2s(r)P_1, \\ -\left(r\dfrac{ds}{dr} + 2s\right)\dfrac{dP_1}{d\theta}, \\ 0, \end{cases}$$

and the velocities of the S_2^0-, S_2^{2c}- and S_2^{2s}-types as

$$\vec{v}_1 = \begin{cases} -6\xi_2(r)rP_2, \\ -\left(r\dfrac{d\xi_2}{dr} + 3\xi_2\right)r\dfrac{dP_2}{d\theta}, \\ 0, \end{cases}$$

$$\vec{v}_2 = \begin{cases} -6\xi_2^{2c}(r)rP_2^2 \cos 2\phi, \\ -\left(r\dfrac{d\xi_2^{2c}}{dr} + 3\xi_2^{2c}\right)r\dfrac{\partial(P_2^2 \cos 2\phi)}{\partial\theta}, \\ -\left(r\dfrac{d\xi_2^{2c}}{dr} + 3\xi_2^{2c}\right)r\dfrac{\partial(P_2^2 \cos 2\phi)}{\sin\theta\partial\phi}, \end{cases}$$

$$\vec{v}_3 = \begin{cases} -6\xi_2^{2s}(r)rP_2^2 \sin 2\phi, \\ -\left(r\dfrac{d\xi_2^{2s}}{dr} + 3\xi_2^{2s}\right)r\dfrac{\partial(P_2^2 \sin 2\phi)}{\partial\theta}, \\ -\left(r\dfrac{d\xi_2^{2s}}{dr} + 3\xi_2^{2s}\right)r\dfrac{\partial(P_2^2 \sin 2\phi)}{\sin\theta\partial\phi}, \end{cases}$$

(1) gives

$$Ds-(4\pi\sigma)^{-1}\left(\frac{d^2s}{dr^2}+\frac{4}{r}\frac{ds}{dr}\right)=-\frac{6}{5}S_1\left(r\frac{d\xi_2}{dr}+5\xi_2\right)-\frac{216}{5}a^{-2}r^2(T_4\xi_2^{2c}-T_3\xi_2^{2s}), \tag{4}$$

while the steady magnetic fields are given as the same as those defined by (22), (23) and (24) in the previous paper[1], so that S_1, T_3 and T_4 in (4) have the same meaning as before. From (2), we also have

$$\left.\begin{aligned}D\rho\xi_2&=(4\pi a^2)^{-1}\left(\frac{1}{3}L_2^0-\frac{2}{3}a^2S_1r^{-1}\frac{ds}{dr}\right),\\ D\rho\xi_2^{2c}&=(4\pi a^2)^{-1}\left(\frac{1}{3}L_2^{2c}+2T_4r^{-5}\int_0^r r^4s\,dr\right)-\frac{2}{3}\rho\omega\left(r\frac{d\xi_2^{2s}}{dr}+3\xi_2^{2s}\right),\\ D\rho\xi_2^{2s}&=(4\pi a^2)^{-1}\left(\frac{1}{3}L_2^{2s}-2T_3r^{-5}\int_0^r r^4s\,dr\right)+\frac{2}{3}\rho\omega\left(r\frac{d\xi_2^{2c}}{dr}+3\xi_2^{2c}\right),\end{aligned}\right\} \tag{5}$$

where ω is defined by

$$\vec{\omega}_0=\begin{cases}\omega\cos\theta\,,\\ -\omega\sin\theta\,,\\ 0\,.\end{cases}$$

L_2^0, L_2^{2c} and L_2^{2s} are to be determined by the conditions that the normal component of the velocities should vanish at the core boundary $r=a$.

Although detailed processes of getting at (4) and (5) are not described here, they are more or less the same as those carried out in the previous parers[1),4)]. We can readily see that (4) and (5) become exactly the same as the equations for a non-rotating earth by making $\omega\to 0$.

3. Solutions

As has been done in the previous paper, let us assume

$$s=\sum_n a_n(r/a)^n,\qquad \xi_2^{2c}=\sum_n A_n(r/a)^n,\qquad \xi_2^{2s}=\sum_n B_n(r/a)^n.$$

When we introduce these relations into (4) from which ξ_2 has been eliminated by use of the first equation of (5), we obtain

$$4\pi\rho D^2a^2\sum_n a_n\left(\frac{r}{a}\right)^n=-\frac{6}{5}S_1\left(\frac{5}{3}L_2^0-\frac{2}{3}S_1\sum_n n(n+3)\left(\frac{r}{a}\right)^{n-2}\right)$$
$$-\frac{216}{5}(4\pi\rho Da^2)\left(T_4\sum_n A_n\left(\frac{r}{a}\right)^{n+2}-T_3\sum_n B_n\left(\frac{r}{a}\right)^{n+2}\right),$$

where the conductivity is assumed to be infinite. We also have

$$D\rho \sum_n A_n\left(\frac{r}{a}\right)^n=(4\pi a^2)^{-1}\left(\frac{1}{3}L_2^{2c}+2T_4\sum_n \frac{a_n}{n+5}\left(\frac{r}{a}\right)^n\right)$$

$$-\frac{2}{3}\rho\omega\sum_n (n+3)B_n\left(\frac{r}{a}\right)^n,$$

$$D\rho \sum_n B_n\left(\frac{r}{a}\right)^n=(4\pi a^2)^{-1}\left(\frac{1}{3}L_2^{2s}-2T_3\sum_n \frac{a_n}{n+5}\left(\frac{r}{a}\right)^n\right)$$

$$+\frac{2}{3}\rho\omega\sum_n (n+3)A_n\left(\frac{r}{a}\right)^n.$$

By equating the coefficients of the corresponding terms of both the sides of these equations, we obtain

$$4\pi\rho D^2a^2a_0=-16S_1^2a_4\,,$$

$$4\pi\rho D^2a^2a_2=\frac{112}{5}S_1^2a_4-\frac{216}{5}(4\pi\rho Da^2)(T_4A_0-T_3B_0)\,,$$

$$4\pi\rho Da^2A_0=-2T_4\left(\frac{a_2}{7}+\frac{a_4}{9}\right)-\frac{2}{3}(4\pi\rho\omega a^2)(3B_0-2B_2-4B_4)\,,$$

$$4\pi\rho Da^2A_2=2T_4\frac{a_2}{7}-\frac{10}{3}(4\pi\rho\omega a^2)B_2\,,$$

$$4\pi\rho Da^2A_4=2T_4\frac{a_4}{9}-\frac{14}{3}(4\pi\rho\omega a^2)B_4\,,$$

$$4\pi\rho Da^2B_0=2T_3\left(\frac{a_2}{7}+\frac{a_4}{9}\right)+\frac{2}{3}(4\pi\rho\omega a^2)(3A_0-2A_2-4A_4)\,,$$

$$4\pi\rho Da^2B_2=-2T_3\frac{a_2}{7}+\frac{10}{3}(4\pi\rho\omega a^2)A_2\,,$$

$$4\pi\rho Da^2B_4=-2T_3\frac{a_4}{9}+\frac{14}{3}(4\pi\rho\omega a^2)A_4\,.$$

where a_n's, A_n's and B_n's for $n>4$ are ignored. It is easily seen that a_n's, A_n's and B_n's for odd n vanish.

There is another equation

$$3a_0-5a_2-7a_4=0$$

which is led from the boundary condition that the magnetic field is continuous at $r=a$.

We can eliminate A_0, A_2, A_4, B_0, B_2 and B_4 from the above equations, whence we have three homogeneous equations for a_0, a_2 and a_4. In order that there is a non-vanishing solution, the following relation should hold good:

$$\begin{vmatrix} 3 & 5 & 7 \\ 4\pi\rho D^2 & 0 & 16S_1^0 a^{-2} \\ 0 & 4\pi\rho D^2-\dfrac{432}{35}a^{-2}T^2\dfrac{D^2}{D^2+\left(\dfrac{10}{3}\omega\right)^2} & -\dfrac{112}{5}S_1^0 a^{-2}-\dfrac{48}{5}a^{-2}T^2\dfrac{D^2}{D^2+\left(\dfrac{14}{3}\omega\right)^2} \end{vmatrix}=0, \tag{6}$$

where

$$T^2=T_3^2+T_4^2 .$$

According to Bullard[7], the T_2^0-field would be very large amounting to the order of probably 10^2 *gauss*, while the T_2^{2c}- and T_2^{2s}-fields would be of intermediate intensity. On the other hand the S_1^0-field is expected to amount to a few *gauss* at the core boundary. Taking this into consideration, the writer would here like to ignore S_1 in (6) for the first approximation. In that case, (6) is written as

$$D^4+\left(\frac{296}{9}\omega^2-\frac{48}{35}\frac{T^2}{a^2\pi\rho}\right)D^2+\omega^2\left(\frac{19600}{81}\omega^2-\frac{15168}{315}\frac{T^2}{a^2\pi\rho}\right)=0 . \tag{7}$$

For the earth's core, we may take $\omega=7.5\times10^{-5}\,sec^{-1}$, $a=3.5\times10^8\,cm$ and $\rho=10\,g/cm^3$. Hence, we see

$$\omega^2 \gg \frac{T^2}{a^2\pi\rho},$$

unless T is much larger than 10^5 *gauss*. Since we do not expect such a large magnetic field in the earth's core, we see that the oscillation is mainly caused by the coupling between the S_2^{2c} and S_2^{2s} motions through the rotation. The period of the free oscillation is estimated from (7) as

5.3 and 7.3 *hours*

for $T \ll 10^5$ *gauss*.

4. Discussion and conclusion

In the light of the above calculation, it becomes clear that the oscillation, which is associated with the S_1^0-type magnetic field, is mainly

7) E. C. BULLARD, *Proc. Roy. Soc. London A*, **197** (1949), 433.

caused by the coupling between the S_2^{2c}- and S_2^{2s}-type motions through the rotation. Unless the magnetic field is larger than the order of 10^5 *gauss*, the free oscillation is simple harmonic. We see, therefore, that the dynamo considered is stable for small disturbances. It is striking that the conclusion for a rotating earth is utterly different from that for a non-rotating earth.

Now, it turns out that there is no reason to doubt the existence of a strong toroidal magnetic field in the earth's core as has been suggested by Bullard. Since the period of the free oscillation amounts to only several *hours*, the secular variations in the earth's magnetic field can not be regarded as the result of free oscillation of that sort. The view that the secular variations are caused by the free magneto-hydrodynamic oscillation as has been tried in the writer's previous paper[1] is not acceptable when we take into account the effect of the Coriolis force.

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19. 地磁気ダイナモの安定性

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Bullard によつて示された S_1^0, T_2^0, T_2^{2c} および T_2^{2s} 磁場ならびに T_1^0 および S_2^{2c} 運動より成り立つダイナモの安定性をしらべた. S_1^0 磁場の小擾乱およびそれに伴う速度変化のみを考察する. この場合流体の運動方程式中には地球廻転の影響が導入された. その結果は S_1^0 磁場をもたらす振動は主として S_2^{2c} および S_2^{2s} 運動の廻転を通じての結合により起されることがわかつた.

この事情は 10^5 *gauss* 程度以上の強磁場が地球核内に存在しない限り変らない. 非廻転地球の場合にトロイダル磁場が大きいと小擾乱が不安定であるという結果があるが, 廻転を考慮した場合は事情は全く異なり, Bullard のいうような強磁場があつてもダイナモは安定である.
