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# *Inverse spectral problem for systems of ordinary differential equations of first order, I.*

Dedicated to Professor Hiroshi Fujita on his 60th birthday

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## § 1. Introduction.

We consider a system of ordinary differential equations of first order in the interval $[0,1]$:

$$\tag{1.1} \begin{cases} \dfrac{du_2(x)}{dx}+p_{11}(x)u_1(x)+p_{12}(x)u_2(x)=\lambda u_1(x) \\ \dfrac{du_1(x)}{dx}+p_{21}(x)u_1(x)+p_{22}(x)u_2(x)=\lambda u_2(x) \qquad (0\leqq x\leqq 1) \end{cases}$$

with boundary conditions

$$\tag{1.2} u_2(0)+hu_1(0)=0$$

and

$$\tag{1.3} u_2(1)+Hu_1(1)=0.$$

Here the $p_{ij}(x)$ $(1\leqq i, j\leqq 2)$ are real-valued $C^1$-functions defined on $[0,1]$ and $h, H\in \boldsymbol{R}\cup\{\infty\}$.
If $h=\infty$ and $H=\infty$ in (1.2) and (1.3), respectively, then we regard (1.2) and (1.3) as $u_1(0)=0$ and $u_1(1)=0$, respectively (cf. Remark 3 below). Moreover the parameter $\lambda$ corresponds to the eigenvalue.

The eigenvalue problem (1.1)-(1.3) describes proper vibrations for various phenomena such as an electric oscillation in a transmission line, a vibration of a string with viscous drag, etc. In Yamamoto [42], we consider such proper vibrations and discuss inverse spectral problems for them.

Let us assume that $p_{ij}$ $(1\leqq i, j\leqq 2)$, $h$ and $H$ are given. Then a problem of finding eigenvalues of (1.1)-(1.3) is one of what are called **forward problems**, and such a problem is nothing but the determination of eigenfrequencies.

On the other hand, the following problem arises from both mathematical and practical interest:

*to study whether a physical system realizing the specified characteristics of vibrations (e.g. eigenfrequencies) is unique.*

This is an identification problem, and, in this problem, assuming that the eigenvalues are given, we determine the coefficients $p_{ij}(x)$ $(1\leqq i,\ j\leqq 2)$ and the real constants $h, H$. Thus in contrast with the forward problem, the identification problem is an **inverse problem**, and more precisely, an **inverse spectral problem** in our case.

The purpose of this paper and the forthcoming one [42] is to study an inverse spectral problem for (1.1)-(1.3).

For the Sturm-Liouville equation

$$(1.4)\qquad \begin{cases} -\dfrac{d^2u(x)}{dx^2}+p(x)u(x)=\lambda u(x) & (0\leqq x\leqq 1)\\ \dfrac{du}{dx}(0)-hu(0)=0 \\ \dfrac{du}{dx}(1)+Hu(1)=0, \end{cases}$$

inverse spectral problems have been studied in detail. We refer to Borg [1], Gel'fand and Levitan [3], Hald [4], [5], [6], Hochstadt [7], [8], [9], Hochstadt and Lieberman [10], Isaacson and Trubowitz [11], Iwasaki [12], Levinson [14], Levitan and Gasymov [15], Mizutani [19], Suzuki [33]-[38], and Willis [41]. Furthermore, for inverse spectral problems for equations of higher order such as $(-1)^m\dfrac{d^{2m}u(x)}{dx^{2m}}+p(x)u(x)=\lambda u(x)$ $(0\leqq x\leqq 1;\ m=2, 3, \cdots)$, we have McLaughlin [17], [18], Sahnovič [30], [31] and Uchiyama [40].

On the other hand, there has been little work for inverse spectral problems for systems of ordinary differential equations as (1.1).

Notice that by eliminating $u_1$ or $u_2$ in the equation (1.1) so as to get a single equation of second order, we cannot reduce our problem (1.1)-(1.3) to the Sturm-Liouville problem, except for particular cases such as $p_{11}(x)=p_{21}(x)=p_{22}(x)=0$ $(0\leqq x\leqq 1)$ and $h=H=0$. Conversely Sturm-Liouville problems of the following form are reduced to our problem (1.1)-(1.3):

$$(1.4)' \qquad \begin{cases} -\dfrac{1}{\rho(x)}\dfrac{d}{dx}\left(E(x)\dfrac{du(x)}{dx}\right)=\lambda u(x) & (0\leqq x\leqq 1) \\ u(0)=0 \quad \text{or} \quad \dfrac{du}{dx}(0)=0 \\ u(1)=0 \quad \text{or} \quad \dfrac{du}{dx}(1)=0. \end{cases}$$

In our forthcoming paper [42], we shall discuss also an inverse spectral problem for (1.4)$'$.

Inverse spectral problems are related to inverse problems for evolutionary systems. For those problems, we refer to Kitamura and Nakagiri [13], Nakagiri [21], Nakagiri and Yamamoto [22]-[25], Murayama [20], Suzuki and Murayama [39], and Suzuki [32], [33], [38]. In [21], [23], [24] and [25], abstract evolution equations of the form $\dfrac{du(t)}{dt}+Au(t)=0$ $(t\geqq 0)$ with $u(0)=u_0$ are considered in Banach spaces, and the unique determination of an operator $A$ from observations of the solution $u(t)$ over a time interval, is studied.

In the rest of this section, we give a formulation of our problem and state our main result.

Let $L^2(0,1)$ be the Hilbert space of square integrable complex-valued functions in the interval $(0,1)$ and let $\{L^2(0,1)\}^2$ be the product space, which is a Hilbert space with an inner product defined by

$$(1.5) \qquad (u,v)=(u,v)_{\{L^2(0,1)\}^2}=\int_0^1 u_1(x)\overline{v_1(x)}dx+\int_0^1 u_2(x)\overline{v_2(x)}dx$$
$$\left(u=\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad v=\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\in\{L^2(0,1)\}^2\right),$$

where $\bar{\alpha}$ denotes the complex conjugate of $\alpha\in\boldsymbol{C}$.

Take real-valued $C^1$-functions $p_{ij}$ and $q_{ij}$ $(1\leqq i, j\leqq 2)$ defined on $[0,1]$ and $h, j, H, H^*, J, J^*\in\boldsymbol{R}\cup\{\infty\}$. We set

$$P(x)=(p_{ij}(x))_{1\leqq i,j\leqq 2}\equiv\begin{pmatrix} p_{11}(x) & p_{12}(x) \\ p_{21}(x) & p_{22}(x) \end{pmatrix},$$
$$Q(x)=(q_{ij}(x))_{1\leqq i,j\leqq 2}\equiv\begin{pmatrix} q_{11}(x) & q_{12}(x) \\ q_{21}(x) & q_{22}(x) \end{pmatrix},$$

and

$$B=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We define an operator $A_{P,h,H}$ as the realization in $\{L^2(0,1)\}^2$ of the differential operator $B\frac{d\cdot}{dx}+P(x)\cdot$ with boundary conditions $u_2(0)+hu_1(0)=0$ and $u_2(1)+Hu_1(1)=0$. That is,

$$(A_{P,h,H}u)(x)=B\frac{du(x)}{dx}+P(x)u(x) \qquad \text{for} \quad u\in\mathscr{D}(A_{P,h,H}), \tag{1.6}$$

where

$$\mathscr{D}(A_{P,h,H})=\left\{u=\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\in\{H^1(0,1)\}^2;\ u_2(0)+hu_1(0)=0,\ \ u_2(1)+Hu_1(1)=0\right\}.$$

Here $H^1(0,1)$ denotes the Sobolev space and $\{H^1(0,1)\}^2$ is its product space. Similarly we define the operators $A_{P,h,H^*}$, $A_{Q,j,J}$, $A_{Q,j,J^*}$, etc.

Let $\sigma(A_{P,h,H})$ denote the spectrum of the operator $A_{P,h,H}$. The following result is known (Russell [28], [29], for example).

PROPOSITION A. *Let $h, H\in \boldsymbol{R}\cup\{\infty\}\diagdown\{-1,1\}$. Then $\sigma(A_{P,h,H})$ consists entirely of countable eigenvalues $\lambda_n$ $(n\in \boldsymbol{Z})$ and the multiplicity of each $\lambda_n$ is one. That is,* $\dim \operatorname{Ker}(\lambda_n-A_{P,h,H})=1$.

REMARK 1. If $|h|=1$ or $|H|=1$, then the conclusion of this proposition does not necessarily hold. For example, we have $\sigma_p(A_{0,1,1})=\sigma_p(A_{0,-1,-1})=\boldsymbol{C}$ and $\sigma_p(A_{0,1,-1})=\sigma_p(A_{0,-1,1})=\varnothing$. Here $\sigma_p(\cdot)$ denotes the point spectrum of an operator under consideration.

Henceforth, for simplicity, we assume that absolute values of real constants in boundary conditions are not equal to 1.

Moreover, without loss of generality, we assume that

$$h\neq\infty, \tag{1.7}$$

in the boundary condition at $x=0$.
In fact, if $h=\infty$, that is, if the boundary condition at $x=0$ is $u_1(0)=0$, then by setting $v=\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\equiv\begin{pmatrix} u_2 \\ u_1 \end{pmatrix}$, we have a system

$$\begin{cases} \dfrac{dv_2(x)}{dx}+p_{22}(x)v_1(x)+p_{21}(x)v_2(x)=\lambda v_1(x) \\[2ex] \dfrac{dv_1(x)}{dx}+p_{12}(x)v_1(x)+p_{11}(x)v_2(x)=\lambda v_2(x) \qquad (0\leqq x\leqq 1) \end{cases} \tag{1.1$'$}$$

with the boundary conditions

$$v_2(0)+0\cdot v_1(0)=0 \tag{1.2)'$$

and

$$v_2(1)+\frac{1}{H}\cdot v_1(1)=0. \tag{1.3)'$$

The system (1.1)′-(1.3)′ is nothing but the system (1.1)-(1.3) where $h$ and $H$ are replaced by 0 and $\frac{1}{H}$, respectively.

Thus, we suppose the following assumption on the real constants in the boundary conditions.

ASSUMPTION.

$$\begin{cases} h, j \in \boldsymbol{R}\setminus\{-1, 1\}, \\ H, H^*, J, J^* \in \boldsymbol{R}\cup\{\infty\}\setminus\{-1, 1\}, \quad \text{and} \quad H\neq H^*. \end{cases} \tag{1.8}$$

In particular, $H\neq H^*$ implies that either $H$ or $H^*$ is finite.

Now our inverse spectral problem or identification problem can be stated as follows:

*Problem* A. Do the conditions

$$\begin{cases} \sigma(A_{Q,j,J})=\sigma(A_{P,h,H}) \\ \sigma(A_{Q,j,J^*})=\sigma(A_{P,h,H^*}) \end{cases} \tag{1.9}$$

imply the equalities $Q(x)=P(x)$ $(0\leqq x\leqq 1)$, $j=h$, $J=H$ and $J^*=H^*$?

Actually we discuss

*Problem* B. Assume that $P, h, H, H^*$ are given. Then characterize $(Q, j, J, J^*)$ satisfying $\sigma(A_{Q,j,J})=\sigma(A_{P,h,H})$ and $\sigma(A_{Q,j,J^*})=\sigma(A_{P,h,H^*})$.

The reason why we have to consider one pair of boundary value problems is because the spectrum of a single problem is not sufficient for the characterization of coefficients. Furthermore, as is seen from Theorem stated below (cf. Proposition 1 in [42]), nothing is added by taking the spectra of more than two boundary value problems for (1.1). Such a formulation was first introduced by Borg [1] for the Sturm-Liouville equation.

REMARK 2. For inverse spectral problems for the Sturm-Liouville equation (1.4), we have two other types of uniqueness theorems. The first is due to Hald [6], Hochstadt and Lieberman [10], Suzuki [33], [38], and Willis [41], where one set of eigenvalues determines $p(x)$ uniquely on the whole interval $[0, 1]$ provided that $p(x)$ is known on the "half" interval $[0, 1/2]$. The second treats the case where $p(x)$ in (1.4) is spatially symmetric (namely, $p(x)=p(1-x)$ for $0\leqq x\leqq 1$). That is, in Borg [1], Hald [4], [5], Hochstadt [8], Levinson [14], and Suzuki [33], [35], it is shown that one set of eigenvalues associated with symmetric boundary conditions determines such a symmetric coefficient $p$. For the latter, we further refer to Iwasaki [12], and Suzuki [36], [37].
The corresponding results can be proved also for our system (1.1)-(1.3).

In order to consider Problem B, we introduce

DEFINITION 1. Let $P=(p_{ij})_{1\leqq i,j\leqq 2}\in\{C^1[0, 1]\}^4$ and $h, H, H^*$ be fixed such that $h\in \boldsymbol{R}\setminus\{-1, 1\}$, $H, H^*\in \boldsymbol{R}\cup\{\infty\}\setminus\{-1, 1\}$, and $H\neq H^*$. We set

$$M(P, h, H, H^*)=\{(Q, j, J, J^*);\ Q=(q_{ij})_{1\leqq i,j\leqq 2}\in\{C^1[0, 1]\}^4,\ j\in \boldsymbol{R}\setminus\{-1, 1\}, J, J^*\in \boldsymbol{R}\cup\{\infty\}\setminus\{-1, 1\},\ \sigma(A_{Q,j,J})=\sigma(A_{P,h,H}) \text{ and } \sigma(A_{Q,j,J^*})=\sigma(A_{P,h,H^*}).\}, \tag{1.10}$$

and

$$\tilde{M}(P, h, H, H^*)=\{(Q, j, J, J^*);\ Q=(q_{ij})_{1\leqq i,j\leqq 2}\in\{C^1[0, 1]\}^4,\ j\in \boldsymbol{R}\setminus\{-1, 1\}, J, J^*\in \boldsymbol{R}\cup\{\infty\}\setminus\{-1, 1\},\ \sigma(A_{Q,j,J})\supset\sigma(A_{P,h,H}) \text{ and } \sigma(A_{Q,j,J^*})\supset\sigma(A_{P,h,H^*}).\}. \tag{1.11}$$

In other words, $M(P, h, H, H^*)$ denotes the totality of operators $A_{Q,j,J}$ and $A_{Q,j,J^*}$ whose spectra coincide with the spectra of the operators $A_{P,h,H}$ and $A_{P,h,H^*}$, respectively.
It is obvious that $(p, h, H, H^*)\in M(P, h, H, H^*)$. If we had $M(P, h, H, H^*)=\{(P, h, H, H^*)\}$, then the two sets of eigenvalues would determine the operators $A_{P,h,H}$ and $A_{P,h,H^*}$ uniquely. (That is, the answer to Problem A would be affirmative.) Thus, for the discussion of uniqueness or non-uniqueness in our inverse problem, it is sufficient to determine the set $M(P, h, H, H^*)$.

Throughout this paper, we note

REMARK 3. Henceforth we set

(1.12) $$\frac{1-H}{1+H}=\frac{1+H}{1-H}=-1, \qquad \text{if} \quad H=\infty.$$

Furthermore, we adopt the following notation:

(1.13) Let $\alpha, \beta \in \boldsymbol{R}$. If $H=\infty$, then the equality $\alpha+\beta H=0$ means $\beta=0$.

Then, without distinguishing the cases $H=\infty$, $J=\infty$, etc. from the cases $H\neq\infty$, $J\neq\infty$, etc., we can formally write and mathematically follow all our discussion in this paper. For example, $H=\infty$ means $u_1(1)=0$ in (1.3).

We can state our main result giving a characterization of $M(P, h, H, H^*)$:

THEOREM. (I) *We have*

(1.14) $$\tilde{M}(P, h, H, H^*)=M(P, h, H, H^*).$$

(II) *We have*

(1.15) $$(Q, j, J, J^*)\in M(P, h, H, H^*)$$

*if and only if* (1.16)-(1.19) *hold;*

(1.16) $$\begin{aligned}&\frac{1-j}{1-h}(q_{11}(x)+q_{12}(x)-q_{21}(x)-q_{22}(x)-p_{11}(x)+p_{12}(x)-p_{21}(x)+p_{22}(x))\\&+\frac{1+j}{1+h}(q_{11}(x)-q_{12}(x)+q_{21}(x)-q_{22}(x)-p_{11}(x)-p_{12}(x)+p_{21}(x)+p_{22}(x))\\&\times\exp\Big(\int_0^x(q_{11}(s)+q_{22}(s)-p_{11}(s)-p_{22}(s))ds\Big)=0 \qquad (0\leqq x\leqq 1),\end{aligned}$$

(1.17) $$\begin{aligned}&\frac{1-j}{1-h}(q_{11}(x)+q_{12}(x)-q_{21}(x)-q_{22}(x)+p_{11}(x)-p_{12}(x)+p_{21}(x)-p_{22}(x))\\&+\frac{1+j}{1+h}(-q_{11}(x)+q_{12}(x)-q_{21}(x)+q_{22}(x)-p_{11}(x)-p_{12}(x)+p_{21}(x)+p_{22}(x))\\&\times\exp\Big(\int_0^x(q_{11}(s)+q_{22}(s)-p_{11}(s)-p_{22}(s))ds\Big)=0 \qquad (0\leqq x\leqq 1),\end{aligned}$$

(1.18) $$\begin{cases}\dfrac{(1+h)(1-H)(1-j)(1+J)}{(1-h)(1+H)(1+j)(1-J)}>0\\[2ex] \log\dfrac{(1+h)(1-H)(1-j)(1+J)}{(1-h)(1+H)(1+j)(1-J)}=\displaystyle\int_0^1(q_{11}(s)+q_{22}(s)-p_{11}(s)-p_{22}(s))ds,\end{cases}$$

$$(1.19)\quad \begin{cases} \dfrac{(1+h)(1-H^*)(1-j)(1+J^*)}{(1-h)(1+H^*)(1+j)(1-J^*)}>0 \\ \log\dfrac{(1+h)(1-H^*)(1-j)(1+J^*)}{(1-h)(1+H^*)(1+j)(1-J^*)}=\displaystyle\int_0^1 (q_{11}(s)+q_{22}(s)-p_{11}(s)-p_{22}(s))ds. \end{cases}$$

From the fact that the equalities (1.16) and (1.17) can be regarded as two nonlinear integral equations of four unknown functions $q_{ij}$ $(1\leqq i,\ j\leqq 2)$, we can show that there are infinitely many $Q=(q_{ij})_{1\leqq i,j\leqq 2}$ satisfying (1.16) and (1.17). That is, the answer to Problem A is negative. We note that for the Sturm-Liouville equation, two sets of eigenvalues determine $p$ in (1.4) uniquely, because there is only one unknown coefficient. Hence our concern turns to how many coefficients in (1.1) can be uniquely determined by two sets of eigenvalues. In fact, since we have only *two* equalities (1.16) and (1.17) on $Q=(q_{ij})_{1\leqq i,j\leqq 2}$, we can determine *at most two* of four components of $Q$. These will be discussed in our forthcoming paper [42], and will be applied to some identification problems for such systems as (1.4)$'$ with more practical interest.

This paper is composed of three sections and three appendixes. In § 2, we derive a formula (a "deformation formula" according to the terminology in Suzuki [33]), which is a key in later discussion. In § 3, we prove Theorem.

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## § 2. Deformation formula.

We begin this section with the following proposition on a system of hyperbolic equations. Let

$$(2.1)\qquad \Omega=\{(x,y);\ 0<y<x<1\},$$

and we recall that $B=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

PROPOSITION 1. *Let* $P=(p_{ij})_{1\leq i,j\leq 2}$ *and* $Q=(q_{ij})_{1\leq i,j\leq 2}$ *be given in* $\{C^1[0, 1]\}^4$, *and let* $h$ *be a given real number such that* $|h|\neq 1$. *Then, for each* $r_1, r_2 \in C^1[0, 1]$, *there exists a unique* $K=K(x, y)=(K_{ij}(x, y))_{1\leq i,j\leq 2} \in \{C^1(\bar{\Omega})\}^4$ *satisfying* (2.2)-(2.5):

$$B\frac{\partial K(x, y)}{\partial x}+Q(x)K(x, y)-K(x, y)P(y)=-\frac{\partial K(x, y)}{\partial y}B \qquad ((x, y)\in\bar{\Omega}). \tag{2.2}$$

$$\begin{cases} K_{12}(x, 0)=hK_{11}(x, 0) \\ K_{22}(x, 0)=hK_{21}(x, 0) \qquad (0\leqq x\leqq 1). \end{cases} \tag{2.3}$$

$$K_{12}(x, x)-K_{21}(x, x)=r_1(x) \qquad (0\leqq x\leqq 1). \tag{2.4}$$

$$K_{11}(x, x)-K_{22}(x, x)=r_2(x) \qquad (0\leqq x\leqq 1). \tag{2.5}$$

Proposition 1 is proved by reducing (2.2)-(2.5) to a system of Volterra's integral equations (cf. Petrovsky [26] and Picard [27]) and we carry out its proof in Appendix I.

For $(p_{ij})_{1\leq i,j\leq 2}$, $(q_{ij})_{1\leq i,j\leq 2}\in\{C^1[0, 1]\}^4$ and $h, j\in \boldsymbol{R}\setminus\{-1, 1\}$, let us set

$$\theta_1(x)=\frac{1}{2}\int_0^x (q_{12}(s)+q_{21}(s)-p_{12}(s)-p_{21}(s))ds \qquad (0\leqq x\leqq 1), \tag{2.6}$$

$$\theta_2(x)=\frac{1}{2}\int_0^x (q_{11}(s)+q_{22}(s)-p_{11}(s)-p_{22}(s))ds \qquad (0\leqq x\leqq 1), \tag{2.7}$$

and

$$\begin{cases} a_1=\dfrac{1-j}{1-h} \\ a_2=\dfrac{1+j}{1+h}. \end{cases} \tag{2.8}$$

Moreover let us put

$$a(x)=\frac{1}{2}\{a_1\exp(-\theta_1(x)-\theta_2(x))+a_2\exp(-\theta_1(x)+\theta_2(x))\} \qquad (0\leqq x\leqq 1), \tag{2.9}$$

$$b(x)=\frac{1}{2}\{a_1\exp(-\theta_1(x)-\theta_2(x))-a_2\exp(-\theta_1(x)+\theta_2(x))\} \qquad (0\leqq x\leqq 1), \tag{2.10}$$

and

$$R(x)=\begin{pmatrix} a(x) & b(x) \\ b(x) & a(x) \end{pmatrix} \qquad (0\leqq x\leqq 1). \tag{2.11}$$

Now we state a "deformation formula" in our case;

LEMMA 1. (I) *For given* $P, Q\in\{C^1[0,1]\}^4$ *and* $h, j\in \boldsymbol{R}$ *such that* $|h|, |j|\neq 1$, *there exists a unique* $K=K(x,y)=(K_{ij}(x,y))_{1\leqq i,j\leqq 2}\in\{C^1(\bar{\Omega})\}^4$ *satisfying* (2.12)-(2.15):

$$B\frac{\partial K(x,y)}{\partial x}+Q(x)K(x,y)-K(x,y)P(y)=-\frac{\partial K(x,y)}{\partial y}B \qquad ((x,y)\in\bar{\Omega}). \tag{2.12}$$

$$\begin{cases} K_{12}(x,0)=hK_{11}(x,0) \\ K_{22}(x,0)=hK_{21}(x,0) \qquad (0\leqq x\leqq 1). \end{cases} \tag{2.13}$$

$$\begin{aligned} & K_{12}(x,x)-K_{21}(x,x) \\ &=\frac{1}{4}a_1\exp(-\theta_1(x)-\theta_2(x)) \\ &\quad\times(q_{11}(x)+q_{12}(x)-q_{21}(x)-q_{22}(x)-p_{11}(x)+p_{12}(x)-p_{21}(x)+p_{22}(x)) \\ &\quad+\frac{1}{4}a_2\exp(-\theta_1(x)+\theta_2(x)) \\ &\quad\times(q_{11}(x)-q_{12}(x)+q_{21}(x)-q_{22}(x)-p_{11}(x)-p_{12}(x)+p_{21}(x)+p_{22}(x)) \\ &\qquad\qquad (0\leqq x\leqq 1). \end{aligned} \tag{2.14}$$

$$\begin{aligned} & K_{11}(x,x)-K_{22}(x,x) \\ &=\frac{1}{4}a_1\exp(-\theta_1(x)-\theta_2(x)) \\ &\quad\times(q_{11}(x)+q_{12}(x)-q_{21}(x)-q_{22}(x)+p_{11}(x)-p_{12}(x)+p_{21}(x)-p_{22}(x)) \\ &\quad+\frac{1}{4}a_2\exp(-\theta_1(x)+\theta_2(x)) \\ &\quad\times(-q_{11}(x)+q_{12}(x)-q_{21}(x)+q_{22}(x)-p_{11}(x)-p_{12}(x)+p_{21}(x)+p_{22}(x)) \\ &\qquad\qquad (0\leqq x\leqq 1). \end{aligned} \tag{2.15}$$

(II) (*a deformation formula*) *For* $\lambda\in\boldsymbol{C}$, *if* $\phi(\cdot)=\phi(\cdot,\lambda)\in\{C^1[0,1]\}^2$ *satisfies*

$$B\frac{d\phi(x)}{dx}+P(x)\phi(x)=\lambda\phi(x) \qquad (0\leqq x\leqq 1) \tag{2.16}$$

*and*

$$\phi(0)=\begin{pmatrix} 1 \\ -h \end{pmatrix}, \tag{2.17}$$

*then* $\psi(\cdot)=\psi(\cdot,\lambda)\in\{C^1[0,1]\}^2$ *defined by*

$$\psi(x,\lambda)=R(x)\phi(x,\lambda)+\int_0^x K(x,y)\phi(y,\lambda)dy \qquad (0\leqq x\leqq 1), \tag{2.18}$$

*satisfies*

$$B\frac{d\psi(x)}{dx}+Q(x)\psi(x)=\lambda\psi(x) \qquad (0\leqq x\leqq 1), \tag{2.19}$$

*and*

$$\psi(0)=\begin{pmatrix}1\\-j\end{pmatrix}. \tag{2.20}$$

REMARK 4. For the Sturm-Liouville problem, a formula of the type of (2.18) is derived in Gel'fand and Levitan [3], Suzuki and Murayama [39], etc., which is stated as follows; Let $p, q\in C^1[0,1]$, and $h, j\in \boldsymbol{R}$ be given. Then there exists a unique $L=L(x,y)\in C^2(\bar{\Omega})$ such that

$$\frac{\partial^2 L(x,y)}{\partial x^2}-\frac{\partial^2 L(x,y)}{\partial y^2}+p(y)L(x,y)=q(x)L(x,y) \qquad ((x,y)\in\bar{\Omega}),$$

$$L(x,x)=j-h+\frac{1}{2}\int_0^x (q(s)-p(s))ds \qquad (0\leqq x\leqq 1),$$

and

$$\frac{\partial L}{\partial y}(x,0)=hL(x,0) \qquad (0\leqq x\leqq 1).$$

Furthermore, for $\lambda\in\boldsymbol{R}$, if $f(\cdot)=f(\cdot,\lambda)\in C^2[0,1]$ satisfies

$$\begin{cases}\left(p(x)-\dfrac{d^2}{dx^2}\right)f(x)=\lambda f(x) & (0\leqq x\leqq 1)\\[2mm] f(0)=1, \qquad \dfrac{df}{dx}(0)=h, & \end{cases}$$

then $g(\cdot)=g(\cdot,\lambda)\in C^2[0,1]$ defined by

$$g(x)=f(x)+\int_0^x L(x,y)f(y)dy \qquad (0\leqq x\leqq 1) \tag{2.21}$$

satisfies

$$\begin{cases}\left(q(x)-\dfrac{d^2}{dx^2}\right)g(x)=\lambda g(x) & (0\leqq x\leqq 1)\\[2mm] g(0)=1, \qquad \dfrac{dg}{dx}(0)=j. & \end{cases}$$

The formula (2.21) is a key for the inverse Sturm-Liouville problem.

On the other hand, in Gasymov and Levitan [2] (cf. Levitan and Sargsjan [16]), a similar formula is shown for the one-dimensional Dirac's system.

REMARK 5. We need to introduce $R(x)$ in our formula (2.18) as a modification factor, because we treat systems of ordinary differential equations. Moreover, with this kind of modification, for more general systems involving $N$ functions on the interval $[0, 1]$, we have formulae similar to (2.18).

PROOF OF LEMMA 1. The part (I) of this lemma is seen by Proposition 1. We can prove the part (II) as follows. In (2.18), we get

$$
\begin{aligned}
(2.22)\qquad & B\frac{d\phi(x)}{dx} \\
&= B\frac{dR(x)}{dx}\phi(x)+BR(x)\frac{d\phi(x)}{dx} \\
&\quad +BK(x, x)\phi(x)+\int_0^x B\frac{\partial K(x, y)}{\partial x}\phi(y)dy \\
&= B\frac{dR(x)}{dx}\phi(x)+R(x)B\frac{d\phi(x)}{dx}+BK(x, x)\phi(x) \\
&\quad +\int_0^x\Big(-\frac{\partial K(x, y)}{\partial y}B\phi(y)+(K(x, y)P(y)-Q(x)K(x, y))\phi(y)\Big)dy \\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(by } BR(x)=R(x)B \text{ and (2.12))} \\
&= B\frac{dR(x)}{dx}\phi(x)+R(x)B\frac{d\phi(x)}{dx}+BK(x, x)\phi(x) \\
&\quad +[-K(x, y)B\phi(y)]_{y=0}^{y=x}+\int_0^x K(x, y)B\frac{d\phi(y)}{dy}dy \\
&\quad +\int_0^x K(x, y)P(y)\phi(y)dy-Q(x)\int_0^x K(x, y)\phi(y)dy \\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(by integration by parts)} \\
&= B\frac{dR(x)}{dx}\phi(x)+R(x)(\lambda-P(x))\phi(x) \\
&\quad +(BK(x, x)-K(x, x)B)\phi(x)+K(x, 0)B\phi(0) \\
&\quad +\int_0^x K(x, y)(\lambda-P(y))\phi(y)dy+\int_0^x K(x, y)P(y)\phi(y)dy
\end{aligned}
$$

$$
\begin{aligned}
&-Q(x)\int_0^x K(x, y)\phi(y)dy \qquad \text{(by (2.16))}\\
&=\Big(B\frac{dR(x)}{dx}-R(x)P(x)+BK(x, x)-K(x, x)B\Big)\phi(x)+\lambda R(x)\phi(x)\\
&\quad+(\lambda-Q(x))\int_0^x K(x, y)\phi(y)dy.
\end{aligned}
$$

In the last equality, we use $K(x, 0)B\phi(0)=0$ by (2.3) and (2.17). Having

$$
Q(x)\phi(x)-\lambda\phi(x)=Q(x)R(x)\phi(x)+(Q(x)-\lambda)\int_0^x K(x, y)\phi(y)dy-\lambda R(x)\phi(x)
$$

by (2.18), we obtain

$$
\begin{aligned}
&B\frac{d\phi(x)}{dx}+Q(x)\phi(x)-\lambda\phi(x)\\
&=\Big(B\frac{dR(x)}{dx}+Q(x)R(x)-R(x)P(x)+BK(x, x)-K(x, x)B\Big)\phi(x).
\end{aligned}
$$

Since, by (2.9)-(2.11) and (2.14), (2.15), we see

$$
B\frac{dR(x)}{dx}+Q(x)R(x)-R(x)P(x)+BK(x, x)-K(x, x)B=0,
$$

we reach the equality (2.19).

Finally, as to the initial condition (2.20), we have only to note that $a(0)=\frac{1-jh}{1-h^2}$, $b(0)=\frac{h-j}{1-h^2}$, and therefore,

$$
R(0)\begin{pmatrix} 1 \\ -h \end{pmatrix}=\begin{pmatrix} a(0) & b(0) \\ b(0) & a(0) \end{pmatrix}\begin{pmatrix} 1 \\ -h \end{pmatrix}=\begin{pmatrix} 1 \\ -j \end{pmatrix}.
$$

## § 3. Proof of Theorem.

First, since $|h|\neq 1$ and $|H|\neq 1$, by Proposition A in § 1, we can set

$$
\sigma(A_{P,h,H})=\{\lambda_n\}_{n\in Z}, \tag{3.1}
$$

where $\lambda_n$ $(n\in Z)$ is an eigenvalue with the multiplicity one. Then by Russell [28], [29], we have

PROPOSITION B. (I) (*the asymptotic behavior of the eigenvalues*) *We put*

$$\gamma=\begin{cases}\dfrac{1}{2}\log\dfrac{(1+h)(1-H)}{(1-h)(1+H)}, & \textit{if} \quad H\neq\infty,\\[2mm] \dfrac{1}{2}\log\dfrac{h+1}{h-1}, & \textit{if} \quad H=\infty,\end{cases} \tag{3.2}$$

*and*

$$\theta=\frac{1}{2}\int_0^1(p_{11}(s)+p_{22}(s))ds. \tag{3.3}$$

*In* (3.2) *the principal values of the logarithms are taken. Then we have*

$$\lambda_n=\gamma+\theta+n\pi\sqrt{-1}+O\left(\frac{1}{n}\right) \quad \textit{as} \quad |n|\to\infty. \tag{3.4}$$

(II) (*the completeness of eigenvectors*) *Let us denote an eigenvector associated with the eigenvalue* $\lambda_n$ *by* $\phi_n(\cdot)=\phi(\cdot,\lambda_n)$. *Then the system* $\{\phi_n\}_{n\in Z}$ *is a* ***Riesz basis*** *in the Hilbert space* $\{L^2(0,1)\}^2$, *that is, each* $\begin{pmatrix}u\\ v\end{pmatrix}\in\{L^2(0,1)\}^2$ *has a unique expansion*

$$\begin{pmatrix}u\\ v\end{pmatrix}=\sum_{n=-\infty}^{\infty}c_n\phi_n \qquad (c_n\in \boldsymbol{C};\ n\in \boldsymbol{Z}) \tag{3.5}$$

*and furthermore*

$$M^{-1}\sum_{n=-\infty}^{\infty}|c_n|^2\leqq\left\|\begin{pmatrix}u\\ v\end{pmatrix}\right\|^2_{\{L^2(0,1)\}^2}\leqq M\sum_{n=-\infty}^{\infty}|c_n|^2 \tag{3.6}$$

*for some positive constant* $M$ *independent of* $\begin{pmatrix}u\\ v\end{pmatrix}$.

In order to prove Theorem, we have only to show

PROPOSITION 2. *We have*

$$(Q,j,J,J^*)\in\tilde{M}(P,h,H,H^*) \tag{3.7}$$

*if and only if the equalities* (1.16)-(1.19) *hold.*

In fact, let Proposition 2 be proved. Then we can derive Theorem as follows; Let us prove the part (II) of Theorem. Firstly, assuming that $(Q,j,J,J^*)\in M(P,h,H,H^*)$, we have to show (1.16)-(1.19). To this end, we note that $M(P,h,H,H^*)\subset\tilde{M}(P,h,H,H^*)$ by (1.10) and (1.11). Hence we have $(Q, j, J, J^*) \in \tilde{M}(P, h, H, H^*)$, which implies the equalities (1.16)-(1.19) by Proposition 2. Conversely we assume that the equalities (1.16)-(1.19) hold. Then we have to show that $(Q, j, J, J^*) \in M(P, h, H, H^*)$. Its proof is carried out in the following manner. By Proposition 2, we get $(Q, j, J, J^*) \in \tilde{M}(P, h, H, H^*)$. On the other hand, we can easily derive the equalities (3.8)-(3.11) from the equalities (1.16)-(1.19), respectively:

$$\begin{aligned} (3.8)\qquad & \frac{1-h}{1-j}(p_{11}(x)+p_{12}(x)-p_{21}(x)-p_{22}(x)-q_{11}(x)+q_{12}(x)-q_{21}(x)+q_{22}(x)) \\ & +\frac{1+h}{1+j}(p_{11}(x)-p_{12}(x)+p_{21}(x)-p_{22}(x)-q_{11}(x)-q_{12}(x)+q_{21}(x)+q_{22}(x)) \\ & \times\exp\left(\int_0^x (p_{11}(s)+p_{22}(s)-q_{11}(s)-q_{22}(s))ds\right)=0 \qquad (0\leqq x\leqq 1), \end{aligned}$$

$$\begin{aligned} (3.9)\qquad & \frac{1-h}{1-j}(p_{11}(x)+p_{12}(x)-p_{21}(x)-p_{22}(x)+q_{11}(x)-q_{12}(x)+q_{21}(x)-q_{22}(x)) \\ & +\frac{1+h}{1+j}(-p_{11}(x)+p_{12}(x)-p_{21}(x)+p_{22}(x)-q_{11}(x)-q_{12}(x)+q_{21}(x)+q_{22}(x)) \\ & \times\exp\left(\int_0^x (p_{11}(s)+p_{22}(s)-q_{11}(s)-q_{22}(s))ds\right)=0 \qquad (0\leqq x\leqq 1), \end{aligned}$$

$$(3.10)\qquad \log\frac{(1+j)(1-J)(1-h)(1+H)}{(1-j)(1+J)(1+h)(1-H)}=\int_0^1 (p_{11}(s)+p_{22}(s)-q_{11}(s)-q_{22}(s))ds,$$

and

$$(3.11)\qquad \log\frac{(1+j)(1-J^*)(1-h)(1+H^*)}{(1-j)(1+J^*)(1+h)(1-H^*)}=\int_0^1 (p_{11}(s)+p_{22}(s)-q_{11}(s)-q_{22}(s))ds.$$

Since (3.8)-(3.11) are nothing but the equalities obtained from (1.16)-(1.19) by replacing $p_{ij}$ $(1\leqq i, j\leqq 2)$, $h, H$ and $H^*$ by $q_{ij}$ $(1\leqq i, j\leqq 2)$, $j, J$ and $J^*$, respectively, we get $(P, h, H, H^*) \in \tilde{M}(Q, j, J, J^*)$, again by Proposition 2. Therefore we prove that $(Q, j, J, J^*) \in M(P, h, H, H^*)$, which shows the "if" part of (II) of Theorem.

Finally the part (I) of Theorem follows from the part (II) and Proposition 2.

Now we proceed to

PROOF OF THE "IF" PART OF PROPOSITION 2. Let us assume that the equalities (1.16)-(1.19) hold. Then we have to show that $(Q, j, J, J^*) \in \tilde{M}(P, h, H, H^*)$.

We recall that $\phi(\cdot)=\phi(\cdot, \lambda)\in\{C^1[0, 1]\}^2$ satisfies

$$B\frac{d\phi(x)}{dx}+P(x)\phi(x)=\lambda\phi(x) \qquad (0\leqq x\leqq 1) \tag{3.12}$$

and

$$\phi(0)=\begin{pmatrix} 1 \\ -h \end{pmatrix}. \tag{3.13}$$

For $\lambda_n\in\sigma(A_{P,h,H})$, we set

$$\psi_n(x)=\begin{pmatrix} \psi_n^{(1)}(x) \\ \psi_n^{(2)}(x) \end{pmatrix}\equiv R(x)\phi(x, \lambda_n), \tag{3.14}$$

where $R(x)$ is the $2\times 2$ matrix given by (2.11). From (1.16)-(1.19) we can see by direct computations

$$B\frac{d\psi_n(x)}{dx}+Q(x)\psi_n(x)=\lambda_n\psi_n(x) \qquad (0\leqq x\leqq 1),$$

$$\psi_n(0)=\begin{pmatrix} 1 \\ -j \end{pmatrix} \qquad \text{and} \qquad \psi_n^{(2)}(1)+J\psi_n^{(1)}(1)=0.$$

Here we note also Remark 3 stated in §1. These imply $\lambda_n\in\sigma(A_{Q,j,J})$. That is, we see that $\sigma(A_{P,h,H})\subset\sigma(A_{Q,j,J})$. We can similarly show that $\sigma(A_{P,h,H^*})\subset\sigma(A_{Q,j,J^*})$ and therefore, we see that $(Q, j, J, J^*)\in\tilde{M}(P, h, H, H^*)$.

PROOF OF THE "ONLY IF" PART OF PROPOSITION 2. Assume that

$$\sigma(A_{P,h,H})\subset\sigma(A_{Q,j,J}) \tag{3.15}$$

and

$$\sigma(A_{P,h,H^*})\subset\sigma(A_{Q,j,J^*}). \tag{3.16}$$

Then we have to prove the equalities (1.16)-(1.19). Let us set

$$\sigma(A_{P,h,H})=\{\lambda_n\}_{n\in Z} \tag{3.17}$$

and

$$\sigma(A_{Q,j,J})=\{\mu_n\}_{n\in Z}. \tag{3.18}$$

Firstly we see, by (3.15), that for each $n\in Z$, there exists some $m(n)\in Z$ such that $\lim_{n\to\infty} m(n)=\infty$ and

$$\lambda_n=\mu_{m(n)} \qquad (n\in \boldsymbol{Z}). \tag{3.19}$$

Therefore it follows from (3.19) and the asymptotic behavior of the eigenvalues ((I) of Proposition B) that we get

$$\begin{aligned}(3.20)\quad &\frac{1}{2}\log\frac{(1+h)(1-H)}{(1-h)(1+H)}+\frac{1}{2}\int_0^1(p_{11}(s)+p_{22}(s))ds+n\pi\sqrt{-1}+O\Big(\frac{1}{n}\Big)\\ &=\frac{1}{2}\log\frac{(1+j)(1-J)}{(1-j)(1+J)}+\frac{1}{2}\int_0^1(q_{11}(s)+q_{22}(s))ds+m(n)\pi\sqrt{-1}+O\Big(\frac{1}{m(n)}\Big)\\ &\qquad\qquad (n\in \boldsymbol{Z}).\end{aligned}$$

Since

$$\frac{1}{2}\int_0^1(p_{11}(s)+p_{22}(s))ds,\quad \frac{1}{2}\int_0^1(q_{11}(s)+q_{22}(s))ds\in \boldsymbol{R}$$

and

$$\operatorname{Im}\frac{1}{2}\log\frac{(1+h)(1-H)}{(1-h)(1+H)},\qquad \operatorname{Im}\frac{1}{2}\log\frac{(1+j)(1-J)}{(1-j)(1+J)}=0,\quad \text{or}\quad \frac{1}{2}\pi,$$

we see that $\lim_{n\to\infty}(m(n)-n)\pi\sqrt{-1}=0$, which implies (1.18). Similarly we can see (1.19).

Secondly, in order to show (1.16) and (1.17), we apply Lemma 1 in § 2. Let $K=K(x, y)\in\{C^1(\bar{\Omega})\}^4$ be the solution to (2.12)-(2.15) and let $R(x)$ be defined by (2.9)-(2.11). We put

$$\begin{aligned}(3.21)\qquad \psi(x,\lambda)&=\begin{pmatrix}\psi_1(x,\lambda)\\ \psi_2(x,\lambda)\end{pmatrix}\\ &=R(x)\phi(x,\lambda)+\int_0^x K(x,y)\phi(y,\lambda)dy \qquad (0\leqq x\leqq 1).\end{aligned}$$

Then, by Lemma 1, we have

$$\begin{cases} B\dfrac{d\psi(x,\lambda)}{dx}+Q(x)\psi(x,\lambda)=\lambda\psi(x,\lambda) \qquad (0\leqq x\leqq 1)\\ \psi(0,\lambda)=\begin{pmatrix}1\\ -j\end{pmatrix}.\end{cases} \tag{3.22}$$

On the other hand, by the assumption (3.15) and the fact that each eigenvalue $\lambda_n$ is simple, we see that $\psi(\cdot,\lambda_n)$ is an eigenvector of $A_{Q,j,J}$ associated with $\lambda_n$. Therefore, from the boundary condition at $x=1$, we get

$$\psi_2(1, \lambda_n)+J\psi_1(1, \lambda_n)=0 \qquad (n\in \boldsymbol{Z}). \tag{3.23}$$

Here we recall that (3.23) means $\psi_1(1, \lambda_n)=0$, if $J=\infty$. Henceforth let us assume that

$$H, H^*, J, J^*\neq\infty. \tag{3.24}$$

Otherwise we can similarly proceed. (In Appendix II, we derive (1.16) and (1.17) in a case where (3.24) does not hold.) Substituting (3.21) into (3.23), we obtain

$$(J-H)a(1)+(1-JH)b(1)+K_2(\lambda_n)+JK_1(\lambda_n)=0 \qquad (n\in \boldsymbol{Z}). \tag{3.25}$$

Here and henceforth, we put for $\lambda\in \boldsymbol{C}$

$$\begin{cases} K_1(\lambda)=\int_0^1 (K_{11}(1, y)\phi_1(y, \lambda)+K_{12}(1, y)\phi_2(y, \lambda))dy \\ K_2(\lambda)=\int_0^1 (K_{21}(1, y)\phi_1(y, \lambda)+K_{22}(1, y)\phi_2(y, \lambda))dy, \end{cases} \tag{3.26}$$

and we recall that $a(x)$ and $b(x)$ are given by (2.9) and (2.10), respectively. By the assumption (3.16), we similarly get

$$(J^*-H^*)a(1)+(1-J^*H^*)b(1)+K_2(\lambda_n^*)+J^*K_1(\lambda_n^*)=0 \qquad (n\in \boldsymbol{Z}). \tag{3.27}$$

Here we set $\sigma(A_{P,h,H^*})=\{\lambda_n^*\}_{n\in \boldsymbol{Z}}$.

As is easily seen, the equalities (1.18) and (1.19) imply

$$\begin{cases} (J-H)a(1)+(1-JH)b(1)=0 \\ (J^*-H^*)a(1)+(1-J^*H^*)b(1)=0. \end{cases} \tag{3.28}$$

Hence by (3.25), (3.27) and (3.28), we reach

$$K_2(\lambda_n)+JK_1(\lambda_n)=0 \qquad (n\in \boldsymbol{Z}), \tag{3.29}$$

and

$$K_2(\lambda_n^*)+J^*K_1(\lambda_n^*)=0 \qquad (n\in \boldsymbol{Z}). \tag{3.30}$$

Since each of the systems $\{\phi(\cdot, \lambda_n)\}_{n\in \boldsymbol{Z}}$ and $\{\phi(\cdot, \lambda_n^*)\}_{n\in \boldsymbol{Z}}$ is complete in $\{L^2(0, 1)\}^2$ ((II) of Proposition B), the equalities (3.29) and (3.30) imply

$$(3.31)\qquad \begin{cases} K_{21}(1, y)+J\cdot K_{11}(1, y)=0 \\ K_{21}(1, y)+J^{*}K_{11}(1, y)=0 \qquad (0\leqq y\leqq 1) \end{cases}$$

and

$$(3.32)\qquad \begin{cases} K_{22}(1, y)+J\cdot K_{12}(1, y)=0 \\ K_{22}(1, y)+J^{*}K_{12}(1, y)=0 \qquad (0\leqq y\leqq 1). \end{cases}$$

Now, by (1.18) and (1.19), we have $\frac{(1-H)(1+J)}{(1+H)(1-J)}=\frac{(1-H^{*})(1+J^{*})}{(1+H^{*})(1-J^{*})}$, which implies

$$(3.33)\qquad J\neq J^{*},$$

by $H\neq H^{*}$ (see (1.8)).
Therefore by (3.31)-(3.33), we obtain

$$(3.34)\qquad K_{ij}(1, y)=0 \qquad (0\leqq y\leqq 1,\ 1\leqq i,\ j\leqq 2).$$

As is proved in Appendix III, we have a result on uniqueness of solutions to a hyperbolic system:

LEMMA 2. *Let $K=K(x, y)$ satisfy*

$$(3.35)\qquad B\frac{\partial K(x, y)}{\partial x}+Q(x)K(x, y)-K(x, y)P(y)=-\frac{\partial K(x, y)}{\partial y}B \qquad ((x, y)\in\bar{\Omega}),$$

$$(3.36)\qquad \begin{cases} K_{12}(x, 0)=hK_{11}(x, 0) \\ K_{22}(x, 0)=hK_{21}(x, 0) \qquad (0\leqq x\leqq 1), \end{cases}$$

*and*

$$(3.37)\qquad K_{ij}(1, y)=0 \qquad (0\leqq y\leqq 1,\ 1\leqq i,\ j\leqq 2).$$

*Then the identity*

$$(3.38)\qquad K(x, y)=0 \qquad ((x, y)\in\bar{\Omega})$$

*holds.*

We return to the proof of the "only if" part. By Lemma 2 and (3.34), noting (2.12) and (2.13), we reach $K(x, y)=0$ $((x, y)\in\bar{\Omega})$. Therefore by (2.14) and (2.15) we see (1.16) and (1.17). Thus the proof of Proposition 2 is completed.

**Appendix I.**

PROOF OF PROPOSITION 1. Setting

$$\text{(I.1)}\qquad \begin{cases} L_1(x, y)=K_{12}(x, y)-K_{21}(x, y), & L_2(x, y)=K_{11}(x, y)-K_{22}(x, y) \\ L_3(x, y)=K_{11}(x, y)+K_{22}(x, y), & L_4(x, y)=K_{12}(x, y)+K_{21}(x, y), \end{cases}$$

we can rewrite (2.2)-(2.5), so that we get

$$\text{(I.2)}\qquad \frac{\partial L_i(x, y)}{\partial x}-\frac{\partial L_i(x, y)}{\partial y}=f_i(x, y, L_1, L_2, L_3, L_4) \qquad ((x, y)\in\bar{\Omega},\ i=1, 2),$$

$$\text{(I.3)}\qquad \frac{\partial L_i(x, y)}{\partial x}+\frac{\partial L_i(x, y)}{\partial y}=f_i(x, y, L_1, L_2, L_3, L_4) \qquad ((x, y)\in\bar{\Omega},\ i=3, 4),$$

$$\text{(I.4)}\qquad \begin{cases} L_3(x, 0)=k\cdot L_1(x, 0)+l\cdot L_2(x, 0) \\ L_4(x, 0)=-l\cdot L_1(x, 0)-k\cdot L_2(x, 0) \qquad (0\leqq x\leqq 1), \end{cases}$$

and

$$\text{(I.5)}\qquad L_i(x, x)=r_i(x) \qquad (0\leqq x\leqq 1,\ i=1, 2).$$

Here and henceforth we put

$$\begin{aligned} f_1(x, y, L_1, L_2, L_3, L_4)=&-\frac{1}{2}(p_{12}(y)+p_{21}(y)+q_{12}(x)+q_{21}(x))L_1(x, y) \\ &-\frac{1}{2}(p_{11}(y)+p_{22}(y)-q_{11}(x)-q_{22}(x))L_2(x, y) \\ &-\frac{1}{2}(p_{11}(y)-p_{22}(y)-q_{11}(x)+q_{22}(x))L_3(x, y) \\ &-\frac{1}{2}(-p_{12}(y)+p_{21}(y)-q_{12}(x)+q_{21}(x))L_4(x, y) \end{aligned}$$

$$\begin{aligned} f_2(x, y, L_1, L_2, L_3, L_4)=&-\frac{1}{2}(p_{11}(y)+p_{22}(y)-q_{11}(x)-q_{22}(x))L_1(x, y) \\ &-\frac{1}{2}(p_{12}(y)+p_{21}(y)+q_{12}(x)+q_{21}(x))L_2(x, y) \\ &+\frac{1}{2}(-p_{12}(y)+p_{21}(y)+q_{12}(x)-q_{21}(x))L_3(x, y) \\ &-\frac{1}{2}(-p_{11}(y)+p_{22}(y)-q_{11}(x)+q_{22}(x))L_4(x, y) \end{aligned}$$

$$
\begin{aligned}
f_3(x, y, L_1, L_2, L_3, L_4) = \; & \frac{1}{2}(-p_{11}(y)+p_{22}(y)-q_{11}(x)+q_{22}(x))L_1(x, y) \\
& +\frac{1}{2}(p_{12}(y)-p_{21}(y)+q_{12}(x)-q_{21}(x))L_2(x, y) \\
& +\frac{1}{2}(p_{12}(y)+p_{21}(y)-q_{12}(x)-q_{21}(x))L_3(x, y) \\
& +\frac{1}{2}(p_{11}(y)+p_{22}(y)-q_{11}(x)-q_{22}(x))L_4(x, y)
\end{aligned}
$$

$$
\begin{aligned}
f_4(x, y, L_1, L_2, L_3, L_4) = \; & \frac{1}{2}(-p_{12}(y)+p_{21}(y)+q_{12}(x)-q_{21}(x))L_1(x, y) \\
& +\frac{1}{2}(p_{11}(y)-p_{22}(y)-q_{11}(x)+q_{22}(x))L_2(x, y) \\
& +\frac{1}{2}(p_{11}(y)+p_{22}(y)-q_{11}(x)-q_{22}(x))L_3(x, y) \\
& +\frac{1}{2}(p_{12}(y)+p_{21}(y)-q_{12}(x)-q_{21}(x))L_4(x, y) \\
& \qquad\qquad ((x, y) \in \bar{\Omega}),
\end{aligned}
$$

and

$$
\text{(I.6)} \qquad k=\frac{-2h}{1-h^2} \quad \text{and} \quad l=\frac{1+h^2}{1-h^2}.
$$

First we integrate (I.2) with (I.5) along the characteristic curve $y+x=const.$, so that we get integral equations (I.7):

$$
\text{(I.7)} \qquad L_i(x, y)=\int_y^{(x+y)/2} f_i(-s+x+y, s, L_1, L_2, L_3, L_4)ds+r_i\left(\frac{x+y}{2}\right) \qquad ((x, y) \in \bar{\Omega},\ i=1, 2).
$$

Second we integrate (I.3) along the characteristic curve $y-x=const.$, so that we obtain

$$
L_i(x, y)=\int_0^y f_i(s+x-y, s, L_1, L_2, L_3, L_4)ds+L_i(x-y, 0) \qquad (i=3, 4).
$$

Therefore by (I.4) and (I.7), we get

$$
(\mathrm{I.8})\quad
\begin{cases}
L_3(x, y) = \displaystyle\int_0^y f_3(s+x-y, s, L_1, L_2, L_3, L_4)ds \\
\qquad + \displaystyle\int_0^{(x-y)/2} (kf_1(-s+x-y, s, L_1, L_2, L_3, L_4) \\
\qquad\qquad + lf_2(-s+x-y, s, L_1, L_2, L_3, L_4))ds \\
\qquad + kr_1\left(\dfrac{x-y}{2}\right) + lr_2\left(\dfrac{x-y}{2}\right) \\
L_4(x, y) = \displaystyle\int_0^y f_4(s+x-y, s, L_1, L_2, L_3, L_4)ds \\
\qquad - \displaystyle\int_0^{(x-y)/2} (lf_1(-s+x-y, s, L_1, L_2, L_3, L_4) \\
\qquad\qquad + kf_2(-s+x-y, s, L_1, L_2, L_3, L_4))ds \\
\qquad - lr_1\left(\dfrac{x-y}{2}\right) - kr_2\left(\dfrac{x-y}{2}\right) \qquad ((x, y) \in \bar{\Omega}).
\end{cases}
$$

Thus, provided that $r_1, r_2 \in C^1[0, 1]$ and $P, Q \in \{C^1[0, 1]\}^4$, the problem (I.2)-(I.5) is equivalent to the Volterra's integral equations (I.7) and (I.8), if $L \in \{C^1(\bar{\Omega})\}^4$ is proved.

A unique solution $L = L(x, y) \in \{C^1(\bar{\Omega})\}^4$ to (I.7) and (I.8) is given by the following iteration method; Let us define approximation sequences $\{L_i^{(n)}(x, y)\}_{n\geqq 0}$ $(1 \leqq i \leqq 4)$ by (I.9)-(I.11):

$$
(\mathrm{I.9})\qquad L_i^{(0)}(x, y) = 0 \qquad ((x, y) \in \bar{\Omega},\ 1 \leqq i \leqq 4),
$$

$$
(\mathrm{I.10})\qquad L_i^{(n)}(x, y) = \int_y^{(x+y)/2} f_i(-s+x+y, s, L_1^{(n-1)}, L_2^{(n-1)}, L_3^{(n-1)}, L_4^{(n-1)})ds
$$
$$
+ r_i\left(\frac{x+y}{2}\right), \qquad ((x, y) \in \bar{\Omega},\ n \geqq 1,\ i = 1, 2),
$$

and

$$
(\mathrm{I}.11)\quad
\left\{
\begin{aligned}
L_3^{(n)}(x, y) = & \int_0^y f_3(s+x-y, s, L_1^{(n-1)}, L_2^{(n-1)}, L_3^{(n-1)}, L_4^{(n-1)})ds \\
& + \int_0^{(x-y)/2} (kf_1(-s+x-y, s, L_1^{(n-1)}, L_2^{(n-1)}, L_3^{(n-1)}, L_4^{(n-1)}) \\
& \qquad + lf_2(-s+x-y, s, L_1^{(n-1)}, L_2^{(n-1)}, L_3^{(n-1)}, L_4^{(n-1)}))ds \\
& + kr_1\left(\frac{x-y}{2}\right) + lr_2\left(\frac{x-y}{2}\right), \\
L_4^{(n)}(x, y) = & \int_0^y f_4(s+x-y, s, L_1^{(n-1)}, L_2^{(n-1)}, L_3^{(n-1)}, L_4^{(n-1)})ds \\
& - \int_0^{(x-y)/2} (lf_1(-s+x-y, s, L_1^{(n-1)}, L_2^{(n-1)}, L_3^{(n-1)}, L_4^{(n-1)}) \\
& \qquad + kf_2(-s+x-y, s, L_1^{(n-1)}, L_2^{(n-1)}, L_3^{(n-1)}, L_4^{(n-1)}))ds \\
& - lr_1\left(\frac{x-y}{2}\right) - kr_2\left(\frac{x-y}{2}\right) \qquad ((x, y) \in \bar{\Omega},\ n \geqq 1).
\end{aligned}
\right.
$$

Setting

$$M = 8(|k| + |l| + 1)\max\left\{\max_{1\leqq i,j\leqq 2} \|p_{ij}\|_{C^0[0,1]},\ \max_{1\leqq i,j\leqq 2} \|q_{ij}\|_{C^0[0,1]}\right\},$$

by induction, we can see the estimates

$$
(\mathrm{I}.12)\qquad
\begin{aligned}
& |L_i^{(n)}(x, y) - L_i^{(n-1)}(x, y)| \\
& \leqq \frac{M^{n-1}(1+x)^{n-1}}{(n-1)!}(|k| + |l| + 1)(\|r_1\|_{C^0[0,1]} + \|r_2\|_{C^0[0,1]}) \\
& \qquad\qquad\qquad ((x, y) \in \bar{\Omega},\ 1 \leqq i \leqq 4),
\end{aligned}
$$

for each $n \geqq 1$.

Thus $L_i(x, y) = \lim_{n\to\infty} L_i^{(n)}(x, y)$ $(1 \leqq i \leqq 4)$ exist uniformly for $(x, y) \in \bar{\Omega}$ and we see that $L_i(x, y)$ $(1 \leqq i \leqq 4)$ satisfy (I.7) and (I.8). Furthermore we can get similar estimates on $\left|\frac{\partial L_i^{(n)}(x, y)}{\partial x} - \frac{\partial L_i^{(n-1)}(x, y)}{\partial x}\right|$ and $\left|\frac{\partial L_i^{(n)}(x, y)}{\partial y} - \frac{\partial L_i^{(n-1)}(x, y)}{\partial y}\right|$, by induction, and therefore we see also that $L \in \{C^1(\bar{\Omega})\}^4$. The uniqueness of solutions to (2.2)-(2.5) is shown by (I.12). Thus the proof of Proposition 1 is completed.

**Appendix II.**

DERIVATION OF (1.16) AND (1.17) IN THE CASE OF $H\neq\infty$, $H^*=\infty$, $J=\infty$ AND $J^*\neq\infty$ IN THE PROOF OF THE "ONLY IF" PART OF PROPOSITION 2.

In this appendix, assuming that

$$H\neq\infty,\ H^*=\infty,\ J=\infty \quad\text{and}\quad J^*\neq\infty, \tag{II.1}$$

we derive (1.16) and (1.17) in the proof of the "only if" part of Proposition 2.

Here we recall that $\phi(x,\lambda)$ and $K_1(\lambda)$, $K_2(\lambda)$ are given by (3.21) and (3.26), respectively and we put

$$\begin{cases}\sigma(A_{P,h,H})=\{\lambda_n\}_{n\in Z}\\ \sigma(A_{P,h,\infty})=\{\lambda_n^*\}_{n\in Z}.\end{cases} \tag{II.2}$$

Furthermore, by the boundary conditions for $\phi$, and $H\neq\infty$, $H^*=\infty$, we note that

$$\phi_2(1,\lambda_n)+H\phi_1(1,\lambda_n)=0 \qquad (n\in \boldsymbol{Z}) \tag{II.3}$$

and

$$\phi_1(1,\lambda_n^*)=0 \qquad (n\in \boldsymbol{Z}). \tag{II.4}$$

Now we have already derived (1.18) and (1.19), and we have obtained

$$\phi_1(1,\lambda_n)=0 \qquad (n\in \boldsymbol{Z}), \tag{II.5}$$

and

$$\phi_2(1,\lambda_n^*)+J^*\phi_1(1,\lambda_n^*)=0 \qquad (n\in \boldsymbol{Z}). \tag{II.6}$$

Then, for the derivation of (1.16) and (1.17), we have only to prove

$$K_{ij}(1,y)=0 \qquad (0\leqq y\leqq 1,\ 1\leqq i,\ j\leqq 2), \tag{II.7}$$

in view of Lemma 2.

Substituting (3.21) into (II.5) and using (II.3), we get

$$(a(1)-Hb(1))\phi_1(1,\lambda_n)+K_1(\lambda_n)=0 \qquad (n\in \boldsymbol{Z}). \tag{II.8}$$

On the other hand, as is easily checked, the equality (1.18) implies

$a(1)-Hb(1)=0$. Therefore by (II.8), we obtain

$$K_1(\lambda_n)=0 \qquad (n\in \boldsymbol{Z}). \tag{II.9}$$

Next, substituting (3.21) into (II.6) and using (II.4), we get

$$(a(1)+J^*b(1))\phi_2(1, \lambda_n^*)+(K_2(\lambda_n^*)+J^*K_1(\lambda_n^*))=0 \qquad (n\in \boldsymbol{Z}). \tag{II.10}$$

Since we see by direct computations that the equality (1.19) implies $a(1)+J^*b(1)=0$, we obtain

$$K_2(\lambda_n^*)+J^*K_1(\lambda_n^*)=0 \qquad (n\in \boldsymbol{Z}) \tag{II.11}$$

by (II.10).

Since each of the systems $\{\phi(\cdot, \lambda_n)\}_{n\in \boldsymbol{Z}}$ and $\{\phi(\cdot, \lambda_n^*)\}_{n\in \boldsymbol{Z}}$ is complete in $\{L^2(0, 1)\}^2$, the equalities (II.9) and (II.11) imply $K_{11}(1, y)=K_{12}(1, y)=0$ $(0\leqq y\leqq 1)$, and $K_{21}(1, y)+J^*K_{11}(1, y)=0$, $K_{22}(1, y)+J^*K_{12}(1, y)=0$ $(0\leqq y\leqq 1)$. Thus we prove (II.7).

**Appendix III.**

PROOF OF LEMMA 2. By an argument similar to the one in Appendix I, we have only to show the following: If $L_i$ $(1\leqq i\leqq 4)$ satisfy (I.2)-(I.4) and

$$L_i(1, y)=0 \qquad (0\leqq y\leqq 1,\ 1\leqq i\leqq 4) \tag{III.1}$$

holds, then $L_i(x, y)=0$ $((x, y)\in \bar{\Omega},\ 1\leqq i\leqq 4)$.

Let us set

$$\begin{cases} \Omega_1=\left\{(x, y);\ 1-x<y<x,\ \dfrac{1}{2}<x<1\right\} \\ \Omega_2=\Omega\diagdown\Omega_1\diagdown\{(x, y);\ 1-x=y\}. \end{cases} \tag{III.2}$$

Then, by a result on uniqueness of solutions to the Cauchy problem for a hyperbolic system (e.g. Petrovsky [26, p. 68]), we see

$$L_i(x, y)=0 \qquad ((x, y)\in \overline{\Omega_1},\ 1\leqq i\leqq 4) \tag{III.3}$$

from (I.2), (I.3) and (III.1).

By (III.3), we have

$$L_i(x, 1-x)=0 \qquad \left(\frac{1}{2}\leqq x\leqq 1,\ 1\leqq i\leqq 4\right). \tag{III.4}$$

Since $L_i$ $(1\leqq i\leqq 4)$ satisfy (I.2)-(I.4) and (III.4), we obtain the integral equations (III.5) and (III.6) for $L_i$ by integrations of (I.2) and (I.3) along the characteristic curves;

$$\text{(III.5)}\quad \begin{cases} L_1(x,y)=-\int_0^y f_1(-s+x+y,s,L_1,L_2,L_3,L_4)ds \\ \qquad +\int_0^{(1-x-y)/2}(kf_3(s+x+y,s,L_1,L_2,L_3,L_4) \\ \qquad\qquad +lf_4(s+x+y,s,L_1,L_2,L_3,L_4))ds \\ L_2(x,y)=-\int_0^y f_2(-s+x+y,s,L_1,L_2,L_3,L_4)ds \\ \qquad -\int_0^{(1-x-y)/2}(lf_3(s+x+y,s,L_1,L_2,L_3,L_4) \\ \qquad\qquad +kf_4(s+x+y,s,L_1,L_2,L_3,L_4))ds \qquad ((x,y)\in\overline{\Omega_2}), \end{cases}$$

and

$$\text{(III.6)}\qquad L_i(x,y)=-\int_y^{(1-x+y)/2} f_i(s+x-y,s,L_1,L_2,L_3,L_4)ds \qquad ((x,y)\in\overline{\Omega_2},\ i=3,4).$$

Setting $m(x,y)=\max_{1\leqq i,j\leqq 2}|L_{ij}(x,y)|$ and

$$M=8(|k|+|l|+1)\max\left\{\max_{1\leqq i,j\leqq 2}\|p_{ij}\|_{C^0[0,1]},\ \max_{1\leqq i,j\leqq 2}\|q_{ij}\|_{C^0[0,1]}\right\},$$

we inductively obtain the estimates

$$\text{(III.7)}\qquad m(x,y)\leqq\frac{M^{n-1}(1-x)^{n-1}}{(n-1)!}\|m\|_{C^0(\overline{\Omega_2})} \qquad ((x,y)\in\overline{\Omega_2}),$$

for each $n\geqq 1$. Since $n$ is arbitrary, the estimates (III.7) prove $m(x,y)=0$ $((x,y)\in\overline{\Omega_2})$. Thus the proof of Lemma 2 is completed.

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