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Precise estimate for large deviation of Donsker-Varadhan type

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0. Introduction.

Let M be a Polish space, $C_b(M)$ be a Banach space consisting of bounded continuous real functions for which its norm is given by $\|f\|_{C_b(M)}=\sup_{x\in M}|f(x)|$, and $\mathcal{P}(M)$ be a metric space consisting of probability measures on M with the Prohorov metric.

Let μ be a probability measure on M whose support is the whole space M and $\{P_x; x\in M\}$ be a family of probability measures on $D([0, \infty)\to M)$ which induces a μ-symmetric Markov process on M. We denote by $\{P_t\}_{t\geq 0}$, $\mathcal{E}$ and $\mathcal{L}$ the induced semigroup, the Dirichlet form and the infinitesimal generator on $L^2(M; d\mu)$, respectively.

Moreover, we impose the following assumptions.

(A-1) For any $t>0$, there is a continuous function $P(t, \cdot, \cdot): M\times M\to(0, \infty)$ such that $P_t f(x)=\int_M P(t, x, y)f(y)\mu(dy)$, μ-a. e. $x\in M$, for all $f\in C_b(M)$.

(A-2) Let $\Pi: L^2(d\mu)\to L^2(d\mu)$ be given by $(\Pi h)(x)=h(x)-\int_M h(y)\mu(dy)$ μ-a. e. x for any $h\in L^2(d\mu)$. Then there is a $\lambda>0$ such that

$$\|\Pi P_t h\|_{L^2(d\mu)}\leqq e^{-\lambda t}\|h\|_{L^2(d\mu)}, \qquad t\geqq 0,\ h\in L^2(d\mu).$$

(A-3) The operator $(I-\mathcal{L})^{-1}: L^2(d\mu)\to L^2(d\mu)$ is a compact operator.

Let us take $\{\varphi_n\}_{n=1}^{\infty}\subset C_b(M)$ and $\{a_n\}_{n=1}^{\infty}\subset(0, \infty)$ satisfying the following and fix them throughout this paper.

(1) $\varphi_n\neq 0$, $n=1, 2, \cdots$, and $\|\varphi_n\|_{C_b(M)}\to 0$ as $n\to\infty$.

(2) $\{\|\varphi_n\|_{L^2(d\mu)}^{-1}\varphi_n\}_{n=1}^{\infty}$ is a complete orthonormal basis in $L^2(M; d\mu)$.

(3) $\sum_{n=1}^{\infty} a_n\cdot\|\varphi_n\|_{L^2(d\mu)}^{-2}<\infty$.

Let $\mathcal{M}$ be the set of signed measures on M with finite total variation. Then for $m\in\mathcal{M}$, we let

$$\|m\|_A \underset{\text{def}}{=} \left\{ \sum_{n=1}^{\infty} a_n \cdot \|\varphi_n\|_{L^2(d\mu)}^{-2} \cdot \left(\int_M \varphi_n dm \right)^2 \right\}^{1/2}$$

$$\leqq \left\{ \sum_{n=1}^{\infty} a_n \cdot \|\varphi_n\|_{L^2(d\mu)}^{-2} \right\}^{1/2} \cdot \sup_n \left| \int_M \varphi_n dm \right|$$

$$\leqq \left\{ \sum_{n=1}^{\infty} a_n \cdot \|\varphi_n\|_{L^2(d\mu)}^{-2} \right\}^{1/2} \left(\sup_n \|\varphi_n\|_{C_b(M)} \right) \cdot \|m\|_{var} < \infty .$$

So $\|\cdot\|_A$ is a norm on $\mathcal{M}$. We denote by $\mathcal{M}_A$ the normed space which is $\mathcal{M}$ as a set and whose norm is $\|\cdot\|_A$.

By the assumption (A-1), we can define Pinned Markov process $P[\cdot|w(0)=x,\ w(T)=y]$ for any $x, y \in M$ and $T>0$. For any $w \in D([0, \infty); M)$ and $T>0$, let $l_T(w) \in \mathcal{P}(M)$ be given by $\int_M f(x) l_T(w)(dx) = (1/T)\int_0^T f(w(t))dt$, $f \in C_b(M)$.

We will assume one of the following assumptions of Donsker-Varadhan type large deviation.

(L-1)
$$\varliminf_{T\to\infty} \frac{1}{T} \cdot \log P[l_T \in G|\ w(0)=x,\ w(T)=y]$$

$$\geqq -\inf\{\mathcal{E}(\varphi, \varphi)\,;\ \varphi \in \mathcal{D}om(\mathcal{E}),\ \varphi^2 d\mu \in G\}$$

for any open set G in $\mathcal{P}(M)$, and $x, y \in M$,
and

$$\varlimsup_{T\to\infty} \frac{1}{T} \cdot \log P[l_T \in C|\ w(0)=x,\ w(T)=y]$$

$$\leqq -\inf\{\mathcal{E}(\varphi, \varphi)\,;\ \varphi \in \mathcal{D}om(\mathcal{E}),\ \varphi^2 d\mu \in C\}$$

for any closed set C in $\mathcal{P}(M)$, and $x, y \in M$.

(L-2)
$$\varliminf_{T\to\infty} \frac{1}{T} \cdot \log P[l_T \in G|\ w(0)=x]$$

$$\geqq -\inf\{\mathcal{E}(\varphi, \varphi)\,;\ \varphi \in \mathcal{D}om(\mathcal{E}),\ \varphi^2 d\mu \in G\}$$

for any open set G in $\mathcal{P}(M)$, and $x \in M$,
and

$$\varlimsup_{T\to\infty} \frac{1}{T} \cdot \log P[l_T \in C|\ w(0)=x]$$

$$\leqq -\inf\{\mathcal{E}(\varphi, \varphi)\,;\ \varphi \in \mathcal{D}om(\mathcal{E}),\ \varphi^2 d\mu \in C\}$$

for any closed set C in $\mathcal{P}(M)$, and $x \in M$.

(L-3) $$\varliminf_{T\to\infty}\frac{1}{T}\cdot\log P_\mu[l_T\in G]\geqq-\inf\{\mathcal{E}(\varphi,\varphi)\,;\,\varphi\in\mathcal{D}om(\mathcal{E}),\varphi^2d\mu\in G\}$$

for any open set G in $\mathcal{P}(M)$,
and

$$\varlimsup_{T\to\infty}\frac{1}{T}\cdot\log P_\mu[l_T\in C]\leqq-\inf\{\mathcal{E}(\varphi,\varphi)\,;\,\varphi\in\mathcal{D}om(\mathcal{E}),\varphi^2d\mu\in C\}$$

for any closed set C in $\mathcal{P}(M)$.

Now let $U:\mathcal{M}_A\to\boldsymbol{R}$ be a bounded smooth function. Then under the assumption (L-1), we see that

$$\lim_{T\to\infty}\frac{1}{T}\cdot\log E^P[\exp(T\cdot U(l_T))|\ w(0)=x,\ w(T)=y]=-f_0,$$

where $f_0=\inf\{\mathcal{E}(\varphi,\varphi)-U(\varphi^2d\mu)\,;\,\varphi\in\mathcal{D}om(\mathcal{E}),\int_M\varphi^2d\mu=1\}$.
Our aim in the present paper is to give more precise estimates to these asymptotics. Main results will be presented in Section 3. But we give one statement which follows from the results in Section 3.

THEOREM. *Suppose that the assumptions* (A-1)-(A-3) *and* (L-1) *are satisfied. Then for any $x,y\in M$, there are a finite dimensional submanifold N in $\mathcal{D}om(\mathcal{E})$ and smooth bounded functions $\rho:N\to[0,\infty)$ and $g:N\to(0,\infty)$ such that*

(1) $N\subset\left\{\varphi\in\mathcal{D}om(\mathcal{E})\,;\,\int_M\varphi^2d\mu=1\right\}$,

(2) $\{\varphi\in N;\,\rho(\varphi)=0\}=V\underset{\mathrm{def}}{=}\left\{\varphi\in\mathcal{D}om(\mathcal{E})\,;\,\int_M\varphi^2d\mu=1,\mathcal{E}(\varphi,\varphi)-U(\varphi^2d\mu)=f_0\right\}$,

and

(3) $$E^P[\exp(T\cdot U(l_T))|\,w(0)=x,\ w(T)=y]$$
$$\sim\exp(-f_0\cdot T)\cdot T^{(\dim N)/2}\cdot\int_N g(z)\cdot\exp(-\rho(z)\cdot T)n_0(dz)$$

as $T\to\infty$. Here $n_0(dz)$ is the Riemannian volume on N, and $k_1(T)\sim k_2(T)$ as $T\to\infty$ denotes that $\lim_{T\to\infty}(k_1(T)/k_2(T))=1$.

Our presentation and results are inspired and strongly infuluenced by Bolthausen [1], [2], [3] and Luttinger [7], [8]. However, we cannot apply our results to the problem raised by Bolthausen [3], since we need strong assumptions to the regularity for the function U, at least C^2-regularity.

Our methods come from Kusuoka-Tamura [6] and Kusuoka-Stroock [5].

We strongly use the advantage that our Markov process is symmetric. We do not know how to handle real non-symmetric Markov process.

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1. Basic Lemma.

Let everything be as in Introduction. The main purpose in this Section is to show the following Lemma.

(1.1) LEMMA. *Let* $\tilde{P}_t=\Pi P_t$, $t\geqq 0$, *and* $\tilde{G}=2\cdot\int_0^\infty \tilde{P}_t dt$. *Then*

$$\left|P(T,x,y)\cdot E^P\left[\exp\left(zT\cdot\int_M \phi(dl_T(w)-d\mu)\right)\right| w(0)=x,\ w(T)=y\right]\right|$$

$$\leqq \exp\left(\frac{|z|^2T}{2}(\phi,\tilde{G}\phi)\right)\cdot\exp\left(|z|(|z|^2T\cdot\lambda R^3(1-|z|R)^{-1})\right)$$

$$\times(1+|z|\cdot Re^{\lambda}(1+P(2,x,x)^{1/2}))(1+|z|\cdot Re^{\lambda}(1+P(2,y,y)^{1/2}))(1-|z|R)^{-2}$$

$$+|z|\cdot R\{P(2,x,x)^{1/2}P(2,y,y)^{1/2}e^{2\lambda}+1\}(1-|z|R)^{-1}$$

$$+P(2,x,x)^{1/2}P(2,y,y)^{1/2}e^{2\lambda}\cdot\exp(-\lambda T)$$

and

$$\left|P(T,x,y)\cdot E^P\left[\exp\left(zT\cdot\int_M \phi(dl_T(w)-d\mu)\right)\right| w(0)=x, w(T)=y\right]$$

$$\left.-\exp\left(\left(\frac{z^2T}{2}\right)\cdot(\phi,\tilde{G}\phi)_{L^2(d\mu)}\right)\right|$$

$$\leqq \exp\left(\frac{|z|^2T}{2}(\phi,\tilde{G}\phi)\right)\cdot[\{\exp\left(|z|(|z|^2T\cdot\lambda R^3(1-|z|R)^{-1})\right)$$

$$\times(1+|z|\cdot Re^{\lambda}(1+P(2,x,x)^{1/2}))(1+|z|\cdot Re^{\lambda}(1+P(2,y,y)^{1/2}))$$

$$\times(1-|z|R)^{-2})\}-1]$$

$$+|z|^2R^2\cdot\exp(R^2\lambda\cdot|z|^2T)$$
$$+|z|\cdot R\{P(2,x,x)^{1/2}P(2,y,y)^{1/2}e^{2\lambda}+1\}(1-|z|R)^{-1}$$
$$+P(2,x,x)^{1/2}P(2,y,y)^{1/2}e^{2\lambda}\cdot\exp(-\lambda T)$$

for any $T>2$, $\phi\in C_b(M)$ *and* $z\in\boldsymbol{C}$ *with* $|z|<R^{-1}$. *Here* $R=4\lambda^{-1}\|\phi\|_{C_b(M)}$.

(1.2) REMARK. In the proof of Lemma (1.1), we only use the assumptions (A-1) and (A-2).

Let $\tilde{\phi}(x)=\phi(x)-\int_M\phi d\mu$ and $\bar{\phi}$ be a multiplication operator $\tilde{\phi}\cdot$ in $L^2(M;d\mu)$. Note that $\|\tilde{\phi}\|_{C_b(M)}\leqq 2\|\phi\|_{C_b(M)}$. Then we have

$$P(T,x,y)\cdot E^P\left[\exp\left(zT\cdot\int_M\phi(dl_T(w)-d\mu)\right)\middle|\, w(0)=x,w(T)=y\right]$$
$$=\sum_{n=0}^{\infty}z^n\cdot P(T,x,y)$$
$$\times E^P\left[\int_{0<s_1<s_2<\cdots<s_n<T}\tilde{\phi}(w(s_1))\cdots\tilde{\phi}(w(s_n))ds_1\cdots ds_n\middle|\, w(0)=x,w(T)=y\right]$$
$$=P(T,x,y)$$
$$+\sum_{n=1}^{\infty}z^n\cdot\int_{\substack{\tau_0,\cdots,\tau_{n-1}>0\\ \tau_0+\cdots+\tau_{n-1}<T}}\Big(P(\tau_0,x,\cdot),$$
$$\bar{\phi}P_{\tau_1}\bar{\phi}\cdots P_{\tau_{n-1}}\bar{\phi}P\Big(T-\sum_{j=0}^{n-1}\tau_j,y,\cdot\Big)\Big)_{L^2(d\mu)}d\tau_0\cdots d\tau_{n-1}.$$

Let $\Pi_0=I-\Pi$ and $\Pi_1=\Pi$. Then for $n\geqq 1$, we have

$$\Big(P(\tau_0,x,\cdot),\bar{\phi}P_{\tau_1}\bar{\phi}\cdots P_{\tau_{n-1}}\bar{\phi}P\Big(T-\sum_{j=0}^{n-1}\tau_j,y,\cdot\Big)\Big)_{L^2(d\mu)}$$
$$=\sum_{l_1,\cdots,l_{n+1}=0,1}\Big(P(\tau_0,x,\cdot),$$
$$\Pi_{l_1}\bar{\phi}P_{\tau_1}\cdots\Pi_{l_{n-1}}\bar{\phi}P_{\tau_{n-1}}\Pi_{l_n}\bar{\phi}\Pi_{l_{n+1}}P\Big(T-\sum_{j=0}^{n-1}\tau_j,y,\cdot\Big)\Big)_{L^2(d\mu)}$$
$$=\Big(\Pi P(\tau_0,x,\cdot),\bar{\phi}\tilde{P}_{\tau_1}\cdots\bar{\phi}\tilde{P}_{\tau_{n-1}}\bar{\phi}\Pi P\Big(T-\sum_{j=0}^{n-1}\tau_j,y,\cdot\Big)\Big)_{L^2(d\mu)}$$
$$+\sum_{k=0}^{[n/2]}\sum_{\substack{i_0+\cdots+i_{k+1}=n+2\\ i_0,i_{k+1}\geqq 1,i_1,\cdots,i_k\geqq 2}}b_{i_0-1}(x;\tau_0,\cdots,\tau_{i_0-2})$$

$$\times \prod_{j=1}^{k} a_{i_j}(\tau_{i_0+\cdots+i_{j-1}}, \cdots, \tau_{i_0+\cdots+i_j-2})$$

$$\times b_{i_{k+1}-1}\left(y\,;\, T-\sum_{l=0}^{n-1}\tau_l, \tau_{n-1}, \tau_{n-2}, \cdots, \tau_{i_0+\cdots+i_k}\right).$$

Here

$$a_n(\tau_1, \cdots, \tau_{n-1}) = (\tilde{\phi}, \tilde{P}_{\tau_1}\tilde{\phi}\tilde{P}_{\tau_2}\cdots\tilde{\phi}\tilde{P}_{\tau_{n-1}}\tilde{\phi})_{L^2(d\mu)}, \quad n \geqq 2,$$

$$b_0(u) = 1, \quad u \in M,$$

and

$$b_n(u\,;\,\tau_0, \cdots, \tau_{n-1}) = (\Pi P(\tau_0, u, \cdot), \tilde{\phi}\tilde{P}_{\tau_1}\cdots\tilde{\phi}\tilde{P}_{\tau_{n-1}}\tilde{\phi})_{L^2(d\mu)}, \quad u \in M, n \geqq 1.$$

Then we have the following.

(1.3) PROPOSITION. (1) $a_2(\tau) = (\phi, \tilde{P}_\tau\phi)_{L^2(d\mu)} \geqq 0, \quad \tau > 0.$

(2) $|a_n(\tau_1, \cdots, \tau_{n-1})| \leqq \exp\left(-\lambda\cdot\sum_{k=1}^{n-1}\tau_k\right)\cdot(2\|\phi\|_{C_b(M)})^n,$

for any $n \geqq 3$ *and* $\tau_1, \cdots, \tau_{n-1} > 0$.

(3) $|b_n(u\,;\,\tau_1, \cdots, \tau_n)| \leqq (4\|\phi\|_{C_b(M)})^n,$

for any $u \in M$, $n \geqq 1$ *and* $\tau_1, \cdots, \tau_n > 0$.

(4) $|b_n(u\,;\,\tau_1, \cdots, \tau_n)| \leqq (4\|\phi\|_{C_b(M)})^n \cdot P(2, u, u)^{1/2}\cdot e^\lambda\cdot\exp\left(-\lambda\left(\sum_{k=1}^{n}\tau_k\right)\right),$

for any $u \in M$, $n \geqq 1$ *and* $\tau_1, \cdots, \tau_n > 0$ *with* $\sum_{k=1}^{n}\tau_k > 1$.

(5)
$$\left|\left(\Pi P(\tau_0, x, \cdot), \tilde{\phi}\tilde{P}_{\tau_1}\cdots\tilde{\phi}\tilde{P}_{\tau_{n-1}}\tilde{\phi}\Pi P\left(T-\sum_{j=0}^{n-1}\tau_j, y, \cdot\right)\right)_{L^2(d\mu)}\right|$$
$$\leqq (4\|\phi\|_{C_b(M)})^n \cdot P(2, x, x)^{1/2}P(2, y, y)^{1/2}\cdot e^{2\lambda}\cdot\exp(-\lambda T),$$

for any $x, y \in M$, $n \geqq 1$, $\tau_0, \cdots, \tau_{n-1} > 0$ *and* $T > 2$.

PROOF. The assertions (1) and (2) are obvious. Let us prove the assertions (3) and (4). Note that

$$|(\Pi P(\tau_1, u, \cdot), \tilde{\phi}\tilde{P}_{\tau_2}\cdots\tilde{\phi}\tilde{P}_{\tau_m}h)_{L^2(d\mu)}|$$
$$= \lim_{T\to\infty}\left|\int_{M^m}(P(\tau_1, u, v_1) - P(T, u, v_1))\tilde{\phi}(v_1)(P(\tau_2, v_1, v_2) - P(T, v_1, v_2))\right.$$
$$\left.\cdots\tilde{\phi}(v_{m-1})(P(\tau_m, v_{m-1}, v_m) - P(T, v_{m-1}, v_m))h(v_m)\mu(dv_1)\cdots\mu(dv_m)\right|$$

$$\leqq 2(4\|\phi\|_{C_b(M)})^{m-1}\cdot\sup\left\{\int_M P(t,u,v)|h(v)|\mu(dv)\ ;\ t\geqq \sum_{k=1}^{m}\tau_k\right\}.$$

$$\leqq 2(4\|\phi\|_{C_b(M)})^{m-1}\cdot\left(\|h\|_{C_b(M)}\wedge P\left(2\cdot\sum_{k=1}^{m}\tau_k, u, u\right)^{1/2}\|h\|_{L^2(d\mu)}\right)$$

for any $h\in C_b(M)$. So we have the assertion (3). If $\sum_{k=1}^{n-1}\tau_k>1$, there is an $m\in\{1,\cdots,n\}$ such that $\sum_{k=1}^{m-1}\tau_k<1\leqq\sum_{k=1}^{m}\tau_k$. Let $\sigma=1-\sum_{k=1}^{m-1}\tau_k$. Then we see that

$$b_n(u\ ;\ \tau_1,\cdots,\tau_n)=(\varPi P(\tau_1,u,\cdot),\bar{\phi}\tilde{P}_{\tau_2}\cdots\bar{\phi}\tilde{P}_{\tau_{m-1}}\bar{\phi}\tilde{P}_\sigma\tilde{h})_{L^2(d\mu)},$$

where $\tilde{h}=\tilde{P}_{\tau_m-\sigma}\bar{\phi}\tilde{P}_{\tau_{m+1}}\cdots\bar{\phi}\tilde{P}_{\tau_n}\bar{\phi}$. Since we have

$$\|\tilde{h}\|_{L^2(d\mu)}\leqq(2\|\phi\|_{C_b(M)})^{n-m+1}\cdot\exp\left(-\lambda\left(\sum_{k=1}^{n}\tau_k-1\right)\right),$$

we have the assertion (4).

Similarly we have the assertion (5).

This completes the proof.

Let

$$A_n=\int_{(0,\infty)^{n-1}}|a_n(\tau_1,\cdots,\tau_{n-1})|d\tau_1\cdots d\tau_{n-1},\quad n\geqq 2,$$

$$B_0(u)=1,\quad u\in M,$$

and

$$B_n(u)=\int_{(0,\infty)^n}|b_n(u\ ;\ \tau_1,\cdots,\tau_n)|d\tau_1\cdots d\tau_n,\quad n\geqq 1,\ u\in M.$$

Then by Proposition (1.3) we have

$$A_2=\frac{1}{2}(\phi,\tilde{G}\phi)_{L^2(d\mu)},$$

$$A_n\leqq\lambda^{-(n-1)}\cdot(2\|\phi\|_{C_b(M)})^n,\quad n\geqq 3,$$

and

$$B_n(u)\leqq\lambda^{-n}\cdot(4\|\phi\|_{C_b(M)})^n\cdot e^\lambda(1+P(2,u,u)^{1/2}),\quad u\in M,\ n\geqq 1.$$

Also for $n\geqq 1$, let

$$R_n(T\ ;\ x,y)=\int_{\substack{\tau_0,\cdots,\tau_{n-1}>0\\ \tau_0+\cdots+\tau_{n-1}<T}}d\tau_0\cdots d\tau_{n-1}$$
$$\times\left(\varPi P(\tau_0,x,\cdot),\bar{\phi}\tilde{P}_{\tau_1}\cdots\bar{\phi}\tilde{P}_{\tau_{n-1}}\bar{\phi}\varPi P\left(T-\sum_{j=0}^{n-1}\tau_j,y,\cdot\right)\right)_{L^2(d\mu)}.$$

Then for $T>2$, by Proposition (1.3) we have

$$(1.4)\qquad |R_n(T;x,y)|\leqq\frac{T^n}{n!}\cdot(4\|\phi\|_{C_b(M)})^n\exp(-\lambda T)\cdot P(2,x,x)^{1/2}P(2,y,y)^{1/2}e^{2\lambda}$$
$$\leqq(4\lambda^{-1}\|\phi\|_{C_b(M)})^nP(2,x,x)^{1/2}P(2,y,y)^{1/2}e^{2\lambda}.$$

For $n\geqq 1$, let $J_n=\{(i_0,\cdots,i_{k+1})\,;\,k\geqq 0,\ i_0,i_{k+1}\geqq 1,\ i_1,\cdots,i_k\geqq 2,\ i_0+\cdots+i_{k+1}=n+2\}$. For $I=(i_0,\cdots,i_{k+1})\in J_n$, let

$$C_n(\tau_0,\cdots,\tau_{n-1};I,T)$$
$$=b_{i_0-1}(x;\tau_0,\cdots,\tau_{i_0-2})\cdot\prod_{j=1}^k a_{i_j}(\tau_{i_0+\cdots+i_{j-1}},\cdots,\tau_{i_0+\cdots+i_j-2})$$
$$\times b_{i_{k+1}-1}\left(y;T-\sum_{l=0}^{n-1}\tau_l,\tau_{n-1},\tau_{n-2},\cdots,\tau_{i_0+\cdots+i_k}\right)$$

and

$$C(I,T)=\int_{\substack{\tau_0,\cdots,\tau_{n-1}>0\\ \tau_0+\cdots+\tau_{n-1}<T}}C_n(\tau_0,\cdots,\tau_{n-1};I,T)d\tau_0\cdots d\tau_{n-1}.$$

Then we have for any $I=(i_0,\cdots,i_{k+1})\in\bigcup_{n=1}^\infty J_n$

$$|C(I,T)|=\left|\int_{\substack{\tau_1,\cdots,\tau_n>0\\ \tau_1+\cdots+\tau_n<T}}b_{i_0-1}(x;\tau_1,\cdots,\tau_{i_0-1})\cdot b_{i_{k+1}-1}(y;\tau_{i_0},\cdots,\tau_{i_0+i_{k+1}-2})\right.$$
$$\left.\times\prod_{j=1}^k a_{i_j}(\tau_{i_0+i_{k+1}+i_1+\cdots+i_{j-1}-j},\cdots,\tau_{i_0+i_{k+1}+i_1+\cdots+i_j-j-2})d\tau_1\cdots d\tau_n\right|$$
$$\leqq\left(\int_{\substack{\sigma_1,\cdots,\sigma_k>0\\ \sigma_1+\cdots+\sigma_k<T}}d\sigma_1\cdots d\sigma_k\right)\cdot B_{i_0-1}(x)B_{i_{k+1}-1}(y)\cdot\prod_{j=1}^k A_{i_j}$$
$$=\frac{T^k}{k!}\cdot B_{i_0-1}(x)B_{i_{k+1}-1}(y)\cdot\prod_{j=1}^k A_{i_j}.$$

Note that $C((1,\underbrace{2,\cdots,2}_{n\text{-times}},1),T)\leqq(T^n/n!)\cdot A_2^n$. Also we see that

$$\frac{T^n}{n!}\cdot A_2^n-C((1,\underbrace{2,\cdots,2}_{n\text{-times}},1),T)$$
$$=\int_0^T ds\frac{s^{n-1}}{(n-1)!}\int_{\substack{\tau_1,\cdots,\tau_n>0\\ \tau_1+\cdots+\tau_n>T-s}}d\tau_1\cdots d\tau_n\prod_{j=1}^n a_2(\tau_j)$$
$$\leqq\|\phi\|_{C_b(M)}^{2n}\cdot\int_0^T ds\int_s^\infty dt\frac{(T-s)^{n-1}t^{n-1}}{(n-1)!\,(n-1)!}\cdot\exp(-\lambda t)$$

$$=(2\|\psi\|_{C_b(M)}/\lambda)^{2n}\cdot\int_0^{\lambda T/2}ds\int_s^\infty dt\frac{((\lambda T/2)-s)^{n-1}t^{n-1}}{(n-1)!(n-1)!}e^{-2t}$$

$$\leqq R^{2n}\cdot\int_0^{\lambda T/2}ds\int_0^\infty dt\frac{((\lambda T/2)-s)^{n-1}}{(n-1)!}e^{-t}$$

$$=R^{2n}\cdot\int_0^{\lambda T/2}ds\frac{((\lambda T/2)-s)^{n-1}}{(n-1)!}e^{-s}.$$

Let $K=\bigcup_{n=1}^\infty J_n$, and set $|I|=n$, if $I\in J_n$. Let $K_0=\{(1,\underbrace{2,\cdots,2}_{k\text{-times}},1)\,;\,k\geqq 1\}$, $K_2=\{(i_0,\cdots,i_{k+1})\in K\,;\,i_0+i_{k+1}\geqq 3\}$, and $K_1=K\backslash(K_0\cup K_2)$. Recall that

$$P(T,x,y)\cdot E^P\Big[\exp\Big(zT\cdot\int_M\psi(dl_T(w)-d\mu)\Big)\Big|\,w(0)=x,w(T)=y\Big] \tag{1.5}$$

$$=1+\sum_{I\in K}z^{|I|}C(I,T)+(P(T,x,y)-1)+\sum_{n=1}^{n}z^nR_n(T\,;\,x,y).$$

Note that

$$\Big|\sum_{I\in K_1}z^{|I|}C(I,T)\Big|$$

$$\leqq\sum_{n=2}^\infty|z|^n\sum_{k=1}^{[(n-1)/2]}\sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\cdots,i_k\geqq 2}}\frac{T^k}{k!}\prod_{j=1}^kA_{i_j}$$

$$\leqq\exp\Big(\frac{|z|^2T}{2}(\psi,\tilde G\psi)_{L^2(d\mu)}\Big)\Big(\exp\Big(|z|^2T\Big(\sum_{k=3}^\infty|z|^{k-2}A_k\Big)\Big)-1\Big),$$

and

$$\Big|\sum_{I\in K_2}z^{|I|}C(I,T)\Big|$$

$$\leqq\sum_{n=1}^\infty|z|^n\sum_{I=(i_0,\cdots,i_{k+1})\in K_2\cap J_n}B_{i_0-1}(x)B_{i_{k+1}-1}\cdot\frac{T^k}{k!}\prod_{j=1}^kA_{i_j}$$

$$\leqq\Big\{\Big(\sum_{k=0}^\infty|z|^kB_k(x)\Big)\Big(\sum_{k=0}^\infty|z|^kB_k(y)\Big)-1\Big\}\cdot\Big(\exp\Big(|z|^2T\Big(\sum_{k=2}^\infty|z|^{k-2}A_k\Big)\Big)\Big).$$

Therefore we have

$$\Big|\sum_{I\in K_1\cup K_2}z^{|I|}C(I,T)\Big| \tag{1.6}$$

$$\leqq\exp\Big(\frac{|z|^2T}{2}(\psi,\tilde G\psi)\Big)$$

$$\times\Big\{\Big(\sum_{k=0}^\infty|z|^kB_k(x)\Big)\Big(\sum_{k=0}^\infty|z|^kB_k(y)\Big)\cdot\exp\Big(|z|^2T\Big(\sum_{k=1}^\infty|z|^kA_{k+2}\Big)\Big)-1\Big\}$$

$$\leqq \exp\left(\frac{|z|^2 T}{2}(\psi, \tilde{G}\psi)\right)\cdot[\{\exp(|z|(|z|^2 T\cdot\lambda R^3(1-|z|R)^{-1}))$$

$$\times(1+|z|\cdot Re^{\lambda}(1+P(2,x,x)^{1/2}))(1+|z|\cdot Re^{\lambda}(1+P(2,y,y)^{1/2}))$$

$$\times(1-|z|R)^{-2}\}-1] \qquad \text{if } |z|<R^{-1}.$$

Also, we have

$$\left|1+\sum_{I\in K_0} z^{|I|}C(I,T)\right| \leqq \exp\left(\frac{|z|^2 T}{2}\cdot(\psi, \tilde{G}\psi)_{L^2(d\mu)}\right) \tag{1.7}$$

and

$$\left|1+\sum_{I\in K_0} z^{|I|}C(I,T)-\exp\left(\frac{z^2 T}{2}\cdot(\psi, \tilde{G}\psi)_{L^2(d\mu)}\right)\right| \tag{1.8}$$

$$\leqq \sum_{n=1}^{\infty}|z|^{2n}R^{2n}\cdot\int_0^{\lambda T/2} ds\frac{((\lambda T/2)-s)^{n-1}}{(n-1)!}\cdot e^{-s}$$

$$\leqq |z|^2R^2\cdot\int_0^{\lambda T/2} ds\exp(((|z|R)^2((\lambda T/2)-s)-s)$$

$$\leqq |z|^2R^2\cdot\exp(\lambda R^2\cdot|z|^2T).$$

By (1.4), we have

$$\left|\sum_{n=1}^{\infty} z^n R_n(T,x,y)\right| \leqq |z|\cdot R\cdot P(2,x,x)^{1/2}P(2,y,y)^{1/2}e^{2\lambda}(1-|z|R)^{-1}, \tag{1.9}$$

if $|z|<R^{-1}$. Also, we have

$$|P(T,x,y)-1| \leqq P(2,x,x)^{1/2}P(2,y,y)^{1/2}e^{2\lambda}\cdot\exp(-\lambda T), \quad T>2. \tag{1.10}$$

These imply our Lemma.

This completes the proof of Lemma (1.1).

2. Uniform estimates.

Let K be a compact metric space and $U: \mathcal{M}_A\times K\to \boldsymbol{R}$ be a continuous function satisfying the following.

(U-1) $\sup\{|U(m,\xi)|;\ m\in\mathcal{P}(M),\ \xi\in K\}<\infty$.

(U-2) $U(\cdot,\xi): \mathcal{M}_A\to\boldsymbol{R}$ is smooth in Fréchet's sense for all $\xi\in K$.

(U-3) $U: \mathcal{M}_A\times K\to\boldsymbol{R}$, $DU: \mathcal{M}_A\times K\to \mathrm{Hom}(\mathcal{M}_A:\boldsymbol{R})$, $D^2U: \mathcal{M}_A\times K\to \mathrm{Hom}(\mathcal{M}_A\times\mathcal{M}_A:\boldsymbol{R})$ are continuous.

Let $f(\xi)=\inf\{\mathcal{E}(u,u)-U(u^2d\mu,\xi);\ u\in\mathcal{D}om(\mathcal{E}),\ \|u\|_{L^2(d\mu)}=1\}$, $\xi\in K$. Also, let $U^{(1)}(m,\xi)(x)=DU(m,\xi)(\delta_x)$, $(m,\xi,x)\in\mathcal{M}_A\times K\times M$. Then by the assumption (U-3), $U^{(1)}: \mathcal{M}_A \times K \times M \to \boldsymbol{R}$ is continuous.

The following is easy.

(2.1) PROPOSITION. *Suppose that* $u \in \mathcal{D}om(\mathcal{E})$, $\|u\|_{L^2(d\mu)}=1$, *and* $f(\xi)= \mathcal{E}(u, u)-U(u^2 d\mu, \xi)$. *Then* $u \in \mathcal{D}om(\mathcal{L})$ *and there is an* $\alpha \in \boldsymbol{R}$ *such that*

$$\mathcal{L}u+U^{(1)}(u^2 d\mu, \xi)(\cdot)u=\alpha u .$$

Moreover,

$$\mathcal{E}(v, v)-\int_M \{U^{(1)}(u^2 d\mu, \xi)(x)-\alpha\} v(x)^2 \mu(dx)$$
$$-2 \cdot D^2 U(u^2 d\mu, \xi)(uv d\mu, uv d\mu) \geqq 0$$

for any $v \in \mathcal{D}om(\mathcal{E})$ *with* $(u, v)_{L^2(d\mu)}=0$.

We also assume the following.

(U-4) For each $\xi \in K$, there is a unique $u(\xi) \in \mathcal{D}om(\mathcal{E})$ satisfying $\|u(\xi)\|_{L^2(d\mu)} =1$ and $\mathcal{E}(u(\xi), u(\xi))-U(u(\xi)^2 d\mu, \xi)=f(\xi)$.

Since $\mathcal{E}(u, u) \geqq \mathcal{E}(|u|, |u|)$, $u \in \mathcal{D}om(\mathcal{E})$, we see that $u(\xi)(x) \geqq 0$, μ-a. e. x for all $\xi \in K$. Also, we have the following.

(2.2) PROPOSITION. $f: K \to \boldsymbol{R}$ *and* $u: K \to \mathcal{D}om(\mathcal{E})$ *are continuous. Moreover, there is a continuous map* $\alpha: K \to \boldsymbol{R}$ *such that*

$$\mathcal{L}u(\xi)+U^{(1)}(u(\xi)^2 d\mu, \xi)(\cdot)u(\xi)=\alpha(\xi)u(\xi) .$$

PROOF. Assume that $\xi_n \in K$, $n=1,2, \cdots$, and $\xi_n \to \xi_\infty$ as $n \to \infty$. It is sufficient to prove $f(\xi_n) \to f(\xi_\infty)$ and $\mathcal{E}_1(u(\xi_\infty)-u(\xi_n),\ u(\xi_\infty)-u(\xi_n)) \to 0$ as $n \to \infty$. Since $C_0=\sup\{-U(m, \xi)\,;\, m \in \mathcal{P}(M), \xi \in K\}<\infty$, we see that $\mathcal{E}(u(\xi), u(\xi)) \leqq C_0$, $\xi \in K$. So taking a subsequence if necessary, we may assume that $u(\xi_n) \to u_\infty$ weakly in $\mathcal{D}om(\mathcal{E})$ as $n \to \infty$ for some $u_\infty \in \mathcal{D}om(\mathcal{E})$. Then we see that $u(\xi_n) \to u_\infty$ in $L^2(M, d\mu)$ and so $\|u_\infty\|_{L^2(d\mu)}=1$. Therefore we have

$$\begin{aligned} & \mathcal{E}(u(\xi_\infty), u(\xi_\infty))-U(u(\xi_\infty)^2 d\mu, \xi_\infty) \\ & =\lim_{n \to \infty}(\mathcal{E}(u(\xi_\infty), u(\xi_\infty))-U(u(\xi_\infty)^2 d\mu, \xi_n)) \\ & \geqq \lim_{n \to \infty}(\mathcal{E}(u(\xi_n), u(\xi_n))-U(u(\xi_n)^2 d\mu, \xi_n)) \\ & \geqq \mathcal{E}(u_\infty, u_\infty)-U(u_\infty^2 d\mu, \xi_\infty) . \end{aligned}$$

This implies that $u_\infty=u(\xi_\infty)$ and $\mathcal{E}(u(\xi_n), u(\xi_n)) \to \mathcal{E}(u_\infty, u_\infty)$ as $n \to \infty$. So we have $u(\xi_n) \to u(\xi_\infty)$ in $\mathcal{D}om(\mathcal{E})$ as $n \to \infty$. Also, we have $f(\xi_n) \to f(\xi_\infty)$, $n \to \infty$. The last assertion follows from Proposition (2.1) and the fact that

$(\mathcal{L}u, u)_{L^2(d\mu)} = -\mathcal{E}(u, u)$, $u \in \mathcal{D}om(\mathcal{E})$.

This completes the proof.

We impose the following assumption on the function U, furthermore.

(U-5) For any $\xi \in K$ and $v \in \mathcal{D}om(\mathcal{E})$ with $(u(\xi), v)_{L^2(d\mu)} = 0$ and $v \neq 0$,

$$\mathcal{E}(v, v) - \int (U^{(1)}(u(\xi)^2 d\mu, \xi)(x) - \alpha(\xi)) v(x)^2 \mu(dx)$$

$$-2 \cdot D^2 U(u(\xi)^2 d\mu, \xi)(v \cdot u(\xi) d\mu, v \cdot u(\xi) d\mu) > 0 .$$

Let $V: M \times K \to \boldsymbol{R}$ be a continuous function given by

$$V(x ; \xi) = U^{(1)}(u(\xi)^2 d\mu, \xi)(x) - \alpha(\xi), \quad x \in M, \xi \in K .$$

Then we have $\mathcal{L}u(\xi) + V(\cdot ; \xi) u(\xi) = 0$ and $\sup\{|V(x ; \xi)| ; x \in M, \xi \in K\}$ $\leqq \left\{\sum_{n=1}^{\infty} a_n \cdot \|\varphi_n\|^2_{C_b(M)} \cdot \|\varphi_n\|^{-2}_{L^2(d\mu)}\right\}^{1/2} \sup\{\|U^{(1)}(u(\xi)^2 d\mu, \xi)\|_A ; \xi \in K\} < \infty$. Let $\lambda(\xi)$ be the second minimum eigenvalue of $-\mathcal{L} - V(\cdot ; \xi)$, $\xi \in K$. Then by Kato [4, Chapter 5, §4] we see that $\lambda: K \to \boldsymbol{R}$ is continuous. Since P_t is ergodic, we see that $\lambda(\xi) > 0$ for any $\xi \in K$. So we see that

$$\lambda_0 \underset{\text{def}}{=} \inf\{\lambda(\xi) ; \xi \in K\} > 0 .$$

We have the following also.

(2.3) PROPOSITION. *Let*

$$F(T, x, y ; \xi) = P(T, x, y) \cdot E^P\left[\exp\left(\int_0^T V(w(t) ; \xi) dt\right) \middle| w(0) = x, \quad w(T) = y\right],$$

for $T > 0$, $x, y \in M$, $\xi \in K$. *Then for any* $T > 0$, $F(T, \cdot, \cdot, \cdot): M \times M \times K \to (0, \infty)$ *is continuous. Also, we have a good version of* $u(\xi)(x)$ *such that* $u(\cdot)(\cdot): K \times M \to (0, \infty)$ *is continuous.*

PROOF. Let $c = \sup\{|V(x, \xi)| ; x \in M, \xi \in K\}$ and let $F_\varepsilon: (0, \infty) \times M \times M \times K \to (0, \infty)$ be given by

$$F_\varepsilon(T, x, y, \xi) = (P(\varepsilon T, x, \cdot), \exp((1-2\varepsilon) T(\mathcal{L} + V(\cdot ; \xi))) P(\varepsilon T, y, \cdot))_{L^2(d\mu)},$$

$$\varepsilon \in (0, 1/2) .$$

By virtue of the assumption (A-3), we see that $F_\varepsilon(T, \cdot, \cdot, \cdot): M \times M \to K \to (0, \infty)$ is continuous. Also, we have

$$F_\varepsilon(T, x, y, \xi) = P(T, x, y) \cdot E^P\left[\exp\left(\int_{\varepsilon T}^{(1-\varepsilon)T} V(w(t) ; \xi) dt\right) \middle| w(0) = x, \quad w(T) = y\right].$$

So we see that $|F(T,x,y;\xi)-F_\varepsilon(T,x,y,\xi)|\leqq P(T,x,y)e^{cT}(e^{2\varepsilon cT}-1)$. Therefore $F_\varepsilon(T,x,y,\xi)\to F(T,x,y,\xi)$ as $\varepsilon\downarrow 0$ uniformly on compacts with respect to $(x,y,\xi)\in M\times M\times K$. This proves the first assertion. Since $u(\xi)(x)=\int_M F(1,x,y,\xi)u(\xi)(y)\,\mu(dy)$ μ-a. e. x, we have the latter assertion.

This completes the proof.

Let ν_ξ, $\xi\in K$ be a probability measure on M given by $d\nu_\xi=u(\xi)^2d\mu$, and let $\{Q_t^{(\xi)}\}_{t\geqq 0}$ be the Markov semigroup in $L^2(M;d\nu_\xi)$ given by

$$Q_t^{(\xi)}h(x)=(u(\xi)(x))^{-1}(\exp(t(\mathcal{L}+V(\cdot\,;\xi)))(u(\xi)h))(x),\quad h\in L^2(M;d\nu_\xi).$$

Also let Π_ξ be the projection operator in $L^2(M;d\nu_\xi)$ given by $(\Pi_\xi h)(x)=h-\int_M h\,d\nu_\xi$, $h\in L^2(M;d\nu_\xi)$. Then we have

$$\|\Pi_\xi Q_t^{(\xi)}\|_{L^2(d\nu_\xi)\to L^2(d\nu_\xi)}\leqq\exp(-\lambda_0 t),\quad t\geqq 0,\ \xi\in K. \tag{2.4}$$

Let $\tilde{Q}_t^{(\xi)}=\Pi_\xi Q_t^{(\xi)}$ and $\tilde{G}_\xi=2\cdot\int_0^\infty\tilde{Q}_t^{(\xi)}dt$. Let

$$Q(t,x,y;\xi)=u(\xi)(x)^{-1}F(t,x,y,\xi)u(\xi)(y)^{-1},\quad t>0,\ x,y\in M.$$

Then $Q(t,\cdot,\cdot\,;\cdot):M\times M\times K\to(0,\infty)$ is continuous, and

$$Q_t^{(\xi)}\phi(x)=\int_M Q(t,x,y;\xi)\phi(y)\nu_\xi(dy).$$

Moreover, if $P_2:L^2(d\mu)\to L^2(d\mu)$ belongs to the trace class, $Q_2^{(\xi)}:L^2(d\nu_\xi)\to L^2(d\nu_\xi)$ also belongs to the trace class and trace $Q_2^{(\xi)}\leqq\exp(2\cdot\sup_{x\in M}V(x;\xi))\cdot$(trace P_2). Let $\{Q_x^{(\xi)};x\in M\}$ be the ν_ξ-symmetric Markov process associated with the semi-group $\{Q_t^{(\xi)}\}_{t\geqq 0}$.

Then we see that for any $x,y\in M$ and $T>0$

$$\begin{aligned} &u(\xi)(x)u(\xi)(y)Q(T,x,y;\xi)\cdot Q^{(\xi)}(dw|\,w(0)=x,\ w(T)=y)|_{\mathcal{F}_0^T}\\ &=P(T,x,y)\cdot\exp\left(\int_0^T V(w(t);\xi)dt\right)\cdot P(dw|\,w(0)=x,\ w(T)=y)|_{\mathcal{F}_0^T} \end{aligned} \tag{2.5}$$

where $\mathcal{F}_S^T=\sigma\{w(t);t\in[S,T]\}$, $0\leqq S\leqq T$.

(2.6) PROPOSITION. *For any $\phi\in C_b(M)$, $(\phi,\tilde{G}_\xi\phi)_{L^2(d\nu_\xi)}$ is continuous in $\xi\in K$.*

PROOF. Note that

$$(\psi, \tilde{Q}_t^{(\xi)}\psi)_{L^2(\nu_\xi)}=\int_{M\times M}\psi(x)u(\xi)(x)F(t, x, y, \xi)u(\xi)(y)\psi(y)\mu(dx)\mu(dy)$$
$$-\int_M \psi(x)^2 u(\xi)(x)^2\mu(dx).$$

So we see that $(\psi, \tilde{Q}_t^{(\xi)}\psi)_{L^2(d\nu_\xi)}$ is continuous in ξ. Since

$$\left|(\psi, \tilde{G}_\xi\psi)_{L^2(d\nu_\xi)}-2\int_0^T(\psi, \tilde{Q}_t^{(\xi)}\psi)_{L^2(d\nu_\xi)}dt\right|\leqq 2\lambda_0^{-1}\|\psi\|_{C_b(M)}^2\cdot e^{-\lambda_0 T},$$

we have our assertion.

(2.7) PROPOSITION. *There is a* $c_0>0$ *such that*

$$D^2U(\nu_\xi\,;\,\xi)((\tilde{G}_\xi h)d\nu_\xi, (\tilde{G}_\xi h)d\nu_\xi)\leqq(1-c_0)\cdot(\tilde{G}_\xi h, h)_{L^2(d\nu_\xi)}$$

for all $\xi\in K$ *and* $h\in L^2(d\nu_\xi)$.

PROOF. By the assumption (U-5) we have

$$\mathcal{E}(v, v)-\int V(x\,;\,\xi)v(x)^2\mu(dx)>2\cdot D^2U(\nu_\xi, \xi)(u(\xi)vd\mu, u(\xi)vd\mu)$$

for any $\xi\in K$ and $v\in\mathcal{D}om(\mathcal{E})$ with $(v, u(\xi))_{L^2(d\mu)}=0$ and $v\neq 0$. Let $E(v, \xi)=\mathcal{E}(v, v)-\int_M V(x\,;\,\xi)v(x)^2\mu(dx)+(v, u(\xi))_{L^2(d\mu)}^2$ for $v\in\mathcal{D}om(\mathcal{E})$, and let $P_\xi v=v-(u(\xi), v)_{L^2(d\mu)}\cdot u(\xi)$, $v\in L^2(M, d\mu)$. Also, let

$$c_1=\sup\{2\cdot D^2U(\nu_\xi, \xi)(u(\xi)(P_\xi v)d\mu, u(\xi)(P_\xi v)d\mu)\,;\,\xi\in K, v\in\mathcal{D}om(\mathcal{E}), E(v, \xi)\leqq 1\}.$$

Then we see that $c_1\leqq 1$. Since $\{(\xi, v)\in K\times\mathcal{D}om(\mathcal{E})\,;\,E(v, \xi)\leqq 1\}$ is a compact subset of $K\times L^2(M\,;\,d\mu)$, we see that $c_1<1$. From the definition, we see that $E(u(\xi)\tilde{G}_\xi h, \xi)=2\cdot\|\tilde{G}_\xi^{1/2}h\|_{L^2(d\nu_\xi)}^2$ and

$$D^2U(\nu_\xi, \xi)(u(\xi)(P_\xi(u(\xi)\tilde{G}_\xi h))d\mu, u(\xi)(P_\xi(u(\xi)\tilde{G}_\xi h))d\mu)$$
$$=D^2U(\nu_\xi, \xi)((\tilde{G}_\xi h)d\nu_\xi, (\tilde{G}_\xi h)d\nu_\xi), \qquad h\in L^2(M\,;\,d\nu_\xi).$$

This implies our assertion.

By virtue of Lemma (1.1), we have the following.

(2.8) PROPOSITION. *Let* $x, y\in M$ *and* $R>0$. *Then there are* $\delta>0$ *and* $C>0$ *such that*

$$\left|Q(T,x,y\,;\,\xi)\cdot E^{Q^{(\xi)}}\left[\exp\left(zT\cdot\int_M\phi(dl_T(w)-d\nu_\xi)\right)\middle|\,w(0)=x,\ w(T)=y\right]\right|$$

$$\leqq C\exp\left(\frac{|z|^2T}{2}\cdot(\phi,\tilde{G}_\xi\phi)_{L^2(d\nu_\xi)}\right)\{|z|+\exp(C\cdot|z|^3T)+\exp(-\lambda_0T)\}$$

and

$$\left|Q(T,x,y\,;\,\xi)\cdot E^{Q^{(\xi)}}\left[\exp\left(zT\cdot\int_M\phi(dl_T(w)-d\nu_\xi)\right)\middle|\,w(0)=x,\ w(T)=y\right]\right.$$

$$\left.-\exp\left(\left(\frac{z^2T}{2}\right)\cdot(\phi,\tilde{G}_\xi\phi)_{L^2(d\nu_\xi)}\right)\right|$$

$$\leqq C\exp\left(\frac{|z|^2T}{2}\cdot(\phi,\tilde{G}_\xi\phi)_{L^2(d\nu_\xi)}\right)$$

$$\times\{|z|+\exp(C\cdot|z|^3T)-1+|z|^2\exp(C\cdot|z|^2T)+\exp(-\lambda_0T)\}$$

for any $T>2$, $\phi\in C(M)$, $\xi\in K$ *and* $z\in\boldsymbol{C}$ *with* $\|\phi\|_{C_b(M)}\leqq R$ *and* $|z|\leqq\delta$.

Note that $\int_M u(\xi)d\mu\leqq 1$, and

$$\int_M u(\xi)(x)\cdot Q(2,x,x\,;\,\xi)^{1/2}\mu(dx)\leqq\int_M Q(2,x,x\,;\,\xi)\nu_\xi(dx)=\mathrm{trace}\,Q_2^{(\xi)}.$$

So we have the following also from Lemma (1.1).

(2.9) PROPOSITION. *Suppose that* $P_2: L^2(d\mu)\to L^2(d\mu)$ *is a nuclear operator, and let* $x\in M$ *and* $R>0$. *Then there are* $\delta>0$ *and* $C>0$ *satisfying the following.*

(1)
$$\left|\int_M\mu(dy)\left(u(\xi)(y)Q(T,x,y\,;\,\xi)\right.\right.$$

$$\left.\left.\times E^{Q^{(\xi)}}\left[\exp\left(zT\cdot\int_M\phi(dl_T(w)-d\nu_\xi)\right)\middle|\,w(0)=x,\ w(T)=y\right]\right)\right|$$

$$\leqq C\exp\left(\frac{|z|^2T}{2}\cdot(\phi,\tilde{G}_\xi\phi)_{L^2(d\nu_\xi)}\right)\{|z|+\exp(C\cdot|z|^3T)+\exp(-\lambda_0T)\},$$

and

$$\left|\int_M\mu(dy)\left(u(\xi)(y)Q(T,x,y\,;\,\xi)\right.\right.$$

$$\left.\times E^{Q^{(\xi)}}\left[\exp\left(zT\cdot\int_M\phi(dl_T(w)-d\nu_\xi)\right)\middle|\,w(0)=x,\ w(T)=y\right]\right)$$

$$-\left(\int_M u(\xi)d\mu\right)\cdot\exp\left(\left(\frac{z^2T}{2}\right)\cdot(\phi,\tilde{G}_\xi\phi)_{L^2(d\nu_\xi)}\right)\Big|$$

$$\leqq C\exp\left(\frac{|z|^2T}{2}\cdot(\phi,\tilde{G}_\xi\phi)_{L^2(d\nu_\xi)}\right)$$

$$\times\{|z|+\exp(C\cdot|z|^3T)-1+|z|^2\exp(C|z|^2T)+\exp(-\lambda_0T)\}$$

for any $T>2$, $\phi\in C(M)$, $\xi\in K$ *and* $z\in\boldsymbol{C}$ *with* $\|\phi\|_{C_b(M)}\leqq R$ *and* $|z|\leqq\delta$.

(2)
$$\Big|\int_{M\times M}\mu(dy_0)\mu(dy_1)\Big(u(\xi)(y_0)u(\xi)(y_1)Q(T,y_0,y_1;\xi)$$

$$\times E^{Q(\xi)}\left[\exp\left(zT\cdot\int_M\phi(dl_T(w)-d\nu_\xi)\right)\Big|\, w(0)=y_0,\ w(T)=y_1\right]\Big)\Big|$$

$$\leqq C\exp\left(\frac{|z|^2T}{2}\cdot(\phi,\tilde{G}_\xi\phi)_{L^2(d\nu_\xi)}\right)\{|z|+\exp(C\cdot|z|^3T)+\exp(-\lambda_0T)\}$$

and

$$\Big|\int_{M\times M}\mu(dy_0)\mu(dy_1)\Big(u(\xi)(y_0)u(\xi)(y_1)Q(T,y_0,y_1;\xi)$$

$$\times E^{Q(\xi)}\left[\exp\left(zT\cdot\int_M\phi(dl_T(w)-d\nu_\xi)\right)\Big|\, w(0)=y_0,\ w(T)=y_1\right]\Big)$$

$$-\left(\int_M u(\xi)d\mu\right)^2\cdot\exp\left(\left(\frac{z^2T}{2}\right)\cdot(\phi,\tilde{G}_\xi\phi)_{L^2(d\nu_\xi)}\right)\Big|$$

$$\leqq C\exp\left(\frac{|z|^2T}{2}\cdot(\phi,\tilde{G}_\xi\phi)_{L^2(d\nu_\xi)}\right)$$

$$\times\{|z|+\exp(C\cdot|z|^3T)-1+|z|^2\exp(C|z|^2T)+\exp(-\lambda_0T)\}$$

for any $T>2$, $\phi\in C(M)$, $\xi\in K$ *and* $z\in\boldsymbol{C}$ *with* $\|\phi\|_{C_b(M)}\leqq R$ *and* $|z|\leqq\delta$.

Also, we have the following.

(2.10) LEMMA. *Let* $x,y\in M$. *For any* $R>0$ *and* $\varepsilon>0$, *there is a* $\delta>0$ *such that*

$$\sup\Big\{Q(T,x,y;\xi)\cdot E^{Q(\xi)}\Big[\exp\Big(\frac{T}{2}\cdot\Big(\int_M\phi(dl_T-d\nu_\xi)\Big)^2\Big),$$

$$\Big|\int_M\phi(dl_T-d\nu_\xi)\Big|<\delta\,\Big|\, w(0)=x,\ w(T)=y\Big];$$

$$T>2,\ \xi\in K,\ \phi\in C_b(M),\ \|\phi\|_{C_b(M)}\leqq R,\ (\phi,\tilde{G}_\xi\phi)_{L^2(d\nu_\xi)}\leqq 1-\varepsilon\Big\}$$

$$<\infty.$$

PROOF. Let X be a random variable. Then we have

$$
\begin{aligned}
&E\left[\exp\left(\frac{T}{2}|X|^2\right), |X|<\delta\right] \\
&=\int_{-\infty}^{\infty} ds\,(2\pi/T)^{-1/2}\exp\left(-\frac{T}{2}\cdot s^2\right)\cdot E[\exp(T\cdot s\cdot X),\ |X|<\delta] \\
&\leqq 2\cdot\int_{2\delta}^{\infty} ds\,(2\pi/T)^{-1/2}\exp\left(-\frac{T}{4}\cdot s^2\right) \\
&\quad+\int_{-2\delta}^{2\delta} ds\,(2\pi/T)^{-1/2}\exp\left(-\frac{T}{2}s^2\right)\cdot E[\exp(T\cdot s\cdot X)].
\end{aligned}
$$

Also, we see that for $T>2$

$$Q(T,x,y\,;\,\xi)\leqq Q(2,x,x\,;\,\xi)^{1/2}Q(2,y,y\,;\,\xi)^{1/2},\qquad \xi\in K.$$

Thus our assertion is an easy consequence of Proposition (2.8).

The following is also an easy consequence of Proposition (2.8).

(2.11) LEMMA. *Let $\psi_1,\cdots,\psi_n\in C_b(M)$. Then for any $R>0$ and $x,y\in M$,*

$$
\begin{aligned}
\sup\Bigg\{\Bigg|&Q(T,x,y\,;\,\xi)\cdot E^{Q^{(\xi)}}\left[\exp\left(\sqrt{-1}\cdot T^{1/2}\cdot\sum_{j=1}^{n}s_j\cdot\int_M\psi_j(dl_T-d\nu_\xi)\right)\right| \\
&\qquad\qquad w(0)=x,\ w(T)=y\Bigg] \\
&-\exp\left(-\frac{1}{2}\cdot\sum_{j,k=1}^{n}s_js_k(\psi_j,\tilde{G}_\xi\psi_k)_{L^2(d\nu_\xi)}\right)\Bigg|\,;\,\xi\in K,\ s_j\in\boldsymbol{R},\ |s_j|\leqq R,\ j=1,\cdots,n\Bigg\}
\end{aligned}
$$

$\longrightarrow 0\qquad$ *as* $T\to\infty$.

Moreover, we have the following.

(2.12) LEMMA. *Let $\psi_1,\cdots,\psi_n\in C(M)$ and $\Phi: D([0,\infty)\,;\,M)\times K\to\boldsymbol{R}$ is a bounded measurable function such that there is an $T_0>0$ for which $\Phi(\cdot,\xi): D([0,\infty)\,;\,M)\to\boldsymbol{R}$ is $\mathscr{F}_0^{T_0}$-measurable. Then for any $x,y\in M$ and $R>0$,*

$$
\begin{aligned}
\sup\Bigg\{\Bigg|&Q(T,x,y\,;\,\xi)\cdot E^{Q^{(\xi)}}\left[\Phi(\cdot,\xi)\cdot\exp\left(\sqrt{-1}\cdot T^{1/2}\cdot\sum_{j=1}^{n}s_j\cdot\int_M\psi_j(dl_T-d\nu_\xi)\right)\right| \\
&\qquad\qquad w(0)=x,\ w(T)=y\Bigg]
\end{aligned}
$$

$$-E^{Q^{(\xi)}}[\Phi(\cdot,\xi)\mid w(0)=x]\cdot\exp\Big(-\frac{1}{2}\sum_{j,k=1}^{n}s_js_k(\phi_j,\tilde{G}_\xi\phi_k)_{L^2(d\nu_\xi)}\Big)\Big|;$$

$$\xi\in K,\ s_j\in\boldsymbol{R},\ |s_j|\leqq R,\ j=1,\cdots,n\Big\}$$

$$\longrightarrow 0 \qquad as\ T\to\infty.$$

PROOF. We may assume that $T_0>1$ and $|f(w,\xi)|\leqq 1$ for all $(w,\xi)\in D([0,\infty);M)\times K$. Let $t>0$, $T>T_0+t+1$, and $g: D([0,\infty);M)\times K\to\boldsymbol{C}$ be a measurable function such that $|g(w,\xi)|\leqq 1$ and $g(\cdot,\xi): D([0,\infty);M)\to\boldsymbol{C}$ is $\mathcal{F}_{T_0+t}^{T}$ measurable for all $\xi\in K$. Let

$$v_0(\xi)(z)=Q(T_0,x,z;\xi)\cdot E^{Q^{(\xi)}}[f\mid w(0)=x,\ w(T_0)=z],$$

and

$$v_1(\xi)(z)=Q(T-(T_0+t),z,y;\xi)\cdot E^{Q^{(\xi)}}[g\mid w(T_0+t)=z,\ w(T)=y],\quad z\in M.$$

Then we have

$$\Big|(v_0(\xi),Q_t^{(\xi)}v_1(\xi))_{L^2(d\nu_\xi)}-\Big(\int_M v_0(\xi)d\nu_u\Big)\Big(\int_M v_1(\xi)d\nu_\xi\Big)\Big|$$

$$\leqq e^{-\lambda_0 t}\|v_0(\xi)\|_{L^2(d\nu_\xi)}\|v_1(\xi)\|_{L^2(d\nu_\xi)}$$

$$\leqq e^{-\lambda_0 t}\cdot Q(2,x,x;\xi)^{1/2}\cdot Q(2,y,y;\xi)^{1/2}.$$

Note that

$$(v_0(\xi),Q_t^{(\xi)}v_1(\xi))_{L^2(d\nu_\xi)}=Q(T,x,y;\xi)\cdot E^{Q^{(\xi)}}[fg\mid w(0)=x,\ w(T)=y],$$

and

$$\Big(\int_M v_0(\xi)d\nu_u\Big)\Big(\int_M v_1(\xi)d\nu_\xi\Big)=E^{Q^{(\xi)}}[f\mid w(0)=x]\cdot E^{Q^{(\xi)}}[g\mid w(T)=y].$$

Thus we have

$$|Q(T,x,y;\xi)\cdot E^{Q^{(\xi)}}[fg\mid w(0)=x,\ w(T)=y]$$

$$-E^{Q^{(\xi)}}[f\mid w(0)=x]\cdot E^{Q^{(\xi)}}[g\mid w(T)=y]|$$

$$\leqq\exp(-\lambda_0 t)\cdot Q(2,x,x;\xi)^{1/2}Q(2,y,y;\xi)^{1/2}.$$

Also, we have that

$$\Big|\int_M\phi(dl_T(w)-d\nu_\xi)-\int_M\phi(dl_T(w(\cdot+t))-d\nu_\xi)\Big|\leqq\frac{2t}{T}\cdot\|\phi\|_{C_b(M)}.$$

These implies our assertion.

This completes the proof.

Let $P_n : \mathcal{M}_A \to \mathcal{M}_A$, $n=1,2,\cdots$, be given by

$$d(P_n m)=\sum_{k=1}^{n}\|\varphi_k\|_{L^2(d\mu)}^{-2}\left(\int_M \varphi_k dm\right)\cdot\varphi_k d\mu, \quad m\in\mathcal{M}.$$

Then it is obvious that

$$\|m-P_n m\|_A^2=\sum_{k=n+1}^{\infty}a_k\cdot\|\varphi_k\|_{L^2(d\mu)}^{-2}\cdot\left(\int_M \varphi_k dm\right)^2, \quad m\in\mathcal{M}.$$

Now let $\tilde{U}: \mathcal{M}_A\times K\to \boldsymbol{R}$ be given by

$$\tilde{U}(m,\xi)=U(m,\xi)-U(\nu_\xi,\xi)-\int_M V(\cdot\,;\xi)(dm-d\nu_\xi), \quad (m,\xi)\in\mathcal{M}\times K.$$

Then we have the following.

(2.13) LEMMA. *For any $p\in(1,\infty)$ and $x,y\in M$, there are $\delta_0>0$ and $n_0\geqq 1$ such that*

$$E^{Q(\xi)}\Big[\exp\Big(pT\cdot\Big|\tilde{U}(l_T,\xi)-\frac{1}{2}D^2U(\nu_\xi,\xi)(P_n(l_T-\nu_\xi),P_n(l_T-\nu_\xi))\Big|\Big),$$
$$\sup_k\Big|\int_M\varphi_k d(l_T-\nu_\xi)\Big|<\delta_0\Big|\, w(0)=x,\ w(T)=y\Big]\leqq 2$$

for all $T>2$, $\xi\in K$ and $n\geqq n_0$.

PROOF. It is easy to see that there is a $C_1>0$ such that

$$\Big|\tilde{U}(m,\xi)-\frac{1}{2}D^2U(\nu_\xi,\xi)(m-\nu_\xi,m-\nu_\xi)\Big|\leqq C_1\|m-\nu_\xi\|_A^3$$

and

$$|D^2U(\nu_\xi,\xi)(m-\nu_\xi,m-\nu_\xi)-D^2U(\nu_\xi,\xi)(P_n(m-\nu_\xi),P_n(m-\nu_\xi))|$$
$$\leqq C_1\|(m-\nu_\xi)-P_n(m-\nu_\xi)\|_A\cdot\|m-\nu_\xi\|_A$$

for any $(m,\xi)\in\mathcal{M}\times K$.

Let $b_n=\left\{\sum_{k=n+1}^{\infty}a_k\cdot\|\varphi_k\|_{L^2(d\mu)}^{-2}\right\}^{1/2}$, $n=1,2,\cdots$. Then we see that

$$\Big|\tilde{U}(m,\xi)-\frac{1}{2}D^2U(\nu_\xi,\xi)(P_n(m-\nu_\xi),P_n(m-\nu_\xi))\Big|$$
$$\leqq C_1\|m-\nu_\xi\|_A^3+C_1b_n\|m-\nu_\xi\|_A^2+C_1b_n^{-1}\|(m-\nu_\xi)-P_n(m-\nu_\xi)\|_A^2$$

$$\leqq (C_0\delta + b_n)C_1 \cdot \sum_{k=1}^{\infty} a_k \cdot \|\varphi_k\|_{L^2(d\mu)}^{-2}\left(\int_M \varphi_k d(m-\nu_\xi)\right)^2$$

$$+ C_1 b_n^{-1} \cdot \sum_{k=n+1}^{\infty} a_k \cdot \|\varphi_k\|_{L^2(d\mu)}^{-2}\left(\int_M \varphi_k d(m-\nu_k)\right)^2$$

for any $\xi \in K$ and $m \in \mathcal{M}$ with $\sup_k \left|\int_M \varphi_k d(m-\nu_\xi)\right| < \delta$. Here

$$C_0 = \left(\sum_{k=1}^{\infty} a_k \cdot \|\varphi_k\|_{L^2(d\mu)}^{-2} \cdot \|\varphi_k\|_{C_b(M)}^2\right)^{1/2}.$$

Let $d_n = (C_0\delta + b_n)C_1 \cdot \sum_{k=1}^{\infty} a_k \cdot \|\varphi_k\|_{L^2(d\mu)}^{-2} + C_1 b_n^{-1} \cdot \sum_{k=n+1}^{\infty} a_k \cdot \|\varphi_k\|_{L^2(d\mu)}^{-2}$. Then we have

$$Q(T, x, y\,;\, \xi) \cdot E^{Q(\xi)}\Big[\exp\Big(pT \cdot \Big|\tilde{U}(l_T, \xi)$$

$$- \frac{1}{2} D^2 U(\nu_\xi, \xi)(P_n(l_T - \nu_\xi), P_n(l_T - \nu_\xi))\Big|\Big),$$

$$\sup_k \left|\int_M \varphi_k d(l_T - \nu_\xi)\right| < \delta \,\Big|\, w(0) = x,\ w(T) = y\Big]$$

$$\leqq d_n^{-1}(C_0\delta + b_n)C_1 \cdot \sum_{k=1}^{\infty} \Big\{a_k \cdot \|\varphi_k\|_{L^2(d\mu)}^{-2}$$

$$\times Q(T, x, y\,;\, \xi) \cdot E\Big[\exp\Big(pd_n T \cdot \Big(\int_M \varphi_k d(l_T - \nu_\xi)\Big)^2\Big),$$

$$\left|\int_M \varphi_k d(l_T - \nu_\xi)\right| < \delta \,\Big|\, w(0) = x,\ w(T) = y\Big]\Big\}$$

$$+ d_n^{-1} C_1 b_n^{-1} \cdot \sum_{k=n+1}^{\infty} \Big\{a_k \cdot \|\varphi_k\|_{L^2(d\mu)}^{-2}$$

$$\times Q(T, x, y\,;\, \xi) \cdot E\Big[\exp\Big(pd_n T \cdot \Big(\int_M \varphi_k d(l_T - \nu_\xi)\Big)^2\Big),$$

$$\left|\int_M \varphi_k d(l_T - \nu_\xi)\right| < \delta \,\Big|\, w(0) = x,\ w(T) = y\Big]\Big\}$$

$$\leqq \sup_k Q(T, x, y\,;\, \xi) \cdot E\Big[\exp\Big(pd_n T \cdot \Big(\int_M \varphi_k d(l_T - \nu_\xi)\Big)^2\Big),$$

$$\left|\int_M \varphi_k d(l_T - \nu_\xi)\right| < \delta \,\Big|\, w(0) = x,\ w(T) = y\Big].$$

Since $d_n \to \delta C_1 b_0^2$ as $n \to \infty$, we have our assertion from Lemma (2.10).

By Proposition (2.7) and the fact that $\|(\tilde{G}_\xi h)\nu_\xi - P_n((\tilde{G}_\xi h)\nu_\xi)\|_A^2 \leqq (\sum_{k=n+1}^{\infty} a_k \cdot \|\varphi_k\|_{L^2(d\mu)}^{-2} \|\varphi_k\|_{C_b(M)}^2) \|\tilde{G}_\xi h\|_{L^2(d\nu_\xi)}^2$, we see that there is an $n_1 \geqq 1$ such that

$$(2.14) \qquad D^2 U(\nu_\xi\,;\,\xi)(P_n((\tilde{G}_\xi h)d\nu_\xi), P_n((\tilde{G}_\xi h)d\nu_\xi)) \leqq (1-c_0/2)\cdot(\tilde{G}_\xi h, h)_{L^2(d\nu_\xi)}$$

for all $n \geqq n_1$, $\xi \in K$ and $h \in L^2(d\nu_\xi)$.

Then we have the following.

(2.15) LEMMA. *For each $n \geqq n_1$ and $x, y \in M$, there is a $\delta_1 > 0$ such that*

$$\sup\Big\{Q(T, x, y\,;\,\xi)$$

$$\times E^{Q(\xi)}\Big[\exp\Big((1-c_0/5)^{-1}\cdot\frac{T}{2}D^2U(\nu_\xi\,;\,\xi)(P_n(l_T-\nu_\xi), P_n(l_T-\nu_\xi))\Big),$$

$$\sup_k\Big|\int_M \varphi_k d(l_T-\nu_\xi)\Big| < \delta_1 \Big| w(0)=x,\ w(T)=y\Big];\ \xi\in K,\ T>2\Big\}$$

$$<\infty.$$

PROOF. Let $a_{ij}(y) = \|\varphi_i\|_{L^2(d\mu)}^{-2}\|\varphi_j\|_{L^2(d\mu)}^{-2}\cdot D^2U(\nu_\xi, \xi)(\varphi_i d\mu, \varphi_j d\mu)$, $i, j = 1, 2, \cdots$. Then $a_{ij}: K \to \boldsymbol{R}$ is continuous. Also, we see that

$$D^2U(\nu_\xi, \xi)(P_n m, P_n m) = \sum_{i,j=1}^{n} a_{ij}(\xi)\Big(\int_M \varphi_i dm\Big)\Big(\int_M \varphi_j dm\Big), \quad m \in \mathcal{M}.$$

Let $\mathcal{M}_0 = \{m \in \mathcal{M}\,;\ m(M) = 0\}$ and $b_n = \sup\{(\tilde{G}_\xi \varphi_k, \varphi_k)_{L^2(d\nu_\xi)}^{1/2}\,;\ \xi \in K,\ k = 1, \cdots, n\}$. Then it is obvious that $b < \infty$. Let $W = \sum_{i=1}^{n} \boldsymbol{R}\varphi_i$. Then by (2.14), we see that

$$D^2U(\nu_\xi, \xi)(P_n m, P_n m) \leqq (1-c_0/2)\cdot\sup\Big\{\Big(\int_M \varphi dm\Big)^2;\ \varphi \in W,\ (\tilde{G}_\xi\varphi, \varphi) \leqq 1\Big\}$$

for all $\xi \in K$ and $m \in \mathcal{M}_0$.

For any $\xi \in K$, it is easy to see (e.g. see the argument in [$K-T$]) that there is a finite subset $\{\tilde{\varphi}_k^{(\xi)}\}_{k=1}^{\tilde{N}(\xi)}$ of W such that

$$D^2U(\nu_\xi, \xi)(P_n m, P_n m) \leqq (1-c_0/3)\cdot\max\Big\{\Big(\int_M \tilde{\varphi}_k^{(\xi)} dm\Big)^2;\ k = 1, \cdots, \tilde{N}(\xi)\Big\}$$

for any $m \in \mathcal{M}_0$, and

$$(\tilde{G}_\xi \tilde{\varphi}_k^{(\xi)}, \tilde{\varphi}_k^{(\xi)})_{L^2(d\nu_\xi)} \leqq 1, \qquad k = 1, \cdots, \tilde{N}(\xi).$$

So noting that $(\tilde{G}_{\xi}(b_n^{-1}\varphi_k), (b_n^{-1}\varphi_k))_{L^2(d\nu_\xi)} \leqq 1$, $\xi \in K$, $k=1, \cdots, n$, and that

$$|D^2U(\nu_\xi, \xi)(P_n m, P_n m) - D^2U(\nu_\eta, \eta)(P_n m, P_n m)|$$

$$\leqq b_n^2\Big(\sum_{i,j=1}^{n} |a_{ij}(\xi) - a_{ij}(\eta)|\Big) \cdot \max\Big\{\Big(\int_M (b_n^{-1}\varphi_k)dm\Big)^2 ; k=1,2,\cdots,n\Big\}, \quad \xi, \eta \in K,$$

$m \in \mathcal{M}$, we see that for each $\xi \in K$, there is a neighbourhood O_ξ of ξ in K and $\{\psi_k^{(\xi)}\}_{k=1}^{N(\xi)} \subset W$ such that

$$D^2U(\nu_\eta, \eta)(P_n m, P_n m) \leqq (1-c_0/4) \cdot \max\Big\{\Big(\int_M \psi_k^{(\xi)} dm\Big)^2 ; \ k=1, \cdots, N(\xi)\Big\}$$

and

$$(\tilde{G}_\xi, \psi_k^{(\xi)}, \psi_k^{(\xi)})_{L^2(d\nu_\xi)} \leqq 1, \qquad k=1, \cdots, N(\xi)$$

for any $\eta \in O_\xi$ and $m \in \mathcal{M}_0$.

Therefore there is a $\delta_1(\xi) > 0$ such that

$$\sup\Big\{Q(T, x, y; \eta) \times E^{Q(\eta)}\Big[\exp\Big((1-c_0/5)^{-1} \cdot \frac{T}{2} D^2U(\nu_\eta ; \eta)(P_n(l_T - \nu_\eta), P_n(l_T - \nu_\eta))\Big), \sup_k \Big|\int_M \varphi_k d(l_T - \nu_\eta)\Big| < \delta_1(\xi) \Big| w(0)=x, \ w(T)=y\Big] ; \ \eta \in O_\xi, \ T>2\Big\} < \infty .$$

Since K is compact, we have our assertion.

As an easy consequence of Lemmas (2.13) and (2.15), we have the following.

(2.16) LEMMA. *For any $x, y \in M$, there are $\delta > 0$ and $\varepsilon > 0$ such that*

$$\sup\Big\{Q(T, x, y ; \xi) \cdot E^{Q(\xi)}\Big[\exp((1+\varepsilon)T \cdot \tilde{U}(l_T ; \xi)), \sup_k \Big|\int_M \varphi_k d(l_T - \nu_\xi)\Big| < \delta \Big| w(0)=x, \ w(T)=y\Big] ; \ \xi \in K, \ T>2\Big\} < \infty .$$

Similarly starting from Proposition (2.9), we have the following.

(2.17) LEMMA. *Suppose that $P_2 : L^2(d\mu) \to L^2(d\mu)$ is a nuclear operator. Then for any $x \in M$, there are $\delta > 0$ and $\varepsilon > 0$ such that*

$$\sup\Big\{\int_M \mu(dy)\Big(u(\xi)(y)Q(T,x,y)\cdot E^{Q^{(\xi)}}\Big[\exp((1+\varepsilon)T\cdot\tilde{U}(l_T;\xi)),$$

$$\sup_k\Big|\int_M \varphi_k d(l_T-\nu_\xi)\Big|<\delta\,\Big|\,w(0)=x,\ w(T)=y\Big]\Big);\ \xi\in K,\ T>2\Big\}$$

$$<\infty,$$

and

$$\sup\Big\{\int_{M\times M}\mu(dy_1)\mu(dy_2)\Big(u(\xi)(y_1)u(\xi)(y_2)Q(T,y_1,y_2)$$

$$\times E^{Q^{(\xi)}}\Big[\exp((1+\varepsilon)T\cdot\tilde{U}(l_T;\xi)),\ \sup_k\Big|\int_M \varphi_k d(l_T-\nu_\xi)\Big|<\delta\Big|$$

$$w(0)=y_0,\ w(T)=y_1\Big]\Big);\ \xi\in K,\ T>2\Big\}$$

$$<\infty.$$

(2.18) PROPOSITION. *For any $x,y\in M$, $P(T,x,y)\to 1$, $T\to\infty$, and $Q(T,x,y;\xi)\to 1$, $T\to\infty$, uniformly in $\xi\in K$.*

PROOF. Note that

$$\begin{aligned}|Q(T+2,x,y;\xi)-1|\leqq&\Big|(Q(1,x,\cdot\,;\xi),Q_T^{(\xi)}Q(1,y,\cdot\,;\xi))_{L^2(d\nu_\xi)}\\&-\Big(\int_M Q(1,x,\cdot\,;\xi)d\nu_\xi\Big)\Big(\int_M Q(1,y,\cdot\,;\xi)d\nu_\xi\Big)\Big|\\&\leqq\exp(-\lambda_0 T)\cdot Q(2,x,x;\xi)^{1/2}\cdot Q(2,y,y;\xi)^{1/2}.\end{aligned}$$

So we have the latter assertion. The proof of the first assertion is similar.

Then we have the following main theorems.

(2.19) THEOREM. *Assume that the assumptions* (A-1)-(A-3), (L-1), *and* (U-1)-(U-5) *are satisfied. Then for any $x,y\in M$, $S>0$ and any $\mathcal{F}_0^S$-measurable bounded function $\Phi: D([0,\infty);M)\to\boldsymbol{R}$,*

$$\begin{aligned}&e^{T\cdot f(\xi)}\cdot E^P[\Phi(w)\cdot\exp(T\cdot U(l_T(w),\xi))|\ w(0)=x,\ w(T)=y]\\&\longrightarrow u(\xi)(x)\cdot u(\xi)(y)\cdot E^{Q^{(\xi)}}[\Phi|\ w(0)=x]\\&\quad\times(\det_{L^2(d\nu_\xi)}(I-D^2U(\nu_\xi,\xi)(\tilde{G}_\xi^{1/2}\cdot,\tilde{G}_\xi^{1/2}\cdot)))^{-1/2}\end{aligned}$$

as $T\to\infty$ uniformly in $\xi\in K$.

PROOF. Since the proof is similar to the proof of [K-T, Theorem (3.14)], we only give a sketch of the proof. By the assumption (L-1), we see that for any $\varepsilon>0$ and $\xi\in K$, there is a $\delta(\varepsilon, \xi)>0$ such that

$$\varlimsup_{T\to\infty}\frac{1}{T}\cdot\log E^{P}\Big[\exp(T\cdot U(l_T, \xi)), \sup_k\Big|\int_M \varphi_k d(l_T-\nu_\xi)\Big|>\varepsilon \Big| w(0)=x, w(T)=y\Big]$$
$$<-f(\xi)-\delta(\varepsilon, \xi)\,.$$

By the definition of $\mathcal{M}_A$, we see that there is a constant $C\in(0, \infty)$ such that

$$|U(m, \xi)-U(m, \xi')|\leqq C\cdot\|U(\cdot, \xi)-U(\cdot, \xi')\|_A, \quad m\in\mathcal{P}(M), \ \xi, \xi'\in K.$$

Therefore we see that for any $\varepsilon>0$, and $\xi\in K$, there is an open neighborhood O_ξ of ξ in K such that

$$\varlimsup_{T\to\infty}\sup\Big\{\frac{1}{T}\cdot\log E^{P}\Big[\exp(T\cdot U(l_T, \xi')), \sup_k\Big|\int_M \varphi_k d(l_T-\nu_{\xi'})\Big|>\varepsilon \Big|$$
$$w(0)=x, \ w(T)=y\Big]+f(\xi')\ ;\ \xi'\in O_\xi\Big\}$$
$$<-\delta(\varepsilon, \xi)/2\,.$$

Since K is compact, we have

$$\varlimsup_{T\to\infty}\sup\Big\{\frac{1}{T}\cdot\log E^{P}\Big[\exp(T\cdot U(l_T, \xi)), \sup_k\Big|\int_M \varphi_k d(l_T-\nu_\xi)\Big|>\varepsilon\Big|$$
$$w(0)=x, \ w(T)=y\Big]+f(\xi)\ ;\ \xi\in K\Big\}$$
$$<0 \qquad \text{for any } \varepsilon>0.$$

By (2.5) we have

$$e^{T\cdot f(\xi)}P(T, x, y)E^{P}\Big[\Phi\cdot\exp(T\cdot U(l_T, \xi)), \sup_k\Big|\int_M \varphi_k d(l_T-\nu_\xi)\Big|<\varepsilon\Big|$$
$$w(0)=x, \ w(T)=y\Big]$$
$$=u(\xi)(x)u(\xi)(y)Q(T, x, y\ ;\ \xi)$$
$$\times E^{Q^{(\xi)}}\Big[\Phi\cdot\exp(T\cdot\tilde{U}(l_T-\nu_\xi\ ;\ \xi)), \sup_k\Big|\int_M \varphi_k d(l_T-\nu_\xi)\Big|<\varepsilon\Big| w(0)=x, \ w(T)=y\Big].$$

By Lemma (2.12), we see that $(\Phi(w), T^{1/2}(l_T(w)-\nu_\xi))$ under $Q^{(\xi)}(dw|\ w(0)=x$, $w(T)=y]$ converges in law to (Φ_ξ, W_ξ), where Φ_ξ and W_ξ are independent Φ_ξ has the same law as $\Phi(w)$ under $Q^{(\xi)}(dw|\ w(0)=x)$ and W_ξ is a Gaussian random measure on M whose characteristic function is given by $E\left[\exp\left(\sqrt{-1}\cdot\int_M \phi(z)\,W_\xi(dz)\right)\right]=\exp\left(-\frac{1}{2}(\tilde{G}_\xi\phi,\phi)\right)$, $\phi\in C_b(M)$. Moreover, Lemma (2.12) shows that this convergence is uniform in $\xi\in K$ in a certain sense. Lemma (2.16) guarantees the uniformly integrability of $\Phi\cdot\exp(T\cdot\tilde{U}(l_T\,;\,\xi))$ under $Q^{(\xi)}(dw|\,w(0)=x,\ w(T)=y)$ with respect to $T\geqq 2$ and $\xi\in K$. Since $T\cdot\tilde{U}(l_T\,;\,\xi)=(1/2)\cdot D^2U(\nu_\xi,\xi)(T^{1/2}(l_T-\nu_\xi),T^{1/2}(l_T-\nu_\xi))+o(1)$, we see that

$$Q(T,x,y\,;\,\xi)\cdot E^{Q^{(\xi)}}[\Phi(w)\cdot\exp(T\cdot\tilde{U}(l_T(w),\xi))|\ w(0)=x,\ w(T)=y]$$

$$\longrightarrow E\left[\Phi_\xi\cdot\exp\left(\frac{1}{2}\cdot D^2U(\nu_\xi,\xi)(W_\xi,W_\xi)\right)\right]$$

$$=E^{Q^{(\xi)}}[\Phi|\ w(0)=x]\cdot(\det_{L^2(d\nu_\xi)}(I-D^2U(\nu_\xi,\xi)(\tilde{G}_\xi^{1/2}\cdot,\tilde{G}_\xi^{1/2}\cdot)))^{-1/2}$$

as $T\to\infty$ uniformly in $\xi\in K$.

Therefore we have our theorem.

Similarly, we have the following.

(2.20) THEOREM. *Assume that the assumptions* (A-1)-(A-3) *and* (U-1)-(U-5) *are satisfied, and that* $P_2: L^2(d\mu)\to L^2(d\mu)$ *is a nuclear operator. Then we have the following.*

(1) *If the assumption* (L-2) *is also satisfied, then for any* $x\in M$, $S>0$ *and any* $\mathcal{F}_0^S$*-measurable bounded function* $\Phi: D([0,\infty)\,;\,M)\to\boldsymbol{R}$,

$$e^{-T\cdot f(\xi)}\cdot E^P[\Phi(w)\cdot\exp(T\cdot U(l_T(w),\xi))|\ w(0)=x]$$

$$\longrightarrow u(\xi)(x)\cdot\left(\int_M u(\xi)d\mu\right)\cdot E^{Q^{(\xi)}}[\Phi|\ w(0)=x]$$

$$\times(\det_{L^2(d\nu_\xi)}(I-D^2U(\nu_\xi,\xi)(\tilde{G}_\xi^{1/2}\cdot,\tilde{G}_\xi^{1/2}\cdot)))^{-1/2}$$

as $T\to\infty$ *uniformly in* $\xi\in K$.

(2) *If the assumption* (L-2) *is also satisfied, then for any* $y\in M$, $S>0$ *and any* $\mathcal{F}_0^S$*-measurable bounded function* $\Phi: D([0,\infty)\,;\,M)\to\boldsymbol{R}$,

$$e^{-T\cdot f(\xi)}\cdot E^P[\Phi(w)\cdot\exp(T\cdot U(l_T(w),\xi))|\ w(T)=y]$$

$$\longrightarrow u(\xi)(y)\cdot E^{Q^{(\xi)}_{\nu_\xi}}[u(\xi)(w(0))\cdot\Phi(w)]$$

$$\times(\det_{L^2(d\nu_\xi)}(I-D^2U(\nu_\xi,\xi)(\tilde{G}_\xi^{1/2}\cdot,\tilde{G}_\xi^{1/2}\cdot)))^{-1/2}$$

as $T\to\infty$ *uniformly in* $\xi\in K$.

(3) *If the assumption* (L-2) *is also satisfied, then for any* $S>0$ *and*

any $\mathcal{F}_0^S$*-measurable bounded function* $\Phi: D([0,\infty); M)\to \boldsymbol{R}$,

$$e^{-T\cdot f(\xi)}\cdot E^{P_\mu}[\Phi(w)\cdot\exp(T\cdot U(l_T(w),\xi))]$$
$$\longrightarrow\left(\int_M u(\xi)d\mu\right)\cdot E^{Q_{\nu_\xi}^{(\xi)}}[u(\xi)(w(0))\cdot\Phi(w)]$$
$$\times(\det_{L^2(d\nu_\xi)}(I-D^2U(\nu_\xi,\xi)(\tilde{G}_\xi^{1/2}\cdot,\tilde{G}_\xi^{1/2}\cdot)))^{-1/2}$$

as $T\to\infty$ *uniformly in* $\xi\in K$.

3. Manifold reflecting singularities and Theorems.

Let us think of the situation given in Section 0. Let $U:\mathcal{M}_A\to\boldsymbol{R}$ be a bounded smooth function and let $\tilde{g}: L^2(d\mu)\to\boldsymbol{R}$ be given by $\tilde{g}(\varphi)=U(\varphi^2d\mu)$. Then $\tilde{g}: L^2(d\mu)\to\boldsymbol{R}$ is a smooth bounded map. Let H be the Hilbert space such that $H=\mathcal{D}om(\mathcal{E})$ as a set and $\|\varphi\|_H^2=2\cdot(\mathcal{E}(\varphi,\varphi)+\|\varphi\|_{L^2(d\mu)}^2)$, $\varphi\in H$. Then H is a subspace of $L^2(d\mu)$ and the inclusion map is a compact operator. Let $g: H\to\boldsymbol{R}$ and $F: H\to\boldsymbol{R}$ be given by $g(h)=\tilde{g}(h)-1$ and $F(h)=\|h\|_{L^2(d\mu)}^2-1$, $h\in H$. Then $g: H\to\boldsymbol{R}$ and $F: H\to\boldsymbol{R}$ are smooth and $DF(h)(k)=(Rh,k)_H$, $h,k\in H$, where $R=\int_0^\infty e^{-t}P_tdt$. Also, we see that

$$\frac{1}{2}\cdot\|h\|_H^2+g(h)=\mathcal{E}(h,h)+U(h^2d\mu)\qquad\text{for } h\in H \text{ with } \int_M h^2d\mu=1.$$

Let $f_0=\inf\{(1/2)\cdot\|h\|_H^2-g(h);\ h\in H,\ F(h)=0\}>-\infty$, and let

$$V=\left\{h\in H;\ \frac{1}{2}\cdot\|h\|_H^2-g(h)=f_0,\ F(h)=0\right\}.$$

Then we have the following.

(3.1) PROPOSITION. *V is a non-void compact set in H.*

PROOF. Assume that $\{h_n\}_{n=1}^\infty\subset H$ such that $F(h_n)=0$, $n\geqq 1$, and $(1/2)\|h_n\|_H^2-g(h_n)\to f_0$, $n\to\infty$. Then we see that $\{h_n\}_{n=1}^\infty$ is a bounded set in H. So taking a subsequence if necessary, we may assume that $h_n\to h_\infty$ weakly in H as $n\to\infty$. Then we see that $F(h_\infty)=0$ and $(1/2)\|h_\infty\|_H^2-g(h_\infty)\leqq f_0$. So we see that $h_\infty\in V$ and $\|h_\infty\|_H^2=2f_0+2g(h_\infty)=\lim_{n\to\infty}\|h_n\|_H^2$. This implies that $h_n\to h_\infty$ in H as $n\to\infty$. These imply our assertion.

Since $Rh\neq 0$ if $h\neq 0$, we see that $DF(h): H\to\boldsymbol{R}$ is non-degenerate for all $h\in V$. Similarly to the argument in Section 2, we see the following.

(3.2) PROPOSITION. (1) *For each* $h \in V$, *there is a unique* $\alpha(h) \in \boldsymbol{R}$ *such that* $h - Dg(h)^* - DF(h)^*\alpha(h) = 0$.

(2) *Let* $h \in V$ *and* $k \in H$ *with* $DF(h)k = 0$. *Then we have*

$$(k, k)_H - D^2g(h)(k, k) - \alpha(h) \cdot D^2F(h)(k, k) \geqq 0 .$$

It is obvious that $DF(h)(h - Dg(h)^* - DF(h)^*\alpha(h)) = 0$, $h \in V$. Note that $DF(h)DF(h)^* = \|Rh\|_H^2 \neq 0$ if $h \neq 0$. So we may assume that the map α is defined in $H \setminus \{0\}$ by $\alpha(h) = (DF(h)DF(h)^*)^{-1}DF(h)(h - Dg(h)^*)$, $h \neq 0$.

Let $\hat{V}$ be a subset in $H \times H$ given by

$$\hat{V} = \{(v, k) \in V \times H \,;\, \|k\|_H = 1,\ DF(v)k = 0,$$
$$\|k\|_H^2 - D^2g(v)(k, k) - \alpha(v) \cdot D^2F(v)(k, k) = 0\} .$$

Then we have the following.

(3.3) PROPOSITION. *Let* $G : (H \setminus \{0\}) \rightarrow H$ *be a smooth map given by* $G(h) = Dg(h)^* + DF(h)^*\alpha(h)$, $h \in H \setminus \{0\}$. *Let* $v \in V$ *and* $k \in H$ *such that* $\|k\|_H = 1$ *and* $DF(v)k = 0$. *Then* $(v, k) \in \hat{V}$ *iff* $k - DG(v)k = 0$.

PROOF. By Proposition (3.2) and Schwartz's inequality, we see that $(v, k) \in \hat{V}$ iff $(k, u)_H - D^2g(v)(k, u) - \alpha(v)D^2F(v)(k, u) = 0$ for any $u \in H$ with $DF(v)u = 0$. So we see $(v, k) \in \hat{V}$ iff

$$k - D^2g(v)(k, \cdot)^* - (D^2F(v)(k, \cdot)\alpha(v))^* - DF(v)^*\beta = 0 \quad \text{for some } \beta \in \boldsymbol{R}.$$

By the definition of α, we have

$$DF(v)(h - D^2g(v)(h, \cdot)^* - (D^2F(v)(h, \cdot)\alpha(v))^* - DF(v)^*D\alpha(v)(h)) = 0 ,$$

for all $h \in H$. Therefore we see that $(v, k) \in \hat{V}$ iff

$$k - D^2g(v)(k, \cdot)^* - (D^2F(v)(k, \cdot)\alpha(v))^* - DF(v)^*D\alpha(v)(k) = 0 .$$

This implies our assertion.

(3.4) DEFINITION. (1) We say that N is a *manifold weakly reflecting singularities* if

(i) N is a finite dimensional submanifold embedded in H,

(ii) $V \subset N$,

and

(iii) $k \in T_v(N)$ if $(v, k) \in \hat{V}$.

(2) We say that N is a *manifold strongly reflecting singularities* if N is a submanifold weakly reflecting singularities and $F(h) = 0$ for all

$h \in N$.

(3.5) THEOREM. *There is a manifold strongly reflecting singularities.*

This theorem is a corollary to the following two lemmas.

(3.6) LEMMA. *There is a manifold weakly reflecting singularities.*

(3.7) LEMMA. *Let N be a manifold weakly reflecting singularities. Then there is a manifold $\tilde{N}$ strongly reflecting singularities of the same dimensions with the manifold N such that there is an embedding $\varphi: \tilde{N} \to N$ with $\varphi(v)=v$, $v \in V$.*

PROOF OF LEMMA (3.6). Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of orthogonal projections in H such that $\dim(\mathrm{Image}\, P_n) < \infty$ and $P_n \uparrow I_H$ strongly as $n \to \infty$. First, we show the following.

(3.8) PROPOSITION. *$P_n|_v : V \to P_n(V)$ is injective if n is sufficiently large.*

PROOF. If not, there are $u_n, v_n \in V$ such that $u_n \neq v_n$ and $P_n u_n = P_n v_n$ for each $n \geqq 1$. Since V is compact, we may assume that $u_n \to u$ and $v_n \to v$ as $n \to \infty$. Then we see that $u=v$. Let $G: H \backslash \{0\} \to H$ be as in Proposition (3.3). Then we see that $u_n = G(u_n)$ and $v_n = G(v_n)$. It is obvious that

$$\begin{aligned} v_n - u_n &= G(v_n) - G(u_n) \\ &= DG(u)(v_n - u_n) + \int_0^1 (DG(u_n + t(v_n - u_n)) - DG(u))(v_n - u_n))dt. \end{aligned}$$

Let $h_n = \|u_n - v_n\|_H^{-1}(u_n - v_n)$. Then taking a subsequence if necessary, we may assume that $h_n \to h_\infty$ weakly in H as $n \to \infty$. Since $h_n - DG(u)h_n \to 0$ in H as $n \to \infty$, and $DG(u): H \to H$ is a compact operator, we see that $h_n \to h_\infty$ in H as $n \to \infty$. So $\|h_\infty\| = 1$. On the other hand, since $P_n h_n = 0$, we have $P_n h_\infty = 0$, $n \geqq 1$. This is a contradiction.

This completes the proof of Proposition (3.8).

Let $G: (H \backslash \{0\}) \to H$ be as in Proposition (3.3). Let $\Phi_n: (H \backslash \{0\}) \to H$ be given by $\Phi_n(h) = h - (I_H - P_n)G(h)$, $h \in (H \backslash \{0\})$. Then we see that $D\Phi_n(v): H \to H$ is invertible for all $v \in V$, if n is sufficiently large. So there is a neighborhood U_n of V such that $\Phi_n|_{U_n}: U_n \to H$ is local diffeomorphism, if n is sufficiently large. Also, we see the following.

(3.9) PROPOSITION. *If n is sufficiently large, there is a neighborhood*

$\tilde{U}_n$ of V such that $\Phi_n|_{\tilde{U}_n}: \tilde{U}_n \to H$ is injective.

PROOF. If not, there are $x_m, y_m \in H$, $m=1,2,\cdots$, such that $x_m \neq y_m$, $\mathrm{dis}(x_m, V)+\mathrm{dis}(y_m, V) \to 0$, $m \to \infty$, and $\Phi_n(x_m)=\Phi_n(y_m)$. Since V is compact, we may assume that $x_m \to x_\infty$ and $y_m \to y_\infty$ as $m \to \infty$. Also, we may assume by Proposition (3.8) that $P_n|_V: V \to P_n(V)$ is injective. Then $x_\infty, y_\infty \in V$, and $\Phi_n(x_\infty)=\Phi_n(y_\infty)$. Since $P_n\Phi_n(h)=P_nh$, we have $P_nx_\infty=P_ny_\infty$, and so $x_\infty=y_\infty$. Since $\Phi_n|_{U_n}$ is local diffeomorphism, we see that $\Phi_n(x_m)=\Phi_n(y_m)$ if m is sufficiently large. This is a contradiction.

This completes the proof of Proposition (3.9).

By Proposition (3.9), if we take a sufficiently large n, $\Phi_n|_{U_n\cap\tilde{U}_n}: U_n\cap\tilde{U}_n \to \Phi_n(U_n\cap\tilde{U}_n)$ is a diffeomorphism. Let $W=P_n(H)\cap\Phi_n(U_n\cap\tilde{U}_n)$ and $\psi: W \to H$ be given by $\psi(x)=(\Phi_n|_{U_n\cap\tilde{U}_n})^{-1}(x)$, $x \in W$. Then $\psi: W \to H$ is an embedding and $V \subset \psi(W)$, because $\Phi_n(v)=P_nv$, $v \in V$, by Proposition (3.2) (1).

Let $(v, k) \in \hat{V}$. Then by Proposition (3.3), we see that $(d/dt)\Phi_n(v+tk)|_{t=0} = k-(I_H-P_n)DG(v)k=P_nk$. So we see that $(d/dt)\psi(P_nv+tP_nk)|_{t=0}=k$, which implies that $k \in \psi_*(v)(P_n(H))$. Therefore we see that $N=\psi(W)$ is a manifold weakly reflecting singularities.

This completes the proof of Lemma (3.6).

PROOF OF LEMMA (3.7). Let N be a manifold weakly reflecting singularities. Let $\{e_n\}_{n=1}^\infty$ be a complete orthonormal basis of H. Let $A_n: \boldsymbol{R} \to H$ be given by $A_n(\xi)=\xi e_n$, $\xi \in \boldsymbol{R}$. Let $\Psi_n: \boldsymbol{R}\times N \to \boldsymbol{R}$ be given by $\Psi_n(\xi, x)=F(x+DF(x)^*\xi+A_n\xi)$, $(\xi, x) \in \boldsymbol{R}\times N$. Since

$$D_\xi\Psi_n(0, v)=DF(v)(DF(v)^*+A_n) \longrightarrow DF(v)DF(v)^*, \quad n \to \infty,$$

we see that $D_\xi\Psi_n(0, v): \boldsymbol{R} \to \boldsymbol{R}$ is invertible for all $v \in V$, if n is sufficiently large. Also, $\Psi_n(0, v)=0$, $v \in V$. Therefore there is a neighborhood U_n of V in N and $\xi_n: U_n \to \boldsymbol{R}$ such that $\xi_n(v)=0$, $v \in V$ and $F(x+DF(x)^*\xi_n(x)+A_n\xi_n(x))=0$. Let $\varphi_n: U_n \to H$ be given by $\varphi_n(x)=x+DF(x)^*\xi_n(x)+A_n\xi_n(x)$. Then we see that $\varphi_{n*}(x)(u)=u+DF(x)^*\xi_{n*}(x)(u)+A_n\xi_{n*}(x)(u)$ for $x \in V$ and $u \in T_x(N) \subset H$.

Note that $A_n^* \to 0$ strongly as $n \to \infty$. So we can take a sufficiently large n such that the following are satisfied.

$$|A_n^*u| \leqq (4(\|DF(x)\|_{H\to R}+1))^{-1} \quad \text{for all } u \in T_x(N),\ x \in V \text{ with } \|u\|_H \leqq 1,$$

and

$$\|DF(x)A_n\|_{R\to R} \leqq 1/2, \qquad x \in V.$$

Suppose that $\varphi_{n*}(x)(u)=0$, $x \in V$, $u \in T_x(N)$. Then we see that

$$|\xi_{n*}(x)u| \leqq |A_n^* u| + |A_n^* DF(x)^* \xi_{n*}(x)u|$$

$$\leqq |A_n^* u| + |\xi_{n*}(x)(u)|/2 .$$

Therefore

$$\|u\|_H \leqq 2 \cdot \|DF(x)^* + A_n\|_{R \to H} \cdot |A_n^* u| \leqq 2^{-1} \|u\|_H ,$$

and so $u=0$. Therefore $\varphi_{n*}(x) : T_x(N) \to H$ has full rank for all $x \in V$. This and similar argument in the proof of Proposition (3.9) imply that there is a neighborhood W_n of V in N such that $\varphi_n|_{W_n} : W_n \to H$ is embedding.

Suppose that $(x, k) \in \hat{V}$. Then $k \in T_x(N)$. Note that $\Psi_{n*}(0, x)(0, k) = DF(x)k = 0$. So $\xi_{n*}(x)k = 0$. This implies that $\varphi_{n*}(x)k = k$. So if we let $\tilde{N} = \varphi_n(W_n)$ and $\varphi : \tilde{N} \to N$ be given by $\varphi = (\varphi_n|_{W_n})^{-1}$, we have our assertion.

This completes the proof of Lemma (3.7).

Now let us think of the situation in Section 0 again, and let N_0 be a manifold strongly reflecting singularities. Let N_1 be a compact set in N_0 which contains V as interior points. Let $\tilde{N}_i = \{h^2 d\mu ; h \in N_i\}$, $i = 0, 1$, and $\tilde{V} = \{h^2 d\mu ; h \in V\}$. Then we may think that $\tilde{N}_0$ is a submanifold in the Hilbert space $\mathcal{M}_A$. Then we see that there are an open neighborhood $\tilde{U}$ of $\tilde{V}$ in $\mathcal{M}_A$ and a smooth map $\Psi : \tilde{U} \to \tilde{N}_0$ such that $\Psi(m)$, $m \in \tilde{U}$, is a unique element in $\tilde{N}_0$ with $\|m - \Psi(m)\|_A = \inf \{\|m - \tilde{m}\|_A ; \tilde{m} \in \tilde{N}_0\}$. Let $\tilde{W}_0 : \tilde{U} \times \tilde{N}_1 \to [0, \infty)$ be given by $\tilde{W}_0(m ; \xi) = (1/2) \cdot \|\Psi(m) - \xi\|_A^2$. Let $\varphi : \mathcal{M}_A \to \boldsymbol{R}$ be a smooth map such that $0 \leqq \varphi \leqq 1$, $\mathrm{dis}_{\mathcal{M}_A}(\mathrm{supp}\, \varphi, \mathcal{M}_A \backslash \tilde{U}) > 0$ and $\varphi(m) = 1$ in a neighborhood of $\tilde{V}$. Let $W_0 : \mathcal{M}_A \times N_1 \to \boldsymbol{R}$ be given by $W_0(m, \xi) = \varphi(m) \tilde{W}_0(m, \xi) + (1 - \varphi(m))$, $(m, \xi) \in \mathcal{M}_A \times N_1$.

Let us regard $\tilde{N}_0$ as a Riemannian manifold with the Riemannian metric induced by $\|\cdot\|_A^2$, and let $n_0(dx)$ be the Riemannian volume. Then we see that there is a neighborhood $\tilde{U}_0$ of $\tilde{V}$ in $\mathcal{M}_A$ such that

$$(2\pi)^{-d/2} T^{d/2} \int_{\tilde{N}_0} \exp(-T \cdot W_0(m, \xi)) n_0(d\xi) \longrightarrow 1, \quad T \to \infty ,$$

uniformly on $m \in \tilde{U}_0$.

Let $U : \mathcal{M}_A \times \tilde{N}_1 \to \boldsymbol{R}$ be given by $U(m, \xi) = U(m) - W_0(m, \xi)$, $m \in \mathcal{M}_A$, $\xi \in N_1$, and let $f(\xi) = \inf \{\mathcal{E}(\varphi, \varphi) - U(\varphi^2 d\mu, \xi) ; \varphi \in \mathcal{D}om(\mathcal{E}), \|\varphi\|_{L^2(d\mu)} = 1\}$. Then we see that $f(\xi) \geqq f_0$ and $f(\xi) = f_0$ if and only if $\xi \in \tilde{V}$. Also, we see that if we let $K = \tilde{V}$, $U|_{\mathcal{M}_A \times \tilde{V}} : \mathcal{M}_A \times \tilde{V} \to \boldsymbol{R}$ satisfies the assumptions (U-1)-(U-3). Then by implicit function theorem we see that there is a compact neighborhood $\tilde{K}_0$ of $\tilde{V}$ in $\tilde{N}_0$ such that if we let $K = \tilde{K}_0$, $U|_{\mathcal{M}_A \times \tilde{K}_0} : \mathcal{M}_A \times \tilde{K}_0 \to \boldsymbol{R}$ satisfies the assumptions (U-1)-(U-3). Let $\tilde{N}$ be the interior open set of $\tilde{K}_0 \cap \tilde{U}_0$. Then $\tilde{V} \subset \tilde{N}$ and again by implicit function theorem, we see that $f(\cdot)|_{\tilde{N}} : \tilde{N} \to \boldsymbol{R}$ is smooth. Let $u(\cdot) : \tilde{K}_0 \to L^2(M ; d\mu)$, $\nu_\xi \in \mathcal{P}(M)$, $\tilde{G}_\xi : L^2(d\nu_\xi)$

$\to L^2(d\nu_\xi)$, $\xi \in \tilde{K}_0$, be as in Section 2.

Now let $\rho: \tilde{N} \to [0, \infty)$ be given by $\rho(\xi) = f(\xi) - f_0$ and $W: \mathcal{M}_A \times \tilde{N}_0 \to \boldsymbol{R}$ be given by $W(m, \xi) = U(m) - W_0(m, \xi)$. Then we have the following.

(3.10) THEOREM. *Suppose that the assumptions* (A-1)-(A-3) *are satisfied, and let* $\Phi: D([0, \infty); M) \to (0, \infty)$ *be an* $\mathcal{F}_0^{T_0}$*-measurable bounded positive continuous function. Then we have the following.*

(1) *If the assumption* (L-1) *is satisfied, then for any* $x, y \in M$

$$e^{T \cdot f_0} \cdot E^P[\Phi \cdot \exp(T \cdot U(l_T)) |\ w(0) = x,\ w(T) = y]$$

$$\sim (T/2\pi)^{(\dim \tilde{N})/2} \cdot \int_{\tilde{N}} g(\xi) \cdot \exp(-T \cdot \rho(\xi)) E^{Q^{(\xi)}}[\Phi |\ w(0) = x] n_0(d\xi)$$

as $T \to \infty$. *Here*

$$g(\xi) = u(\xi)(x) \cdot u(\xi)(y) \cdot (\det_{L^2(d\nu_\xi)}(I - D^2 W(\nu_\xi, \xi)(\tilde{G}_\xi^{1/2} \cdot, \tilde{G}_\xi^{1/2} \cdot)))^{-1/2}.$$

(2) *If* $P_2: L^2(d\mu) \to L^2(d\mu)$ *is a nuclear operator and the assumption* (L-2) *is satisfied, then for any* $x \in M$

$$e^{T \cdot f_0} \cdot E^P[\Phi \cdot \exp(T \cdot U(l_T)) |\ w(0) = x]$$

$$\sim (T/2\pi)^{(\dim \tilde{N})/2} \cdot \int_{\tilde{N}} g(\xi) \cdot \exp(-T \cdot \rho(\xi)) E^{Q^{(\xi)}}[\Phi |\ w(0) = x] n_0(d\xi)$$

as $T \to \infty$. *Here*

$$g(\xi) = u(\xi)(x) \cdot \left(\int_M u(\xi) d\nu_\xi\right) \cdot (\det_{L^2(d\nu_\xi)}(I - D^2 W(\nu_\xi, \xi)(\tilde{G}_\xi^{1/2} \cdot, \tilde{G}_\xi^{1/2} \cdot)))^{-1/2}.$$

(3) *If* $P_2: L^2(d\mu) \to L^2(d\mu)$ *is a nuclear operator and the assumption* (L-2) *is satisfied, then for any* $y \in M$

$$e^{T \cdot f_0} \cdot E^{P_\mu}[\Phi \cdot \exp(T \cdot U(l_T)) |\ w(T) = y]$$

$$\sim (T/2\pi)^{(\dim \tilde{N})/2} \cdot \int_{\tilde{N}} g(\xi) \cdot \exp(-T \cdot \rho(\xi)) E^{Q_{\nu_\xi}^{(\xi)}}[u(\xi)(w(0)) \cdot \Phi] n_0(d\xi)$$

as $T \to \infty$. *Here*

$$g(\xi) = u(\xi)(y) \cdot (\det_{L^2(d\nu_\xi)}(I - D^2 W(\nu_\xi, \xi)(\tilde{G}_\xi^{1/2} \cdot, \tilde{G}_\xi^{1/2} \cdot)))^{-1/2}.$$

(4) *If* $P_2: L^2(d\mu) \to L^2(d\mu)$ *is a nuclear operator and the assumption* (L-3) *is satisfied, then for any* $x \in M$

$$e^{T\cdot f_0}\cdot E^{P_\mu}[\Phi\cdot\exp(T\cdot U(l_T))]$$

$$\sim(T/2\pi)^{(\dim\tilde{N})/2}\cdot\int_{\tilde{N}}g(\xi)\cdot\exp(-T\cdot\rho(\xi))E^{Q_{\nu_\xi}^{(\xi)}}[u(\xi)(w(0))\cdot\Phi]n_0(d\xi)$$

as $T\to\infty$. *Here*

$$g(\xi)=\left(\int_M u(\xi)d\nu_\xi\right)\cdot(\det_{L^2(d\nu_\xi)}(I-D^2W(\nu_\xi,\xi)(\tilde{G}_\xi^{1/2}\cdot,\tilde{G}_\xi^{1/2}\cdot)))^{-1/2}.$$

PROOF. Since the proofs are similar, we only prove the assertion (1). Note that

$$\overline{\lim_{T\to\infty}}\frac{1}{T}\log E^P[\exp(T\cdot U(l_T)),\ l_T\in\mathcal{M}_A\setminus\tilde{U}_0|\ w(0)=x,\ w(T)=y]<-f_0.$$

Let $W:\mathcal{M}_A\times\tilde{N}_0\to\boldsymbol{R}$ be given by $W(m,\xi)=U(m)-W_0(m,\xi)$. Then we see that

$$E^P[\exp(T\cdot U(l_T))|\ w(0)=x,\ w(T)=y]$$

$$\sim E^P[\exp(T\cdot U(l_T)),\ l_T\in\tilde{U}_0|\ w(0)=x,\ w(T)=y]$$

$$\sim(2\pi)^{-d/2}\cdot T^{d/2}\cdot\int_{\tilde{N}_0}n_0(d\xi)E^P[\exp(T\cdot W(l_T,\xi))|\ w(0)=x,\ w(T)=y]$$

as $T\to\infty$. By Theorem (2.19), we see that

$$\exp(T\cdot f(\xi))\cdot E^P[\exp(T\cdot W(l_T,\xi))|\ w(0)=x,\ w(T)=y]\longrightarrow g(\xi),\quad\text{as } T\to\infty,$$

uniformly in $\xi\in\tilde{K}_0$. Therefore we have our theorem.

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