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*Applications of the Fourier transform to the boundary element method**)

Dedicated to Professor Hiroshi Fujita on his 60-th birthday

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Abstract

We give two applications of the Fourier transform to the boundary element method. One of the problems which we consider is the Dirichlet problems for the Laplace operator $\Delta=\partial^2/\partial x_1^2+\cdots+\partial^2/\partial x_n^2$ and the other is the initial boundary value problem for the heat operator $\partial/\partial t-\Delta$. We derive an inequality which implies a coercivity for the boundary element algorithm for Δ. The proof gives another view-point to the coercivity which is known. The second application is made to the heat operator. We prove that a certain quantity which is expressed by a certain integral of the Fourier transform of the solution is nonincreasing. This fact shows an unconditional stability of the boundary element approximation for the heat operator with respect to the L^2-norm. We also prove a conditional convergence.

§ 1. Introduction.

The objective of this paper is to show the effectiveness of the Fourier transform in the theory of the boundary element method (BEM). This paper is divided into two parts. In the first part (§§ 2–4), we consider a Dirichlet problem for the Laplace equation in a two or three dimensional bounded domain. As is shown in Le Roux [10] or Nedelec and Planchard [12], the solvability of the linear equation given by the BEM discretization is reduced to the coercivity of a certain bilinear form, which we describe in § 2. We give in § 3 a proof of the coercivity which is slightly different from those in [10, 12]. Our proof is essentially the same as those in Costabel and Wendland [6] or Hsiao and Wendland [7]. We, however, believe that our presentation is more compact, although a proof which is more compact than ours are shown to the author by Prof. D. N. Arnold.

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To show the effectiveness of the Fourier transform, we prove in the second part (§§ 5–8) an unconditional stability of a certain BEM algorithm for the initial boundary value problem of the heat equation. We consider a heat equation under the Dirichlet boundary condition in a two or three dimensional bounded domain. Using the spline elements which are piecewise constant both in the time variable and the space variable, we discretize in the step-by-step manner (see § 5 for the details). We prove that this scheme is unconditionally stable in the L^2-norm. Our result can be regarded as a high dimensional version of Kesavan and Vasudevamurthy [9] where the BEM in the case of one space variable is considered. On the other hand, Iso, Takahashi and Onishi [8] and Costabel, Onishi and Wendland [5] consider the discretization of the time dependent problems under the Neumann or the third kind boundary condition. They prove a conditional stability assuming that the mesh sizes are sufficiently small. Their proof, however, does not seem to work in the case of the Dirichlet boundary condition.

The present paper consists of 8 sections. In § 2 we write down concretely the BEM algorithm for Δ. In § 3 we state and prove a theorem which ensures the solvability of the BEM scheme. § 4 contains an example which supplements the theorem in § 3. In § 5 we state the formulation of the BEM approximation of the heat operator $\partial/\partial t-\Delta$ under the Dirichlet boundary condition. In § 6 we prove that the algorithm given in § 5 is unconditionally stable in the L^2-norm. In § 7 we derive an error estimate which shows a conditional convergence. We give concluding remarks in § 8.

§ 2. The Laplace equation.

Let us consider a Dirichlet problem for the Laplace operator:

$$\Delta u=0 \quad \text{in } \Omega, \tag{2.1}$$

$$u=\phi \quad \text{on } \Gamma. \tag{2.2}$$

Here Ω ia a bounded domain in $\boldsymbol{R}^2$ or $\boldsymbol{R}^3$ with a boundary Γ of C^2-class. The function ϕ is given on Γ. To compute u numerically, the boundary element method (BEM) is widely used and is found very effective (see, e.g., Brebbia [3], Brebbia and Walker [4] or Nakayama [11]). We give a simplest form of the BEM to (2.1–2) as follows. First we choose a fundamental solution $E(x)$ for Δ:

$$E(x)=-\frac{1}{2\pi}\log|x| \qquad \text{in } \boldsymbol{R}^2,$$

or

$$E(x)=\frac{1}{4\pi|x|} \qquad \text{in } \boldsymbol{R}^3.$$

Then, by Green's formula, we have

$$u(x)=\int_\Gamma \frac{\partial u}{\partial \nu}(y)E(x-y)\,d\Gamma_y-\int_\Gamma u(y)\frac{\partial E}{\partial \nu_y}(x-y)\,d\Gamma_y \quad (x\in\Omega), \tag{2.3}$$

$$\frac{1}{2}u(x)=\int_\Gamma \frac{\partial u}{\partial \nu}(y)E(x-y)\,d\Gamma_y-\mathrm{PV.}\int_\Gamma u(y)\frac{\partial E}{\partial \nu_y}(x-y)\,d\Gamma_y \quad (x\in\Gamma), \tag{2.4}$$

where $\frac{\partial}{\partial \nu}$ is the outward normal derivative, $d\Gamma_y$ is the line element on Γ, and PV. means the principal value. As is known, the equality (2.3) implies that we can compute u if we are given the values of $\frac{\partial u}{\partial \nu}$ on Γ. The spirit of the BEM lies in computing $\frac{\partial u}{\partial \nu}$ by (2.4) which we regard as an integral equation for $\frac{\partial u}{\partial \nu}$. In this way, the solvability of schemes by the BEM is reduced to the solvability of the problem below:

(P) *Given ϕ on Γ, find q on Γ such that*

$$\int_\Gamma q(y)E(x-y)\,d\Gamma_y=\phi(x) \qquad (x\in\Gamma).$$

As usual, let $L^2(\Gamma)$ denote the function space of all real valued square summable functions on Γ. Then this problem is further reduced as follows: Let us define a quadratic form $A(\ ,\ )$: $L^2(\Gamma)\times L^2(\Gamma)\to\boldsymbol{R}$ by

$$A(q,p)=\int_\Gamma\int_\Gamma E(x-y)q(y)p(x)\,d\Gamma_x d\Gamma_y. \tag{2.5}$$

(Actually the defining domain of $A(\ ,\ )$ can be wider than $L^2(\Gamma)$, see Remark 2 in § 3 below). We denote the inner product in $L^2(\Gamma)$ by $(\ ,\ )$. Then we reformulate (P) as

(P)′ *Given $\phi\in L^2(\Gamma)$, find $q\in L^2(\Gamma)$ such that*

$$A(q,p)=(\psi,p) \qquad (p\in L^2(\Gamma)). \tag{2.6}$$

Therefore we can obtain the unique solvability of the BEM scheme by the coercivity of $A(\ ,\ )$.

§ 3. Theorem and proof.

To state the result rigorously, we define a function space X:

$$X=\left\{p\in L^2(\Gamma);\ \int_\Gamma p(x)d\Gamma_x=0\right\} \qquad \text{if} \quad \Omega\subset \boldsymbol{R}^2,$$
$$X=L^2(\Gamma) \qquad \text{if} \quad \Omega\subset \boldsymbol{R}^3.$$

We use usual Sobolev spaces $H^s(\Gamma)$ $(s\in \boldsymbol{R})$. The norm in $H^s(\Gamma)$ is denoted by $\|\ \|_s$. For the definition of $H^s(\Gamma)$, see, e.g., Adams [1]. Now our goal is to show the following

THEOREM 1. *There exists a positive constant c depending only on Γ such that*

$$A(q,q)\geqq c\|q\|^2_{-1/2} \qquad (q\in X). \tag{3.1}$$

REMARK 1. In the applications the solution q corresponds to the Neumann data $\dfrac{\partial u}{\partial \nu}$. Hence coercivity in X is sufficient for our purpose. Note that three dimensional version is stronger in conclusion than the two dimensional one by the definition of X.

REMARK 2. Actually (3.1) holds for any q in a function space which is the completion of X by the norm $\|\ \|_{-1/2}$. However, it is not trivial that the definition (2.5) is meaningful for arbitrary $p, q\in H^{-1/2}(\Gamma)$. Hence we first consider in $L^2(\Gamma)$ and make a completion, if necessary.

Before giving the proof of Theorem 1, we explain the meaning of (3.1) briefly. A discretization of (2.6) is formulated as follows: Let X_h be a finite dimensional subspace of X, and consider the problem below:
Find $q_h\in X_h$ such that

$$A(q_h,p_h)=(\psi,p_h) \qquad (p_h\in X_h). \tag{2.6_h}$$

Then our theorem ensures the solution q_h exists uniquely and satisfies a priori estimate $c\|q_h\|_{-1/2}\leqq \|\psi\|_{1/2}$.

The proof of (3.1) is based on a certain identity ((3.3) below). Before introducing the identity, we need some symbols. Denoting the dimension of Ω by n, we first define a distribution in $\boldsymbol{R}^n$:

$$\delta_q\colon \phi \longrightarrow \int_\Gamma q(x)\phi(x)d\Gamma_x \qquad (\phi \in C_0^\infty(\boldsymbol{R}^n)).$$

For $q \in L^2(\Gamma)$, this is a distribution with compact support $(\mathrm{supp}(\delta_q) \subset \Gamma)$. Let $\hat{\delta}_q$ denote the Fourier transform of δ_q. Then it is a smooth function in $\boldsymbol{R}^n$ given by

$$\hat{\delta}_q(\xi) = (2\pi)^{-n/2}\int_\Gamma e^{-ix\cdot\xi}q(x)d\Gamma_x \qquad (\xi \in \boldsymbol{R}^n) \tag{3.2}$$

(see, e.g., Schwartz [13] or Yosida [14]).
We derive Theorem 1 from the following

PROPOSITION 3.1. *For all $p, q \in X$, it holds that*

$$A(q, p) = \int_{\boldsymbol{R}^n} \frac{1}{|\xi|^2}\hat{\delta}_q(\xi)\overline{\hat{\delta}_p(\xi)}d\xi. \tag{3.3}$$

REMARK 3. The integral in (3.3) is convergent for any q, p in X. For, the integrability near infinity will be seen in the proof of (3.3) below. The integrability near the origin is easy to see: if $n=3$, this is clear. If $n=2$, we assume that $q, p \in X$, which implies $\hat{\delta}_q(0) = \hat{\delta}_p(0) = 0$. Therefore the integral is convergent.

PROOF OF THEOREM 1. By (3.3) we have

$$\begin{aligned} A(q, q) &= \int_{\boldsymbol{R}^n} \frac{1}{|\xi|^2}|\hat{\delta}_q(\xi)|^2 d\xi \\ &\geqq \int_{\boldsymbol{R}^n} (1+|\xi|^2)^{-1}|\hat{\delta}_q(\xi)|^2 d\xi \\ &= \|\delta_q\|^2_{H^{-1}(\boldsymbol{R}^n)}. \end{aligned} \tag{3.4}$$

On the other hand, it is known that the norm in $H^{-1}(\boldsymbol{R}^n)$ is equivalent to

$$\sup\{\langle \delta_q, w\rangle;\ w \in H^1(\boldsymbol{R}^n),\ \|w\|_{H^1(\boldsymbol{R}^n)} \leqq 1\}.$$

As is easily seen, this norm is equivalent to

$$\sup\left\{\int_\Gamma q(x)v(x)d\Gamma_x;\ v \in H^{1/2}(\Gamma),\ \|v\|_{1/2} \leqq 1\right\},$$

which is equivalent to $\|q\|_{-1/2}$. Hence $\|\delta_q\|_{H^{-1}(R^n)} \geqq c\|q\|_{-1/2}$, with a constant c depending only on Γ. This inequality and (3.4) prove Theorem 1.

Now it remains to prove Proposition 3.1. First we prove it formally. We note that

$$\int_\Gamma E(x-y)q(y)d\Gamma_y = E*\delta_q(x),$$

where $*$ means the convolution. Hence, by the Plancherel theorem, it holds that

$$A(q, p) = \langle E*\delta_q, \delta_p\rangle = \langle E*\delta_q, \overline{\overline{\delta_p}}\rangle = (2\pi)^{n/2}\langle \hat{E}, \hat{\delta}_q\overline{\hat{\delta}_p}\rangle.$$

The Fourier transform $\hat{E}$ of E is given by

$$\hat{E}(\xi) = \frac{1}{2\pi}\mathrm{Pf.}\frac{1}{|\xi|^2} + \hat{c}\delta(\xi) \qquad \text{if} \quad n=2, \tag{3.5}$$

$$\hat{E}(\xi) = \frac{1}{(2\pi)^{3/2}}\frac{1}{|\xi|^2} \qquad \text{if} \quad n=3,$$

where Pf. implies the finite part, $\hat{c}$ is an absolute constant and δ is Dirac's delta function (see Schwartz [13]). Therefore we have (3.3), since $\hat{\delta}_q(0) = \hat{\delta}_p(0) = 0$ if $n=2$.

Although this is very formal, we can justify it:

PROOF OF PROPOSITION 3.1. The function $E(x)$ can be regarded as a tempered distribution in R^n. δ_q is a distribution of compact support. Hence $E*\delta_q$ is a tempered distribution in R^n. If $n=2$, its Fourier transform is

$$\begin{aligned} 2\pi\hat{E}(\xi)\hat{\delta}_q(\xi) &= 2\pi\left\{\frac{1}{2\pi}\mathrm{Pf.}\,(1/|\xi|^2) + \hat{c}\delta(\xi)\right\}\hat{\delta}_q(\xi) \\ &= (\mathrm{Pf.}\,1/|\xi|^2)\hat{\delta}_q(\xi) = \hat{\delta}_q(\xi)/|\xi|^2, \end{aligned}$$

since the smooth function $\hat{\delta}_q(\xi)$ vanishes at the origin. By the definition (3.2), all the derivatives of $\hat{\delta}_q$ are bounded. Therefore, for any real-valued rapidly decreasing function ϕ in R^2, it holds that

$$\begin{aligned} \int_{R^2}\int_\Gamma E(x-y)q(y)d\Gamma_y\phi(x)dx &= \langle E*\delta_q, \phi\rangle = \langle E*\delta_q, \hat{\tilde{\phi}}\rangle \\ &= \langle 2\pi\hat{E}\hat{\delta}_q, \tilde{\phi}\rangle = \int_{R^2}\frac{1}{|\xi|^2}\hat{\delta}_q(\xi)\tilde{\phi}(\xi)d\xi, \end{aligned}$$

where $\tilde{\phi}$ is the inverse Fourier transform. Similarly we can obtain an equality for $n=3$. We now have

$$\int_{R^n}\int_{\Gamma} E(x-y)q(y)d\Gamma_y \phi(x)dx = \int_{R^n} \frac{1}{|\xi|^2}\hat{\delta}_q(\xi)\overline{\hat{\phi}(\xi)}d\xi. \tag{3.6}$$

(Note that $\tilde{\phi}=\bar{\hat{\phi}}$, since ϕ is real-valued.)
We, then, approximate δ_p by smooth functions. Let $\rho \in C_0^\infty(R^n)$ be a function with the following properties:

$$0 \le \rho \le 1 \text{ in } R^n, \ \rho(x)=0 \text{ if } |x|>1, \ \rho(x)=1 \text{ near } x=0,$$

$$\int_{R^n} \rho(x)dx = 1.$$

As usual, we put $\rho_j = j^n \rho(jx)$ $(j=1, 2, \cdots)$ and define ϕ_j by $\phi_j = \delta_p * \rho_j$. Then it is known that $\phi_j \in C_0^\infty(R^n)$ and that $\phi_j \to \delta_p$ as a distribution. Furthermore we can easily verify that

$$\int_{R^n} \phi_j(x)F(x)dx \longrightarrow \int_{\Gamma} p(x)F(x)d\Gamma_x \qquad \text{for all} \quad F \in C(R^n), \text{ as } j \to \infty,$$

and that

$$|\hat{\phi}_j(\xi)| \le |\hat{\delta}_p(\xi)|. \tag{3.7}$$

The former is easy to see and the latter holds by virtue of $\hat{\phi}_j(\xi) = (2\pi)^{n/2}\hat{\delta}_p(\xi)\hat{\rho}_j(\xi)$ and $|\hat{\rho}_j(\xi)| \le (2\pi)^{-n/2}\int_{R^n} \rho_j(x)dx = (2\pi)^{-n/2}$.

We now replace ϕ in (3.6) by ϕ_j and let j tend to the infinity. Then the left hand side tends to $A(q, p)$. On the other hand, the sequence $\{\hat{\phi}_j(\xi)\}$ converges to $\hat{\delta}_p(\xi)$ for every $\xi \in R^n$. Since (3.7) holds and since the function $\frac{1}{|\xi|^2}|\hat{\delta}_q(\xi)\hat{\delta}_p(\xi)|$ is summable, the right hand side converges to

$$\int_{R^n} \frac{1}{|\xi|^2}\hat{\delta}_q(\xi)\overline{\hat{\delta}_p(\xi)}d\xi$$

by virtue of Lebesgue's dominated convergence theorem. This completes the proof of Proposition 3.1. Q.E.D.

As we can see from the proof, we have the following corollary:

COROLLARY 1. *If we define* $Y=\{q \in H^{-1/2}(\Gamma); \langle q, 1\rangle = 0\}$ *for* $n=2$ *and* $Y=H^{-1/2}(\Gamma)$ *if* $n=3$, *then* (3.1) *holds for any* $q \in Y$ *and* $A(\ ,\ )$ *is a bounded bilinear form on* Y.

PROOF. By the positivity of $A(\ ,\ )$, we have $A(q,p)\leqq(A(q,q))^{1/2}\times(A(p,p))^{1/2}$. Therefore we have to show $A(q,q)\leqq c\|q\|^2_{-1/2}$ with a constant c depending only on Γ. Put

$$A(q,q)=\int_{|\xi|\leq 1}\frac{|\hat{\delta}_q(\xi)|^2}{|\xi|^2}d\xi+\int_{|\xi|\geq 1}\frac{|\hat{\delta}_q(\xi)|^2}{|\xi|^2}d\xi=I_1+I_2.$$

Clearly $|I_2|\leqq 2\int_{|\xi|\geq 1}\frac{|\hat{\delta}_q(\xi)|^2}{1+|\xi|^2}d\xi\leqq c\|q\|^2_{-1/2}$. On the other hand, we have

$$\hat{\delta}_q(\xi)-\hat{\delta}_q(0)=(2\pi)^{-n/2}\int_\Gamma(e^{-ix\cdot\xi}-1)q(x)d\Gamma_x,$$

which implies

$$|\hat{\delta}_q(\xi)-\hat{\delta}_q(0)|\leqq(2\pi)^{-1/2}\|e^{ix\cdot\xi}-1\|_{1/2}\|q\|_{-1/2}.$$

Since it is clear that $\|e^{ix\cdot\xi}-1\|_{1/2}\leqq c|\xi|$, we have $|I_1|\leqq c\|q\|^2_{-1/2}$ if n=2. If $n=3$, we note that

$$|\hat{\delta}_q(0)|\leqq(2\pi)^{-1/2}\|1\|_{1/2}\|q\|_{-1/2}$$

and that $\int_{|\xi|\leq 1}\frac{1}{|\xi|^2}d\xi$ is finite. By these facts we have $|I_1|\leqq c\|q\|^2_{-1/2}$. We now obtain

$$A(q,p)\leqq c\|q\|_{-1/2}\|p\|_{-1/2}$$

for any q,p in X. By the completion we have the conclusion. Q.E.D.

REMARK 4. The inequality (3.1) appears in Le Roux [10] for $n=2$ and in Nedelec and Planchard [12] for $n=3$. Although our proof is, in its essential part, the same as theirs and those in [6, 7, 15], it seems to the author that the presentation given here is more intuitive and compact.

§ 4. Supplement to Theorem.

In Theorem 1 we considered $X=L^2(\Gamma)/R$ if $n=2$. Here we give an example which shows that (3.1) does not, in general, hold on $\underline{L^2(\Gamma)}$ when $n=2$. Let Ω be a disk of radius R with the origin as its center. For an arbitrary $f\in L^2(\Gamma)$ we expand it by the Fourier series:

$$f=\sum_n \alpha_n e^{in\theta}\qquad(0\leqq\theta<2\pi).$$

Then we have

$$A(f, f)=R^2 \sum_{n,m} \int_0^{2\pi}\int_0^{2\pi} \frac{-1}{2\pi} \log|\mathrm{Re}^{i\theta}-\mathrm{Re}^{i\phi}| \alpha_n \overline{\alpha_m} e^{in\theta} e^{-im\phi} d\theta d\phi$$
$$=\pi R^2 \sum_{n\neq 0} \frac{1}{|n|} |\alpha_n|^2 - 2\pi R^2 |\alpha_0|^2 \log R.$$

Therefore the coercivity of $A(\ ,\ )$ holds if and only if $0<R<1$.

§ 5. Nonstationary problem.

From now on we discuss the stability and convergence of the boundary element approximation for the heat equation. We solve it under the Dirichlet boundary condition and the initial condition:

$$\frac{\partial u}{\partial t}=\Delta u \qquad (x\in\Omega,\ 0<t), \tag{5.1}$$

$$u(t, x)=0 \qquad (x\in\Gamma,\ 0<t), \tag{5.2}$$

$$u(0, x)=a(x) \qquad (x\in\Omega). \tag{5.3}$$

Here Ω is a bounded region in $\boldsymbol{R}^2$ or $\boldsymbol{R}^3$. Γ is the boundary of Ω. To discretize (5.1–3) we use the fundamental solution to the heat equation:

$$E(t, x)=(4\pi t)^{-n/2} \exp\left(-\frac{|x|^2}{4t}\right),$$

where n is the dimension of the domain. Let $u(t, x)$ satisfy (5.1). Then Green's formula yields for $x\in\Omega$ and $t>0$

$$u(t, x)=\int_\Omega E(t, x-y)u(0, y)dy+\int_0^t\int_\Gamma E(t-s, x-y)\frac{\partial u}{\partial \nu_y}(s, y)d\Gamma_y ds - \int_0^t\int_\Gamma \frac{\partial}{\partial \nu_y}E(t-s, x-y)u(s, y)d\Gamma_y ds \tag{5.4}$$

where $d\Gamma_y$ is the line (or surface) element on Γ. For $x\in\Gamma$ it holds that

$$\frac{1}{2}u(t, x)=\int_\Omega E(t, x-y)u(0, y)dy+\int_0^t\int_\Gamma E(t-s, x-y)\frac{\partial u}{\partial \nu_y}(s, y)d\Gamma_y ds - \mathrm{PV.}\int_0^t\int_\Gamma \frac{\partial}{\partial \nu_y}E(t-s, x-y)u(s, y)d\Gamma_y ds \tag{5.5}$$

since Γ is of C^2-class by the assumption. Now we use the boundary condition to obtain

$$(5.6) \qquad 0=\int_{\Omega} E(t, x-y)u(0, y)dy+\int_0^t\int_{\Gamma} E(t-s, x-y)\frac{\partial u}{\partial \nu_y}(s, y)d\Gamma_y ds.$$

We discretize this equation and obtain a discrete version of $\dfrac{\partial u}{\partial \nu}$. Then we substitute it for $\dfrac{\partial u}{\partial \nu}$ in (5.4). In this way we compute approximate value of $u(t, x)$. The precise algorithm is stated as follows: Let τ and h be positive numbers. τ is a mesh size for the time discretization and h is a mesh size for space discretization. We divide the boundary Γ into curved segments (or triangles if $n=3$) in a usual way. Here we remark that we use curved elements directly, hence we do not take into account of the effects caused by the approximation of Γ by a polygon or polyhedron. The maximum size of the segments (or triangles) is h by assumption. We introduce a function space X_h which is the set of all piecewise constant functions on Γ. We also introduce a bilinear form $A(\ ,\ )$: For functions q and p defined on Γ, we set

$$(5.7) \qquad A(q, p)=\int_0^{\tau}\int_{\Gamma}\int_{\Gamma} E(\tau-s, x-y)q(y)p(x)d\Gamma_y d\Gamma_x ds.$$

Now we determine $q_h \in X_h$ by requiring that q_h satisfies

$$(5.8) \qquad A(q_h, p_h)=-\int_{\Gamma}\int_{\Omega} E(\tau, x-y)a(y)p_h(x)dyd\Gamma_x$$

for all $p_h \in X_h$. Then we define the approximate function $u_h(\tau, x)$ by

$$(5.9) \qquad u_h(\tau, x)=\int_{\Omega} E(\tau, x-y)a(y)dy+\int_0^{\tau}\int_{\Gamma} E(\tau-s, x-y)q_h(y)d\Gamma_y ds.$$

This function is, by definition, the approximation of u at $t=\tau$. Regarding $u_h(\tau, x)$ as an initial value, we repeat the same procedure to obtain $u_h(2\tau, x)$, and so on. In this way, we can compute $u_h(n\tau, x)$ $(n=1, 2, \cdots)$.

In what follows we prove that we can perform the procedure above, i.e., q_h exists uniquely. Furthermore we prove that the L^2-norm of $u_h(n\tau, \cdot)$ is bounded from above by a constant which is independent of n, τ and h. This means that the scheme given above is unconditionally L^2-stable. We then prove that the approximate solution converges to the true solution under a certain condition.

§ 6. Unconditional stability.

In this section we prove the unconditional stability of the scheme given in the preceding section. Since we must solve q_h in each time step, the equation (5.8) should be generalized in the following way. Let u_0 be an arbitrary element in $H^1(\boldsymbol{R}^n)$. The function a is extended to $\boldsymbol{R}^n$ by setting it to zero outside Ω. Then a can be regarded as an element of $H^1(\boldsymbol{R}^n)$ under the assumption that $a \in H_0^1(\Omega)$. Now we consider a problem to find a q_h in X_h satisfying

$$A(q_h, p_h) = -\int_\Gamma \int_{\boldsymbol{R}^n} E(\tau, x-y) u_0(y) p_h(x) dy d\Gamma_x. \tag{6.1}$$

PROPOSITION 6.1. *The quadratic form A is positive definite. Hence* (6.1) *determines q_h uniquely. More precisely, we have*

$$A(q, q) \geqq c\tau |q|_{-1/2}^2 \qquad (q \in L^2(\Gamma)), \tag{6.2}$$

for any $\tau \in (0, 1)$, where c is a constant depending only on Γ.

PROOF. As in § 3 we have $\int_\Gamma E(\tau-s, x-y) q(y) d\Gamma_y = E(\tau-s, \cdot) * q$. (In this section we write simply q instead of δ_q.) In a similar way, we obtain

$$A(q, p) = (2\pi)^{n/2} \int_0^\tau \int_{\boldsymbol{R}^n} \hat{E}(\tau-s, \xi) \hat{q}(\xi) \overline{\hat{p}(\xi)} d\xi ds,$$

where $\hat{E}(\tau-s, \xi) = (2\pi)^{-n/2} \int_{\boldsymbol{R}^n} e^{-ix\cdot\xi} E(\tau-s, x) dx$. This Fourier transform is known to be equal to $(2\pi)^{-n/2} \exp(-(\tau-s)|\xi|^2)$. Hence we have

$$\begin{aligned} A(q, p) &= \int_0^\tau \int_{\boldsymbol{R}^n} e^{-(\tau-s)|\xi|^2} \hat{q}(\xi) \overline{\hat{p}(\xi)} d\xi ds \\ &= \int_{\boldsymbol{R}^n} \frac{1-e^{-\tau|\xi|^2}}{|\xi|^2} \hat{q}(\xi) \overline{\hat{p}(\xi)} d\xi ds. \end{aligned}$$

On the other hand, it holds that $\frac{1}{1+x} \leqq \frac{1-e^x}{x} \leqq \frac{2}{1+x}$ for $x>0$. Therefore we obtain for $0<\tau<1$

$$A(q, q) \geqq \int_{\boldsymbol{R}^n} \frac{\tau}{1+\tau|\xi|^2} |\hat{q}(\xi)|^2 d\xi \geqq c\tau |q|_{-1/2}^2.$$

This inequality shows that $A(q, q) = 0$ is equivalent to $\hat{q}(\xi) \equiv 0$, which is equivalent to $q=0$ in the sense of distribution in $\boldsymbol{R}^n$. Hence $A(q, q)$ vanishes if and only if $q=0$ almost everywhere in Γ. Q.E.D.

We now prove the following theorem which is the core of the stability criterion. To state it compactly we define the following symbol:

$$\langle u\rangle_\tau=\left(\int_{\boldsymbol{R}^n}\frac{\tau|\xi|^2}{1-e^{-\tau|\xi|^2}}|\hat{u}(\xi)|^2 d\xi\right)^{1/2}.$$

THEOREM 2. *Let u_0 be an element given in $H^1(\boldsymbol{R}^n)$. Let q_h be a function in X_h determined by* (6.1) *and u_h be defined by*

$$u_h(\tau, x)=\int_{\boldsymbol{R}^n} E(\tau, x-y)u_0(y)dy+\int_0^\tau\int_\Gamma E(\tau-s, x-y)q_h(y)d\Gamma_y ds. \tag{6.3}$$

Then we have

$$\langle u_h(\tau, \cdot)\rangle_\tau \leqq \langle u_0\rangle_\tau. \tag{6.4}$$

PROOF. As is seen easily, the right hand side of (6.3) is a continuous function of $x\in \boldsymbol{R}^n$. Hence its Fourier transform with respect to x is computed as follows:

$$\hat{u}_h(\tau, \xi)=e^{-\tau|\xi|^2}\hat{u}_0(\xi)+\frac{1-e^{-\tau|\xi|^2}}{|\xi|^2}\hat{q}_h(\xi). \tag{6.5}$$

On the other hand, the function q_h is characterized by (6.1), which is written equivalently

$$\int_{\boldsymbol{R}^n}\frac{1-e^{-\tau|\xi|^2}}{|\xi|^2}\hat{q}_h(\xi)\overline{\hat{p}_h(\xi)}d\xi=-\int_{\boldsymbol{R}^n} e^{-\tau|\xi|^2}\hat{u}_0(\xi)\overline{\hat{p}_h(\xi)}d\xi \quad (\hat{p}_h\in \hat{X}_h). \tag{6.6}$$

Here we have introduced a function space $\hat{X}_h\equiv\{\hat{p}_h;\ p_h\in X_h\}$. Equality (6.6) naturally leads us to a function space of all functions which are square summable with respect to the following weighted measure:

$$H=\left\{f;\ \int_{\boldsymbol{R}^n}|f(\xi)|^2\frac{1-e^{-\tau|\xi|^2}}{|\xi|^2}d\xi<+\infty\right\}.$$

Of course the inner product $(f, g)_\tau=\int_{\boldsymbol{R}^n} f(\xi)\overline{g(\xi)}\frac{1-e^{-\tau|\xi|^2}}{|\xi|^2}d\xi$ is equipped to this space.

Now we introduce an orthogonal projection $\Pi_{\tau,h}$ from H onto $\hat{X}_h$.

Then (6.6) shows that $\hat{q}_h = -\Pi_{\tau,h}\left(\frac{|\xi|^2}{1-e^{-\tau|\xi|^2}}e^{-\tau|\xi|^2}\hat{u}_0(\xi)\right)$. If we put $v(\xi) = \frac{|\xi|^2}{1-e^{-\tau|\xi|^2}}e^{-\tau|\xi|^2}\hat{u}_0(\xi)$, we have

$$\hat{u}_h(\tau, \xi) = \frac{1-e^{-\tau|\xi|^2}}{|\xi|^2}(1-\Pi_{\tau,h})\left(\frac{|\xi|^2 e^{-\tau|\xi|^2}}{1-e^{-\tau|\xi|^2}}\hat{u}_0(\xi)\right) = \frac{1-e^{-\tau|\xi|^2}}{|\xi|^2}(1-\Pi_{\tau,h})(v(\xi)). \tag{6.7}$$

From this equality we easily obtain (6.4) in the following way:

$$\begin{aligned}\langle u_h(\tau, \cdot)\rangle_\tau^2 &= \tau\int_{R^n}\frac{1-e^{-\tau|\xi|^2}}{|\xi|^2}|(I-\Pi_{\tau,h})v|^2 d\xi \\ &\leqq \tau\int_{R^n}\frac{1-e^{-\tau|\xi|^2}}{|\xi|^2}|v(\xi)|^2 d\xi \\ &= \int_{R^n}\frac{\tau|\xi|^2}{1-e^{-\tau|\xi|^2}}e^{-2\tau|\xi|^2}|\hat{u}_0(\xi)|^2 d\xi \\ &\leqq \langle u_0\rangle_\tau^2.\end{aligned}$$

Thus the proof is completed. Q.E.D.

For a given u_0 we define $u_h(\tau, \cdot)$ as above. Replacing u_0 by $u_h(\tau, \cdot)$, we obtain $u_h(2\tau, \cdot)$. By repeating this procedure, we obtain a sequence $\{u_h(n\tau, \cdot);\ n=1, 2, 3, \cdots\}$. As a corollary of Theorem 2 we have the following

COROLLARY 2. *For any positive integer n, it holds that*

$$\|u_h(n\tau, \cdot)\|_0 \leqq \|u_0\|_1, \tag{6.8}$$

where $\|\ \|_0$ is the norm in $L^2(R^n)$ and $\|\ \|_1$ is the norm in $H^1(R^n)$.

PROOF. Using Theorem 2 repeatedly, we obtain $\langle u_h(n\tau, \cdot)\rangle_\tau \leqq \langle u_0\rangle_\tau$. On the other hand, it holds that

$$1 \leqq \frac{x}{1-e^{-x}} \leqq 1+x \qquad (0<x<+\infty).$$

This inequality, together with Plancherel's theorem, yields that

$$\|f\|_0 \leqq \langle f\rangle_\tau, \qquad \langle g\rangle_\tau \leqq \|g\|_1.$$

Hence we have (6.8). Q.E.D.

The inequality (6.8) holds, whatever τ and h may be. This is our main result which shows unconditional stability with respect to the L^2-norm.

§ 7. Convergence.

In this section we prove that the approximate solution constructed in §§ 5 and 6 converges to the true solution if $h\tau^{-1/2}\to 0$.

THEOREM 3. *Let a satisfy $a\in H_0^1(\Omega)\cap H^2(\Omega)$ and $\Delta a\in H_0^1(\Omega)$ and u be the solution to* (5.1–3). *Let u_0 be a function in $H^1(\boldsymbol{R}^n)$ which approximate a and let $u_h(n\tau, x)$ be functions given by the algorithm* (6.3). *We fix an arbitrary $T>0$ and a natural number N and set $\tau=T/N$. Then we have*

$$\|u(n\tau,\cdot)-u_h(n\tau,\cdot)\|_{0,\Omega}\leqq\|u_0-a\|_{1,\boldsymbol{R}^n}+c(\tau^{1/2}+h\tau^{-1/2})(\|a\|_{2,\Omega}+\|\Delta a\|_{1,\Omega}) \quad \textit{for} \quad n=1,2,\cdots,N, \tag{7.1}$$

where c is a constant depending only on Ω and T. $\|\ \|_{s,\Omega}$ and $\|\ \|_{s,\boldsymbol{R}^n}$ are the norms in $H^s(\Omega)$ and $H^s(\boldsymbol{R}^n)$, respectively.

PROOF. We prove that

$$\langle u(\tau,\cdot)-u_h(\tau,\cdot)\rangle_\tau\leqq\langle u_0-a\rangle_\tau+c(\tau^{1/2}+h\tau^{-1/2})\tau(\|a\|_{2,\Omega}+\|\Delta a\|_{1,\Omega}). \tag{7.2}$$

Then the same proof yields

$$\begin{aligned}\langle u(m\tau,\cdot)-u_h(m\tau,\cdot)\rangle_\tau&\leqq\langle u((m-1)\tau,\cdot)-u_h((m-1)\tau,\cdot)\rangle_\tau\\&\quad+c(\tau^{1/2}+h\tau^{-1/2})\tau(\|u((m-1)\tau,\cdot)\|_{2,\Omega}+\|\Delta u((m-1)\tau,\cdot)\|_{1,\Omega})\\&\leqq\langle u((m-1)\tau,\cdot)-u_h((m-1)\tau,\cdot)\rangle_\tau+c'(\tau^{1/2}+h\tau^{-1/2})\tau(\|a\|_{2,\Omega}+\|\Delta a\|_{1,\Omega}).\end{aligned}$$

Summing up these inequalities from $m=1$ to $m=n$, we have (7.1). To show (7.2), we note that the true solution u satisfies

$$\hat{u}(\tau,\xi)=e^{-\tau|\xi|^2}\hat{a}(\xi)+\int_0^\tau e^{-(\tau-s)|\xi|^2}\hat{q}(s,\xi)ds,$$

where $q=\dfrac{\partial u}{\partial\nu}$. We define a function $r_h\in H^{-1/2}(\Gamma)$ by

$$A(r_h,p_h)=-\int_\Gamma\int_\Omega E(\tau,x-y)a(y)p_h(x)dyd\Gamma_x \qquad (p_h\in X_h)$$

or equivalently $\hat{r}_h=-\Pi_{\tau,h}\left(\dfrac{|\xi|^2e^{-\tau|\xi|^2}}{1-e^{-\tau|\xi|^2}}\hat{a}(\xi)\right)$. Then it holds that

$$\begin{aligned}\hat{u}(\tau,\xi)-\hat{u}_h(\tau,\xi)&=e^{-\tau|\xi|^2}(\hat{a}(\xi)-\hat{u}_0(\xi))+\int_0^\tau e^{-(\tau-s)|\xi|^2}(\hat{q}(s,\xi)-\hat{r}_h(\xi))ds\\&\quad+\frac{1-e^{-\tau|\xi|^2}}{|\xi|^2}(\hat{r}_h(\xi)-\hat{q}_h(\xi))\\&=\frac{1-e^{-\tau|\xi|^2}}{|\xi|^2}(I-\Pi_{\tau,h})\left(\frac{|\xi|^2e^{-\tau|\xi|^2}}{1-e^{-\tau|\xi|^2}}(\hat{a}-\hat{u}_0)\right)\\&\quad+\int_0^\tau e^{-(\tau-s)|\xi|^2}(\hat{q}(s,\xi)-\hat{r}_h(\xi))ds\\&\equiv I_1+I_2.\end{aligned}$$

Therefore we have

$$\begin{aligned}\langle u(\tau,\cdot)-u_h(\tau,\xi)\rangle_\tau&\leqq\langle I_1\rangle_\tau+\langle I_2\rangle_\tau\\&\leqq\langle u_0-a\rangle_\tau+\langle I_2\rangle_\tau.\end{aligned}$$

Therefore it is enough to show that

$$\langle I_2\rangle_\tau\leqq c\tau(\tau^{1/2}+h\tau^{-1/2})(\|a\|_{2,\Omega}+\|\Delta a\|_{1,\Omega}). \tag{7.3}$$

This is verified as follows: Let $B(\ ,\ )$ be a bilinear form defined by

$$B(f,g)=\int_0^\tau\int_{R^n}e^{-(\tau-s)|\xi|^2}\hat{f}(s,\xi)\overline{\hat{g}(s,\xi)}d\xi ds.$$

Its defining domain is a set of functions on $(0,\tau)\times\Gamma$ satisfying $B(f,f)<\infty$. If we regard an element of X_h as a function on $(0,\tau)\times\Gamma$ which does not depend on s, the space X_h is a closed subspace of the defining domain of $B(\ ,\ )$. Note that $B(q-r_h,p_h)=0$ for any $p_h\in X_h$. This fact means that

$$B(q-r_h,q-r_h)\leqq B(q-p_h,q-p_h)\qquad\text{for any}\quad p_h\in X_h.$$

On the other hand, we have

$$\begin{aligned}|I_2|^2&\leqq\int_0^\tau e^{-(\tau-s)|\xi|^2}ds\int_0^\tau e^{-(\tau-s)|\xi|^2}|\hat{q}(s,\xi)-\hat{r}_h(s,\xi)|^2ds\\&=\frac{1-e^{-\tau|\xi|^2}}{|\xi|^2}\int_0^\tau e^{-(\tau-s)|\xi|^2}|\hat{q}(s,\xi)-\hat{r}_h(s,\xi)|^2ds.\end{aligned}$$

Therefore

$$\begin{aligned}\int_{R^n}\frac{\tau|\xi|^2}{1-e^{-\tau|\xi|^2}}|I_2|^2d\xi&\leqq\tau\int_{R^n}\int_0^\tau e^{-(\tau-s)|\xi|^2}|\hat{q}(s,\xi)-\hat{r}_h(\xi)|^2dsd\xi\\&=\inf_{p_h\in X_h}\tau\int_{R^n}\int_0^\tau e^{-(\tau-s)|\xi|^2}|\hat{q}(s,\xi)-\hat{p}_h(\xi)|^2dsd\xi.\end{aligned} \tag{7.4}$$

To estimate the integrant, we use the identity

$$\hat{q}(s,\xi)=-\int_s^\tau \frac{\partial \hat{q}}{\partial t}(t,\xi)dt+\hat{q}(\tau,\xi).$$

Then, by (7.4), we have

$$\langle I_2\rangle_\tau^2 \leqq \inf_{p_h\in X_h} 2\tau\int_{R^n}\frac{1-e^{-\tau|\xi|^2}}{|\xi|^2}|\hat{q}(\tau,\xi)-\hat{p}_h(\xi)|^2 d\xi \qquad (7.5)$$
$$+2\tau\int_{R^n}\int_0^\tau e^{-(\tau-s)|\xi|^2}\left|\int_s^\tau \frac{\partial \hat{q}}{\partial t}(t,\xi)\,dt\right|^2 ds d\xi.$$

Since $\dfrac{1-e^{-\tau|\xi|^2}}{|\xi|^2}\leqq\dfrac{1}{1+|\xi|^2}$ for $0<\tau<1$, the first term is majorized by

$$\inf_{p_h\in X_h}\tau\|\hat{q}(\tau,\cdot)-\hat{p}_h\|^2_{H^{-1}(R^n)}\leqq c\tau\inf_{p_h\in X_h}\|q(\tau,\cdot)-p_h\|^2_{-1/2}$$
$$\leqq c_1\tau h^2\|q(\tau,\cdot)\|^2_{1/2}\leqq c_2\tau h^2\|a\|^2_{2,\Omega}.$$

Here the second and the third inequalities follow from

$$\|w-p_h\|_{-1/2}\leqq ch\|w\|_{1/2} \qquad (w\in H^{1/2}(\Gamma)) \qquad (7.6)$$

and $\|q(\tau,\cdot)\|_{1/2}\leqq c\|u(\tau,\cdot)\|_{2,\Omega}\leqq c_1\|a\|_{2,\Omega}$, respectively. For the proof of (7.6), see page 122 of [12]. On the other hand, the second term of (7.5) is majorized by

$$2\tau\int_{R^n}\int_0^\tau e^{-(\tau-s)|\xi|^2}(\tau-s)\int_s^\tau\left|\frac{\partial\hat{q}}{\partial t}(t,\xi)\right|^2 dtdsd\xi$$
$$\leqq 2\tau^2\int_{R^n}\int_0^\tau\left|\frac{\partial\hat{q}}{\partial t}(t,\xi)\right|^2 dt\int_0^\tau e^{-(\tau-s)|\xi|^2}dsd\xi$$
$$=2\tau^2\int_{R^n}\int_0^\tau\frac{1-e^{-\tau|\xi|^2}}{|\xi|^2}\left|\frac{\partial\hat{q}}{\partial t}(t,\xi)\right|^2 dtd\xi$$
$$\leqq c\tau^2\int_0^\tau\left\|\frac{\partial\hat{q}}{\partial t}(t,\cdot)\right\|^2_{-1,R^n}dt.$$

Since it holds that

$$\left\|\frac{\partial\hat{q}}{\partial t}\right\|_{-1,R^n}\leqq c\left\|\frac{\partial q}{\partial t}\right\|_{-1/2}\leqq c_1\left\|\frac{\partial u}{\partial t}\right\|_{1,\Omega}$$
$$\leqq c(\|a\|_{2,\Omega}+\|\Delta a\|_{1,\Omega}) \qquad (0<t),$$

we finally obtain

$$\langle I_2\rangle_\tau^2\leqq c(\tau^3+\tau h^2)(\|a\|_{2,\Omega}+\|\Delta a\|_{1,\Omega})^2.$$

Thus we get to (7.3). Q.E.D.

§ 8. Concluding remarks.

Concerning BEM approximation, we proved two inequalities (3.1) and (6.4). These are shown by the intuitive use of the Fourier transform. The inequality (3.1) appears in [6, 7, 10, 12] but the inequality (6.4) seems to be new for us. Unconditional stability is a direct consequence of (6.4). We showed convergence under the condition that $h^2/\tau \to 0$. The proof of convergence in the case of Dirichlet problem in a domain of dimension $\geqq 2$ seems also to be new. We do not know how close this is to the optimal rate. We, however, can not improve this in our elementary framework.

We do not know our approach is effective in the collocation method, since, in this case, the resulting equation is no longer self-adjoint. We know Arnold and Wendland [2] where the solvability and convergence of a collocation method for two dimensional stationary problem is rigorously proved.

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