4. *On the Theory of Elastic Waves in Granular Substance. II.*

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In the preceding paper, we had no space to go into details. We wish here to treat some of them.

1. *On the secular equation and its solutions of three dimensional model.* In the Paper I[1] we first considered lattice structure, then by taking the average in all directions we obtained an isotropic model. But it is not meaningless to deal directly with lattice structure, for our granular substance might have locally this lattice structure and show aeolotropic nature and we might observe some of its velocities.

In I-§ 2 we gained the equation of motion for the three dimensional granular lattice

$$(v/v_0)^2 \mathfrak{A} - \sum_{j=1}^{n/2} (\mathfrak{A}\,\mathfrak{t}_j)(s\,\mathfrak{t}_j)^2\,\mathfrak{t}_j = 0, \qquad (1)$$

or

$$\begin{cases} \{\lambda_{11}-(v/v_0)^2\}A_1 + \lambda_{12}A_2 + \lambda_{13}A_3 = 0, \\ \lambda_{12}A_1 + \{\lambda_{22}-(v/v_0)^2\}A_2 + \lambda_{23}A_3 = 0, \\ \lambda_{13}A_1 + \lambda_{23}A_2 + \{\lambda_{33}-(v/v_0)^2\}A_3 = 0, \end{cases} \qquad (2)$$

where $\mathfrak{A}=(A_1,\ A_2,\ A_3)$, $\mathfrak{t}_j=(t_{j1},\ t_{j2},\ t_{j3})$

$$\lambda_{11}=\sum_{j=1}^{n/2} t_{j1}^2 (s\,\mathfrak{t}_j)^2 \ , \quad \lambda_{12}=\sum_{j=1}^{n/2} t_{j1}\,t_{j2}\,(s\,\mathfrak{t}_j)^2, \qquad (3)$$

Then the secular equation can be immediately written as follows

$$\begin{vmatrix} \lambda_{11}-(v/v_0)^2 & \lambda_{12} & \lambda_{13} \\ \lambda_{12} & \lambda_{22}-(v/v_0)^2 & \lambda_{23} \\ \lambda_{13} & \lambda_{23} & \lambda_{33}-(v/v_0)^2 \end{vmatrix} = 0 \qquad (4)$$

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1) *Bull. Earthq. Res. Inst.*, 27 (1949), 11. Cited as I hereafter.

If the three roots are v_1, v_2 and v_3 ($v_1 > v_2 > v_3$), from (4) we get

$$\sum_{i=1}^{3}(v_i/v_0)^2=\lambda_{11}+\lambda_{12}+\lambda_{33}=\sum_{j=1}^{n/2}(s\,\mathfrak{t}_j)^2 \quad\ldots\ldots\ldots\ldots(5)$$

1.1 *Simple cubic type. (n=6)*

	$\mathfrak{t}_1$	$\mathfrak{t}_2$	$\mathfrak{t}_3$
t_{j1}	1	0	0
t_{j2}	0	1	0
t_{j3}	0	0	1

$$(s\mathfrak{t}_j)=s_j,$$

$$\lambda_{jj}=s_j^2\ ,\quad \lambda_{ij}=0\quad (i \neq j).$$

From $$(v_i/v_0)^2=s_j^2\quad (i,\ j=1,\ 2,\ 3), \quad\ldots\ldots\ldots(6)$$

$$(v_p/v_0)^2=0.6 \quad\ldots\ldots\ldots\ldots(7)$$

When $s=\mathfrak{t}_i$, v_1 has its maximum value v_0. In this case only the longitudinal wave exists. (From (2) $Ai=0$ for $i \neq j$.)

1.2. *Body-centred cubic type. (n=8)*

	$\mathfrak{t}_1$	$\mathfrak{t}_2$	$\mathfrak{t}_3$	$\mathfrak{t}_4$
$\sqrt{3}\ t_{j1}$	1	-1	1	1
$\sqrt{3}\ t_{j2}$	1	1	-1	1
$\sqrt{3}\ t_{j3}$	1	1	1	-1

$$\lambda_{jj}=4/9\ ,\quad \lambda_{ij}=(8/9)\ s_i s_j\ (i \neq j).$$

$$\sum_{i=1}^{3}(v_i/v_0)^2=4/3 \quad\ldots\ldots\ldots\ldots(8)$$

The secular equation becomes

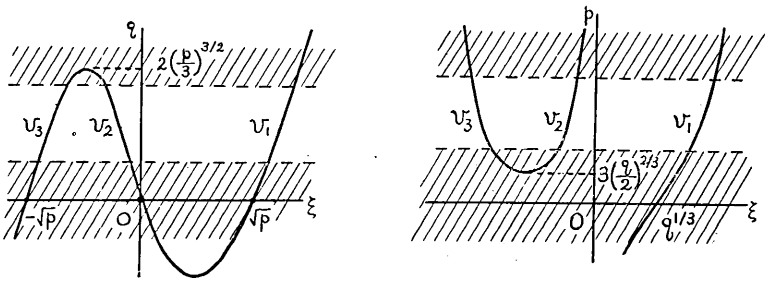
$$\{1-(9/4)\,(v/v_0)\}^3-4\,(s_1^2\,s_2^3+s_2^2\,s_3^2+s_3^2\,s_1^2)\,\{1-(9/4)\,(v/v_0)^2\} +16\,s_1^2\,s_2^2\,s_3^2=0 \quad\ldots\ldots\ldots\ldots(9)$$

and

$$(v_p/v_0)^2=0.8 \quad\ldots\ldots\ldots\ldots(10)$$

When $s=\mathfrak{t}_i$, v_1 has its maximum value.[2)]

2) If we put $\xi=\{(9/4)(v/v_0)^2-1\}/2$, $p=s_1^2s_2^2+s_2^2s_3^2+s_3^2s_1^2$, $q=2s_1^2s_2^2s_3^2$,
(9) becomes $\xi^3-p\xi-q=0$
By definition $p/3 \geq (q/2)^{2/3} \geq 0$.



Both p and q have their greatest value when $s_1^2=s_2^2=s_3^2=1/3$. Fig. 1 shows $(q-\xi)$ and $(p-\xi)$ curves, where shadowed domains are forbidden. From these curves we see that the greatest value of ξ corresponds to the greatest values of p and q.

In this case, we have

$$p=1/3\ ,\quad q=2/27.$$

putting it in (14)

$$(\xi+1/3)^2(\xi-2/3)=0$$

or $$(v_1/v_0)^2=28/27\ ,\quad (v_2/v_0)^2=(v_3/v_0)^2=4/27 \quad\ldots\ldots\ldots\ldots(11)$$

If $s=\mathfrak{t}$, from (2) we have

for v_1 , $A_1=A_2=A_3$, or $\mathfrak{A}\,\|\,s$. (longitudinal wave)

for v_2 and v_3 , $A_1+A_2+A_3=0$, or $(\mathfrak{A}\ s)=0$. (transversal wave)

1.3. *Face centered cubic type. (n=12)*

	$\mathfrak{t}_1$	$\mathfrak{t}_2$	$\mathfrak{t}_3$	$\mathfrak{t}_4$	$\mathfrak{t}_5$	$\mathfrak{t}_6$
$\sqrt{2}\ t_{i1}$	0	0	1	1	1	-1
$\sqrt{2}\ t_{i2}$	1	-1	0	0	1	1
$\sqrt{3}\ t_{i3}$	1	1	1	-1	0	0

$$\lambda_{jj}=(1+s_j^2)/2\ ,\quad \lambda_{ij}=s_is_j\ ,$$

$$(v_p/v_0)^2=1.2,\ \ldots\ldots\ldots\ldots(12)$$

$$\sum_{i=1}^{3}(v_i/v_0)^2=2\ .\ \ldots\ldots\ldots\ldots(13)$$

The secular equation becomes

$$\begin{vmatrix} (1+s_1^2)/2-(v/v_0)^2 & s_1s_2 & s_1s_3 \\ s_2s_1 & (1+s_2^2)/2-(v/v_0)^2 & s_2s_3 \\ s_3s_1 & s_3s_2 & (1+s_3^2)/2-(v/v_0)^2 \end{vmatrix}=0\ldots\ldots(14)$$

When $s=\mathfrak{t}_i$, v_1 has its maximum value.

If we put $s=\mathfrak{t}_1$, from the secular equation we have

$$(v_1/v_0)^2=312\ ,\quad (v_2/v_0)^2=(v_3/v_0)^2=1/2\ \ldots\ldots\ldots\ldots(15)$$

From (2) we know that v_1 and v_2 (or v_3) correspond to longitudinal and transversal waves respectively.

2. *Decomposition of a symmetric tensor.*

Next we must complete the calculation involved in § 3 of the first report. That is, to accomplish the following integration,

$$C^0=\frac{1}{\Omega}\int V^*(r)\,CV(r)\,d\Omega\ \ldots\ldots\ldots\ldots(1)$$

where $V(r)$ is the representation of the rotation group by means of a symmetric tensor of rank 2, or a 6–6 matrix of the coefficients of the following linear transformation[3)]

3) In the paper I we used the symbol x for the strain tensor, which suffers same transformation as the one used here, i. e. $V\,(r)$ is same in both cases.

$$\begin{pmatrix} x'^2 \\ y'^2 \\ z'^2 \\ 2y'z' \\ 2x'z' \\ 2x'y' \end{pmatrix} = V(r) \begin{pmatrix} x^2 \\ y^2 \\ z^2 \\ 2yz \\ 2xz \\ 2xy \end{pmatrix}, \quad \text{with } x'^2+y'^2+z'^2=x^2+y^2+z^2. \qquad \text{(2), (3)}$$

(2) may be written briefly $x'=V(r)x,$ ....(2)′
x and x' being the above 1-6 matrices or vectors in a 6-dimensional space. We have transformed $V(r)$ to a orthogonal matrix $U(r)$

$$U(r)=T^{-1}V(r)T, \quad U^*(r)U(r)=1. \qquad \text{(4)}$$

And (1) was written

$$C^{0T}=\frac{1}{\Omega}\int U^*(r)\,C^T U(r)\,d\Omega\,, \qquad \text{(5)}$$

where

$$C^{0T}=T^*\,C^0 T\,, \quad C^T=T^*CT \qquad \text{(6), (7)}$$

(3) means that in the considered 6-dimensional vector space $x^2+y^2+z^2$ forms an invariant one dimensional subspace. As we have no other restrictions to the six functions x^2, ..., $2yz$, ...we may expect $U(r)$ to be the sum of two irreducible parts of 1st and 5th order. (This is also a consequence of general theory of representations)
Then $U(r)$ has the following form

$$U(r)=\begin{pmatrix} 1 & 0\cdots 0 \\ 0 & \\ \vdots & U_1(r) \\ 0 & \end{pmatrix} \qquad \text{(8)}$$

where $U_1(r)$ is a 5-5 matrix, irreducible and orthogonal. If C_1^T and C_1^{0T} are cofactors of c_{11}^T and c_{11}^{0T} (1-1 elements of C_0^T and C_0^{0T}) the integrand of (5) becomes

$$U^*(r)C^T U(r)=\begin{pmatrix} c_{11}^T & 0\cdots\cdots 0 \\ 0 & \\ \vdots & U_1^*(r)C_1^T U_1(r) \\ 0 & \end{pmatrix}$$

Then from (5) we get

$$c_{11}^{0T}=c_{11}^T\,, \quad c_{1j}^{0T}=c_{j1}^{0T}=0 \quad (j=2, 3, \ldots\ldots, 6) \qquad \text{(9)}$$

$$C_1^{0T}=\frac{1}{\Omega}\int U_1^*(r)\,C_1^T U_1(r)d\Omega \qquad \text{(10)}$$

Now C_1^{0T} commutes with all $U_1(s)$. (s means an arbitrary element of rotation group.) For

$$U_1(s)^{-1}C_1^{0T}U_1(s)=\frac{1}{\Omega}\int U_1(s)^*U_1^*(r)C_{11}^TU_1(r)U_1(s)d\Omega,\ (t=rs)$$

$$=\frac{1}{\Omega}\int U_1^*(t)C_1^TU_1(t)d\Omega=C_1^{0T}\ ,$$

Then C_1^{0T} must be a multiple of unit matrix as $U_1(r)$ is irreducible. (Schur's lemma[4])

$$C_1^{0T}=\begin{pmatrix} c_{22}^{0T} & & 0 \\ & \ddots & \\ 0 & & c_{66}^{0T}\end{pmatrix},\quad c_{22}^{0T}=\cdots\cdots=c_{66}^{0T}\ \ldots\ldots\ldots\ldots(11)$$

The trace (sum of all diagonal elements) is invariant under orthogonal transformation. Comparing both sides of (10) we have

$$c_{jj}^{0T}=\frac{1}{5}\sum_{k=2}^{6}c_{kk}^T\ ,\quad (j\geqq 2)\ \ldots\ldots\ldots\ldots(12)$$

Before determining T, it is necessary to remark that T cannot be orthogonal. For the square sum of $x^2,\ldots\ldots,\ 2yz,\ldots\ldots$ is not invariant under three dimensional rotations, but the sum of $x^2,\ldots\ldots,\ \sqrt{2}\,yz,\ldots\ldots$ is invariant. We express by χ the vector with these components: $\chi^*=(x^2,\ldots\ldots,\ \sqrt{2}\,yz,\ldots\ldots)$

Then $$x=S\chi\ \ldots\ldots\ldots\ldots(13)$$

where

$$S=\begin{pmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & 1 & & & \\ & & & \sqrt{2} & & \\ & & & & \sqrt{2} & \\ 0 & & & & & \sqrt{2}\end{pmatrix}\ \ldots\ldots\ldots\ldots(14)$$

(4) means that the vector x is transformed by T to another vector ξ.

$$x=T\xi\ ,\quad \xi'=U(r)\xi\ \ldots\ldots\ldots\ldots(15)$$

From (13) and (15)

$$\chi=R\xi\ ,\quad T=SR\ \ldots\ldots\ldots\ldots(16)$$

As the transformation of χ is orthogonal

$$\chi'=W(r)\chi\ ,\quad W(r)W^*(r)=1\ ,\ \ldots\ldots\ldots\ldots(17)$$

from (15) and (16) $$W(r)=R^{-1}U(r)R$$

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4) For the proof of Schur's lemma, literature on group representations should be consulted; e. g., *H. Weyl, Classical Groups,* Chap. III. (1939).

From (4) and (17) $$I=WW^*=R^{-1}U(r)(RR^*)U(r)^*(R^*)^{-1}$$

Thus RR^* commutes with all $U(r)$, therefore it must be a multiple of unit matrix. To make calculation easier, we choose RR^* for unit matrix.

$$RR^*=1 \qquad \text{.........(18)}$$

As said before, one component of ξ must be $\alpha\ (x^2+y^2+z^2)$. Then (16) is written as

$$\begin{pmatrix} x^2 \\ \vdots \\ \sqrt{2}\ yz \\ \vdots \end{pmatrix}=R\begin{pmatrix} \alpha\ (x^2+y^2+z^2) \\ \vdots \\ \vdots \end{pmatrix}$$

or

$$R^{-1}=\begin{pmatrix} \alpha & \alpha & \alpha & 0 & 0 & 0 \\ & & \cdots\cdots & & & \\ & & \cdots\cdots & & & \end{pmatrix}$$

From (19) $\alpha^2=1/3$

and

$$R^*=\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & 0 & 0 & 0 \\ & & \cdots\cdots & & & \\ & & \cdots\cdots & & & \end{pmatrix} \qquad \text{.........(20)}$$

Putting (16) in (7)

$$C^T=R^*S^*CSR=R^*C^sR\ , \qquad \text{.........(21)}$$

where

$$C^s=S^*C\,S=\begin{pmatrix} c_{11} & c_{12} & c_{13} & 2c_{14} & 2c_{15} & 2c_{16} \\ & & \cdots\cdots & & & \\ & & \cdots\cdots & & & \\ c_{61} & c_{62} & c_{63} & 2c_{64} & 2c_{65} & 2c_{66} \end{pmatrix} \qquad \text{.........(22)}$$

From (21) $$c_{11}^T=\sum_{i,\,j=1}^{6} r_{i1}r_{j1}c_{ij}^s=\frac{1}{3}(c_{11}+c_{22}+c_{33})+\frac{2}{3}(c_{12}+c_{13}+c_{23}) \qquad \text{......(23)}$$

and $$t_r.C^T=t_r.C^s=c_{11}+c_{22}+c_{33}+2\,(c_{44}+c_{55}+c_{66}) \qquad \text{.........(24)}$$

Then from (9) and (11) we get

$$c_{11}^{0T}=(1/3)\,(c_{11}+c_{22}+c_{33})+(2/3)\ (c_{12}+c_{23}+c_{31}) \qquad \text{.........(24)}$$

$$c_{jj}^{0T}=(1/5)\,(t_r.C^T-C_{11}^T)=(2/15)\,(c_{11}+c_{22}+c_{33})-(2/15)\,(c_{12}+c_{13}+c_{31}) +(2/5)\,(c_{44}+c_{55}+c_{66})$$

(21) may be also applied to C^0.

$$C^{0S}=RC^{0T}R^*$$

Considering that C^{0T} is diagonal, we calculate the elements as follows,

$$c_{ij}^{0s} = \sum_k r_{ik}\, r_{jk}\, c_{kk}^{0T} = (c_{11}^{0T} - c_{22}^{0T})\, r_{i1}\, r_{j1} + c_{22}^{0T} \sum_k r_{ik}\, r_{jk}$$

$$= (c_{11}^{0T} - c_{22}^{0T})\, r_{i1} r_{i1} + c_{22}^{0T}\, \delta_{ij} \;,$$

$$\left\{\begin{array}{l} c_{11}^{0} = c_{22}^{0} = c_{33}^{0} = (1/3)\, c_{11}^{0T} + (2/3)\, c_{22}^{0T} \\ c_{12}^{0} = c_{13}^{0} = c_{23}^{0} = (1/3)\, (c_{11}^{0T} - c_{22}^{0T}) \\ c_{44}^{0} = c_{55}^{0} = c_{66}^{0} = (1/2)\, c_{22}^{0T} = (1/2)\, (c_{11}^{0} - c_{12}^{0}) \end{array}\right. \qquad \ldots\ldots (25)$$

Putting (24) in (25) we get final results. (I. § 3 (12))

Complete form of T was not necessary in our calculation, but it is of some interest to determine it.

As already said in the previous paper, five components of ξ are solid harmonics. And $\xi^*\xi$ must be invariant, therefore

$$\xi = \begin{pmatrix} (1/\sqrt{3}\; (x^2+y^2+z^2) \\ \pm\beta(-x^2-y^2+2z^2) \\ \pm\sqrt{3}\beta\, (x^2-y^2) \\ \pm 2\sqrt{3}\;\beta\, yz \\ \pm 2\sqrt{3}\;\beta\, xz \\ \pm 2\sqrt{3}\;\beta\, xy \end{pmatrix}$$

From (16) and (19) we can determine β and R (up to signs).

$$R = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} & 0 & 0 & 0 \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} & 0 & 0 & 0 \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \xi = \begin{pmatrix} (1/\sqrt{3}\; (x^2+y^2+z^2) \\ (-1/\sqrt{6})(x^2+y^2) + (2/\sqrt{6})z^2 \\ (1/\sqrt{2})\; (x^2-y^2) \\ \sqrt{2}\; yz \\ \sqrt{2}\; xz \\ \sqrt{2}\; xy \end{pmatrix}$$

T can be immediately gained by multiplying S to R.

4. 粒狀物体を傳はる彈性波の理論II

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第一報に於ては紙面の關係でのせる事のできなかつた計算を示す．即ち
三次元の模型で速度を決定する永年方程式を導き，その解を求めること．
彈性テンソルを既約部分に分けること．