4. *Elastic Waves from a Point in an Isotropic Heterogeneous Sphere. Part 2.*

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Introduction. In his previous paper[1] the writer studied theoretically the waves propagated through a heterogeneous medium, in which λ, one of Lame's elastic constants, is a function of the distance r from the centre of symmetry, such that $\sqrt{\frac{\lambda+2\mu}{\rho}}=a-br^2$, rigidity μ, and density ρ are constant. He there worked out a general solution in special coordinates (t, β, φ), the origin of which coincides with the centre of the disturbance.

From the calculations, it appears that waves of irrotational dilatation can be propagated independently of waves of equivoluminal rotation, as in the case of a homogeneous medium, and that the velocity of propagation of dilatational waves varies with its wave length. Here a question arises with respect to the physical meaning of t. While t, calculated by the integral (3), is independent of wave length, the same integral is accepted as giving the travel time of a dilatational wave that originated at the coordinate origin O', and observed at that point, of which t is one of the above mentioned coordinates.

So long as waves of dilatation and waves of rotation are able to travel independently of each other, it can be easily proved by a process of discussion similar to that adopted by Love in the case of a homogeneous medium,[2] that the velocity of propagation of the surface of discontinuity is either $\sqrt{\frac{\lambda+2\mu}{\rho}}$ or $\sqrt{\frac{\mu}{\rho}}$, acccording as there is no rotation or there is no dilatation. It is, therefore, evident, from Fermat's Principle of the wave path, that t is the travel time of a disturbance of dilatational character, so long as the rigidity and the density of the medium are constant. However, the wave motions through a heterogeneous medium, in general, including such as those considered in this

1) R. YOSIYAMA, *Bull. Earthq. Res. Inst.*, **11** (1933), 1~13.
2) A. E. H. LOVE, *The Mathematical Theory of Elasticity*, 4th ed., p. 297.

paper, are neither purely dilatational nor purely rotational, so that the above mentioned process of discussion seems inapplicable.

Now, in this paper, the writer discusses the waves propagated through another particular heterogeneous medium, in which rigidity $\mu=0$, while λ and ρ vary with r. The propriety of interpreting t as travel time of disturbance will be shown analytically. The solution of waves thus obtained will be helpful not only in elucidating the question of the waves in the earth's core of the nature as usually accepted, but also that of sound waves in the atmosphere and that of atmospheric oscillations.

The writer hopes also that some possible applications of the present process of analysis will be found in future to the problems concerning a heterogeneous medium of more general character.

1. Putting $\mu=0$ in the equation of motion (2),

$$\rho\frac{\partial^2 \boldsymbol{D}}{\partial T^2}=\operatorname{grad}\{(\lambda+2\mu)\Delta\}-2\operatorname{rot}\mu\boldsymbol{W}+2[\operatorname{grad}\mu,\boldsymbol{W}]$$

$$-2\Delta\operatorname{grad}\mu+2(\mathfrak{E}\operatorname{grad}\mu), \tag{2}$$

we have

$$\rho\frac{\partial^2 \boldsymbol{D}}{\partial T^2}=\operatorname{grad}\lambda\Delta, \tag{56}$$

and it will be seen from the equation that certain components of rotation of $\frac{\partial^2 \boldsymbol{D}}{\partial T^2}$ do not vanish when the medium is heterogeneous in its density, so that the wave motion through the medium cannot be purely dilatational.

By performing the divergence operation on both members of the equation we obtain

$$\frac{d\rho}{dr}\frac{\partial^2 u_r}{\partial T^2}+\rho\frac{\partial^2 \Delta}{\partial T^2}=\nabla^2(\lambda\Delta), \tag{57}$$

where u_r represents the r-component of displacement.

Eliminating u_r from (57) and the r-component of the vector equation (56), we get the equation

$$\rho\frac{\partial^2}{\partial T^2}\lambda\Delta=\lambda\nabla^2(\lambda\Delta)-\frac{\lambda}{\rho}\frac{d\rho}{dr}\frac{\partial}{\partial r}(\lambda\Delta), \tag{58}$$

and in virtue of the relation,

$$\nabla^2\phi\psi=\psi\nabla^2\phi+2(\operatorname{grad}\phi\operatorname{grad}\psi)+\phi\nabla^2\psi, \tag{59}$$

the equation transforms to

$$\frac{\partial^2}{\partial T^2}\frac{\lambda\Delta}{\sqrt{\rho}}=\frac{\lambda}{\rho}\left(\nabla^2-\sqrt{\rho}\,\nabla^2\frac{1}{\sqrt{\rho}}\right)\frac{\lambda\Delta}{\sqrt{\rho}}. \tag{60}$$

The integration of equation (60) is effected on some initial or boundary conditions which, we may assume, are given with regard to voluminal change or stress. In our present problem, there are three normal stress components equal to $\lambda\Delta$, but no tangential stress. The resultant integral, on being put into equation (56), gives the requisite solution of the problem under discussion, although, unless some initial conditions are given to $\boldsymbol{D}$ and $\dfrac{\partial \boldsymbol{D}}{\partial T}$, the results will, to a certain extent, be indeterminate, because any additional displacement may be superposed such that both acceleration and divergence vanish, since such displacement satisfies equation (56) and is unaffected by the initial or boundary conditions on the voluminal change or stress.

Putting

$$\boldsymbol{D}=\frac{1}{\rho}\operatorname{grad}\phi, \tag{61}$$

we get

$$\Delta=\frac{1}{\rho}\nabla^2\phi-\frac{1}{\rho^2}\frac{d\rho}{dr}\frac{\partial\phi}{\partial r}=\frac{1}{\sqrt{\rho}}\left\{\nabla^2\frac{\phi}{\sqrt{\rho}}-\phi\nabla^2\frac{1}{\sqrt{\rho}}\right\} \tag{62}$$

Let ϕ_0 be a function independent of time T, such that

$$\nabla^2\frac{\phi_0}{\sqrt{\rho}}-\phi_0\nabla^2\frac{1}{\sqrt{\rho}}=0. \tag{63}$$

Then the displacement

$$\boldsymbol{D}_0=(A+BT)\frac{1}{\rho}\operatorname{grad}\phi_0, \tag{64}$$

in which A and B are constant, satisfies the equation of motion (56) with $\dfrac{\partial^2\boldsymbol{D}}{\partial T^2}=0$ and $\operatorname{div}\boldsymbol{D}=O$, and has no bearing on the conditions referred to voluminal changes or stress. The indeterminateness occurs also when the problem is concerned with a homogeneous medium, and the displacement is, as may be seen from (64), everlasting or everincreasing.

As regards the everlasting displacement component, $\dfrac{A}{\rho}\operatorname{grad}\phi_0$.

The equation of elastic equilibrium in statics, when there is no body force, being

$$\operatorname{grad} \lambda \Delta = 0, \tag{65}$$

this displacement component will appear when a constant traction acts on a boundary surface. For example, let us examine the results of the investigation concerning "Elastic waves animated by the potential energy of initial strain" by Dr. H. Kawasumi and the writer.[3] In case (b), therein stated, a constant traction P_0 acts on the boundary surface $r=a$, and the displacement component expressed by $\frac{a^3P_0}{r^2}$ in (19) and other equations, is obviously quite similar in nature to that mentioned above.

Now, concerning the ever-increasing displacement component, $BT\frac{1}{\rho}\operatorname{grad}\phi_0$, the imperfection of the equation of motions should be taken into consideration. In the equation, for example in the r-component, terms such as $\frac{\partial u_r}{\partial T}\frac{\partial^2 u_r}{\partial r\partial T}$, etc, are ignored on the assumption that these are negligible compared with $\frac{\partial^2 u_r}{\partial T^2}$, and $\frac{\partial^2 \boldsymbol{D}}{\partial T^2}$ is written instead of $\frac{d^2\boldsymbol{D}}{dT^2}$. However, if $\boldsymbol{D}=BT\frac{1}{\rho}\operatorname{grad}\phi_0$, $\frac{\partial u_r}{\partial T}\frac{\partial^2 u_r}{\partial r\partial T}$ does not vanish, whereas $\frac{\partial^2 u_r}{\partial T^2}$ vanishes. Therefore, it is desirable to prescribe the initial or boundary conditions properly, so that the displacement under consideration will at least be negligible compared with the other components.

2. Now in order to discuss the problem more concretely, we shall assume that

$$\frac{\lambda}{\rho} = (a-br^2)^2, \tag{66}$$

$$\frac{\lambda}{\rho}\sqrt{\rho}\,\nabla^2\frac{1}{\sqrt{\rho}} = a, \tag{67}$$

and, substituting (61) and (62) in equation (60), we get

$$\frac{\partial^2}{\partial T^2}\frac{\phi}{\sqrt{\rho}} = \frac{\lambda}{\rho}\left(\nabla^2 - \sqrt{\rho}\,\nabla^2\frac{1}{\sqrt{\rho}}\right)\frac{\phi}{\sqrt{\rho}}. \tag{68}$$

3) H. KAWASUMI and R. YOSIYAMA, *Bull. Earthq. Res. Inst.*, **13** (1935), 496~503.

This equation can be solved by the method given in the writer's previous paper, with the sole substitution of $p^2-\alpha$ for p^2. The result is

$$\phi=\frac{\sqrt{\rho}}{\sqrt{a-br^2}}\sqrt{\frac{2\sqrt{ab}}{\sinh\,2\sqrt{ab}\,t}}\,U_n^{(l)}(\sqrt{p^2-\alpha-ab}\,t)$$

$$\cdot\ P_n^m(\cos\beta)\begin{matrix}\sin m\varphi\\ \cos m\varphi\end{matrix}\,e^{\mp ipT},\quad l=1,\ 2, \tag{69}$$

in which p denotes the frequency in 2π sec. and $U_n^{(l)}(\sqrt{p^2-\alpha-ab}\,t)$ is the function worked out by the writer in his previous paper, which corresponds to Hankel's cylindrical function in the case of a homogeneous medium, and we have, moreover,

$$\varDelta=\frac{1}{\sqrt{\rho}}\left\{\nabla^2\frac{\phi}{\sqrt{\rho}}-\phi\nabla^2\frac{1}{\sqrt{\rho}}\right\}=\frac{1}{\lambda}\frac{\partial^2\phi}{\partial T^2}\,. \tag{70}$$

When t is large, we can approximately put

$$U_n^{(1)}\approx f_1\sqrt{\frac{2}{\pi p}}\sqrt{\frac{2\sqrt{ab}}{\sinh\,2\sqrt{ab}t}}\,e^{i(\sqrt{p^2-\alpha-ab}t-\frac{n+1}{2}\pi)}, \tag{71}$$

$$U_n^{(2)}\approx f_2\sqrt{\frac{2}{\pi p}}\sqrt{\frac{2\sqrt{ab}}{\sinh\,2\sqrt{ab}t}}\,e^{-i(\sqrt{p^2-\alpha-ab}t-\frac{n+1}{2}\pi)}, \tag{72}$$

where f_1 and f_2 are respectively a certain function of p, ab, α and n.[4] Since a, b, and α are constants, the surface of the same phase of a wave is given by $t=const.$, which is a wave front. The phase velocity V, normal to the wave front when t is large, is given by

$$V=(a-br^2)\frac{p}{\sqrt{p^2-\alpha-ab}}\,, \tag{73}$$

and,

$$\frac{\partial V}{\partial p}=-(a-br^2)\frac{\alpha+ab}{\sqrt{p^2-\alpha-ab}}\,. \tag{74}$$

Thus we find that the phase velocity depends on the period of the wave, and the larger the absolute value of $\alpha+ab$ the larger the degree of the dispersion. Normal or anomalous dispersion occurs according as $\alpha+ab$ is positive or negative. In either case, as frequency p increases, $\alpha+ab$ becomes comparatively negligible, and the velocity

4) Concerning the asymptotic expansion of U_n, the previous paper contains an error with respect to the terms f_1, f_2.

approaches $a-br^2$, which is greatest when $\alpha+ab$ is negative, and is least when $\alpha+ab$ is positive.

It will be noticed that the degree of dispersion depends on $\alpha+ab$, and not on α or ab separately. If $\alpha+ab=0$, although the medium is heterogeneous, the wave is not dispersive, while the reverse is the case even when $b=0$, that is, even when $\sqrt{\frac{\lambda}{\rho}}$ is constant throughout the medium, the wave is dispersive, provided α is not zero. For example, assume $b=0$, and that

$$\rho=\rho_0\frac{r^2}{\sinh^2\frac{\sqrt{\alpha}}{a}r}, \tag{75}$$

then

$$\phi=\frac{\sqrt{\rho}}{\sqrt{t}}H^{(i)}_{n+\frac{1}{2}}(\sqrt{p^2-\alpha}\,t)P^m_n(\cos\beta)\begin{matrix}\sin m\varphi\\ \cos m\varphi\end{matrix}e^{\pm ipT}, \tag{76}$$

where $H^{(i)}_{n+\frac{1}{2}}$ is Hankel's cylindrical function, and the phase velocity

$$V=\frac{ap}{\sqrt{p^2-\alpha}}, \tag{77}$$

and is dependent on frequency p.

From (66) and (67) we get ρ as a function of r in the form

$$\frac{1}{\sqrt{\rho}}=\frac{2\sqrt{ab}\,\operatorname{cosech}2\sqrt{ab}\,t_0}{\sqrt{a-br^2}}\{A\sinh\sqrt{\alpha+ab}\,t_0$$
$$+B\cosh\sqrt{\alpha+ab}\,t_0\}, \tag{78}$$

where

$$t_0=\frac{1}{2\sqrt{ab}}\operatorname{arcsinh}\frac{2\sqrt{ab}\,r}{a-br^2}, \tag{79}$$

A and B being arbitrary constants. For certain assumed values of α we get ρ;

$$\alpha=-ab;\qquad \rho=\frac{r^2}{B^2(a-br^2)}$$

$$\alpha=0;\qquad \rho=\frac{r^2}{(A+Br)^2}$$

$$\alpha=3ab;\qquad \rho=\frac{r^2(a-br^2)}{\{Ar+B(a+br^2)\}^2}$$

$$\alpha = 15ab\,; \qquad \rho = \frac{r^2(a-br^2)^3}{[Ar(a+br^2)+B\{(a-br^2)^2+8abr^2\}]^2}\,.$$

3. We shall now discuss the propagation of a disturbance. A slight modification of the notations in (69) gives

$$\phi = \frac{\sqrt{\rho}}{\sqrt{a-br^2}}\sqrt{\frac{2\sqrt{ab}}{\sinh 2\sqrt{ab}\,t}}\,U_n^{(i)}(pt)P_n^m(\cos\beta)\begin{matrix}\sin m\varphi\\ \cos m\varphi\end{matrix}\,e^{\pm i\sqrt{p^2+\alpha'^2}T}, \qquad (80)$$

where $\alpha'^2 = \alpha + ab$ is the constant of the medium, the period of the wave being $2\pi/\sqrt{p^2+\alpha'^2}$, whence the phase velocity is

$$V = (a-br^2)\frac{\sqrt{p^2+\alpha'^2}}{p}\,. \qquad (81)$$

This is, remembering (51) and (52), generalized into the form

$$\phi = \frac{\sqrt{\rho}}{\sqrt{a-br^2}}\left(\frac{\sinh 2\sqrt{ab}\,t}{2\sqrt{ab}}\right)^n P_n^m(\cos\beta)\begin{matrix}\sin m\varphi\\ \cos m\varphi\end{matrix}$$

$$\cdot\left(\frac{2\sqrt{ab}}{\sinh 2\sqrt{ab}\,t}\,\frac{d}{dt}\right)^n\frac{2\sqrt{ab}}{\sinh 2\sqrt{ab}\,t}\int_0^\infty\{f_1(p)e^{-ipt+i\sqrt{p^2+\alpha'^2}T}$$

$$+g_1(p)e^{ipt-i\sqrt{p^2+\alpha'^2}T}+f_2(p)e^{-ipt-i\sqrt{p^2+\alpha'^2}T}+g_2(p)e^{ipt+i\sqrt{p^2+\alpha'^2}T}\}\,dp. \qquad (82)$$

Let us first suppose the initial condition

$$\left.\begin{aligned}\Delta &= \frac{2\sqrt{ab}\,\mathrm{cosech}\,2\sqrt{ab}\,t}{\sqrt{\rho}\,(a-br^2)^{\frac{5}{2}}}\,\psi_1(t)\\ \frac{\partial\Delta}{\partial T} &= \frac{2\sqrt{ab}\,\mathrm{cosech}\,2\sqrt{ab}\,t}{\sqrt{\rho}\,(a-b\,r^2)^{\frac{5}{2}}}\,\psi_2(t)\end{aligned}\right\} \qquad (83)$$

at $T=0$.

To determine $f_1(p)$, $f_2(p)$, $g_1(p)$, $g_2(p)$ in order that (82) may satisfy (83), we have relation (70) and Fourier's double integral theorem,

$$\psi(t) = \frac{1}{2\pi}\left\{\int_0^\infty dp\int_{-\infty}^\infty\psi(\omega)e^{ip(t-\omega)}d\omega + \int_0^\infty dp\int_{-\infty}^\infty\psi(\omega)e^{-ip(t-\omega)}d\omega\right\} \qquad (84)$$

Thus we have

$$\Delta = \frac{2\sqrt{ab}\,\mathrm{cosech}\,2\sqrt{ab}\,t}{\pi\sqrt{\rho}\,(a-br^2)^{\frac{5}{2}}}\left\{\int_{-\infty}^\infty\psi_1(\omega)\,d\omega\int_0^\infty\cos\sqrt{p^2+\alpha'^2}T\cos p(t-\omega)\,dp\right.$$

$$\left.+\int_{-\infty}^\infty\psi_2(\omega)\,d\omega\int_0^\infty\frac{\sin\sqrt{p^2+\alpha'^2}T}{\sqrt{p^2+\alpha'^2}}\cos p(t-\omega)\,dp\right\}$$

$$=\frac{2\sqrt{ab}\,\mathrm{cosech}\,2\sqrt{ab}\,t}{\pi\sqrt{\rho}\,(a-br^2)^{\frac{5}{2}}}\left\{\frac{\partial}{\partial T}\int_{-\infty}^{\infty}\varphi_1(\omega)\,d\omega\int_0^{\infty}\frac{\sin\sqrt{p^2+a'^2}\,T}{\sqrt{p^2+a'^2}}\cos p(t-\omega)\,dp\right.$$

$$\left.+\int_{-\infty}^{\infty}\varphi_2(\omega)\,d\omega\int_0^{\infty}\frac{\sin\sqrt{p^2+a'^2}T}{\sqrt{p^2+a'^2}}\cos p(t-\omega)\,dp\right\}. \qquad (85)$$

Putting $x=T$, $y=p$, $z=a'$, $\nu=\frac{1}{2}$ and $T\cos\varphi=u$ in the known formula,[5]

$$\int_0^{\frac{\pi}{2}}\cos\,(xy\cos\varphi)J_{\nu-\frac{1}{2}}(xz\sin\varphi)\,(\sin\varphi)^{\nu+\frac{1}{2}}d\varphi$$

$$=\sqrt{\frac{\pi}{2x}}\,\frac{z^{\nu-1}}{(y^2+z^2)^{\frac{\nu}{2}}}J_\nu(x\sqrt{y^2+z^2}),\quad \Re(\nu)>-\frac{1}{2} \qquad (86)$$

we get

$$\frac{\sin\sqrt{p^2+a'^2}T}{\sqrt{p^2+a'^2}}=\int_0^T\cos pu\,J_0(a'\sqrt{T^2-u^2})\,du. \qquad (87)$$

We have, therefore,

$$\frac{1}{\pi}\int_0^{\infty}\frac{\sin\sqrt{p^2+a'^2}T}{\sqrt{p^2+a'^2}}\cos p(t-\omega)\,dp$$

$$=\frac{1}{\pi}\int_0^{\infty}\cos p(t-\omega)\,dp\int_0^T\cos pu\,J_0(a'\sqrt{T^2-u^2})\,du. \qquad (88)$$

With this equation, let us compare Fourier's integral theorem

$$f(x)=\frac{2}{\pi}\int_0^{\infty}\cos\alpha x\,d\alpha\int_0^{\infty}f(\lambda)\cos\alpha\lambda\,d\lambda, \qquad (89)$$

which holds when $f(x)$ is an even function of x, whence we get

$$\frac{1}{\pi}\int_0^{\infty}\frac{\sin\sqrt{p^2+a'^2}T}{\sqrt{p^2+a'^2}}\cos p(t-\omega)\,dp=\frac{1}{2}J_0(a'\sqrt{T^2-(t-\omega)^2}) \quad |t-\omega|<T$$

$$=0, \qquad |t-\omega|>T$$

By means of this relation, we get from (85) as the final result,

5) See, for example, "Ooyôsugaka" p. 348 by Dr. S. SANO.

$$\varDelta = \frac{\sqrt{ab}\,\mathrm{cosech}\,2\sqrt{ab}\,t}{\sqrt{\rho}\,(a-br^2)^{\frac{5}{2}}}\left\{\frac{\partial}{\partial T}\int_{t-T}^{t+T} J_0(a'\sqrt{T^2-(t-\omega)^2})\,\psi_1(\omega)\,d\omega\right.$$

$$\left.+\int_{t-T}^{t+T} J_0(a'\sqrt{T^2-(t-\omega)^2})\,\psi_2(\omega)\,d\omega\right\}. \qquad (90)$$

For convenience in physically interpreting this result, let us suppose that, at first, when $T=0$, the disturbance is restricted to that region of space within a closed surface $t=t_0$, and that it is observed at a point outside the closed surface where the coordinate $t>t_0$. Since $\psi_1(t)$ and $\psi_2(t)$ vanish when $t>t_0$, the upper limit of the integration becomes t_0. And as far as $t-T>t_0$, that is, $T<t-t_0$, $\psi_1(\omega)$ and $\psi_2(\omega)$ are zero, so that the integrals also vanish. It is when the time T is so large that $t-T=t_0$ or $T=t-t_0$, that $\psi_1(t_0)$ or $\psi_2(t_0)$, the disturbance on the surface $t=t_0$ at $T=0$, first begins to contribute to the integral. Consequently, in current terminology, $t-t_0$ is equal to the travel-time of the disturbance.

Since $\psi_1(\omega)$ and $\psi_2(\omega)$ respectively have their meaning only when $\omega\geq 0$, the lower limit of ω is zero, and when $t-T=0$ or $T=t$, the effects of $\psi_1(0)$ and $\psi_2(0)$ apparent.

In problems of sound waves in the atmosphere, we frequently meet with cases in which a'^2 is negative. In such cases, certain restrictions to the initial conditions will be necessary in order that the results, mathematically obtained, as mentioned above, may preserve their reasonableness when interpreted physically. Besides, discussions with respect to those restrictions may lead to the study of the problem of stability of the state of the medium. In this paper, however, we shall confine ourselves to a study of the effects of the dispersion assuming a'^2 positive, that is a' real.

When the constraint in the medium is released by itself, $\psi_2(t)=0$. And

$$\varDelta = \frac{\sqrt{ab}\,\mathrm{cosech}\,2\sqrt{ab}\,t}{\sqrt{\rho}\,(a-br^2)^{\frac{5}{2}}}\,\frac{\partial}{\partial T}\int_{t-T}^{t_0} J_0(a'\sqrt{T^2-(t-\omega)^2})\,\psi_1(\omega)\,d\omega$$

$$= \frac{\sqrt{ab}\,\mathrm{cosech}\,2\sqrt{ab}\,t}{\sqrt{\rho}\,(a-br^2)^{\frac{5}{2}}}\left\{\psi_1(t-T)\right.$$

$$\left.-\int_{t-T}^{t_0}\frac{a'TJ_1(a'\sqrt{T^2-(t-\omega)^2})}{\sqrt{T^2-(t-\omega)^2}}\,\psi_1(\omega)\,d\omega\right\}. \qquad (91)$$

The second term represents the effect of dispersion. Further study will be made of the following two cases (a) and (b).

(a) When $T \gg t(>t_0)$, the lower limit of the integration becomes zero, and, moreover, we can put, approximately,

$$\frac{J_1(a'\sqrt{T^2-(t-\omega)^2})}{\sqrt{T^2-(t-\omega)^2}}=\sqrt{\frac{2}{\pi}}\frac{\cos(a'T-\frac{3}{4}\pi)}{\sqrt{a'T}\cdot T},$$

So that the effect of dispersion, which remains as a damping oscillatory motion after the disturbance of the form prescribed at the origin passed by, will be estimated in virtue of the expression,

$$\int_{t-T}^{t_0}\frac{a'TJ_1(a'\sqrt{T^2-(t-\omega)^2})}{\sqrt{T^2-(t-\omega)^2}}\psi_1(\omega)d\omega$$

$$\doteqdot\sqrt{\frac{2}{\pi}}\sqrt{\frac{a'}{T}}\cos\left(a'T-\frac{3}{4}\pi\right)\int_0^{t_0}\psi_1(\omega)d\omega. \tag{92}$$

(b) When the initial motions, observed at a position sufficiently distant from the origin of the disturbance, are concerned, $T\doteqdot t$ and $a'\sqrt{T^2-(t-\omega)^2}\ll 1$.

Then we may put, without serious error,

$$\int_{t-T}^{t_0}\frac{a'TJ_1(a'\sqrt{T^2-(t-\omega)^2})}{\sqrt{T^2-(t-\omega)^2}}\psi_1(\omega)d\omega=0.4a'^2\,t\int_{t-T}^{t_0}\psi_1(\omega)d\omega, \tag{93}$$

because $\frac{J_1(x)}{x}$ decreases uniformly from 0·5 to about 0·4 as x increases from zero to unity.

In order to get a concrete knowledge of the effect, we assume that

$$\begin{aligned}\sqrt{\frac{\lambda}{\rho}}&=(a-br^2)\\&=\ 7{\cdot}5\ \text{km/see} &&\text{at } r=6300\ \text{km}\\&=12{\cdot}5\ \text{km/see} &&\text{at } r=5100\ \text{km}\\ \rho&=a'-b'r^2\\&=\ 3{\cdot}0 &&\text{at } r=6300\ \text{km}\\&=\ 4{\cdot}7 &&\text{at } r=5100\ \text{km},\end{aligned}$$

and we get $a'^2=0{\cdot}4\times10^{-4}$, and, by writing τ for the time elapsed after the first arrival of the disturbance, (93) may be written

$$0{\cdot}16\times10^{-4}t\int_{t_0-\tau}^{t_0}\psi_1(\omega)d\omega.$$

If t_0, to which the dimensions of the initially disturbed region is proportional, be 5 seconds or so, even when $t=500$ sec., that is when the wave is observed at a position on the surface $r=6500$ km. and about 10 degrees distant from the centre of the disturbance, the effect of dispersion on the maximum value of voluminal change, as well as that of acceleration, velocity and displacement of the motion, caused by the disturbance propagated, does not amount, in rough estimation, to even 5 per cent. It is not possible to arrive at any conclusion regarding the effect of dispersion at every instant without assuming the form of $\psi_1(t)$.

4. When, instead of (83), the initial condition are given in the form

$$\Delta \text{ or } \frac{\partial\Delta}{\partial T}=\frac{(2\sqrt{ab}\,\mathrm{cosech}\,2\sqrt{ab}\,t)^{-n}}{\sqrt{\rho}\,(a-br^2)^{\frac{5}{2}}}P_n^m(\cos\beta)\frac{\sin m\varphi}{\cos m\varphi}$$

$$\cdot\left(\frac{d}{d(\sinh^2\sqrt{ab}\,t)}\right)^n 2\sqrt{ab}\,\mathrm{cosech}\,2\sqrt{ab}\,t\cdot\psi(t)\,, \tag{94}$$

the problem can be studied, remembering (82), and applying a process similar to that above mentioned to the function under differentiation.

5. The suface, $t=$const., will now be examined. Eliminating i and β from (25) and (26), we get

$$\frac{\sinh\sqrt{ab}\,t}{\sqrt{ab}}=\sqrt{\frac{(r-h)^2+4hr\sin^2\frac{\theta}{2}}{(a-br^2)(a-bh^2)}}\,, \tag{95}$$

or

$$r^2\left\{1+(a-bh^2)\frac{\sinh^2\sqrt{ab}\,t}{a}\right\}-2rh\cos\theta=\frac{\sinh^2\sqrt{ab}\,t}{b}(a-bh^2), \tag{96}$$

from which it will be seen that the surface is spherical, the centre of which moves on the line $\theta=0$.

6. When a is not constant, equation (68) is not solvable in the form (69). However, if the gradient of a is small, or if our attention is confined to a sufficiently restricted region, the solution may be approximately expressed by (69). It then appears that $t=$const. does not represent a surface of the same phase; the form of the surface of the phase front of a wave depending on its period. Even in that case, by confining our attention to a sufficiently small region at every instant, and by applying similar reasoning to that just given to such successive small portions, we may conclude that the velocity of propagation of a disturbance is $\sqrt{\frac{\lambda}{\rho}}$.

Appendix.

For convenienee of future application, we shall examine the nature of R_n or X_n -function.

For simplicity of notation, p^2-a-ab is substituted by p^2. Consider then the two differential equations

$$\frac{d^2\phi}{dt^2}+\left\{p^2-\frac{4ab\,n(n+1)}{\sinh^2 2\sqrt{ab}\,t}\right\}\phi=0 \tag{97}$$

$$\frac{d^2\psi}{dt^2}+\left\{q^2-\frac{4ab\,n(n+1)}{\sinh^2 2\sqrt{ab}\,t}\right\}\psi=0. \tag{98}$$

From (97) multiplied by ψ, subtract (98) multiplied by ϕ, and integrate with respect to t from zero to λ,

$$\int_0^\lambda \phi\psi dt=\frac{1}{p^2-q^2}\left[\phi\frac{d\psi}{dt}-\psi\frac{d\phi}{dt}\right]_0^\lambda. \tag{99}$$

Putting

$$\phi=X_n^{(1)}(pt),\quad \psi=X_n^{(1)}(qt),\quad n\geq 0,^{6)}$$

$$\int_0^\lambda X_n(pt)X_n(qt)\,dt=\frac{1}{p^2-q^2}\left\{X_n(pt)\frac{dX_n(qt)}{dt}-X_n(qt)\frac{dX_n(pt)}{dt}\right\}_{t=\lambda}. \tag{100}$$

As will be seen from the results worked out in the previous paper, we have, when t is large,

$$u\equiv 2\sqrt{ab}\coth 2\sqrt{ab}\,t=2\sqrt{ab}\{1+O(e^{-4\sqrt{ab}\,t})\}$$

$$X_n(pt)=\frac{1}{p^n}\left\{f_n(p)\sin(pt-\frac{n-1}{2}\pi)-g_n(p)\cos\left(pt-\frac{n-1}{2}\pi\right)\right\}$$
$$\cdot\left\{1+O(e^{-4\sqrt{ab}\,t})\right\}, \tag{101}$$

$$\frac{dX_n(pt)}{dt}=\frac{1}{p^{n-1}}\left\{f_n(p)\cos(pt-\frac{n-1}{2}\pi)+g_n(p)\sin\left(pt-\frac{n-1}{2}\pi\right)\right\}$$
$$\cdot\left\{1+O(e^{-4\sqrt{ab}\,t})\right\}, \tag{102}$$

in which

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6) Of the two X_n-functions, studied in the previous paper and given in (42), we have taken $X_n^{(1)}$. Suffix (1) will hereafter be omitted.

$$\begin{array}{ll} f_0(p)=0 \qquad , & g_0(p)=1 \\ f_1(p)=2\sqrt{ab} \qquad , & g_1(p)=p \\ f_2(p)=6\sqrt{ab}\,p \qquad , & g_2(p)=p^2-16ab \\ f_3(p)=12\sqrt{ab}\,p^2-48(ab)^{\frac{3}{2}} & g_3(p)=p^3-44abp \end{array}$$

$$\text{etc.} \tag{103}$$

Substituting (101), (102), and similar expressions related to $X_n(qt)$ in the right hand side of (100),

$$\begin{aligned}\int_0^\lambda X_n(pt)X_n(qt)\,dt=&\frac{1}{2(p^2-q^2)}\frac{1}{p^nq^n}\\ &\cdot\Big[(p+q)\{f_n(p)f_n(q)+g_n(p)g_n(q)\}\sin(p-q)\lambda\\ &+(p-q)\{f_n(p)f_n(q)-g_n(p)g_n(q)\}\sin\{(p+q)\lambda-n\pi\}\\ &-(p-q)\{g_n(p)f_n(q)+f_n(p)g_n(q)\}\cos\{(p+q)\lambda-n\pi\}\\ &-(p-q)\{g_n(p)f_n(q)-f_n(p)g_n(q)\}\cos(p-q)\lambda\Big]\{1+O(e^{-4\sqrt{ab}\lambda})\}\end{aligned} \tag{104}$$

It will readily be seen that, as shown in (103), of the two functions $f_n(p)$ and $g_n(p)$, the one is even and the other odd, so that

$$g_n(p)f_n(q)+f_n(p)g_n(q)=(p+q)G_1(p,\ q) \tag{105}$$

$$g_n(p)f_n(q)-f_n(p)g_n(q)=(p-q)G_2(p,\ q) \tag{106}$$

where $G_1(p,q)$ and $G_2(p,q)$ are respectively certain polynomials of p and q. Let us multiply both members of (104) by

$$\frac{\phi(q)\,dq}{|X_n(p,\infty)|\,|X_n(q,\infty)|}, \tag{107}$$

where

$$\left.\begin{aligned}|X_n(p,\infty)|&=\frac{1}{p^n}\sqrt{f_n^2(p)+g_n^2(p)}\\ |X_n(q,\infty)|&=\frac{1}{q^n}\sqrt{f_n^2(q)+g_n^2(q)}\end{aligned}\right\}, \tag{108}$$

and integrate from α to β, and then consider the limit to be $\lambda\to\infty$. Now it is known that

If $\phi(x)$ be continuous, except at a finite number of discontinuities, and if it have limited total fluctuation in the range $(\alpha,\ \beta)$, then, as $\lambda\to\infty$,

$$\text{I.}\quad \int_\alpha^\beta \psi(\theta)\sin\lambda\theta\, d\theta = O\left(\frac{1}{\lambda}\right)$$

$$\int_\alpha^\beta \psi(\theta)\cos\lambda\theta\, d\theta = O\left(\frac{1}{\lambda}\right) \tag{109}$$

[7]

$$\text{II.}\quad \int_\alpha^\beta \psi(\theta)\frac{\sin\lambda\theta}{\theta}\,d\theta = 0 \qquad ;\ \alpha\beta>0$$

$$= \frac{\pi}{2}\left\{f(+0)+f(-0)\right\};\ \alpha<0,\ \beta>0$$

$$= \frac{\pi}{2}f(+0) \qquad ;\ \alpha=0,\ \beta>0$$

$$= \frac{\pi}{2}f(-0) \qquad ;\ \alpha<0,\ \beta=0 \tag{110}$$

[8]

Therefore, with a suitable restriction on the form of $\phi(q)$,

$$\lim_{\lambda\to\infty}\int_\alpha^\beta \frac{\phi(q)}{|X_n(p,\infty)|\,|X_n(q,\infty)|}(p-q)\{g_n(p)f_n(q)+f_n(p)g_n(q)\}\cdot\cos\{(p+q)\lambda-n\pi\}dq$$

$$=\lim_{\lambda\to\infty}\int_\alpha^\beta \frac{\phi(q)\cdot G_1(p,q)}{|X_n(p,\infty)|\,|X_n(q,\infty)|}\cos\{(p+q)\lambda-n\pi\}dq=0.$$

In like manner, the integration involving $\cos(p-q)\lambda$ vanishes. Thus

$$2\int_\alpha^\beta \phi(q)dq\int_0^\infty \frac{X_n(pt)X_n(qt)}{|X_n(p,\infty)|\,|X_n(q,\infty)|}dq$$

$$=\lim_{\lambda\to\infty}\int_\alpha^\beta \frac{f_n(p)f_n(q)+g_n(p)g_n(q)}{\sqrt{f_n^2(p)+g_n^2(p)}\ \sqrt{f_n^2(q)+g_n^2(q)}}\ \frac{\sin(p-q)\lambda}{p-q}\phi(q)dq$$

$$+(-1)^n\lim_{\lambda\to\infty}\int_\alpha^\beta \frac{f_n(p)f_n(q)-g_n(p)g_n(q)}{\sqrt{f_n^2(p)+g_n^2(p)}\ \sqrt{f_n^2(q)+g_n^2(q)}}\ \frac{\sin(p+q)\lambda}{p+q}\phi(q)dq$$

which in simplified notation is

$$=\lim_{\lambda\to\infty}\int_\beta^\alpha \Phi_1(p,q)\frac{\sin(p-q)\lambda}{p-q}dq+(-1)^n\lim_{\lambda\to\infty}\int_\alpha^\beta \Phi_2(p,q)\frac{\sin(p+q)\lambda}{p+q}dq \tag{111}$$

7) Modern Analysis, 4th ed., p. 172.

8) A modification of Dirichlets integral. See, for example, "*Sugaku-gairon.*" p. 115, by Dr. K. TERAZAWA.

Putting, now, $p-q=\theta$, in the first integral, and $p+q=\theta$ in the second integral, we have

$$\lim_{\lambda\to\infty}\int_{\alpha}^{\beta}\Phi_1(p,q)\frac{\sin(p-q)\lambda}{p-q}dq=\lim_{\lambda\to\infty}\int_{p-\beta}^{p-\alpha}\Phi_1(p,p-\theta)\frac{\sin\lambda\theta}{\theta}d\theta \tag{112}$$

$$\lim_{\lambda\to\infty}\int_{\alpha}^{\beta}\Phi_2(p,q)\frac{\sin(p+q)\lambda}{p+q}dq=\lim_{\lambda\to\infty}\int_{p+\alpha}^{p+\beta}\Phi_2(p,\theta-p)\frac{\sin\lambda\theta}{\theta}d\theta \tag{113}$$

If $0\leq\alpha<p<\beta$, integration (113) vanishes, while (112) becomes

$$\pi\Phi_1(p,\ p)=\pi\phi(p).$$

We have thus proved that by putting $\alpha=0$, $\beta\to\infty$,

$$\phi(p)=\frac{2}{\pi}\int_0^\infty\phi(q)dq\int_0^\infty\frac{X_n(pt)X_n(qt)}{|X_n(p,\infty)|\,|X_n(q,\infty)|}dt, \tag{114}$$

provided

$$\int_0^\infty|\phi(q)|dq=\text{finite.}$$

This shows that

$$\left.\begin{aligned}&\textit{if} &\psi(t)&=\sqrt{\frac{2}{\pi}}\int_0^\infty\phi(p)\frac{X_n(pt)}{|X_n(p,\infty)|}dp\\ &\textit{then} &\phi(p)&=\sqrt{\frac{2}{\pi}}\int_0^\infty\psi(t)\frac{X_n(pt)}{|X_n(p,\infty)|}dt\end{aligned}\right\}. \tag{115}$$

This functional relation corresponds to Hankel's reciprocal theorem in the Bessel function,

$$\left.\begin{aligned}&\textit{if} &\psi(x)&=\int_0^\infty\phi(x)J_\nu(\lambda x)\lambda d\lambda\\ &\textit{then} &\phi(x)&=\int_0^\infty\psi(x)J_\nu(\lambda x)\lambda d\lambda,\end{aligned}\right\} \tag{116}$$

although $X_n(pt)$ is not a function of pt alone, depending on p and t respectively in a different manner, this cannot be written in a strictly reciprocal form.

Although when $b<0$, the relation which cerresponds to (115) cannot be easily obtained, it will be noticed that $r\to\infty$ corresponds, by (95), to

$t \rightarrow \frac{1}{\sqrt{ab'}} \sin^{-1}\sqrt{\frac{a}{a+b'h^2}}$, where $b' \equiv -b$. Therefore, the integration with respect to t should be performed within a finite range.

In virtue of the relation (115) with the known theorems referred to Legendre's function, the problems of propagation of disturbance will be discussed in a future paper under more general conditions.

In conclusion, the author desires to express his sincere thanks to Professor T. Matuzawa and Dr. H. Kawasumi for their kind criticisms.

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4. 等方不均一球内の一點より起る彈性波 (第2報)

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剛性が無い媒質に就いて計算を行つた．斯様な媒質中では唯一種類の彈性波のみが傳播するこさになり運動方程式も簡單になるので比較的に嚴密な吟味を行ひ易い．從つて不均質性が彈性波に及ぼす影響の一つ一つを抽出して考へて行くのに斯様な計算の結果が何等かの手段を與へるのではないかさ思ふ．

主な結果を擧ぐれば

(1) 體積變化さ捩れさを同時に伴つた彈性波が傳播する．

(2) 波面が平面に近くても (數學的に云へば $t \rightarrow \infty$ さすることによつて考へらるのであるが) 週期によつて傳播速度が異る．然も媒質の性質の變り方が特殊でない限り同位相の波面の形も週期によつて異る．が

(3) 擾亂の傳はる速度は媒質に固有な値 $\sqrt{\frac{\lambda}{\rho}}$ さなる．

最後に今後任意の境界條件の下に解く爲の準備さして第一報に述べた X_n－函數 (從つて R_n－函數さ云つてもよいのであるが) の性質に就いて計算を行つた．

又これによつて見るに t に無關係な $|X_n(p, \infty)|$ で割つたものを新しく X_n－ 函數さ定義した方が便利な樣である．
