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Some inequalities for some increasing additive functionals of planar Brownian motion and an application to Nevanlinna theory

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§ 0. Introduction.

In this paper we are interested in the relation between the growth of some increasing additive functionals of complex Brownian motion and Nevanlinna theory. On the growth of martingales we know, first of all, Burkholder-Davis-Gundy inequality, that is, let M_t be a continuous martingale, $M_t^* \equiv \sup_{0\leq s\leq t} M_s$, and $\langle M\rangle_t$ be its quadratic variation process. If $p\in(0,\infty)$, then there exist constants c_1, c_2 depending only on p such that for any stopping time T,

$$c_1 E\langle M\rangle_T^{p/2} \leqq E(M_T^*)^p \leqq c_2 E\langle M\rangle_T^{p/2},$$

holds. Moreover Barlow-Yor [1] showed,

$$c_1 E\langle M\rangle_T^{p/2} \leqq E(L_T^*)^p \leqq c_2 E\langle M\rangle_T^{p/2},$$

where $L_t^* \equiv \sup_x L(x, t)$. $L(x, t)$ is the local time of M_t. Recently Bass [2] extended such an inequality to more general functionals. In order to prove Nevanlinna theory probabilistically, we need establish such a type of inequality for some increasing additive functionals of planar Brownian motion $\{Z_t\}$ $(Z_0=1)$, like

$$\int_0^t |Z_s|^{-2} ds \quad \text{and} \quad \int_0^t g(Z_s) ds,$$

with $g\in L^1(\boldsymbol{C})$. In § 1 we state our inequalities and prove them in § 2.

What fills the role to join our inequalities to Nevanlinna theory is the lemma on logarithmic derivative which R. Nevanlinna applied to his second main theorem. Here we give a little information of Nevanlinna theory on $\boldsymbol{C}$. (See [6].) Let f be a meromorphic function on $\boldsymbol{C}$ and $a\in\boldsymbol{P}_1$ (one dimensional complex projective space).
Set

$$m(a, r)=\begin{cases}\displaystyle\int_0^{2\pi}\log^+\left|\frac{1}{f(re^{i\theta})-a}\right|d\theta & \text{if } a\neq\infty \\ \displaystyle\int_0^{2\pi}\log^+|f(re^{i\theta})d\theta & \text{if } a=\infty\end{cases}$$

with $\log^+ a=\log\{\max(a, 1)\}$,

$$N(a, r)=\sum\log\frac{r}{|\zeta|} \qquad \text{(counting function)},$$

where the sum is taken over all roots of $f(\zeta)=a$ within $\{|\zeta|\leqq r\}$ repeated according to their multiplicity, and

$$T(r)=\int_{|z|\leq r} g_r(0, z)\frac{|f'(z)|^2}{(1+|f(z)|^2)^2}dV(z)$$

(Ahlfors-Shimizu characteristic function), where $g_r(x, y)$ is the Green function on $\{|z|\leqq r\}$, and $dV(z)$ is Lebesgue measure.

THE FIRST MAIN THEOREM.

$$m(a, r)+N(a, r)=T(r)+O(1).$$

We can show this equality probabilistically from Dynkin's formula or Ito's formula.

THE SECOND MAIN THEOREM (abr. SMT.). *Let* $a_1, \cdots, a_q$ *be distinct points of* $\boldsymbol{P}_1$, $q\geqq 3$ *and* $N_1(r)$ *be the counting function of the critical points of* f. *Then*

$$\sum_{\lambda=1}^{q} m(a_\lambda, r)+N_1(r)\leqq 2T(r)+O(\log T(r)+\log r)$$

is valid except for finite length of r.

This is a very deep result. For example Picard's theorem is verified from this. The lemma on logarithmic derivative is the key to his original proof of SMT. We show that this lemma is deduced from our inequalities in § 3. Carne also noticed the relation between Brownian motion and Nevanlinna theory in [3] where he proved SMT by differential geometric aspect of Brownian motion.

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§ 1. Some inequalities for some increasing additive functionals of planar Brownian motion.

Let $(Z_t, \Omega, P_1, \mathcal{F}, \mathcal{F}_t)$ be a Brownian motion on $\boldsymbol{C}$ starting from 1, $g(z)$ be any non-negative function on $\boldsymbol{C}$ satisfying $\int_C g(z)dV(z)<\infty$ and $G(r)$ be any non-negative function on $[0, \infty)$ satisfying $\int_0^\infty G(r)dr<\infty$. Let us consider the additive functionals such as $\int_0^t |Z_s|^{-2}ds$, $\int_0^t g(Z_s)ds$ and $\int_0^t G(|Z_s|)ds$. We define a set of $\mathcal{F}_t$-stopping times as

$$S(\mathcal{F})\equiv\{\sigma:\text{ an }\mathcal{F}_t\text{-stopping time satisfying }\sigma\circ\theta_t=\sigma-t\\ \text{on }\sigma\geqq t\text{ for any }t\geqq 0.\},$$

where θ_t is the shift operator of Brownian motion.

Now the following inequalities are valid.

THEOREM. *Let $Z_t, g(z), G(|z|)$ be as above.*

i) *If $\alpha\in(0, 1/2)$, there exist constants $c_1>0, c_2>0$ such that for any $\sigma\in S(\mathcal{F})$*

$$E_1\left(\int_0^\sigma \frac{ds}{|Z_s|^2}\right)^\alpha \leqq c_1E_1(\log^+\sigma)^{2\alpha}+c_2. \tag{1.1}$$

ii) *If $\alpha\in(0, \infty)$, there exist constants $c_3>0, c_4>0$ such that for any $\mathcal{F}_t$-stopping time σ*

$$c_3E_1(\log^+\sigma)^{2\alpha}\leqq E_1\left(\int_0^\sigma \frac{ds}{|Z_s|^2}\right)^\alpha +c_4. \tag{1.2}$$

iii) *If $\alpha\in(0, 1)$, there exist constants $c_5>0$, $c_6>0$ such that for any $\mathcal{F}_t$-stopping time σ*

$$E_1\left(\int_0^\sigma g(Z_s)ds\right)^\alpha \leqq c_5E_1(\log^+\sigma)^\alpha+c_6. \tag{1.3}$$

iv) *If $\alpha\in(0, \infty)$, there exist constants $c_7>0$, $c_8>0$ such that for any $\mathcal{F}_t$-stopping time σ*

$$c_7E_1(\log^+\sigma)^\alpha\leqq E_1\left(\int_0^\sigma G(|Z_s|)ds\right)^\alpha +c_8. \tag{1.4}$$

The above theorem is immediately verified by integrating the following Burkholder type good λ inequalities.

PROPOSITION. *For any $\beta>3, 0<\delta<1$, there exist $h_i(\delta)>0$ and $H_i(\lambda)>0$, $i=1,\cdots,4$ satisfying that $h_i(\delta)$ tends to zero as $\delta\downarrow 0$ and*

$\int_0^\infty H_i(\lambda)d\lambda^\alpha<\infty$ *where* $\alpha\in(0,\infty)$ *if* $i\neq 3$ *and* $\alpha\in(0,1)$ *if* $i=3$, *such that the following is valid.*

i) *If* $\sigma\in S(\mathcal{F})$,

$$P_1\left(\int_0^\sigma |Z_s|^{-2}ds>\beta\lambda,\ (\log^+\sigma)^2\leqq\delta\lambda\right)\leqq h_i(\delta)P_1\left(\int_0^\sigma |Z_s|^{-2}ds>\lambda\right)+H_1(\lambda) \tag{1.5}$$

ii) *If* σ *is an* $\mathcal{F}_t$*-stopping time,*

$$P_1\left((\log^+\sigma)^2>\beta\lambda,\ \int_0^\sigma |Z_s|^{-2}ds\leqq\delta\lambda\right)\leqq h_2(\delta)P_1((\log^+\sigma)^2>\lambda)+H_2(\lambda)\,. \tag{1.6}$$

iii) *If* σ *is an* $\mathcal{F}_t$*-stopping time,*

$$P_1\left(\int_0^\sigma g(Z_s)ds>\beta\lambda,\ \log^+\sigma\leqq\delta\lambda\right)\leqq h_3(\delta)P_1\left(\int_0^\sigma g(Z_s)ds>\lambda\right)+H_3(\lambda)\,. \tag{1.7}$$

iv) *If* σ *is an* $\mathcal{F}_t$*-stopping time,*

$$P_1\left(\log^+\sigma>\beta\lambda,\ \int_0^\sigma G(|Z_s|)ds\leqq\delta\lambda\right)\leqq h_4(\delta)P_1(\log^+\sigma>\lambda)+H_4(\lambda)\,. \tag{1.8}$$

In the above proposition the condition that σ must belong to $S(\mathcal{F})$ is technical. But I don't know whether this could be suppressed or there could be a counterexample.

We wish to apply the theorem to Nevanlinna's theorem so that it motivates us to have the following corollary. Let Z_t be a Brownian motion on $\boldsymbol{C}$ and define

$$\tau_r=\inf\{t\geqq 0:|Z_t|=r\}\,.$$

COROLLARY 1. *Suppose* $Z_0=0$. *Let* g *and* G *as above and* $0<\alpha<1$. *Then there exist constants* $c_i>0$, $i=1,\cdots,8$ *such that for any non-constant meromorphic function* f *on* $\boldsymbol{C}$ (*we assume* $f(0)=1$ *for simplicity.*)

i)
$$c_1E\left(\log^+\int_0^{\tau_r}|f'(Z_s)|^2ds\right)^\alpha-c_2\leqq E\left(\int_0^{\tau_r}\frac{|f'(Z_s)|^2}{|f(Z_s)|^2}ds\right)^{\alpha/2}$$
$$\leqq c_3E\left(\log^+\int_0^{\tau_r}|f'(Z_s)|^2ds\right)^\alpha+c_4\,. \tag{1.9}$$

ii)
$$c_5E\left(\log^+\int_0^{\tau_r}|f'(Z_s)|^2ds\right)^\alpha-c_6\leqq E\left(\int_0^{\tau_r}G(|f(Z_s)|)|f'(Z_s)|^2ds\right)^\alpha$$
$$\leqq c_7E\left(\log^+\int_0^{\tau_r}|f'(Z_s)|^2ds\right)^\alpha+c_8\,. \tag{1.10}$$

PROOF. We have only to make a time change argument. There exists a Brownian motion $\tilde{Z}_t$ on $\boldsymbol{C}$ with $\tilde{Z}_0=f(0)=1$ such that $f(Z_t)=\tilde{Z}_{\rho_t}$ with $\rho_t=\int_0^t|f'(Z_s)|^2ds$. $\tilde{Z}_t$ has a filtration $\tilde{\mathcal{F}}_t$ defined by $\tilde{\mathcal{F}}_t=\mathcal{F}_{\phi_t}$, $\phi_t=\rho_t^{-1}$, so that ρ_t is an $\tilde{\mathcal{F}}_t$-stopping time and $\rho_{\tau_r}\in S(\tilde{\mathcal{F}})$. □

Corollary 1 is available for the proof of Nevanlinna's lemma on logarithmic derivative. We will see this in § 3.

At the close of this section we mention the relation between Barlow-Yor's theorem and ours. Set $M_t\equiv\log|Z_t|$. Then $\langle M\rangle_t=\int_0^t|Z_s|^{-2}ds$ and M_t is a continuous local martingale. Using Barlow-Yor's inequality, we have

$$\begin{aligned}E\left(\int_0^\sigma G(|Z_s|)ds\right)^\alpha &\leqq E\left(\int_{-\infty}^\infty G(e^x)L(\sigma,x)dx\right)^\alpha\\ &\leqq \|G\|_{L^1(C)}E\left(\sup_x L(\sigma,x)\right)^\alpha\\ &\leqq \text{const. } E\langle M\rangle_\sigma^{\alpha/2}=\text{const. } E\left(\int_0^\sigma|Z_s|^{-2}ds\right)^{\alpha/2}.\end{aligned}$$

In conjunction with our inequality i) of Theorem, we get iii) for $\sigma\in S(\mathcal{F})$.

§ 2. Proof of Proposition.

First we must remark the following fact that we will use frequently to prove the claim. Let Z_t be a complex Brownian motion with $Z_0=z$. Then there exists a one-dimensional Brownian motion B_u starting from zero such that

$$\log|Z_t|-\log|z|=B(u_t),$$

with $u_t=\int_0^t|Z_s|^{-2}ds$. From this we have

$$u_{\tau_r}=\int_0^{\tau_r}|Z_s|^{-2}ds=\sigma_{\log r/|z|}\overset{d}{=}(\log r/|z|)^2\sigma_1,$$

with $\sigma_a=\inf\{t\geqq 0\,;\ B_t=a\}$. (The last equality is valid by the Brownian scaling property.) We use this fact without further notice.

We prepare lemmas for the proof of Proposition.

LEMMA 1. *Let $\tau_r\equiv\inf\{t\geqq 0: |z_t|=r\}$. If $|z|\leqq r$, then we have for $0<\varepsilon<2$ and $t<r$,*

i) $$P_z(\tau_r > r^{2+\varepsilon}) \leqq r^{-\varepsilon}, \tag{2.1}$$

ii) $$P_z(\tau_r \leqq t) \leqq I_0(\sqrt{2}\,|z| t^{-1/2}) / \boldsymbol{I}_0(\sqrt{2} r t^{-1/2}), \tag{2.2}$$

where $\boldsymbol{I}_0(x) = \sum_{n=0}^{\infty} \dfrac{(x/2)^{2n}}{(n!)^2}$.

PROOF. i) $P_z(\tau_r > r^{2+\varepsilon}) \leqq r^{-2-\varepsilon} E_z \tau_r = r^{-2-\varepsilon}(r^2 - |z|^2) \leqq r^{-\varepsilon}$.
ii) follows from the passage time formula. ([5]). □

LEMMA 2. *Let* $0 < \varepsilon < 2$. *If* $r^{-1+\varepsilon/2} \leqq |z| \leqq r^{-1+\varepsilon/2}$ *and* $r > 1$,

i) $$P_z\left(\int_0^{\tau_r} |Z_s|^{-2} ds > u\right) \leqq \frac{2 \log r}{\sqrt{u}}, \tag{2.3}$$

ii) $$P_z\left(\int_0^{r^{2-\varepsilon}} |Z_s|^{-2} ds > u\right) \leqq \frac{2 \log r}{\sqrt{u}} + C \cdot \boldsymbol{I}_0(r^{\varepsilon/2})^{-1}. \tag{2.4}$$

(*C is a constant independent of r.*)
If $|z| \leqq r^{1-\varepsilon/2}$,

iii) $$P_z\left(\int_0^{r^{2+\varepsilon}} |Z_s|^{-2} ds < u\right) \leqq P((\log r^{\varepsilon/2})^2 \sigma_1 < u) + r^{-\varepsilon}. \tag{2.5}$$

PROOF. i) $$P_z\left(\int_0^{\tau_r} |Z_s|^{-2} ds > u\right) = P((\log r/|z|)^2 \sigma_1 > u) \leqq \frac{2 \log r}{\sqrt{u}}.$$

ii) follows from i) and Lemma 1. iii) is proved similarly to ii). □

LEMMA 3. *Let* $0 < \varepsilon < 2$ *and* $r > 1$. *There exist positive constants* c_1 *and* c_2 *such that*
i) *if* $|z| < r^{1-\varepsilon/2}$,

$$P_z\left(\int_0^{\tau_r} G(|Z_s|) ds \leqq k\right) \leqq \frac{1}{c_1 k^{-1} \log r + 1}, \tag{2.6}$$

ii) *if* $t > 1$,

$$P_z\left(\int_0^t g(Z_s) ds > k\right) \leqq \frac{c_2(\log t + 1)}{k}. \tag{2.7}$$

PROOF. i) Let $l(t, x)$ be the local time of B_t ($B_0 = 0$). The scaling property of Brownian motion implies

$$\int_0^{\tau_r} G(|Z_s|)ds = \int_0^{\sigma \log r/|z|} G(|z| \exp B_s) \exp 2B_s ds$$

$$= \int_{-\infty}^{\infty} G(e^x) e^{2x} l\left(\sigma_1, \frac{x-\log|z|}{\log r/|z|}\right) dx \cdot \log r/|z| .$$

For $Q \subset \boldsymbol{R}$ set

$$X_Q \equiv \int_Q G(e^x) e^{2x} l\left(\sigma_1, \frac{x-\log|z|}{\log r/|z|}\right) dx ,$$

$$K_Q \equiv \int_Q G(e^x) e^{2x} dx .$$

Then by Jensen's inequality,

$$E[\exp(-\alpha X_Q)] \leqq K_Q^{-1} E\left[\int_Q G(e^x) e^{2x} \cdot \exp\left\{-\alpha K_Q l\left(\sigma_1, \frac{x-\log|z|}{\log r/|z|}\right)\right\} dx\right]$$

$$= K_Q^{-1} \int_Q G(e^x) e^{2x} E\left[\exp\left\{-\alpha K_Q l\left(\sigma_1, \frac{x-\log|z|}{\log r/|z|}\right)\right\}\right] dx .$$

If $0 < s \leqq 1$, then it is well-known that

$$E[\exp\{-\alpha l(\sigma_1, s)\}] = \frac{1}{2\pi(2s\alpha+1)} . \tag{2.8}$$

Now we take $Q = \left\{x ;\ 1/2 \leqq \frac{x-\log|z|}{\log r/|z|} \leqq 1\right\}$. From (2.7) and (2.8) we have

$$E[\exp(-\alpha X_Q)] \leqq \frac{1}{2\pi(K_Q\alpha+1)} .$$

If $X \equiv X_R$, then $X \geqq X_Q$.

$$P_{|z|}(X \leqq k) \leqq e^{\alpha k} E[e^{-\alpha X}] \leqq e^{\alpha k} E[e^{-\alpha X} Q]$$

$$\leqq \frac{e^{\alpha k}}{2\pi(K_Q\alpha+1)} .$$

Put $\alpha = k^{-1}$. Then we have

$$P_{|z|}(X \leqq k) \leqq \text{const.} \frac{1}{K_Q k^{-1}+1} . \tag{2.9}$$

On the other hand if $0 < |z| \leqq r^\gamma$ with $0 < \gamma < 1$,

$$\int_0^{\tau_r} G(|Z_s|)ds \geqq (1-\gamma) \log r \cdot X . \tag{2.10}$$

From (2.9) and (2.10), we have

$$P_z\Big(\int_0^{\tau_r} G(|Z_s|)ds\Big) \leqq P\Big(X \leqq \frac{1}{(1-\gamma)\log r}\Big)$$

$$\leqq \text{const.} \frac{1}{K_Q(1-\gamma)k^{-1}\log r+1}.$$

ii) From Chebyshev's inequality, it follows

$$P_z\Big(\int_0^t g(Z_s)ds > k\Big) \leqq k^{-1}E_z\Big[\int_0^t g(Z_s)ds\Big].$$

We see easily that this is dominated by $c_2 k^{-1}(\log t+1)$. □

LEMMA 4. *Suppose* $r>1$ *and* f *is bounded on* $S_r=\{z: |z|=r\}$. *If* $|z|<1$, *then uniformly with* z, *we have*

$$E_z[f(Z_{\tau_r})] = E_1[f(Z_{\tau_r})] + O(r^{-1}). \tag{2.11}$$

PROOF. This results from the estimation of Poisson kernel, that is,

$$\left|\frac{r^2-|z|^2}{|y-z|^2} - \frac{r^2-1}{|y-1|^2}\right| = O(r^{-1}) \quad \text{if } |y|=r. \quad \square$$

PROOF OF i) OF PROPOSITION. Set $S_\lambda \equiv \inf\Big\{t>0\,;\ \int_0^t |Z_s|^{-2}ds > \lambda\Big\}$ and $\mu \equiv 1/2 \times \delta^{1/6}\sqrt{\lambda}$. We can carry out the proof by dividing the left hand side of (1.5) into three parts which correspond to the position of Z_{S_λ}.

1°) $S_\lambda \leqq \tau_{e^\mu} \wedge \tau_{e^{-\mu}}$ case.

$$P_1\Big(\int_0^\sigma |Z_s|^{-2}ds > \beta\lambda,\ (\log^+\sigma)^2 \leqq \delta\lambda,\ S_\lambda \leqq \tau_{e^\mu} \wedge \tau_{e^{-\mu}}\Big)$$

$$\leqq P_1\Big(\int_0^{\exp\sqrt{\delta\lambda}} |Z_s|^{-2}ds > \beta\lambda,\ S_\lambda \leqq \tau_{e^\mu} \wedge \tau_{e^{-\mu}},\ S_\lambda < \sigma\Big).$$

Since $\int_0^{\exp\sqrt{\delta\lambda}} |Z_s|^{-2}ds \leqq \int_0^{S_\lambda} |Z_s|^{-2}ds + \int_{S_\lambda}^{e^{2\mu}+S_\lambda} |Z_s|^{-2}ds$, the strong Markov property implies that the above is dominated by

$$E_1\Big[P_{Z_{S_\lambda}}\Big(\int_0^{e^{2\mu}} |Z_s|^{-2}ds > (\beta-1)\lambda\Big) : S_\lambda \leqq \tau_{e^\lambda} \wedge \tau_{e^{-\lambda}},\ S_\lambda < \sigma\Big]. \tag{2.12}$$

From Lemma 2 we have on $\{S_\lambda \leqq \tau_{e^\mu} \wedge \tau_{e^{-\mu}}\}$,

$$P_{Z_{S_\lambda}}\left(\int_0^{e^{2\mu}}|Z_s|^{-2}ds>(\beta-1)\lambda\right)\leqq \text{const.}\frac{\delta^{1/6}\sqrt{\lambda}}{\sqrt{(\beta-1)\lambda}}+c\cdot \boldsymbol{I}_0(e^q)^{-1}$$
$$=\text{const.}\delta^{1/6}+c\cdot\boldsymbol{I}_0(e^q)^{-1}$$

with $q=(\varepsilon/(2-\varepsilon))\mu$. Therefore

$$(2.12)\leqq \text{const. }\delta^{1/6}P_1(S_\lambda<\sigma)+\text{const. }\boldsymbol{I}_0(e^q)^{-1}.$$

2°) $\tau_{e^\mu}<S_\lambda$ case.

$$P_1\left(\int_0^\sigma|Z_s|^{-2}ds>\beta\lambda,\ (\log^+\sigma)\leqq\delta\lambda,\ \tau_{e^\mu}<S_\lambda\right)$$
$$\leqq P_1\left(\int_0^\sigma|Z_s|^{-2}ds>\beta\lambda,\ (\log^+\sigma)\leqq\delta\lambda,\ \tau_{e^\mu}<S_\lambda\right)$$
$$\leqq P_1(\tau_{e^\mu}<e^{\sqrt{\delta\lambda}}).$$

Lemma 1 implies the above is dominated by

$$C\cdot\boldsymbol{I}_0(e^{\mu-\sqrt{\delta\lambda}/2})^{-1}\quad(\mu-\sqrt{\delta\lambda}>0),$$

which is $d\lambda^\alpha$-integrable.

3°) $\tau_{e^{-\mu}}<S_\lambda$ case.

Before estimating the third part, we should notice the invariance of complex Brownian motion for a conformal transformation on $\boldsymbol{C}$, i.e. $z\mapsto 1/z$. This maps Z_t to $\tilde{Z}_{F_t}$ where $\tilde{Z}_t$ is a complex Brownian motion starting from 1 and $F_t=\int_0^t|Z_s|^{-4}ds$. $\tilde{Z}_t$ has a filtration $\tilde{\mathscr{F}}_t$ defind by $\tilde{\mathscr{F}}_t\equiv\mathscr{F}_{F_t^{-1}}$ so that F_σ is an $\tilde{\mathscr{F}}_t$-stopping time if σ is an $\mathscr{F}_t$-stopping time. Moreover F_σ belongs to $S(\tilde{\mathscr{F}})$ if $\sigma\in S(\mathscr{F})$. From now on we put tildes on heads of quantities relating to $\tilde{Z}_t$. Then $F_{S_\lambda}=\tilde{S}_\lambda$ and $F_{\tau_a}=\tilde{\tau}_{a^{-1}}$ are valid. Here we had better notice the following lemma.

LEMMA 5. *Put* $I_i(\omega)=1_{(\sigma<S_{(\beta-i)\lambda})}(\omega)$ *with* $\sigma\in S(\mathscr{F})$ *and* $i=0,1,2,\cdots\leqq\beta$. *Let* T *be any stopping time. Then we have*

i) $$S_{\beta\lambda}\circ\theta_T\leqq S_{(\beta+1)\lambda}-T\quad \textit{if } S_{\beta\lambda}\geqq T\tag{2.13}$$

ii) $$S_\lambda-T\leqq S_\lambda\circ\theta_T\quad \textit{if } S_\lambda\geqq T\tag{2.14}$$

iii) $$I_{i+1}(\theta_T\omega)\leqq I_i(\omega)\quad \textit{if } \sigma\geqq T\tag{2.15}$$

iv) $$I_i(\theta_T\omega)\geqq I_i(\omega)\quad \textit{if } \sigma\geqq T\tag{2.16}$$

PROOF. i) and ii) follow from the definition of S_λ. (iii) and (iv) are obvious from i) and ii). $\square$

Now, we start with the estimation of the third part. The case $S_{\beta\lambda} > \tau_{e\mu}$ has been considered already in 2°). Hence we can assume $S_{\beta\lambda} \leqq \tau_{e\mu}$.

$$
\begin{aligned}
&P(S_{\beta\lambda} < \sigma < e^{\sqrt{\delta\lambda}},\ \tau_{e-\mu} < S_\lambda,\ S_{\beta\lambda} \leqq \tau_{e\mu}) \\
&\leqq P_1(S_{\beta\lambda} < \sigma \wedge \tau_{e\mu},\ \tau_{e-\mu} < S_\lambda,\ \tau_{e-\mu} < \tau_{e\tilde{\mu}}) + P_1(\tau_{e\tilde{\mu}} \leqq e^{\sqrt{\delta\lambda}}) \\
&= J_1 + J_2\,, \qquad (2.17)
\end{aligned}
$$

where $\tilde{\mu} = (1/2)\delta^{5/12}\sqrt{\lambda}$. Converting quantities relative to $\{Z_t\}$ into the corresponding quantities relative to $\{\tilde{Z}_t\}$, we have

$$
J_1 = P_1(\tilde{S}_{\beta\lambda} < \tilde{\sigma} \wedge \tilde{\tau}_{e-\mu},\ \tilde{\tau}_{e\mu} < \tilde{S}_\lambda,\ \tilde{\tau}_{e\mu} < \tilde{\tau}_{e-\tilde{\mu}})\,.
$$

Since $\tilde{\sigma}$ and $\tilde{\tau}_{e-\mu} \in S(\tilde{\mathfrak{F}})$, we have, using i) of Lemma 5,

$$
\begin{aligned}
J_1 &\leqq E_1[P_{\tilde{Z}_{\tilde{\tau}_{e\mu}}}(\tilde{S}_{(\beta-1)\lambda} < \tilde{\sigma} \wedge \tilde{\tau}_{e-\mu}) \cdot I_0(\omega)\,;\ \tilde{\tau}_{e\mu} < \tilde{\tau}_{e-\tilde{\mu}}] \\
&= E_1[P_{\tilde{Z}_{\tilde{\tau}_{e\mu}}}(\tilde{S}_{(\beta-1)\lambda} < \tilde{\sigma} \wedge \tilde{\tau}_{e-\mu}) \cdot I_0(\omega)] \\
&\quad - E_1[P_{\tilde{Z}_{\tilde{\tau}_{e\mu}}}(\tilde{S}_{(\beta-1)\lambda} < \tilde{\sigma} \wedge \tilde{\tau}_{e-\mu}) \cdot I_0(\omega)\,;\ \tilde{\tau}_{e\mu} \geqq \tilde{\tau}_{e-\tilde{\mu}}]\,. \qquad (2.18)
\end{aligned}
$$

Using the strong Markov property and iii) of Lemma 5 we have

$$
\begin{aligned}
&E_1[P_{\tilde{Z}_{\tilde{\tau}_{e\mu}}}(\tilde{S}_{(\beta-1)\lambda} < \tilde{\sigma} \wedge \tilde{\tau}_{e-\mu}) \cdot I_0(\omega)\,;\ \tilde{\tau}_{e\mu} \geqq \tilde{\tau}_{e-\tilde{\mu}}] \\
&\geqq E_1[E_{\tilde{Z}_{\tilde{\tau}_{e-\tilde{\mu}}}}[P_{\tilde{Z}_{\tilde{\tau}_{e\mu}}}(\tilde{S}_{(\beta-1)\lambda} < \tilde{\sigma} \wedge \tilde{\tau}_{e-\mu}) \cdot I_1(\omega)]\,;\ \tilde{\tau}_{e\mu} \geqq \tilde{\tau}_{e-\hat{\mu}}]\,.
\end{aligned}
$$

If $\hat{\mu} < \mu$, similarly we see

$$
\begin{aligned}
&E_{\tilde{Z}_{\tilde{\tau}_{e-\tilde{\mu}}}}[P_{\tilde{Z}_{\tilde{\tau}_{e\mu}}}(\tilde{S}_{(\beta-1)\lambda} < \tilde{\sigma} \wedge \tilde{\tau}_{e-\mu}) \cdot I_1(\omega)] \\
&\geqq E_{\tilde{Z}_{\tilde{\tau}_{e-\tilde{\mu}}}}[E_{\tilde{Z}_{\tilde{\tau}_{e\hat{\mu}}}}[P_{\tilde{Z}_{\tilde{\tau}_{e\mu}}}(\tilde{S}_{(\beta-1)} < \tilde{\sigma} \wedge \tilde{\tau}_{e-\mu}) \cdot I_2(\omega)]]\,.
\end{aligned}
$$

Lemma 4 implies this equals

$$
E_1[E_{\tilde{Z}_{\tilde{\tau}_{e\hat{\mu}}}}[P_{\tilde{Z}_{\tilde{\tau}_{e\mu}}}(\tilde{S}_{(\beta-1)\lambda} < \tilde{\sigma} \wedge \tilde{\tau}_{e-\mu}) \cdot I_2(\omega)]] + O(e^{-\hat{\mu}})\,.
$$

However, using iv) of Lemma 5, this is dominated from below by

$$
E_1[P_{\tilde{Z}_{\tilde{\tau}_{e\mu}}}(\tilde{S}_{(\beta-1)\lambda} < \tilde{\sigma} \wedge \tilde{\tau}_{e-\mu}) \cdot I_2(\omega)] + O(e^{-\hat{\mu}})\,.
$$

Therefore (2.18) is dominated by

$$E_1[P_{\tilde{Z}_{\tilde{\tau}_{e^\mu}}}(\tilde{S}_{(\beta-1)\lambda}<\tilde{\sigma}\wedge\tilde{\tau}_{e^{-\mu}})\cdot I_2(\omega)]\cdot P_1(\tilde{\tau}_{e^\mu}<\tilde{\tau}_{e^{-\hat{\mu}}})$$
$$+E_1[P_{\tilde{Z}_{\tilde{\tau}_{e^\mu}}}(\tilde{S}_{(\beta-1)\lambda}<\tilde{\sigma}\wedge\tilde{\tau}_{e^{-\mu}})\cdot\{I_0(\omega)-I_2(\omega)\}]+O(e^{-\hat{\mu}})\,.$$

It is easily seen that by using strong Markov property conversely we have

$$\begin{aligned}\text{the first term} &\leqq P_1(\tilde{\tau}_{e^\mu}<\tilde{\tau}_{e^{-\hat{\mu}}})\cdot P_1(\tilde{S}_\lambda<\tilde{\sigma})\\ &=\frac{\hat{\mu}}{\hat{\mu}+\mu}\cdot P_1(S_\lambda<\sigma)\\ &=\frac{\delta^{5/12}}{\delta^{5/12}+\delta^{1/6}}\cdot P_1(S_\lambda<\sigma)\\ &\leqq\delta^{1/4}\cdot P_1(S_\lambda<\sigma)\,. \end{aligned}\tag{2.19}$$

Similarly we have

$$\begin{aligned}\text{the second term} &\leqq P_1(\tilde{S}_{(\beta-1)\lambda}<\tilde{\sigma}\wedge\tilde{\tau}_{e^{-\mu}},\ \tilde{S}_{(\beta-2)\lambda}<\tilde{\tau}_{e^\mu}<\tilde{S}_{(\beta-1)\lambda}\wedge\sigma)\\ &\leqq P_1(\tilde{S}_{(\beta-1)\lambda}<\tilde{\sigma},\ \tilde{S}_\lambda<\tilde{\tau}_{e^{-\mu}}\wedge\tau_{e^\mu},\ \tilde{S}_{(\beta-2)\lambda}<\tilde{\tau}_{e^{2\mu}})\\ &\leqq\delta^{1/6}\cdot P_1(S_\lambda<\sigma)\,. \end{aligned}\tag{2.20}$$

On the other hand Lemma 1 shows

$$J_2\leqq C\cdot \boldsymbol{I}_0(e^\nu)^{-1}\tag{2.21}$$

with $\nu=\frac{1}{2}\delta^{5/12}\sqrt{\lambda}-\frac{1}{2}\sqrt{\delta\lambda}>0$. From (2.18)$\sim$(2.21)

$$\begin{aligned}&P_1(S_{\beta\lambda}<\sigma<e^{\sqrt{\delta\lambda}},\ \tau_{e^{-\mu}}<S_\lambda,\ S_{\beta\lambda}<\tau_{e^\mu})\\ &\leqq(\delta^{1/4}+\delta^{1/6})\cdot P_1(S_\lambda<\sigma)+C\cdot\boldsymbol{I}_0(e^\nu)^{-1}+O(e^{-\hat{\mu}})\,. \end{aligned}\tag{2.22}$$

Putting together 1°)$\sim$3°) and setting $h_1(\delta)=\text{const.}\ \delta^{1/6}$ and $H_1(\lambda)=C\cdot\boldsymbol{I}_0(e^\nu)^{-1}+O(e^{-\hat{\mu}})$ lead i).

PROOF OF ii). There exists λ_0 such that $e^{\sqrt{(\beta-1)\lambda}}\leqq e^{\sqrt{\beta\lambda}}-e^{\sqrt{\lambda}}$ for any $\lambda\geqq\lambda_0$. Choose γ and ε as $\frac{1}{2-\varepsilon}\leqq\gamma\leqq\frac{2-\varepsilon}{2(2+\varepsilon)}\sqrt{\beta-1}$ and $0<\varepsilon<2$. If $\lambda\geqq\lambda_0$,

$$P_1\left((\log^+\sigma)^2>\beta\lambda,\ \int_0^\sigma|Z_s|^{-2}ds\leqq\delta\lambda,\ e^{\sqrt{\lambda}}<\tau_{e^{\gamma\sqrt{\lambda}}}\right)$$

$$=P_1\left(\sigma>e^{\sqrt{\beta\lambda}},\ \int_0^\sigma|Z_s|^{-2}ds\leqq\delta\lambda,\ e^{\sqrt{\lambda}}<\tau_{e^{\gamma\sqrt{\lambda}}},\ e^{\sqrt{\lambda}}<\sigma\right)$$

$$=E_1\left[P_{Z_{e^{\sqrt{\lambda}}}}\left(\int_0^{e^{\sqrt{\beta\lambda}-\sqrt{\lambda}}}|Z_s|^{-2}ds\leqq\delta\lambda\right);\ e^{\sqrt{\lambda}}<\tau_{e^{\gamma\sqrt{\lambda}}},e^{\sqrt{\lambda}}<\sigma\right]$$

$$\leqq E_1\left[P_{Z_{e^{\sqrt{\lambda}}}}\left(\int_0^{e^{\sqrt{(\beta-1)\lambda}}}|Z_s|^{-2}ds\leqq\delta\lambda\right);\ e^{\sqrt{\lambda}}<\tau_{e^{\gamma\sqrt{\lambda}}},e^{\sqrt{\lambda}}<\sigma\right]$$

$$\leqq E_1\left[P_{Z_{e^{\sqrt{\lambda}}}}(e^{\sqrt{(\beta-1)\lambda}}\leqq\tau_{e^a});\ e^{\sqrt{\lambda}}<\tau_{e^{\gamma\sqrt{\lambda}}}\right]$$

$$+E_1\left[P_{Z_{e^{\sqrt{\lambda}}}}\left(\int_0^{\tau_{e^a}}|Z_s|^{-2}ds\leqq\delta\lambda\right);\ e^{\sqrt{\lambda}}<\tau_{e^{\gamma\sqrt{\lambda}}},e^{\sqrt{\lambda}}<\sigma\right]$$

with $a=\sqrt{(\beta-1)\lambda}/(2+\varepsilon)$. From Lemma 1

$$\text{the above first term}\leqq\exp\left(-\frac{1}{2+\varepsilon}\sqrt{(\beta-1)\lambda}\right),$$

$$\text{and the second term}=P((a-\gamma)^2\sigma_1\leqq\delta\lambda)$$

$$\leqq P(\sigma_1\leqq b\delta)$$

with $b=4(2+\varepsilon)\varepsilon^{-2}(\beta-1)^{-1}$. Therefore if $\lambda_0\leqq\lambda$,

$$P_1\left((\log^+\sigma)^2>\beta\lambda,\int_0^\sigma|Z_s|^{-2}ds\leqq\delta\lambda,e^{\sqrt{\lambda}}<\tau_{e^{\gamma\sqrt{\lambda}}}\right)$$

$$\leqq P(\sigma_1\leqq\delta)P_1(\sigma>e^{\sqrt{\lambda}})+\exp\left(-\frac{\varepsilon}{2+\varepsilon}\sqrt{(\beta-1)\lambda}\right).$$

From this and Lemma 1 if we set

$$h_2(\delta)=P(\sigma_1\leqq b\delta),$$

$$H_2(\lambda)=\exp\left(-\frac{\varepsilon}{2+\varepsilon}\sqrt{(\beta-1)\lambda}\right)+c_1\boldsymbol{I}_0(\exp c_2\sqrt{\lambda})^{-1}+I_{[0,\lambda_0)}(\lambda),$$

where $I_{[0,\lambda_0)}(\lambda)$ is the indicator function of $[0,\lambda_0)$, then we have ii).

PROOF OF iii). Set $S_\lambda\equiv\inf\left\{t\geqq0:\int_0^t g(Z_s)ds>\lambda\right\}$. If $\lambda\geqq1$,

$$P_1\left(\int_0^\sigma g(Z_s)ds>\beta\lambda,\log^+\sigma\leqq\delta\lambda\right)\leqq P_1\left(\int_0^{e^{\delta\lambda}}g(Z_s)ds>\beta\lambda,S_\lambda<\sigma\right)$$

$$\leqq E_1\left[P_1\left(\int_0^{e^{\delta\lambda}}g(Z_s)ds>(\beta-1)\lambda\right):S_\lambda<\sigma\right]$$

$$\leqq \frac{C}{(\beta-1)\lambda}(\delta\lambda+1)P_1(S_\lambda<\sigma) \quad \text{(By Lemma 3)}$$

$$\leqq \frac{C}{\beta-1}\delta\cdot P_1(S_\lambda<\sigma)+\frac{C}{\beta-1}\cdot\lambda^{-1}.$$

The second term is integrable for $d\lambda^\alpha$ $(0<\alpha<1)$ on $[1,\infty)$. Therefore if we put the $h_3(\delta)=(C/(\beta-1))\delta$ and $H_3(\lambda)=$the second term$\times I_{[1,\infty)}(\lambda)+I_{[0,1)}(\lambda)$, we have iii).

As for the proof of iv), we carry out the same procedure as ii) using Lemma 3 instead of Lemma 2. Then we have iv).

§ 3. Application to Nevanlinna theory.

R. Nevanlinna studied the value distribution of meromorphic functions and established his theory which we call usually Nevanlinna theory. This theory has been contributed to and extended by many authors (cf. [4,7]), but we here consider the simplest case, that is, meromorphic functions on $\boldsymbol{C}$. His SMT is usually proved with differential geometric method. We, however, return to his original proof and give a probabilistic proof to his lemma on logarithmic derivative which played the main role in his proof.

LEMMA ON LOGARITHMIC DERIVATIVE ([4,6]). *Let f be meromorphic function on $\boldsymbol{C}$. Then we have that*

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{|f'(re^{i\theta})|}{|f(re^{i\theta})|}d\theta\leqq O(\log T(r)+\log r)$$

is valid except for finite length of r.

We show this lemma via our inequalities in § 1. We need the following lemma.

LEMMA 6. *Let $K(z)$ be a positive function on $\boldsymbol{C}^n$ and Z_t be a Brownian motion on $\boldsymbol{C}^n$ $(Z_0=0)$. For any $\delta>0$ and $o<\alpha<1$, there exist positive constants c_1 and c_2 such that*

i) $$EK(Z_{\tau_r})\leqq c_1 r^{(2n-1)\delta}\left\{E\left[\int_0^{\tau_r}K(Z_s)ds\right]\right\}^{(1+\delta)2} \quad //\delta(r).$$

ii) $$EK(Z_{\tau_r})\leqq c_2 r^{\gamma}\left\{E\left(\int_0^{\tau_r}K(Z_s)ds\right)^{2\alpha}\right\}\frac{(1+\delta)^2}{2} \quad //\delta(r),$$

with $\gamma=(2n-1)\delta+2(1-\alpha)(1+\delta)^2$. *The notation "*$//\delta(r)$*" means that the inequalities hold except for the set of finite length of* r.

This lemma is a simple corollary of the following lemma on monotone increasing functions.

LEMMA 7 [6, 7]. *Let* $h(r)>0$ *for* $r\geqq 0$ *and be monotone increasing. Then* $h(r)$ *is differentiable at almost all points and for* $\delta>0$

$$\frac{d}{dr}h(r)\leqq\{h(r)\}^{1+\delta}\quad //\delta(r),$$

holds.

PROOF OF LEMMA ON LOGARITHMIC DERIVATIVE. Let $0<\alpha<1/2$. Jensen's inequality and Lemma 7 imply

$$\begin{aligned}E\left[\log^+\frac{|f'(Z_{\tau_r})|}{|f(Z_{\tau_r})|}\right]&=\frac{1}{\alpha}E\left[\log^+\frac{|f'(Z_{\tau_r})|^\alpha}{|f(Z_{\tau_r})|^\alpha}\right]\\&\leqq\frac{1}{\alpha}\log E\left[\frac{|f'(Z_{\tau_r})|^\alpha}{|f(Z_{\tau_r})|^\alpha}\right]+O(1)\\&\leqq\text{const.}\log E\left(\int_0^{\tau_r}\frac{|f'(Z_s)|^2}{|f(Z_s)|^2}ds\right)^{2\alpha}+O(\log r)\quad //\delta(r).\end{aligned}$$

Using Corollary 1 in § 1, we have

$$\begin{aligned}\text{the R.H.S.}&\leqq\text{const.}\log^+E\left(\log^+\int_0^{\tau_r}|f'(Z_s)|^2ds\right)^{2\alpha}+O(\log r)\quad //\delta(r)\\&\leqq\text{const.}\log^+E\left(\int_0^{\tau_r}\frac{|f'(Z_s)|^2}{(1+|f(Z_s)|^2)^2}ds\right)^{2\alpha}+O(\log r)\quad //\delta(r).\end{aligned}$$

Since $0<2\alpha<1$ and

$$T(r)=E\left[\int_0^{\tau}\frac{|f'(Z_s)|^2}{(1+|f(Z_s)|^2)^2}ds\right],$$

then using Jensen's inequality we get finally

$$E\log^+\frac{|f'(Z_{\tau_r})|}{|f(Z_{\tau_r})|}\leqq\text{const.}\log^+T(r)+O(\log r)\quad //\delta(r).\qquad\square$$

Although we have devoted ourselves to the case of meromorphic functions on $\boldsymbol{C}$, our methods are available for the case of ones defined on a disk of $\boldsymbol{C}$, $\boldsymbol{C}^n$ or a ball of $\boldsymbol{C}^n$. Let F be a meromorphic function on

$\{z\in \boldsymbol{C}^n;\ |z|<R\}$ $(n\geqq 1$ and $R\leqq\infty)$ and Z_t be a Brownian motion on $\boldsymbol{C}^n$ with $Z_0=0$. There exists a complex Brownian motion $\tilde{Z}_t$ on $\boldsymbol{C}$ such that

$$F(Z_t)=\tilde{Z}_{\psi_t} \qquad \text{if } t<\tau_R\,,$$

with $\psi_t=\int_0^t \sum_{i=1}^n \left|\frac{\partial F}{\partial z_i}\right|^2 (Z_s)ds$. Hence applying our theorem again to this case, we can rewrite Corollary 1 more generally.

COROLLARY 2. *Let $G(r)$ be as in Corollary 1. For $0<\alpha<1$ there exist positive constants C_i, $i=1,\cdots,8$ such that for any meromorphic functions F on $\{z\in C^n;\ |z|<R\}$ $(n\geqq 1,\ R\leqq\infty)$*

i)
$$C_1E(\log^+\psi_{\tau_r})^\alpha-C_2\leqq E\left(\sum_{i=1}^n\int_0^{\tau_r}\frac{|F_{z_i}(Z_s)|^2}{|F(Z_s)|^2}ds\right)^{\alpha/2}$$
$$\leqq C_3E(\log^+\psi_{\tau_r})^\alpha+C_4\,,$$

ii)
$$C_5E(\log^+\psi_{\tau_r})^\alpha-C_6\leqq E\left(\sum_{i=1}^n\int_0^{\tau_r}G(|F(Z_s)|)|F_{z_i}(Z_s)|^2ds\right)^\alpha$$
$$\leqq C_7E(\log^+\psi_{\tau_r})^\alpha+C_8\,,$$

with $r<R$ and $F_{z_i}(z)=\frac{\partial F}{\partial z_i}(z)$.

In the case of $\boldsymbol{C}^n$ we can use Lemma 7. Then we carry out the same procedure as the case $\boldsymbol{C}$ to prove the lemma on logarithmic derivative which was proved by Vitter [8]. Let F be a meromorphic function on $\boldsymbol{C}^n$ with $F\equiv f_1/f_0$; f_0, f_1 are some holomorphic functions on $\boldsymbol{C}^n$. Define characteristic function of F;

$$T_F(r)=\int_{\partial B(r)}\log(|f_0|^2+|f_1|^2)d\pi_r$$

where $\partial B(r)=\{z\in\boldsymbol{C}^n;\ |z|=r\}$ and $d\pi_r$ is the normalized uniform measure on $\partial B(r)$. We can show easily that

$$T_F(r)=E\left[\sum_{i=1}^n\int_0^{\tau_r}\frac{|F_{z_i}(Z_s)|^2}{(1+|F(Z_s)|^2)^2}ds\right].$$

LEMMA ON LOGARITHMIC DERIVATIVE FOR SEVERAL VARIABLES [8].

$$\int_{\partial B(r)}\log^+\frac{|F_{z_i}|}{|F|}d\pi_r\leqq O(\log T_F(r)+\log r) \quad //\delta(r)\,.$$

Vitter proved this via differential geometric method and showed the second main theorem for meromorphic maps from $\boldsymbol{C}^n$ to $\boldsymbol{P}^m(\boldsymbol{C})$ with this lemma.

In the case of a disk or a ball of $\boldsymbol{C}^n$ we have only to modify Lemma 6 such that the exceptional set should be measured by the hyperbolic length.

At the end of this paper we remark on the proof of SMT. Our inequalities avail the usual proof as follows.

1) Ahlfors type. [4].
Set for $a \in \boldsymbol{P}_1$

$$\rho_a(w)dw\wedge d\bar{w} \equiv \frac{1}{(\log[w, a]^2)^2[w, a]^2}\cdot\frac{dw\wedge d\bar{w}}{(|w|^2+1)^2},$$

where $[w, a]=\dfrac{|w-a|}{\sqrt{|w|^2+1}\sqrt{|a|^2+1}}$. And set for a meromorphic function f

$$f^*\left(\prod_{i=1}^{q}\rho_{a_i}(w)dw\wedge d\bar{w}\right)\equiv\zeta(z)dz\wedge d\bar{z},\ \ \zeta(z)=\prod_{i=1}^{q}\rho_{a_i}(f(z))|f'(z)|^2,$$

with $a_1, \cdots, a_q \in \boldsymbol{P}_1$ distinct points. It is easy to see that SMT is reduced to

$$E[\log\zeta(Z_{\tau_r})]\leqq O(\log T(r)+\log r)\quad //\delta(r).$$

Since $\prod_{i=1}^{q}\rho_{a_i}(z)\in L^1(\boldsymbol{C})$, we can apply our theorem and carry out the similar proof of this to the lemma on logarithmic derivative.

2) [6]. Let $o<\alpha<1$, and let f and a_i, $i=1, \cdots, q$ be as 1). Set

$$\phi(w)dw\wedge d\bar{w}\equiv\prod_{i=1}^{q}\frac{(|w|^2+1)^\alpha(|a_i|^2+1)^\alpha}{|w-a_i|^{2\alpha}}\cdot\frac{dw\wedge d\bar{w}}{(|w|^2+1)^2},$$

$$\xi(z)dz\wedge d\bar{z}\equiv f^*(\phi(w)dw\wedge d\bar{w}).$$

Then $\phi(z)\in L^1(\boldsymbol{C})$. For the same reason of 1), we can show

$$E[\log\xi(Z_{\tau_r})]\leqq O(\log T(r)+\log r)\quad //\delta(r).$$

From this estimate we have

$$\textstyle\sum_{i=1}^{q}\alpha\cdot m(a_i, r)+N_1(r)\leqq 2T(r)+O(\log T(r)\log r)\quad //\delta(r).$$

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