The nonlinear transformation of Gaussian measure on Banach space and its absolute continuity (I)

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§1. Introduction.

To begin with, let us introduce some preliminary notions. For real Banach spaces E and F, E^* denotes the dual space of E with the strong topology, I_E denotes the identical map on E and $\mathcal{L}^\infty(E, F)$ denotes the Banach space consisting of all bounded linear operators from E into F with the operator norm. For real Hilbert spaces H and K, $\mathcal{L}^2(H, K)$ denotes the Hilbert space of all Hilbert-Schmidt operators from H into K with a Hilbert-Schmidt norm.

We say that (μ, H, B) is an abstract Wiener space if μ, H and B satisfy the following condition (W-1) and (W-2).

(W-1) B is a real separable Banach space, and H is a real separable Hilbert space densely and continuously included in B.

We identify H^* with H, then B^* is naturally regarded as a dense subset of H and the inclusion map from B^* into H is continuous. And moreover the relation ${}_{B^*}\langle u, v\rangle_B=(u, v)_H$ holds for any $u\in B^*$ and $v\in H$.

(W-2) μ is a Gaussian probability measure on B such that

$$\int_B \exp(\sqrt{-1}\,{}_{B^*}\langle u, z\rangle_B)\mu(dz)=\exp\left(-\frac{1}{2}\|u\|_H^2\right)$$

for each $u\in B^*$.

Throughout this paper we promise that (μ, H, B) denotes an abstract Wiener space.

In this paper we study the following problem. Let F be a measurable map from B into H. Our problem is to study when the image measure $(I_B-F)\mu$ on B through I_B-F: $B\to B$ is absolutely continuous relative to μ, and to give the explicit form of its density function.

R. H. Cameron and W. T. Martin are the first to study this problem. In their paper [2], they dealt in the case that B is the space of all continuous functions on the interval [0, 1] and μ is an ordinary Wiener measure. L. Gross [5] and H. H. Kuo [8] extended the work of R. H. Cameron and W. T. Martin for general abstract Wiener spaces.

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From a viewpoint of stochastic differential equation, this problem was studied by V. Girsanov [4] and M. Motoo [11], and they showed that Ito integral appeared essentially in the density function.

In 1974, R. Ramer [13] introduced an abstract version of Ito integral (we will call it Ito-Ramer integral), and he solved our problem under some assumptions, one of which is that $F: B \to H$ is continuous. But his assumptions are sometimes so strong that we cannot apply his theorem to stochastic differential equations. He conjectured in his paper that we could solve our problem under some weaker assumptions than his. We give one of answers to his conjecture in this paper.

Now let us shortly summarize the content of our paper.

We give some tools for later use in Sections 2, 3 and 4. We also introduce an Ito-Ramer integral as an abstract version of Ito integral in Section 5 following the idea of Ramer [13].

The main results are stated in Sections 6, 7 and 8. We say that a measurable map $F: B \to H$ is an $\mathcal{H}-C^1$ map, if there exists a Hilbert Schmidt operator $DF(z): H \to H$ for each $z \in B$ such that (1) $\|F(z+h)-F(z)-DF(z)h\|_H = o(\|h\|_H)$ for each $z \in B$ as $\|h\|_H \to 0$, and (2) $DF(z+\cdot): H \to \mathcal{L}^2(H, H)$ is continuous for each $z \in B$. We prove the following in Section 6.

THEOREM 6.2. *If $F: B \to H$ is an $\mathcal{H}-C^1$ map and $I_H - DF(z): H \to H$ is invertible for μ-a.e. z, then $(I_B - F)\mu$ is absolutely continuous relative to μ.*

THEOREM 6.4. *Let $F: B \to H$ be an $\mathcal{H}-C^1$ map, and assume that $I_B - F: B \to B$ is bijective and $I_H - DF(z): H \to H$ is invertible for any $z \in B$. Then $(I_B - F)^{-1}\mu$ and μ are mutually absolutely continuous.*

We shall also give the explicit form of the density function $\dfrac{d(I_B-F)^{-1}\mu}{d\mu}$ in Theorem 6.4.

Here let us explain the difference between Gross-Kuo's result, Ramer's result and ours. In Kuo [8], Theorem 6.4 above has been shown under the assumption that the image of F is contained in B^* and $F: B \to B^*$ is continuously Frechet differentiable. (The presentation of Gross [5] is different from this.) In Kuo's case, $DF(z): H \to H$, $z \in B$, becomes a nuclear operator automatically. In Ramer [13], Theorem 6.4 has been shown under the assumption that $F: B \to H$ and $DF: B \to \mathcal{L}^2(H, H)$ are continuous. But we shall prove Theorem 6.4 without the assumption of the continuity of $F: B \to H$ and $DF: B \to \mathcal{L}^2(H, H)$. This is the different point from Ramer [13].

We give some extended theorems of Theorems 6.2 and 6.4 in Section 7. There we do not need to assume that $F(z+\cdot): H \to H$, $z \in B$, is continuous any more.

In Section 8, we prove a certain Sard type theorem and give a certain sufficient condition under which μ is absolutely continuous relative to $(I_B-F)\mu$.

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§ 2. Preliminary material.

In this section we note some known results about an abstract Wiener space (μ, H, B). Let $\mathcal{P}(E)$ denote the set of all projections of H with a finite dimensional range included in E for each vector subspace E of H. Then $\mathcal{P}(E)$ is a directed set with usual order induced by the inclusion of ranges. It is obvious that any projection P belonging to $\mathcal{P}(B^*)$ is extensible to a bounded linear operator $\bar{P}: B\to B^*$. R. Carmona [3] proved

PROPOSITION 2.1. *Let $\{P_n\}_{n=1}^{\infty}$ be an increasing sequence of elements of $\mathcal{P}(B^*)$ strongly converging to I_H, then $\|z-\bar{P}_n z\|_B\to 0$, $n\to\infty$ for μ-a.e. z. Moreover, for any $1<q<\infty$, $\int_B\|z-\bar{P}_n z\|_B^q\mu(dz)\to 0$, $n\to\infty$.*

The following is well-known. (See for an example R. Ramer [13].)

PROPOSITION 2.2. *For any Hilbert Schmidt operator K on H, there exists a measurable map $K: B\to H$ such that $\|\tilde{K}z-K\bar{P}z\|_H\to 0$, $P\in\mathcal{P}(B^*)$, in probability with respect to μ, and $\int_B\|\tilde{K}z\|_H^2\mu(dz)=\mathrm{trace}\, KK^*$.*

We call $\tilde{K}$ the stochastic extension of K. For each $P\in\mathcal{P}(B^*)$, it is easy to see that $\tilde{P}z=\bar{P}z$ for μ-a. e. z, and so we identify $\tilde{P}$ with $\bar{P}$.

The following are also well-known. For the proof, see H. Kuo [9] Chap. 1 Section 4 and Chap. 2 Section 5.

PROPOSITION 2.3. *The inclusion map from H to B is a compact operator.*

PROPOSITION 2.4. *For each $u\in B$, let N_u be an operator on B defined by $N_u z=u$ for any $z\in B$. Then the image measure $(I_B-N_u)^{-1}\mu$ is absolutely continuous relative to μ, if and only if u belongs to H. Furthermore, for each $u\in B^*$, $\frac{d(I_B-N_u)^{-1}\mu}{d\mu}(z)=\exp\left({}_{B^*}\langle u, z\rangle_B-\frac{1}{2}\|u\|_H^2\right)$.*

PROPOSITION 2.5. *Let K be a bounded linear operator from B into B^*, and assume that I_H-K is invertible as a map from H into itself. Then K is a nuclear operator on H, $I_B-K: B\to B$ is invertible and $(I_B-K)^{-1}\mu$ is absolutely continuous relative to μ. Furthermore,*

$$\frac{d(I_B-K)^{-1}\mu}{d\mu}(z)=|\det(I_H-K)|\exp\left({}_{B^*}\langle Kz, z\rangle_B-\frac{1}{2}\|Kz\|_H^2\right).$$

§ 3. H^1-class maps.

In this section we introduce an infinite dimensional analogue of the classical Sobolev space. Let E denote a real Banach space.

Definition 3.1. (1) We say that a map $f: \boldsymbol{R}\to E$ is absolutely continuous if, for any $-\infty<a<b<\infty$ and $\varepsilon>0$, there exists some $\delta(\varepsilon, a, b)>0$ such that $\sum_{i=1}^{n}\|f(t_i)-f(s_i)\|_E<\varepsilon$ holds for any integer n and $a\leqq t_1<s_1\leqq t_2<s_2\cdots t_n<s_n\leqq b$, $\sum_{i=1}^{n}|t_i-s_i|<\delta(\varepsilon, a, b)$.

(2) We say that a map $f: \boldsymbol{R}\to E$ is strictly absolutely continuous, if $f: \boldsymbol{R}\to E$ is continuous, $f(t)$ is strongly differentiable for almost every t and it satisfies that $\int_a^b\left\|\frac{df}{dt}(t)\right\|_E dt<\infty$ and $f(b)-f(a)=\int_a^b\frac{df}{dt}(t)dt$ for any $-\infty<a<b<\infty$, where $\frac{df}{dt}(t)$ denotes strong derivative of f at t.

Proposition 3.1. (1) *Given a map $f: \boldsymbol{R}\to E$, f is absolutely continuous if f is strictly absolutely continuous.*

(2) *Assume that E is reflexive and $f: \boldsymbol{R}\to E$ is absolutely continuous, then f is strictly absolutely continuous.*

See V. Barbu and Th. Precupanu [1], for the proof.

We say that $F: B\to E$ is strongly measurable, if there exists a Borel subset Ω of B and a separable closed subspace E_0 of E such that $F(\Omega)\subset E_0$, $F|_\Omega: \Omega\to E_0$ is measurable and $\mu(\Omega)=1$.

Definition 3.2. We say that a strongly measurable map $F: B\to E$ is *stochastic Gâteaux H-differentiable* (abbreviated by $S.G.D.$) if there exists a strongly measurable map $DF: B\to\mathcal{L}^\infty(H, E)$ such that $\frac{1}{t}{}_{E^*}\langle u, F(z+th)-F(z)\rangle_E$ $\to{}_{E^*}\langle u, DF(z)h\rangle_E$ in probability with respect to μ, $t\to 0$, for each $u\in E^*$ and $h\in H$. DF is called a *stochastic Gâteaux H-derivative* of F.

Remark 3.1. By virtue of Proposition 2.4, $F(z+th)$ is determinate for μ-a.e. z. without depending on a version of F, so is $DF(z)$.

Definition 3.3. We say that a strongly measurable map $F: B\to E$ is *ray absolutely continuous with probability one* (abbreviated by $R.A.C.$), if there exists a strongly measurable map $\tilde{F}_h: B\to E$ for each $h\in H$ such that $\tilde{F}_h(z)=F(z)$ for μ-a. e. $z\in B$ and $\tilde{F}_h(z+th)$ is strictly absolutely continuous in t for each $z\in B$.

DEFINITION 3.4. We say that a map $F: B\to E$ belongs to $H^1(B\to E; d\mu)$ if $F: B\to E$ is strongly measurable, S. G. D. and R. A. C.

PROPOSITION 3.2. *Let $F: B\to E$ be an element of $H^1(B\to E; d\mu)$. Then for each $h\in H$ and $-\infty<a<b<\infty$, $\int_a^b \|DF(z+sh)h\|_E\, ds<\infty$ for μ-a. e. $z\in B$ and*

$$\tilde{F}_h(z+bh)-\tilde{F}_h(z+ah)=\int_a^b DF(z+sh)h\,ds \qquad \text{for } \mu\text{-a. e. } z.$$

Here $\tilde{F}_h$ is a version of F as in Definition 3.3.

Before proving our proposition, we shall introduce some notion. For each finite dimensional subspace K of H, we define the probability measures μ_K and $\mu_K\perp$ on B by $\mu_K=\tilde{P}_K\mu$ and $\mu_K\perp=(I_B-\tilde{P}_K)\mu$, where P_K is the orthogonal projection from H onto K and $\tilde{P}_K$ is the stochastic extension of P_K. Let $\{k_1, \cdots, k_n\}$ be an orthogonal base of K. Then it is easy to see that

$$\begin{aligned}(3.1)\qquad \int_B f(z)\mu(dz)&=\int_{B\times B} f(z+\tilde{z})\mu_K\perp(dz)\otimes\mu_K(d\tilde{z})\\ &=\int_{B\times R^n} f\Big(z+\sum_{j=1}^n x_jk_j\Big)\mu_K\perp(dz)\otimes\Big(\frac{1}{2\pi}\Big)^{n/2}\exp\Big(-\frac{1}{2}\sum_{j=1}^n x_j^2\Big)dx\end{aligned}$$

for each bounded measurable function f on B.

Now let us prove our proposition. We may assume that $\|h\|_H=1$ without loss of generality. Let $K=\boldsymbol{R}h$. Since $\tilde{F}_h(z+th)$ is strongly differentiable in t for a. e. t and $\int_a^b\Big\|\frac{d}{dt}\tilde{F}_h(z+th)\Big\|_E dt<\infty$, it is easy to see that

$$(3.2)\qquad \frac{1}{\tau}\{\tilde{F}_h(z+(t+\tau)h)-\tilde{F}_h(z+th)\}\to\frac{d}{dt}\tilde{F}_h(z+th) \qquad \text{for a. e.}(z, t)$$

with respect to $\mu_K\perp(dz)\otimes\Big(\frac{1}{2\pi}\Big)^{1/2}e^{-t^2/2}dt$ as $\tau\to0$. It follows from Definition 3.2 and (3.1) that

$$\frac{1}{\tau}{}_{E^*}\langle u, \tilde{F}_h(z+(t+\tau)h)-\tilde{F}_h(z+th)\rangle_E\to{}_{E^*}\langle u, DF(z+th)h\rangle_E, \qquad \tau\to0,$$

in probability with respect to $\mu_K\perp(dz)\otimes\Big(\frac{1}{2\pi}\Big)^{1/2}e^{-t^2/2}dt$, which implies that

$$(3.3)\qquad \frac{d}{dt}\tilde{F}_h(z+th)=DF(z+th)h \qquad \text{for a. e. } (z, t)$$

with respect to $\mu_K\perp(dz)\otimes dt$. By the definition of $\mu_K\perp$, it is easy to see that

(3.3) holds for a. e. (z, t) with respect to $\mu(dz)\otimes dt$. This completes the proof.

REMARK 3.1. By (3.3) we get

$$\tilde{F}_h(z+bh)-\tilde{F}_h(z+ah)=\int_a^b DF(z+th)h\,dt \qquad \text{for } \mu_{K^\perp}\text{-a. e. } z.$$

DEFINITION 3.5. We say that a measurable function w defined on B is *strictly positive*, if there exists a measurable function $\tilde{w}_h$ for each $h\in H$ such that

(1) $\tilde{w}_h(z)=w(z)$ for μ-a. e. z and
(2) $\inf\{\tilde{w}_h(z+th)\,;\ -T<t<T\}>0$ for any $z\in B$ and $T>0$.

The following theorem shows the stability of $H^1(B\to E\,;\,d\mu)$.

THEOREM 3.1. *Let F_n $(n=1, 2, \cdots)$ be an element of $H^1(B\to E\,;\,d\mu)$, let F be a strongly measurable map from B to E, G be a strongly measurable map from B to $\mathcal{L}^\infty(H, E)$, and let w be a strictly positive measurable function on B. Assume furthermore that*

(1) $\|F(z)-F_n(z)\|_E\to 0$ *in probability with respect to* μ *as* $n\to\infty$ *and*

(2) $\int_B\|G(z)h\|_E w(z)\mu(dz)<\infty$ *and* $\int_B\|G(z)h-DF_n(z)h\|_E w(z)\mu(dz)\to 0$, $n\to\infty$, *for each* $h\in H$. *Then F belongs to $H^1(B\to E\,;\,d\mu)$ and $DF(z)=G(z)$ for* μ-a. e. z.

PROOF. For simplicity, we assume $E=\boldsymbol{R}$. Take a measurable map $\tilde{F}_{n,h}$: $B\to\boldsymbol{R}$ as in Definition 3.3 for F_n. Let $K=\boldsymbol{R}h$ as in the proof of Proposition 3.2, then by Remark 3.1 we obtain

$$\tilde{F}_{n,h}(z+bh)-\tilde{F}_{n,h}(z+ah)=\int_a^b DF_n(z+sh)h\,ds \qquad \text{for } \mu_{K^\perp}\text{-a. e. } z. \tag{3.4}$$

The right and left hands of (3.4) are continuous in a and b. So there exists a σ-compact subset Ω_1 of B such that $\mu_{K^\perp}(\Omega_1)=1$ and (3.4) holds for any $z\in\Omega_1$ and $a<b$.

Taking a subsequence if necessary, we may assume $\tilde{F}_{n,h}(z)\to F(z)$ for μ-a. e. z. So there exists a σ-compact subset Ω_2 of B such that $\mu_{K^\perp}(\Omega_2)=1$ and for each $z\in\Omega_2$

$$\tilde{F}_{n,h}(z+th)\to F(z+th)\,,\quad n\to\infty\,, \qquad \text{for a. e. } t. \tag{3.5}$$

By assumption (2), we get

$$\int_B\mu_{K^\perp}(dz)\int_R|G(z+th)h-DF_n(z+th)h|\,\tilde{w}_h(z+th)e^{-t^2/2}dt\to 0$$

where $\tilde{w}_h$ is a function on B as in Definition 3.5. So we may assume that there exists a σ-compact subset Ω_3 of B such that $\mu_{K^\perp}(\Omega_3)=1$ and

$$\int_{R} |G(z+th)h-DF_n(z+th)h|\,\tilde{w}_h(z+th)e^{-t^2/2}dt \to 0\,,$$

$n\to\infty$, for each $z\in\Omega_3$. By virtue of Definition 3.5 (2), we obtain

$$\int_a^b |G(z+th)h-D\tilde{F}_{n,h}(z+th)h|\,dt\to 0 \tag{3.6}$$

for any $a<b$ and $z\in\Omega_3$.

Let $\Omega_4=\Omega_1\cap\Omega_2\cap\Omega_3$, then by (3.4), (3.5) and (3.6) it is easy to see that $\mu_{K}\bot(\Omega_4)=1$, $\tilde{F}_{n,h}(z+th)$ are convergent as $n\to\infty$ for any t and $z\in\Omega_4$ and $\lim \tilde{F}_{n,h}(z+th)=F(z+th)$ for a.e. (z, t) with respect to $\mu_{K}\bot(dz)\otimes dt$.

Moreover we can see that

$$\lim_{n\to\infty}\tilde{F}_{n,h}(z+bh)-\lim_{n\to\infty}\tilde{F}_{n,h}(z+ah)=\int_a^b G(z+th)h\,dt$$

for any $a<b$ and $z\in\Omega_4$. Let $\Omega_0=\Omega_4+\boldsymbol{R}h$, then Ω_0 is σ-compact in B and $\mu(\Omega_0)=1$. Define a map $\tilde{F}_h: B\to\boldsymbol{R}$ by

$$\tilde{F}_h(z)=\begin{cases}\lim_{n\to\infty}\tilde{F}_{n,h}(z) & \text{if } z\in\Omega_0\\ 0 & \text{otherwise.}\end{cases}$$

It is easy to see that $\tilde{F}_h(z)=F(z)$ for μ-a.e. z and

$$\tilde{F}_h(z+th)=\begin{cases}\tilde{F}_h(z)+\int_0^t G(z+sh)h\,ds & \text{if } z\in\Omega_0\\ 0 & \text{otherwise.}\end{cases}$$

Therefore F is R.A.C. On the other hand, it is easy to see that

$$\frac{1}{t}\{\tilde{F}_h(z+th)-\tilde{F}_h(z)\}\to G(z)h\,,\quad t\to 0\,,\qquad \text{for}\quad \mu\text{-a.e. } z.$$

This completes the proof.

§ 4. The partition of unity.

DEFINITION 4.1. For any subset A of B, we define a function $\rho(\cdot\,;A)$: $B\to[0,\infty]$ by

$$\rho(z\,;A)=\begin{cases}\inf\{\|h\|_H\,;\,h\in(A-z)\cap H\} & \text{if } (A-z)\cap H\neq\emptyset\,,\\ \infty & \text{otherwise.}\end{cases}$$

Then we get the following.

PROPOSITION 4.1. (1) *If subsets A and A' of B satisfy $A\subset A'$, then $\rho(z\,;A)\geqq\rho(z\,;A')$ for each $z\in B$.*

(2) *For any subset A of B and $h\in H$, $\rho(z+h\,;A)\leqq\|h\|_H+\rho(z\,;A)$ for each $z\in B$.*

(3) *Let* $\{A_n\}_{n=1}^{\infty}$ *is increasing subsets of* B *and* $A=\bigcup_{n=1}^{\infty} A_n$, *then* $\rho(z; A_n) \downarrow \rho(z; A)$, $n\to\infty$, *for each* $z\in B$.

PROOF. (1) and (2) are obvious, so it suffices us to show

$$\varliminf_{n\to\infty} \rho(z; A_n) \leqq \rho(z; A) \quad \text{for each} \quad z\in B. \tag{4.1}$$

However, (4.1) is obvious when $(A-z)\cap H=\emptyset$. Therefore we will show (4.1) when $(A-z)\cap H\neq\emptyset$. For any $\varepsilon>0$, there exists $h\in(A-z)\cap H$ such that $\|h\|_H \leqq\rho(z; A)+\varepsilon$. Since $h+z\in A$ and $A_n\uparrow A$, there exists an integer n_0 such that $h+z\in A_n$ for each $n\geqq n_0$. So $h\in(A_n-z)\cap H$, which implies $\rho(z; A_n)\leqq\rho(z; A)+\varepsilon$ for any $n\geqq n_0$. This completes the proof.

THEOREM 4.1. (1) *If* K *is a compact subset of* B, *then* $\rho(\cdot\,; K): B\to[0, \infty]$ *is lower semi-continuous.*

(2) *If* G *is a* σ*-compact subset of* B, *then* $\rho(\cdot\,; G): B\to[0, \infty]$ *is measurable.*

PROOF. (2) is an immediate consequence of (1) and Proposition 4.1 (3). So it is sufficient to show (1). Let $A_a=\{z\in B; \rho(z; K)\leqq a\}$ and $S_a=\{h\in H; \|h\|_H\leqq a\}$ for each $a\geqq 0$. It is obvious that $A_a\supset K+S_a$. On the other hand let $z\in A_a$, then there exists a sequence $\{h_n\}_{n=1}^{\infty}\subset(K-z)\cap H$ such that $\|h_n\|_H\leqq a+\frac{1}{n}$. By virtue of Proposition 2.3, taking a subsequence if necessary, there exists $h\in H$ such that h_n converges to h in B as $n\to\infty$ and $\|h\|_H\leqq a$. In view of the closedness of K, we get $z+h\in K$, which shows that $z\in K+S_a$. So we obtain $A_a\subset K+S_a$. According to Proposition 2.3 S_a is compact in B, so is A_a. This completes the proof.

REMARK 4.1. Let ϕ be a smooth function on $\boldsymbol{R}$ with compact support and G be a σ-compact subset of B. Let $g(z)=\phi(\rho(z; G))$ for each $z\in B$ with the convention that $\phi(\infty)=0$. Then g is a measurable function on B and

$$|g(z+h)-g(z)|\leqq c\|h\|_H$$

for each $z\in B$ and $h\in H$, where

$$c=\sup\left\{\left|\frac{d\phi}{dt}(t)\right|;\ t\in\boldsymbol{R}\right\}.$$

THEOREM 4.2. *Let* E *be a separable reflexive Banach space and* F *be a measurable map from* B *to* E *and suppose furthermore that there exists a positive constant* c *such that* $\|F(z+h)-F(z)\|_E\leqq c\|h\|_H$ *for each* $z\in B$ *and* $h\in H$. *Then there exists a measurable subset* D_0 *of* B *and a map* $DF: B\to\mathcal{L}^{\infty}(H, E)$ *such that*

(1) $\mu(D_0)=1$

(2) $\lim_{t\to 0}\frac{1}{t}(F(z+th)-F(z))=DF(z)h$ *for any* $z\in D_0$ *and* $h\in H$, *and*

(3) $DF(\cdot)h: B\to E$ *is measurable for each* $h\in H$.

In particular if $DF: B\to\mathcal{L}^{\infty}(H, E)$ *is strongly measurable,* F *belongs to* $H^1(B\to E; d\mu)$.

PROOF. Let V be a countable subset of B^* such that V is Q-module and dense in H. Then $F(z+tv)$ is strictly absolutely continuous in t for any $z\in B$ and $v\in V$ by virtue of Proposition 3.1, and so $F(z+tv)$ is strongly differentiable in t for a.e. t. Let $D_v=\left\{z\in B; \frac{1}{r}\{F(z+rv)-F(z)\}\text{ is convergent as } r\to 0, r\in Q\right\}$, then D_v is measurable and $\mu(D_v)=1$. Let $G(z; v)=\lim_{\substack{r\to 0\\ r\in Q}}\frac{1}{r}\{F(z+rv)-F(z)\}$ for each $v\in V$ and $z\in D_v$, then $G(\cdot; v): D_v\to E$ is measurable. Since $F(z+tv)$ is continuous in t, we obtain $G(z; v)=\lim_{t\to 0}\frac{1}{t}\{F(z+tv)-F(z)\}$.

We claim the following (4.2) and (4.3).

(4.2) $G(z; rv)=rG(z; v)$ for each $r\in Q$ and $z\in D_v=D_{rv}$.

(4.3) Let $D_{v_1,v_2}=\{z\in D_{v_1}\cap D_{v_2}\cap D_{v_1+v_2}; G(z; v_1)+G(z; v_2)=G(z; v_1+v_2)\}$ for each $v_1, v_2\in V$, then $\mu(D_{v_1,v_2})=1$.

(4.2) is obvious. (4.3) is also obvious whenever v_1 and v_2 are linearly dependent over $\boldsymbol{R}$. So we prove (4.3) when v_1 and v_2 are linearly independent over $\boldsymbol{R}$. Since E is separable and reflexive, E^* is also separable. Let $\{u_n\}_{n=1}^{\infty}$ be a countable dense subset of E^* and let $f_n(x, y; z)={}_{E^*}\langle u_n, F(z+xv_1+yv_2)\rangle_E$ for each $x, y\in\boldsymbol{R}$ and $z\in B$. Then $f_n(x, y; z)$ is Lipschitz continuous in (x, y). According to H. Radmacher [12], $f_n(x, y; z)$ is totally differentiable in (x, y) for a.e. (x, y). So let

$$A^n=\{z\in D_{v_1}\cap D_{v_2}\cap D_{v_1+v_2}; {}_{E^*}\langle u_n, G(z; v_1+v_2)\rangle_E$$
$$={}_{E^*}\langle u_n, G(z; v_1)\rangle_E+{}_{E^*}\langle u_n, G(z; v_2)\rangle_E\}$$

and $A_z^n=\{(x, y)\in\boldsymbol{R}^2; z+xv_1+yv_2\in A^n\}$, then $\boldsymbol{R}^2\backslash A_z^n$ is of Lebesgue measure zero for each $z\in B$, and accordingly

$$\mu(A^n)=\int_B\mu_{K^{\perp}}(dz)\mu_K(\{xv_1+yv_2; (x, y)\in A_z^n\})=1,$$

where $K=\boldsymbol{R}v_1+\boldsymbol{R}v_2$. This shows (4.3).

Now we can prove Theorem 4.2. Let $D_0=\cap\{D_{v_1,v_2}; v_1, v_2\in V\}$, then D_0 is measurable and $\mu(D_0)=1$. For each $z\in D_0$, $G(z, \cdot): V\to E$ is a Q-linear map by virtue of (4.2) and (4.3), and furthermore

$$\|G(z; v)\|_E=\lim_{t\to 0}\frac{1}{t}\|F(z+tv)-F(z)\|_E\leqq c\|v\|_H.$$

So we may extend $G(z;\cdot)$ to a bounded linear operator $DF(z): H \to E$. Let $DF(z)=0$ for $z \in B \backslash D_0$, then $DF(\cdot)h$ is a measurable map from B to E for each $h \in H$. Moreover for any $h \in H$, $v \in V$ and $z \in D_0$,

$$\begin{aligned}
&\overline{\lim_{t\to 0}}\left\|\frac{1}{t}(F(z+th)-F(z))-DF(z)h\right\|_E \\
&\leqq \overline{\lim_{t\to 0}}\left\|\frac{1}{t}(F(z+tv)-F(z))-G(z;v)\right\|_E \\
&\quad + \overline{\lim_{t\to 0}}\left\|\frac{1}{t}(F(z+th)-F(z+tv)\right\|_E + \|DF(z)(h-v)\|_E \\
&\leqq 2c\|h-v\|_H .
\end{aligned}$$

Since V is dense in H, we get

$$\overline{\lim_{t\to 0}}\left\|\frac{1}{t}(F(z+th)-F(z))-DF(z)h\right\|_H = 0 .$$

This completes the proof.

The following is immediate conclusion of Theorem 4.2 and Remark 4.1.

COROLLARY TO THEOREM 4.2. *Let G be a σ-compact subset of B and ϕ be a smooth function on $\boldsymbol{R}$ with compact support. Then $g(\cdot)=\phi(\rho(\cdot\,;G)): B \to \boldsymbol{R}$ belongs to $H^1(B \to \boldsymbol{R}; d\mu)$ and*

$$\|Dg(z)\|_{\mathcal{L}^\infty(H,\boldsymbol{R})} \leqq \sup\left\{\left|\frac{d\phi}{dt}(t)\right|;\ t \in \boldsymbol{R}\right\} \qquad \textit{for } \mu\text{-a.e. } z.$$

§ 5. Ito-Ramer integral and $\mathcal{H}$-C^1 maps.

In this section we introduce Ito-Ramer integral which is an extension of Ito's stochastic integral in some sense. This was first introduced by R. Ramer [13] and the result in this section owes much to Ramer's results.

Let F be an element of $H^1(B \to H; d\mu)$. We define a measurable function $L_P F$ on B by

$$L_P F(z) = (F(z), \tilde{P}z)_H - \text{trace}\, PDF(z) \qquad \text{for each } \ P \in \mathcal{P}(H),$$

where $\tilde{P}$ is a stochastic extension of P. Then $L_P F(z)$ is defined for μ-a.e. z.

DEFINITION 5.1. We say that a map $F: B \to H$ belongs to $\mathcal{D}(L)$, the domain of L, if

(1) F belongs to $H^1(B \to H; d\mu)$,
(2) $DF(z)$ belongs to $\mathcal{L}^2(H, H)$ for μ-a.e. z and
(3) there exists a measurable function LF on B such that $L_P F(z) \to LF(z)$, $P \in \mathcal{P}(H)$, in probability with respect to μ.

We call LF the Ito-Ramer integral of F.

REMARK 5.1. If F belongs to $\mathcal{D}(L)$, then DF is a strongly measurable map from B to $\mathcal{L}^2(H, H)$.

The following is due to R. Ramer [13].

THEOREM 5.1. *Let F be an element of $H^1(B\to H; d\mu)$ and assume that $DF(z)$ belongs to $\mathcal{L}^2(H, H)$ for μ-a.e. z and*

$$\int_B \{\|F(z)\|_H^2+\|DF(z)\|_{\mathcal{L}^2(H,H)}^2\}\,\mu(dz)<\infty\,.$$

Then F belongs to $\mathcal{D}(L)$ and

$$\int_B |LF(z)|^2\mu(dz)\leqq\int_B \{\|F(z)\|_H^2+\|DF(z)\|_{\mathcal{L}^2(H,H)}^2\}\,\mu(dz)\,.$$

DEFINITION 5.2. We say that a measurable function w on B is a positive weight function, if

(1) $w(z)>0$ for each $z\in B$ and

(2) $w(z+\cdot): H\to \boldsymbol{R}$ is continuous for each $z\in B$.

REMARK 5.2. Any positive weight function is strictly positive.

THEOREM 5.2. *Let F be an element of $H^1(B\to H; d\mu)$ and w be a positive weight function. Assume that $DF(z)$ belongs to $\mathcal{L}^2(H, H)$ for μ-a.e. z and*

$$\int_B \{\|F(z)\|_H^2+\|DF(z)\|_{\mathcal{L}^2(H,H)}^2\}\,w(z)\mu(dz)<\infty\,.$$

Then F belongs to $\mathcal{D}(L)$. Furthermore there exists a positive measurable function k defined on B dependent only on w such that

$$\int_B |LF(z)|^2k(z)\mu(dz)\leqq\int_B \{\|F(z)\|_H^2+\|DF(z)\|_{\mathcal{L}^2(H,H)}^2\}\,w(z)\mu(dz)\,.$$

PROOF. Let $A_n=\left\{z\in B;\ w(z+h)\geqq\frac{1}{n}\text{ for any } h\in H \text{ such that } \|h\|_H\leqq\frac{1}{n}\right\}$, $n=1, 2, \cdots$. The continuity of $w(z+\cdot): H\to(0, \infty)$ assures the measurability of A_n. Since μ is a Radon measure on B, there exists a σ-compact subset G_n of B such that $G_n\subset A_n$ and $\mu(A_n\backslash G_n)=0$.

Now let ϕ be a smooth function on $\boldsymbol{R}$ such that $|\phi(t)|\leqq1$ and $|\phi'(t)|\leqq4$ for any $t\in\boldsymbol{R}$, $\phi(t)=1$ for $|t|\leqq\frac{1}{3}$, and $\phi(t)=0$ for $|t|\geqq\frac{2}{3}$. Let $\phi_n(z)=\phi(n\rho(z; G_n))$, then ϕ_n belongs to $H^1(B\to\boldsymbol{R}; d\mu)$ and $\|D\phi_n(z)\|_{\mathcal{L}^\infty(H,\boldsymbol{R})}\leqq4n$ for μ-a.e. z by virtue of Corollary to Theorem 3.2. Let $F_n(z)=\phi_n(z)F(z)$ furthermore. Obviously F_n belongs to $H^1(B\to H; d\mu)$ and it is easy to see that $DF_n(z)h=(D\phi_n(z)h)F(z)+$

$\phi_n(z)DF(z)h$ for each $h\in H$ and μ-a.e. z. Since $D\phi_n(z)=0$ and $\phi_n(z)=1$ for μ-a.e. $z\in G_n$, we get

(5.1) $$F_n(z)=F(z) \quad \text{and} \quad DF_n(z)=DF(z) \qquad \text{for} \quad \mu\text{-a.e.}\, z\in G_n.$$

It is easily seen that $\|F_n(z)\|_H\leqq\|F(z)\|_H$ and

$$\begin{aligned}\|DF_n(z)\|_{\mathcal{L}^2(H,H)}&\leqq\|F(z)\|_H\|D\phi_n(z)\|_{\mathcal{L}^\infty(H,R)}+\|DF(z)\|_{\mathcal{L}^2(H,H)}\\ &\leqq 4n\|F(z)\|_H+\|DF(z)\|_{\mathcal{L}^2(H,H)}.\end{aligned}$$

On the other hand, $\phi_n(z)=0$ and $D\phi_n(z)=0$ for μ-a.e. $z\in B$ satisfying $\rho(z\,;G_n)>\frac{2}{3n}$. Therefore we get $F_n(z)=0$ and $DF_n(z)=0$ for μ-a.e. $z\in B$ satisfying $w(z)<\frac{1}{n}$.

Hence we obtain

$$\begin{aligned}&\int_B\{\|F_n(z)\|_H^2+\|DF_n(z)\|_{\mathcal{L}^2(H,H)}^2\}\,\mu(dz)\\ &\leqq n\int_B\{\|F_n(z)\|_H^2+\|DF_n(z)\|_{\mathcal{L}^2(H,H)}^2\}\,w(z)\mu(dz)\\ &\leqq 33n^2\int_B\{\|F(z)\|_H^2+\|DF(z)\|_{\mathcal{L}^2(H,H)}^2\}\,w(z)\mu(dz)<\infty,\end{aligned}$$

which implies $F_n\in\mathcal{D}(L)$ in view of Theorem 5.1. Moreover let

$$k_n(z)=\begin{cases}\dfrac{1}{2^n}\,\dfrac{1}{33n^2} & \text{if} \quad z\in G_n,\\ 0 & \text{otherwise.}\end{cases}$$

Then we have

$$\begin{aligned}&\int_B|LF_n(z)|^2k_n(z)\mu(dz)\\ &\leqq\frac{1}{2^n}\int_B\{\|F(z)\|_H^2+\|DF(z)\|_{\mathcal{L}^2(H,H)}^2\}\,w(z)\mu(dz).\end{aligned}$$

However, (5.1) shows that

(5.2) $$\chi_{G_n}(z)L_PF(z)=\chi_{G_n}(z)L_PF_n(z) \qquad \text{for} \quad \mu\text{-a.e.}\, z\in B$$

and any $P\in\mathcal{P}(H)$. We define a measurable function LF on B by

$$LF(z)=\begin{cases}LF_1(z) & \text{if} \quad z\in G_1,\\ LF_{n+1}(z) & \text{if} \quad z\in G_{n+1}\backslash G_n,\ n=1,2,\cdots,\\ 0 & \text{otherwise.}\end{cases}$$

Noticing that $\mu(\bigcup_n G_n)=1$, we see that $L_PF(z)$ converges to $LF(z)$, $P\in\mathcal{P}(H)$, in probability with respect to μ. So F belongs to $\mathcal{D}(L)$. Furthermore let $k(z)=$

$\sum_{n=1}^{\infty} k_n(z)$, then $k(z)>0$ for μ-a.e. z, and it is easy to see that

$$\int_B |LF(z)|^2 k(z)\mu(dz) \leqq \int_B \{\|F(z)\|_H^2 + \|DF(z)\|_{\mathcal{L}^2(H,H)}^2\} w(z)\mu(dz).$$

This completes the proof.

DEFINITION 5.3. We say that a measurable map $F: B \to H$ is an $\mathcal{H}$-C^1 map, if
(1) for each $z \in B$, there exists a Hilbert Schmidt operator $DF(z): H \to H$ such that $\|F(z+h)-F(z)-DF(z)h\|_H = o(\|h\|_H)$ as $\|h\|_H \to 0$, and
(2) for any $z \in B$, $DF(z+\cdot): H \to \mathcal{L}^2(H, H)$ is continuous.

COROLLARY TO THEOREM 5.2. *Any $\mathcal{H}$-C^1 map belongs to $\mathcal{D}(L)$.*

PROOF. For any $\mathcal{H}$-C^1 map $F: B \to H$, let

$$w(z) = \{1 + \|F(z)\|_H^2 + \|DF(z)\|_{\mathcal{L}^2(H,H)}^2\}^{-1}.$$

Then it is easy to see that w is a positive weight function and

$$\int_B \{\|F(z)\|_H^2 + \|DF(z)\|_{\mathcal{L}^2(H,H)}^2\} w(z)\mu(dz) < \infty.$$

So our assertion has been proved.

§ 6. Nonlinear transformation of μ and its absolute continuity.

DEFINITION 6.1. For each element F of $\mathcal{D}(L)$, we define

$$d(z; F) = \delta(I_H - DF(z)) \exp\left\{LF(z) - \frac{1}{2}\|F(z)\|_H^2\right\}.$$

Here $\delta(A)$ denotes the Carleman-Fredholm determinant of an operator $A: H \to H$. For $K \in \mathcal{L}^2(H, H)$, the Carleman-Fredholm determinant of $I_H - K$ is defined by

$$\delta(I_H - K) = \prod_{i=1}^{\infty} \lambda_i \exp(1-\lambda_i),$$

where the λ_i's are the eigenvalues of $I_H - K$ counted with their multiplicities. It is easy to see that

(i) $\delta(I_H - \cdot): \mathcal{L}^2(H, H) \to \boldsymbol{R}$ is continuous,
(ii) $\delta(I_H - K) = \det(I_H - K)\exp(\text{trace}\, K)$ for any nuclear operator $K: H \to H$, and
(iii) $\delta((I_H - K_1)(I_H - K_2)) = \delta(I_H - K_1)\delta(I_H - K_2)\exp(-\text{trace}\, K_1 K_2)$ for any $K_1, K_2 \in \mathcal{L}^2(H, H)$.

THEOREM 6.1. *Let $F: B \to H$ be a measurable map belonging to $\mathcal{D}(L)$. Assume that there exists a constant c, $0<c<1$, such that $\|F(z+h)-F(z)\|_H \leqq c\|h\|_H$ for each $z \in B$ and $h \in H$.*

Then

(1) *$I_B-F: B\to B$ is bijective,*

(2) *the image measure $(I_B-F)\mu$ induced by μ through I_B-F is absolutely continuous relative to μ, and*

(3) $\int_B f(z)\mu(dz)\geqq\int_B f((I_B-F)z)|d(z;F)|\mu(dz)$ *for any positive measurable function f on B.*

PROOF. Step 1. Let $\mathcal{F}=\{G: B\to H;\ G$ is measurable and it satisfies that $\|G(z+h)-G(z)\|_H\leqq c\|h\|_H$ for any $z\in B$ and $h\in H\}$, and for any $G\in\mathcal{F}$, we put inductively

$$\begin{cases} u_0(z;G)=0, \\ u_{n+1}(z;G)=G(z+u_n(z;G)) \quad \text{for each } z\in B \text{ and } n=1,2,\cdots. \end{cases}$$

It is easy to see that

$$\begin{aligned}\|u_{n+1}(z;G)-u_n(z;G)\|_H&\leqq c\|u_n(z;G)-u_{n-1}(z;G)\|_H\\ &\leqq c^n\|G(z)\|_H.\end{aligned}$$

So there exists $u_\infty(z;G)=\lim_{n\to\infty}u_n(z;G)$ in H. We also get

$$\|u_\infty(z;G)\|_H\leqq\frac{1}{1-c}\|G(z)\|_H \quad\text{and} \tag{6.1}$$

$$\|u_\infty(z;G)-u_n(z;G)\|_H\leqq\frac{c^n}{1-c}\|G(z)\|_H. \tag{6.2}$$

Obviously $u_\infty(z;G)=G(z+u_\infty(z;G))$, which implies

$$(I_B-G)(z+u_\infty(z;G))=z. \tag{6.3}$$

This shows that $I_B-G: B\to B$ is surjective.

On the other hand, suppose that $(I_B-G)z_1=(I_B-G)z_2$ for some $z_1, z_2\in B$. Then $z_1-z_2=G(z_1)-G(z_2)\in H$ and

$$\|z_1-z_2\|_H=\|G(z_2+z_1-z_2)-G(z_2)\|_H\leqq c\|z_1-z_2\|_H.$$

This proves $z_1=z_2$, and so $I_B-G: B\to B$ is injective.

Hence we have proved that

$$I_B-G: B\to B \text{ is bijective for each } G\in\mathcal{F}, \text{ and} \tag{6.4}$$

$$(I_B-G)^{-1}z=z+u_\infty(z;G). \tag{6.5}$$

This proves (1) in our assertion.

Step 2. Take $P_n\in\mathcal{P}(B^*)$, $n=1,2,\cdots$, such that $P_n\uparrow I_H$ strongly and $L_{P_n}F(z)\to LF(z)$ for μ-a.e. z, and let $F_n(z)=P_nF(z)$. It is obvious that $F_n\in\mathcal{F}$.

Since $u_0(z;F)=u_0(z;F_n)=0$ and

$$u_{m+1}(z;F)-u_{m+1}(z;F_n)=F(z+u_m(z;F))-F_n(z+u_m(z;F_n)),$$

we see inductively that $u_m(z;F_n)\to u_m(z;F)$ in H, $n\to\infty$, for each m. By virtue of (6.2), we get

$$\|u_\infty(z;F)-u_\infty(z;F_n)\|_H\leqq\frac{2c^m}{1-c}\|F(z)\|_H+\|u_m(z;F)-u_m(z;F_n)\|_H.$$

This proves that $u_\infty(z;F_n)\to u_\infty(z;F)$ in H, $n\to\infty$. Therefore by (6.5) we obtain

$$(I_B-F_n)^{-1}z\to(I_B-F)^{-1}z \tag{6.6}$$

in B, $n\to\infty$, for each $z\in B$.

Step 3. We will prove the following.

Claim 1. For any continuous bounded function g on B,

$$\begin{aligned}&\int_B g((I_B-F_n)^{-1}z)\mu(dz)\\&=\int_B g(z)|\delta(I_H-P_nDF(z))|\exp\left\{L_{P_n}F(z)-\frac{1}{2}\|P_nF(z)\|_H^2\right\}\mu(dz).\end{aligned}$$

Let $\{e_1,\cdots,e_n\}$ be an orthonormal base of $E=P_nH$, and let $F_n(z_1,z_2)=F_n(z_1+z_2)$ for each $z_1\in E$ and $z_2\in B$. Then $F_n(\cdot,\cdot):E\times B\to E$ is measurable. It is easy to see that

$$I_E-F_n(\cdot,z_2):E\to E \tag{6.7}$$

is bijective and Lipschitz continuous for each $z_2\in B$, and

$$(I_E-F_n(\cdot,z_2))^{-1}z_1=(I_B-F_n)^{-1}(z_1+z_2)-z_2 \tag{6.8}$$

for each $z_1\in E$ and $z_2\in B$. Therefore we obtain

$$\begin{aligned}&\int_B g((I_B-F_n)^{-1}z)\mu(dz)\\&=\int_{B\times R^m}g\Big((I_B-F_n)^{-1}\Big(\sum_{j=1}^m x_je_j+z\Big)\Big)\mu_{E^\perp}(dz)\otimes\Big(\frac{1}{2\pi}\Big)^{m/2}\exp\Big\{-\frac{1}{2}\sum_{j=1}^m x_j^2\Big\}dx_1\cdots dx_m\\&=\Big(\frac{1}{2\pi}\Big)^{m/2}\int_B\mu_{E^\perp}(dz)\int_{R^m}g\Big((I_E-\tilde{F}_n(\cdot,z))^{-1}\Big(\sum_{j=1}^m x_je_j\Big)+z\Big)\exp\Big\{-\frac{1}{2}\sum_{j=1}^m x_j^2\Big\}dx_1\cdots dx_m.\end{aligned}$$

Since $I_E-\tilde{F}_n(\cdot,z_2):E\to E$ is homeomorphic and Lipschitz continuous, by virtue of R. Radmacher [12]

$$\begin{aligned}&\int_{R^m}g\Big((I_E-F_n(\cdot,z))^{-1}\Big(\sum_{j=1}^m x_je_j\Big)+z\Big)\exp\Big\{-\frac{1}{2}\sum_{j=1}^m x_j^2\Big\}dx_1\cdots dx_m\\&=\int_{R^m}g\Big(\sum_{j=1}^m x_je_j+z\Big)\Big|\det\Big(\delta_{i,j}-\frac{\partial f_i}{\partial x_j}(x;z)\Big)_{i,j=1,\cdots,m}\Big|\\&\qquad\times\exp\Big\{-\frac{1}{2}\sum_{j=1}^m(x_j-f_j(x;z))^2\Big\}dx_1\cdots dx_m\end{aligned}$$

where $\delta_{i,j}$ is Kronecker's delta and

$$f_i(x\,;z)=\Big(e_i,\ F_n\Big(\sum_{j=1}^m x_je_j+z\Big)\Big)_H.$$

According to Proposition 3.2, we get

$$\frac{\partial f_i}{\partial x_j}(x\,;z)=\Big(e_i,\ DF_n\Big(\sum_{k=1}^m x_ke_k+z\Big)e_j\Big)_H \quad \text{for a.e.}\,(x,\,z)$$

with respect to $dx_1\cdots dx_m\otimes\mu_{E\perp}(dz)$.

So we obtain

$$\Big|\det\Big(\delta_{i,j}-\frac{\partial f_i}{\partial x_j}\Big)\Big|\exp\Big\{-\frac{1}{2}\sum_{j=1}^m (x_j-f_j(x\,;z))^2\Big\}$$

$$=\Big|\det\Big(I_H-P_nDF\Big(\sum_{j=1}^m x_je_j+z\Big)\Big)\Big|\exp\Big(-\frac{1}{2}\sum_{j=1}^m x_j^2\Big)$$

$$\times\exp\Big\{\Big(P_nF\Big(\sum_{j=1}^m x_je_j+z\Big),\ P_n\Big(\sum_{j=1}^m x_je_j\Big)\Big)_H-\frac{1}{2}\Big\|P_nF\Big(\sum_{j=1}^m x_je_j+z\Big)\Big\|_H^2\Big\}$$

$$=\Big|\delta\Big(I_H-P_nDF\Big(\sum_{j=1}^m x_je_j+z\Big)\Big)\Big|\exp\Big(-\frac{1}{2}\sum_{j=1}^m x_j^2\Big)$$

$$\times\exp\Big\{L_{P_n}F\Big(\sum_{j=1}^m x_je_j+z\Big)-\frac{1}{2}\Big\|P_nF\Big(\sum_{j=1}^m x_je_j+z\Big)\Big\|_H^2\Big\}$$

for a.e. $(x,\,z)$ with respect to $dx_1\cdots dx_m\times\mu_{E\perp}(dz)$. This proves our assertion.

Step 4. Let g be a positive bounded continuous function on B. Then we get

$$\int_B g((I_B-F)^{-1}z)\mu(dz)$$

$$=\varliminf_{n\to\infty}\int_B g((I_B-F_n)^{-1}z)\mu(dz)$$

$$=\varliminf_{n\to\infty}\int_B g(z)\,|\,\delta(I_H-P_nDF(z))\,|\exp\Big\{L_{P_n}F(z)-\frac{1}{2}\|P_nF(z)\|_H^2\Big\}\mu(dz)$$

$$\geqq\int_B g(z)\,|\,d(z\,;F)\,|\,\mu(dz)$$

by (6.6) and Claim 1.

So for any positive bounded measurable function f on B we get

$$\int_B f((I_B-F)^{-1}z)\mu(dz)\geqq\int_B f(z)\,|\,d(z\,;F)\,|\,\mu(dz).$$

Replacing $f(z)$ by $f((I_B-F)z)$, we obtain

$$\int_B f(z)\mu(dz)\geqq\int_B f((I_B-F)z)\,|\,d(z\,;F)\,|\,\mu(dz) \tag{6.9}$$

for any positive measurable function f on B. This proves (3) in our assertion.

In view of Theorem 4.2, we get $\|DF(z)\|_{\mathcal{L}^\infty(H,H)}\leqq c$ for μ-a.e. z, and so

$\delta(I_H-DF(z))>0$ for μ-a.e. z. Hence $|d(z;F)|>0$ for μ-a.e. z. Now (2) in our assertion is an easy conclusion of (6.9). This completes the proof.

LEMMA 6.1. *Let F_1 and F_2 be elements of $\mathcal{D}(L)$ and F_3 be an element of $H^1(B\to H; d\mu)$. Assume furthermore that*

(1) *$(I_B-F_2)\mu$ is absolutely continuous relative to μ,*
(2) *$(I_B-F_3)z=(I_B-F_1)(I_B-F_2)z$ for μ-a.e. z, and*
(3) *$I_H-DF_3(z)=(I_H-DF_1((I_B-F_2)z))(I_H-DF_2(z))$ for μ-a.e. z.*

Then F_3 belongs to $\mathcal{D}(L)$ and it satisfies that $d(z;F_3)=d((I_B-F_2)z;F_1)d(z;F_2)$ for μ-a.e. z.

PROOF. Notice that $DF_1((I_B-F_2)z)$ and $d((I_B-F_2)z;F_1)$ are well-defined for μ-a.e. z, because $(I_B-F_2)\mu$ is absolutely continuous relative to μ.

According to our assumptions (2) and (3),

$$F_3(z)=F_1((I_B-F_2)z)+F_2(z)$$

and

$$DF_3(z)=DF_1((I_B-F_2)z)(I_H-DF_2(z))+DF_2(z)\in\mathcal{L}^2(H, H) \qquad \text{for } \mu\text{-a.e. } z.$$

So

$$\begin{aligned}L_PF_3(z)&=(F_3(z), \tilde{P}z)_H-\operatorname{trace} PDF_3(z)\\&=L_PF_1((I_B-F_2)z)+L_PF_2(z)+(F_1((I_B-F_2)z), PF_2(z))_H\\&\quad+\operatorname{trace} PDF_1((I_B-F_2)z)DF_2(z).\end{aligned}$$

Since $L_PF_1(z)\to LF_1(z)$, $P\in\mathcal{P}(H)$, in probability with respect to μ and $(I_B-F_2)\mu$ is absolutely continuous relative to μ, $L_PF_1((I_B-F_2)z)\to LF_1((I_B-F_2)z)$, $P\in\mathcal{P}(H)$, in probability with respect to μ. Hence we obtain

$$\begin{aligned}L_PF_3(z)\to LF_1((I_B-F_2)z)+LF_2(z)+(F_1((I_B-F_2)z, F_2(z))_H\\+\operatorname{trace} DF_1((I_B-F_2)z)DF_2(z),\end{aligned}$$

$P\in\mathcal{P}(H)$, in probability with respect to μ. This shows that F_3 belongs to $\mathcal{D}(L)$ and

$$\begin{aligned}LF_3(z)=LF_1((I_B-F_2)z)+LF_2(z)+(F_1((I_B-F_2)z), F_2(z))_H\\+\operatorname{trace} DF_1((I_B-F_2)z)DF_2(z).\end{aligned}$$

Using this fact, we can obtain $d(z;F_3)=d((I_B-F_2)z;F_1)d(z;F_2)$ by easy calculation.

THEOREM 6.2. *Let $F\colon B\to H$ be an $\mathcal{H}$-C^1 map and D be a measurable subset of B, and assume that $I_H-DF(z)\colon H\to H$ is invertible for μ-a.e. $z\in D$. Then $(I_B-F)(\mu|_D)$ is absolutely continuous relative to μ, where $\mu|_D$ is the restricted measure of μ to D. In particular, if $(I_B-F)|_D\colon D\to B$ is injective, then it satisfies that*

$$\int_{(I_B-F)D} f(z)\mu(dz) \geqq \int_D f((I_B-F)z)\,|d(z\,;F)|\,\mu(dz)$$

for any positive measurable function f on B.

PROOF. Step 1. Let $\mathcal{K}_0$ be a countable dense subset of $\mathcal{L}^2(H, H)$, and let $\{P_n\}_{n=1}^{\infty}$ is an increasing subsequence of $\mathcal{P}(B^*)$ strongly converging to I_H, and put

$$\mathcal{K}=\{P_nKP_n\,;K\in\mathcal{K}_0,\ n=1, 2, \cdots\}\cap\{K\in\mathcal{L}^2(H, H)\,;\ I_H-K\in GL(H)\},$$

where $GL(H)$ is a set of all invertible bounded operator from H onto H. Then it is obvious that $\mathcal{K}$ is dense in $\mathcal{L}^2(H, H)\cap(I_H-GL(H))$ and every element K of $\mathcal{K}$ is extensible to a bounded linear operator $\tilde{K}$ from B to B^*. Moreover let $\mathcal{V}$ be a countable subset of B^* such that $\mathcal{V}$ is dense in H.

For any $K\in\mathcal{K}$ and $n=1, 2, \cdots$, we define a subset $A_K^{1,n}$ of B by

$$\text{(6.10)}\qquad A_K^{1,n}=\Big\{z\in B\,;\ \|DF(z+h)-DF(z)\|_{\mathcal{L}^2(H,H)}\leqq\frac{1}{24}\|(I_H-K)^{-1}\|^{-1}_{\mathcal{L}^\infty(H,H)}$$
$$\text{for any } h\in H \text{ such that } \|h\|_H\leqq\frac{1}{n}\Big\}.$$

Since $DF(z+\cdot): H\to\mathcal{L}^2(H, H)$ is continuous for each $z\in B$, $A_K^{1,n}$ is a measurable subset of B.

We also define a measurable subset $A_K^{2,n}$ of B by

$$\text{(6.11)}\qquad A_K^{2,n}=\Big\{z\in B\,;\ \|F(z+h)-F(z)-DF(z)h\|_H\leqq\frac{1}{24n}\|(I_H-K)^{-1}\|^{-1}_{\mathcal{L}^\infty(H,H)}$$
$$\text{for any } h\in H \text{ with } \|h\|_H\leqq\frac{1}{n}\Big\}.$$

It is obvious that $A_K^{1,n}\uparrow B$ and $A_K^{2,n}\uparrow B$ as $n\to\infty$ for each $K\in\mathcal{K}$. Furthermore, for each $K\in\mathcal{K}$, $v\in\mathcal{V}$ and $n=1, 2, \cdots$, we define a measurable subset $\tilde{A}(n, K, v)$ of B by

$$\text{(6.12)}\qquad \tilde{A}(n, K, v)=A_K^{1,n}\cap A_K^{2,n}$$
$$\cap\Big\{z\in B\,;\ \|DF(z)-K\|_{\mathcal{L}^2(H,H)}\leqq\frac{1}{24}\|(I_H-K)^{-1}\|^{-1}_{\mathcal{L}^\infty(H,H)}$$
$$\text{and } \|F(z)-Kz-v\|_H\leqq\frac{1}{24n}\|(I_H-K)^{-1}\|^{-1}_{\mathcal{L}^\infty(H,H)}\Big\}.$$

It is easy to see that $D\subset\cup\{\tilde{A}(n, K, v)\,;\ K\in\mathcal{K},\ v\in\mathcal{V},\ n=1, 2, \cdots\}$. By (6.10), (6.11) and (6.12), for any $z\in\tilde{A}(n, K, v)$ and $h\in H$ with $\|h\|_H\leqq\dfrac{1}{n}$, we get

$$\text{(6.13)}\qquad \|DF(z+h)-K\|_{\mathcal{L}^2(H,H)}$$
$$\leqq\|DF(z+h)-DF(z)\|_{\mathcal{L}^2(H,H)}+\|DF(z)-K\|_{\mathcal{L}^2(H,H)}$$

$$\leqq \frac{1}{12}\|(I_H-K)^{-1}\|^{-1}_{\mathcal{L}^\infty(H,H)}$$

and

(6.14)
$$\begin{aligned}&\|F(z+h)-K(z+h)-v\|_H\\ &\leqq\|F(z+h)-F(z)-DF(z)h\|_H+\|F(z)-Kz-v\|_H\\ &\quad+\|DF(z)-K\|_{\mathcal{L}^\infty(H,H)}\|h\|_H\\ &\leqq\frac{1}{8n}\|(I_H-K)^{-1}\|^{-1}_{\mathcal{L}^\infty(H,H)}.\end{aligned}$$

Step 2. By virtue of Proposition 2.5, $I_B-\tilde{K}: B\to B$ is homeomorphism for each $K\in\mathcal{K}$ and

(6.15)
$$(I_B-\tilde{K})^{-1}\mu(dz)=|d(z;K)|\mu(dz).$$

For any $K\in\mathcal{K}$ and $v\in\mathcal{V}$, let

$$\begin{aligned}\tilde{F}_{K,v}(z)&=F((I_B-\tilde{K})^{-1}z)-\tilde{K}(I_B-\tilde{K})^{-1}z-v\\ &=(F-\tilde{K}-N_v)(I_B-\tilde{K})^{-1}z,\end{aligned}$$

where $N_vz=v$ for each $z\in B$. It is obvious that $\tilde{F}_{K,v}: B\to H$ is an $\mathcal{H}$-C^1 map and that

(6.16)
$$(I_B-F)(z)=(I_B-N_v)(I_B-\tilde{F}_{K,v})(I_B-\tilde{K})z$$

for each $z\in B$. It is also easy to see that

(6.17)
$$D\tilde{F}_{K,v}(z)=(DF((I_B-\tilde{K})^{-1}z)-K)(I_H-K)^{-1} \quad \text{for each } z\in B.$$

Assume that $z\in(I_B-\tilde{K})\tilde{A}(n, K, v)$ and $\|h\|_H\leqq\frac{1}{n}\|(I_H-K)^{-1}\|^{-1}_{\mathcal{L}^\infty(H,H)}$, then in view of (6.13) and (6.14) we get

(6.18)
$$\|\tilde{F}_{K,v}(z+h)\|_H\leqq\frac{1}{8n}\|(I_H-K)^{-1}\|^{-1}_{\mathcal{L}^\infty(H,H)} \quad \text{and}$$

(6.19)
$$\|D\tilde{F}_{K,v}(z+h)\|_{\mathcal{L}^2(H,H)}\leqq\frac{1}{12}.$$

Since $\tilde{A}(n, K, v)$ is measurable, there exists a σ-compact subset $A(n, K, v)$ of B such that $A(n, K, v)\subset\tilde{A}(n, K, v)$ and $\mu(\tilde{A}(n, K, v)\backslash A(n, K, v))=0$. Then we obtain

(6.20)
$$\mu(D\backslash\cup\{A(n, K, v); K\in\mathcal{K}, v\in\mathcal{V}, n=1, 2, \cdots\})=0.$$

Let ϕ be a smooth function on $\boldsymbol{R}$ such that $\phi(t)=1$ for $|t|\leqq\frac{1}{3}$, $\phi(t)=0$ for $|t|\geqq\frac{2}{3}$ and $|\phi'(t)|\leqq4$ for any $t\in\boldsymbol{R}$, and let

$$\psi(z)=\psi_{n,K,v}(z)=\phi(n\|(I_H-K)^{-1}\|_{\mathcal{L}^\infty(H,H)}\rho(z;(I_B-\tilde{K})A(n, K, v))).$$

Noticing that $\psi(z+th)\tilde{F}_{K,v}(z+th)$ is strictly absolutely continuous in t for any $z\in B$ and $h\in H$, by virtue of (6.16), (6.17) and (6.18) we get

(6.21)
$$\begin{aligned}&\|\phi(z+h)\tilde{F}_{K,v}(z+h)-\phi(z)\tilde{F}_{K,v}(z)\|_H\\&\leqq\int_0^1\left\|\frac{d}{dt}\phi(z+th)\tilde{F}_{K,v}(z+th)+\phi(z+th)D\tilde{F}_{K,v}(z+th)h\right\|_H dt\\&\leqq\int_0^1|\phi'(n\|(I_H-K)^{-1}\|_{\mathcal{L}^\infty(H,H)}\rho(z;(I_B-\tilde{K})A(n,K,v)))|\\&\quad\times\|\tilde{F}_{K,v}(z+th)\|_H dt\ n\|(I_H-K)^{-1}\|_{\mathcal{L}^\infty(H,H)}\|h\|_H\\&\quad+\int_0^1\phi(z+th)\|D\tilde{F}_{K,v}(z+th)\|_{\mathcal{L}^2(H,H)}\|h\|_H dt\\&\leqq\frac{7}{12}\|h\|_H \text{ for any } z\in B \text{ and } h\in H.\end{aligned}$$

According to Theorem 4.2, $\phi\tilde{F}_{K,v}\colon B\to H$ belongs to $H^1(B\to H;d\mu)$ and

$$D(\phi\tilde{F}_{K,v})(z)h=(D\phi(z)h)\tilde{F}_{K,v}(z)+\phi(z)D\tilde{F}_{K,v}(z)h$$

for any $h\in H$ and μ-a.e. z. So by virtue of (6.17) and (6.18) we obtain

$$\begin{aligned}&\|D(\phi\tilde{F}_{K,v})(z)\|_{\mathcal{L}^2(H,H)}\\&\leqq\|D\phi(z)\|_{\mathcal{L}^\infty(H,R)}\|\tilde{F}_{K,v}(z)\|_H+|\phi(z)D\tilde{F}_{K,v}(z)\|_{\mathcal{L}^2(H,H)}\\&\leqq\frac{7}{12}\qquad\text{for}\quad \mu\text{-a.e. } z.\end{aligned}$$

Hence in view of Theorem 5.1 $\phi\tilde{F}_{K,v}$ belongs to $\mathcal{D}(L)$. Thus $\phi\tilde{F}_{K,v}$ satisfies the assumption of Theorem 6.1, and accordingly

(6.22)
$$\int_B f(z)\mu(dz)\geqq\int_B f((I_B-\phi\tilde{F}_{K,v})z)\,|d(z;\phi\tilde{F}_{K,v})|\,\mu(dz)$$

for any positive measurable function f on B, and $(I_B-\phi\tilde{F}_{K,v})\mu$ is absolutely continuous relative to μ. So it follows from Propositions 2.4 and 2.5 that $(I_B-N_v)(I_B-\phi\tilde{F}_{K,v})(I_B-\tilde{K})\mu$ is absolutely continuous relative to μ.

For each Borel measurable set A of B, by virtue of (6.20) we get

$$\mu((I_B-F)^{-1}A\cap D)\leqq\sum_{n,K,v}\mu((I_B-F)^{-1}A\cap A(n,K,v)).$$

Since $(I_B-F)z=(I_B-N_v)(I_B-\phi\tilde{F}_{K,v})(I_B-\tilde{K})z$ for each $z\in A(n,K,v)$ by the definition of $\phi_{n,K,v}$ and $\tilde{F}_{K,v}$, $\mu((I_B-F)A\cap D)=0$ provided that $\mu(A)=0$. This shows that $(I_B-F)(\mu|_D)$ is absolutely continuous relative to μ.

Step 3. Let us define a measurable map $G_{n,K,v}\colon B\to H$ by

$$G_{n,K,v}z=z-(I_B-N_v)(I_B-\phi_{n,K,v}\tilde{F}_{K,v})(I_B-\tilde{K})z$$

for each $z\in B$. Then Lemma 6.1 proves that $G_{n,K,v}$ belongs to $\mathcal{D}(L)$ and

(6.23)
$$d(z;G_{n,K,v})=d((I_B-\phi\tilde{F}_{K,v})(I_B-\tilde{K})z;N_v)d((I_B-\tilde{K})z;\phi\tilde{F}_{K,v})d(z;\tilde{K}).$$

By the definition of $\phi_{n,K,v}$ and $\tilde{F}_{K,v}$, we also obtain $G_{n,K,v}(z)=F(z)$ and $DG_{n,K,v}(z)=DF(z)$ for μ-a.e. $z\in A(n,K,v)$, and consequently we get

(6.24) $$d(z\,;G_{n,K,v})=d(z\,;F) \quad \text{for} \quad \mu\text{-a.e. } z\in A(n, K, v).$$

Propositions 2.4 and 2.5 show that

$$\int_B g(z)\mu(dz)=\int_B g((I_B-N_v)z)\,|\,d(z\,;N_v)|\,\mu(dz) \quad \text{and}$$

$$\int_B g(z)\mu(dz)=\int_B g((I_B-\tilde{K})z)\,|\,d(z\,;\tilde{K})|\,\mu(dz)$$

for any positive measurable function g on B. Using (6.22) and (6.23), we get

$$\begin{aligned}\int_B g(z)\mu(dz)&=\int_B g((I_B-N_v)z)\,|\,d(z\,;N_v)|\,\mu(dz)\\ &\geqq\int_B g((I_B-N_v)(I_B-\phi\tilde{F}_{K,v})z)\,|\,d((I_B-\phi\tilde{F}_{K,v})z\,;N_v)|\\ &\quad\times|\,d(z\,;\phi\tilde{F}_{K,v})|\,\mu(dz)\\ &\geqq\int_B g((I_B-G_{n,K,v})z)\,|\,d(z\,;G_{n,K,v})|\,\mu(dz)\end{aligned}$$

for any positive measurable function g on B.

Assume that $I_B-F|_D: D\to B$ is injective. Then $(I_B-F)(A\cap D)$ is measurable for any measurable set A. By virtue of (6.20), there exists mutually disjoint sets $C(n, K, v)$, $K\in\mathcal{K}$, $v\in\mathcal{V}$, $n=1, 2, \cdots$, such that $C(n, K, v)\subset A(n, K, v)\cap D$, $C(n, K, v)$'s are measurable and $\mu(D\setminus \bigcup_{n,K,v} C(n, K, v))=0$. Then according to (6.24), we get

$$\begin{aligned}&\int_{(I_B-F)D} f(z)\mu(dz)\\ &\geqq\sum_{n,K,v}\int_B \chi_{(I_B-F)C(n,K,v)}(z)f(z)\mu(dz)\\ &\geqq\sum_{n,K,v}\int_B \chi_{(I_B-F)C(n,K,v)}((I_B-G_{n,K,v})z)f((I_B-G_{n,K,v})z)\\ &\quad\times|\,d(z\,;G_{n,K,v})|\,\mu(dz)\\ &\geqq\sum_{n,K,v}\int_B \chi_{C(n,K,v)}(z)f((I_B-F)z)\,|\,d(z\,;F)|\,\mu(dz)\\ &\geqq\int_D f((I_B-F)z)\,|\,d(z\,;F)|\,\mu(dz)\end{aligned}$$

for any positive measurable function f on B.

This completes the proof.

THEOREM 6.3. *Let F and G be $\mathcal{H}$-C^1 maps from B to H, and assume that $(I_B-F)(I_B-G)z=z$ and $(I_B-G)(I_B-F)z=z$ for μ-a.e. z. Assume furthermore that $I_H-DF(z): H\to H$ and $I_H-DG(z): H\to H$ are invertible for μ-a.e. z. Then $(I_B-F)\mu=|\,d(z\,;G)|\,\mu(dz)$ and $(I_B-G)\mu=|\,d(z\,;F)|\,\mu(dz)$.*

PROOF. Let $D=\{z\in B\,;(I_B-F)(I_B-G)z=(I_B-G)(I_B-F)z=z\}$. Then $\mu(D)=1$, and it is easy to see that $(I_B-F)|_D$ and $(I_B-G)|_D$ are injective on D. By our assumption, we get

$$G((I_B-F)z)+F(z)=z-(I_B-G)(I_B-F)z=0 \qquad \text{for} \quad \mu\text{-a.e. } z\in B. \tag{6.25}$$

Since the left hand of (6.25) is Fréchet differentiable along H-direction and the right hand is stochastic Gâteaux-H differentiable, we obtain

$$DG((I_B-F)z)(I_H-DF(z))+DF(z)=0\,,$$

which implies $I_H=(I_H-DG((I_B-F)z))(I_H-DF(z))$ for μ-a.e. $z\in B$. By Theorem 6.2 and Lemma 6.1, we get $1=d(z\,;0)=d((I_B-F)z\,;G)d(z\,;F)$ for μ-a.e. z. Similarly we get $d((I_B-G)z\,;F)d(z\,;G)=1$ for μ-a.e. z.

In view of Theorem 6.2, for any positive measurable function f on B, we obtain

$$\begin{aligned}\int_B f(z)\mu(dz) &\geqq \int_{(I_B-F)D} f(z)\mu(dz)\\ &\geqq \int_D f((I_B-F)z)|d(z\,;F)|\mu(dz)\\ &= \int_B f((I_B-F)z)|d(z\,;F)|\mu(dz)\\ &\geqq \int_B f((I_B-F)(I_B-G)z)|d((I_B-G)z\,;F)||d(z\,;G)|\mu(dz)\\ &= \int_B f(z)\mu(dz)\,.\end{aligned}$$

So

$$\int_B f(z)\mu(dz)=\int_B f((I_B-F)z)|d(z\,;F)|\mu(dz)\,.$$

Replacing $f(z)$ by $f((I_B-G)z)$, we get

$$\int_B f((I_B-G)z)\mu(dz)=\int_B f(z)|d(z\,;F)|\mu(dz)\,.$$

This proves our assertion.

THEOREM 6.4. *Let F be an $\mathcal{H}$-C^1 map from B to H, and assume that $I_B-F\colon B\to B$ is bijective and $I_H-DF(z)\colon H\to H$ is invertible for each $z\in B$. Then $(I_B-F)^{-1}\mu(dz)=|d(z\,;F)|\mu(dz)$.*

PROOF. Let $G(z)=z-(I_B-F)^{-1}z=-F((I_B-F)^{-1}z)$ for each $z\in B$. Then the implicit functional theorem (see J. Schwartz [14] for an example) assures us that $G\colon B\to H$ is an $\mathcal{H}$-C^1 map and $I_H-DG(z)=(I_H-DF((I_B-F)^{-1}z))^{-1}$ for each $z\in B$. So our assertion is an easy consequence of Theorem 6.3.

§7. Some more results about nonlinear transformation of μ and its absolute continuity.

DEFINITION 7.1. We say that a measurable map $F: B \to H$ is regular, if

(1) F belongs to $H^1(B \to H; d\mu)$,
(2) $I_H - DF(z): H \to H$ is invertible and $DF(z): H \to H$ is a Hilbert-Schmidt operator for μ-a.e. z, and
(3) there exist a positive weight function w, a measurable set D_0 of μ-measure one and a sequence $\{F_n\}_{n=1}^{\infty}$ of $\mathcal{H}$-C^1 maps, such that

(i) $\int_B \{\|F(z)\|_H^2 + \|DF(z)\|_{\mathcal{L}^2(H,H)}^2\} w(z)\mu(dz) < \infty$,

(ii) $\int_B \{\|F(z) - F_n(z)\|_H^2 + \|DF(z) - DF_n(z)\|_{\mathcal{L}^2(H,H)}^2\} w(z)\mu(dz) \to 0$ as $n \to \infty$,

(iii) $I_H - DF_n(z): H \to H$ is invertible for μ-a.e. z, and

(iv) $I_B - F_n|_{D_0}: D_0 \to B$ is injective.

REMARK 7.1. If $F: B \to H$ is regular, then F belongs to $\mathcal{D}(L)$ by virtue of Theorem 5.2.

THEOREM 7.1. *Suppose that $F: B \to H$ is regular. Then $(I_B - F)\mu$ is absolutely continuous relative to μ, and*

$$\int_B f(z)\mu(dz) \geqq \int_B f((I_B - F)z)\,|d(z;F)|\,\mu(dz)$$

for any positive measurable function f on B.

PROOF. Let F_n's be $\mathcal{H}$-C^1 maps and D_0 be a measurable set as in Definition 7.1. Then in view of Theorem 5.2, we get $LF_n(z) \to LF(z)$ in probability with respect to $\mu(dz)$. According to Theorem 6.2, we obtain

$$\begin{aligned}\int_B f(z)\mu(dz) &\geqq \int_{(I_B - F_n)D_0} f(z)\mu(dz) \\ &\geqq \int_{D_0} f((I_B - F_n)z)\,|d(z;F_n)|\,\mu(dz) \\ &\geqq \int_B f((I_B - F_n)z)\,|d(z;F_n)|\,\mu(dz).\end{aligned}$$

So using Fatou's lemma, we have

$$\int_B f(z)\mu(dz) \geqq \int_B f((I_B - F)z)\,|d(z;F)|\,\mu(dz)$$

for any positive bounded measurable function f on B. By the similar argument to the proof of Theorem 6.1 we can easily see that $(I_B - F)\mu$ is absolutely continuous relative to μ. This completes the proof.

THEOREM 7.2. *Suppose that* $F: B\to H$ *and* $G: B\to H$ *are regular and that* $(I_B-F)(I_B-G)z=z$ *and* $(I_B-G)(I_B-F)z=z$ *for* μ-a.e. z. *Suppose furthermore that* $d(\cdot\,; F): B\to \boldsymbol{R}$ *and* $d(\cdot\,; G): B\to \boldsymbol{R}$ *are strictly positive functions. Then*

$$(I_B-F)\mu(dz)=|d(z\,; G)|\mu(dz)$$

and

$$(I_B-G)\mu(dz)=|d(z\,; F)|\mu(dz)\,.$$

PROOF. If we show that

$$I_H=(I_H-DF((I_B-G)z))(I_H-DG(z)) \quad \text{for } \mu\text{-a.e. } z \text{ and} \tag{7.1}$$

$$I_H=(I_H-DG((I_B-F)z))(I_H-DF(z)) \quad \text{for } \mu\text{-a.e. } z, \tag{7.2}$$

then our theorem will be proved by the similar argument to the proof of Theorem 6.3. (7.1) and (7.2) are similar, so we will prove (7.1).

According to Definition 7.1, there exist a sequence $\{F_n\}_{n=1}^{\infty}$ of $\mathcal{H}$-C^1 maps and a positive weight function $w: B\to\boldsymbol{R}$ such that

$$\int_B \{\|F(z)-F_n(z)\|_H^2+\|DF(z)-DF_n(z)\|_{\mathcal{L}^2(H,H)}^2\}\, w(z)\mu(dz)\to 0$$

as $n\to\infty$. On the other hand there exists a positive weight function $v: B\to\boldsymbol{R}$ such that

$$\int_B \{\|G(z)\|_H^2+\|DG(z)\|_{\mathcal{L}^2(H,H)}^2\} v(z)\mu(dz)<\infty\,.$$

Replacing $v(z)$ by $\min\{v(z), 1\}$ if necessary, we may assume that $v(z)\leqq 1$ for each $z\in B$.

Since $(I_B-F)(I_B-G)z=z$ for μ-a.e. z, we get

$$G(z)=-F((I_B-G)z) \quad \text{for } \mu\text{-a.e. } z. \tag{7.3}$$

By virtue of Theorem 7.1 $(I_B-G)\mu$ is absolutely continuous relative to μ, which shows that

$$F_n((I_B-G)z)\to F((I_B-G)z) \tag{7.4}$$

in probability with respect to μ.

Take an arbitrary element h of H and fix it for a moment. Since $G: B\to H$ is R.A.C., there exists a measurable map $\tilde{G}_h: B\to H$ such that $\tilde{G}_h(z)=G(z)$ for μ-a.e. z and $\tilde{G}_h(z+th)$ is strictly absolutely continuous in t for each $z\in B$. Then we get

$$F_n((I_B-\tilde{G}_h)z)=F_n((I_B-G)z) \quad \text{for } \mu\text{-a.e. } z$$

and

$$F_n((I_B-\tilde{G}_h)(z+th))=F_n(z+th-\tilde{G}_h(z+th))$$

for each $z\in B$ and $t\in\boldsymbol{R}$. According to the assumption of F_n, $F_n(z+\cdot): H\to H$ is continuously Fréchet differentiable, which shows that $F_n((I_B-\tilde{G}_h)(z+th))$ is strictly absolutely continuous in t for each $z\in B$. So $F_n((I_B-G)(\cdot)): B\to H$ is

R. A. C. By the similar argument we can easily see that $w((I_B-G)(\cdot)): B\to\boldsymbol{R}$ is strictly positive. On the other hand, we get

$$\frac{d}{dt}F_n((I_B-\tilde{G}_h)(z+th))=DF_n((I_B-\tilde{G}_h)(z+th))\frac{d}{dt}\{z+th-\tilde{G}_h(z+th)\}$$
$$=DF_n((I_B-G)(z+th))(I_H-DG(z+th))h$$

for a. e. $(z, t)\in B\times\boldsymbol{R}$ with respect to $\mu(dz)\times dt$. This shows that $F_n((I_B-G)(\cdot)): B\to H$ is S. G. D. and

$$D(F_n(I_B-G)(\cdot))(z)=DF_n((I_B-G)z)(I_H-DG(z))\in\mathcal{L}^2(H, H) \tag{7.5}$$

for μ-a. e. z.

Let $\rho(z)=\{|d(z; G)|w((I_B-G)z)v(z)\}^{1/2}$ for each $z\in B$. It is easy to see that $\rho(z)$ is strictly positive. Using Theorem 7.1, we get

$$\begin{aligned}
&\left\{\int_B\|D(F_n((I_B-G)(\cdot))(z)-DF((I_B-G)z)(I_H-DG(z))\|_{\mathcal{L}^2(H,H)}\rho(z)\mu(dz)\right\}^2 \\
&\leqq\left\{\int_B\|DF_n((I_B-G)z)-DF((I_B-G)z)\|_{\mathcal{L}^2(H,H)}(1+\|DG(z)\|_{\mathcal{L}^2(H,H)})\rho(z)\mu(dz)\right\}^2 \\
&\leqq\int_B\|DF_n((I_B-G)z)-DF((I_B-G)z)\|^2_{\mathcal{L}^2(H,H)}w((I_B-G)z)|d(z; G)|\mu(dz) \\
&\quad\times\int_B(1+\|DG(z)\|^2_{\mathcal{L}^2(H,H)})v(z)\mu(dz) \\
&\leqq\int_B\|DF_n(z)-DF(z)\|^2_{\mathcal{L}^2(H,H)}w(z)\mu(dz)\left\{2+2\int_B\|DG(z)\|^2_{\mathcal{L}^2(H,H)}v(z)\mu(dz)\right\} \\
&\to 0 \quad \text{as} \quad n\to\infty.
\end{aligned} \tag{7.6}$$

By virtue of Theorem 3.1, (7.4) and (7.6), we can see that $F((I_B-G)(\cdot)): B\to H$ belongs to $H^1(B\to H; d\mu)$ and

$$D(F((I_B-G)(\cdot))(z)=DF((I_B-G)z)(I_H-DG(z)) \quad \text{for} \quad \mu\text{-a. e. } z. \tag{7.7}$$

This and (7.3) prove that $DG(z)=-DF((I_B-G)z)(I_H-DG(z))$ for μ-a. e. z, which implies (7.1). This completes the proof.

§ 8. The Sard type theorem and its application.

We prove the following Sard type theorem.

THEOREM 8.1. *Let $F: B\to H$ be an $\mathcal{H}$-C^1 map, and let A be a measurable subset of B defined by*

$$A=\{z\in B; I_H-DF(z): H\to H \text{ is invertible}\}$$
$$=\{z\in B; \delta(I_H-DF(z))\neq 0\}.$$

Assume that ν is a probability measure on B such that the restricted measures $\nu|_A$ and $\mu|_A$ on A are mutually singular. Then $(I_B-F)\nu$ and μ are mutually singular.

PROOF. Step 1. We prove the following.

Claim 1. There exists a σ-compact subset Ω_0 of B such that $\mu(\Omega_0)=\nu(\Omega_0)=1$ and $(I_B-F)(\Omega_0\cap G)$ is σ-compact for any σ-compact subset G of B.

By virtue of Lusin's theorem, there exists a compact subset K_n for each $n=1, 2, \cdots$ such that $F|_{K_n}: K_n\to H$ is continuous, $\mu(K_n)>1-\dfrac{1}{n}$ and $\nu(K_n)>1-\dfrac{1}{n}$. Then $(I_B-F)(K_n\cap K)$ is compact for any compact subset K of B. Let $\Omega_0=\bigcup\limits_n K_n$, then $\mu(\Omega_0)=\nu(\Omega_0)=1$ and $(I_B-F)(\Omega_0\cap K)$ is σ-compact for each compact subset K of B. This proves our assertion.

Step 2. Let ε be an arbitrary positive number and fix it throughout the proof. Then $\mu(\Omega_m)>1-\dfrac{\varepsilon}{2}$ and $\nu(\Omega_m)>1-\dfrac{\varepsilon}{2}$ for some integer m, where

$$\Omega_m=\Big\{z\in B\,;\ \|DF(z+h)-DF(z)\|_{\mathcal{L}^2(H,H)}\leqq\frac{1}{120}\ \text{for any } h\in H \text{ such that } \|h\|_H\leqq\frac{8}{m}\Big\}.$$

Notice that Ω_m is measurable because F is an $\mathcal{H}$-C^1 map.

Take a subsequence $\{P_n\}_{n=1}^{\infty}$ of $\mathcal{P}(B^*)$ such that $P_n\uparrow I_H$ strongly, $n\to\infty$. Then $\mu(\Omega_{n,m})>1-\varepsilon$ and $\nu(\Omega_{n,m})>1-\varepsilon$ for some integer n, where

$$\Omega_{n,m}=\Omega_m\cap\Big\{z\in B\,;\ \|(I_H-P_n)F(z)\|_H\leqq\frac{1}{120m}\ \text{and}\ \|(I_H-P_n)DF(z)\|_{\mathcal{L}^2(H,H)}\leqq\frac{1}{120}\Big\}.$$

Let Ω be a σ-compact subset of B such that

$$\Omega\subset\Omega_{n,m}\cap\Omega_0\,,\quad \mu(\Omega)>1-\varepsilon\quad\text{and}\quad \nu(\Omega)>1-\varepsilon\,. \tag{8.1}$$

It is easy to see that

$$\|(I_H-P_n)DF(z+h)\|_{\mathcal{L}^2(H,H)}\leqq\frac{1}{60}\ \text{for any } h\in H,\ \|h\|_H\leqq\frac{8}{m}\ \text{and } z\in\Omega. \tag{8.2}$$

We also obtain

$$\|(I_H-P_n)F(z+h)\|_H\leqq\frac{17}{120m}\ \text{for any } h\in H,\ \|h\|_H\leqq\frac{8}{m}\ \text{and } z\in\Omega, \tag{8.3}$$

because

$$\|(I_H-P_n)F(z+h)\|_H\leqq\|(I_H-P_n)F(z)\|_H+\int_0^1\|(I_H-P_n)DF(z+th)\|_{\mathcal{L}^2(H,H)}\|h\|_H dt\,.$$

Take a smooth function ϕ on $\boldsymbol{R}$ such that $0 \leqq \phi(t) \leqq 1$ and $|\phi'(t)| \leqq 2$ for $t \in \boldsymbol{R}$, $\phi(t)=1$ for $|t| \leqq 6$ and $\phi(t)=0$ for $|t| \geqq 7$, and let $g(z)=\phi(m\rho(z\,;\Omega))$ and $G(z)=g(z)\cdot(I_H-P_n)F(z)$. It is easy to see that G belongs to $H^1(B\to H\,; d\mu)$ and

$$DG(z)h=(Dg(z)h)\cdot(I_H-P_n)F(z)+g(z)\cdot(I_H-P_n)DF(z)h \tag{8.4}$$

for μ-a.e. z and each $h \in H$.

Hence we get by (8.1) and (8.2)

$$\begin{aligned} \|DG(z)\|_{\mathcal{L}^2(H,H)} &\leqq \|Dg(z)\|_{\mathcal{L}^\infty(H,R)}\cdot\|(I_H-P_n)F(z)\|_H \\ &\quad +\|(I_H-P_n)DF(z)\|_{\mathcal{L}^2(H,H)} \\ &\leqq \frac{3}{10} \qquad \text{for} \quad \mu\text{-a.e.}\, z, \quad \text{and} \end{aligned} \tag{8.5}$$

$$\|G(z)\|_H \leqq \frac{17}{120m} < \frac{1}{6m} \qquad \text{for each} \quad z \in B. \tag{8.6}$$

In view of Theorem 5.1, (8.5) and (8.6) imply that G belongs to $\mathcal{D}(L)$. We can also see that $I_B-G: B\to B$ is bijective by virtue of Theorem 6.1.

Let $\tilde{E}=(I_B-G)\Omega$, then $\tilde{E}$ is measurable because $I_B-G: B\to B$ is injective. Suppose that $\rho(x\,;\tilde{E}) \leqq \frac{3}{m}$ for some $x \in B$. Then there exists some $z \in \Omega$ such that $x-(I_B-G)z \in H$ and $\|x-(I_B-G)z\|_H < \frac{17}{5m}$. By (6.1) and (6.5) in Theorem 6.1 and (8.5), we obtain

$$\|(I_B-G)^{-1}x-z\|_H \leqq \frac{1}{1-3/10}\|x-(I_B-G)z\|_H < \frac{5}{m},$$

which shows that $\rho((I_B-G)^{-1}x,\,\Omega) < \frac{5}{m}$. Thus we get

$$(I_B-(I_H-P_n)F)(I_B-G)^{-1}x=x \tag{8.7}$$

for any $x \in B$ such that $\rho(x\,;\tilde{E}) \leqq \frac{3}{m}$.

Let E be a σ-compact subset of B such that $E \subset \tilde{E}$ and $\mu(\tilde{E}\backslash E)=(I_B-G)\nu(\tilde{E}\backslash E)=0$ and take a smooth function ψ on $\boldsymbol{R}$ such that $0 \leqq \psi(t) \leqq 1$ and $|\psi'(t)| \leqq 2$ for $t \in \boldsymbol{R}$, $\psi(t)=1$ for $|t| \leqq 1$, and $\psi(t)=0$ for $|t| \geqq 2$. Let $k(x)=\psi(m\rho(x\,;E))$ and $K(x)=k(x)\{x-(I_B-G)^{-1}x\}=-k(x)G((I_B-G)^{-1}x)$ for each $x \in B$. Since $F: B\to H$ is an $\mathcal{H}$-C^1 map, it is easy to see that K belongs to $H^1(B\to H\,; d\mu)$ by virtue of (8.7), and we obtain

$$\begin{aligned} DK(x)h &= (Dk(x)h)\{x-(I_B-G)^{-1}x\} \\ &\quad +k(x)[I_H-\{I_H-(I_H-P_n)DF((I_B-G)^{-1}x)\}^{-1}]h \\ &= -(Dk(x)h)G((I_B-G)^{-1}x) \\ &\quad -k(x)(I_H-P_n)DF((I_B-G)^{-1}x)\{I_H-(I_H-P_n)DF((I_B-G)^{-1}x)\}^{-1}h \end{aligned}$$

for any $h\in H$ and μ-a.e. x. So we get

$$\|K(x)\|_H\leqq\frac{1}{6m}\qquad\text{for any}\quad x\in B\quad\text{and}\tag{8.8}$$

$$\|DK(x)\|_{\mathcal{L}^2(H,H)}\leqq 2m\cdot\frac{1}{6m}+\frac{1/60}{1-1/60}<\frac{1}{2}\qquad\text{for}\quad \mu\text{-a.e. } x.\tag{8.9}$$

This shows that K belongs to $\mathcal{D}(L)$ and

$$\|K(x+h)-K(x)\|_H\leqq\frac{1}{2}\|h\|_H\qquad\text{for any}\quad x\in B\quad\text{and}\quad h\in H.\tag{8.10}$$

Step 3. Let $S=-(I_B-F)(I_B-K)+I_B$, then it is easy to see that S belongs to $\mathcal{D}(L)$ and $d(z;S)=d((I_B-K)z;F)d(z;K)$ for μ-a.e. z by virtue of Lemma 6.1. Let M be a σ-compact subset such that $M\subset(I_B-G)^{-1}E\subset\Omega$, $\mu(M\cap A)=0$ and $\nu(M)=\nu(\Omega)$. The existence of such M is guaranteed by the singularity between $\mu|_A$ and $\nu|_A$. Let $N=(I_B-G)M=(I_B-K)^{-1}M$. According to Theorem 6.1 and (8.10), we get

$$\mu(N\cap(I_B-K)^{-1}A)=(I_B-K)\mu(M\cap A)=0\,.\tag{8.11}$$

Since $M\subset\Omega$, $(I_B-S)N=(I_B-F)M$ is σ-compact. Notice that

$$Sx=-(I_B-F)(I_B-K)x+x=P_nF(I_B-K)x\in P_nH$$

for any $x\in B$ such that $\rho(x;E)<\frac{1}{m}$. By Fubini's theorem, we get

$$\begin{aligned}\mu((I_B-S)N)&=\int_{B\times B}\chi_{(I_B-S)N}(z_1+z_2)\mu_{P_nH^\perp}(dz_1)\mu_{P_nH}(dz_2)\\&=\int_B\mu_{P_nH^\perp}(dz_1)\int_{P_nH}\chi_{(I_B-S)N-z_1}(z_2)\mu_{P_nH}(dz_2)\\&=\int_B\mu_{P_nH^\perp}(dz_1)\mu_{P_nH}((I_{P_nH}-S(\cdot+z_1))((N-z_1)\cap P_nH))\,.\end{aligned}$$

Thus using usual Sard's lemma (see J. Schwartz [14] for an example),

$$\begin{aligned}\mu((I_B-S)N)\leqq&\int_B\mu_{P_nH^\perp}(dz_1)\int_{(N-z_1)\cap P_nH}|\delta(I_{P_nH}-DS(z_1+z_2)|_{P_nH})|\\&\times\exp\Big[(S(z_1+z_2),\,z_2)_H-\operatorname{trace}DS(z_1+z_2)|_{P_nH}\\&-\frac{1}{2}\|S(z_1+z_2)\|_H^2\Big]\mu_{P_nH}(dz_2)\\\leqq&\int_B\mu_{P_nH^\perp}(dz_1)\int_{(N-z_1)\cap P_nH}|d(z_1+z_2;S)|\mu_{P_nH}(dz_2)\\\leqq&\int_N|d(z;S)|\mu(dz)\leqq\int_N|d((I_B-K)z;F)||d(z;K)|\mu(dz)\,.\end{aligned}$$

In view of (8.11), $d((I_B-K)z\,;F)=0$ for μ-a.e. $z\in N$, and accordingly $\mu((I_B-F)M)=\mu((I_B-S)N)=0$. On the other hand, $(I_B-F)\nu((I_B-F)M)\geqq\nu(M)=\nu(\Omega)>1-\varepsilon$. Since ε is arbitrary, this shows that μ and $(I_B-F)\nu$ are mutually singular. This completes the proof.

Using Theorem 8.1, we can prove the following.

THEOREM 8.2. *Suppose that* $F: B\to H$ *is an* $\mathcal{H}$-C^1 *map such that*
(1) $F(z+h_n)\to F(z)$ *in* H, $n\to\infty$, *for each* $z\in B$, *whenever* $h_n\to 0$ *weakly in* H, *and*
(2) $\lim_{r\to\infty}\sup\left\{\frac{\|F(z+h)\|_H}{\|h\|_H}\,;\ h\in H \text{ and } \|h\|_H\geqq r\right\}<1$ *for each* $z\in B$. *Then* μ *is absolutely continuous relative to* $(I_B-F)\mu$.

PROOF. Step 1. Let $U_r=\{h\in H\,;\|h\|_H\leqq r\}$. Since U_r is weakly compact in H, the image of U_r through $F(z+\cdot)$ is compact in H. So $\sup\{\|F(z+h)\|_H\,;h\in U_r\}<\infty$ for each $z\in B$. According to our assumption (2), there exists some $r>0$ for each $z\in B$ such that $\|F(z+h)\|_H\leqq r$ for any $h\in U_r$. Since $F(z+\cdot)|_{U_r}: U_r\to U_r$ is continuous and the image is compact, there exists some $h\in H$ such that $F(z+h)=h$ by virtue of Shauder's fixed point theorem.

Step 2. Let $K_z=\{h\in H\,;F(z+h)=h\}$ for each $z\in B$. Then $K_z\neq\emptyset$ for any $z\in B$ by Step 1. In view of our assumption (2), there exists some $r>0$ such that $K_z\subset U_r$.

Suppose that $\{h_n\}_{n=1}^{\infty}\subset K_z$, then $\{h_n\}_{n=1}^{\infty}\subset U_r$, and so there exists a subsequence $\{h_{n_j}\}$ and $h_0\in H$ such that $h_{n_j}\to h_0$, $j\to\infty$, weakly in H. This implies that $F(z+h_{n_j})\to F(z+h_0)$, $j\to\infty$, strongly in H. Since $h_n=F(z+h_n)$, we get $h_{n_j}\to h_0$, $j\to\infty$, strongly in H and $h_0\in K_z$. So K_z is compact in the strong topology of H. Let K be a map from B to comp(H) such that z corresponds to K_z through K, where comp(H) is a space of all compact subsets of H. (See D.W. Stroock and S.R.S. Varadhan [16] Chapter 12 about the topology of comp(H) and its property.)

Step 3. We prove the following in this step.

Claim 2. Let G be an open set in H. Then

$$\{z\in B\,;\ K_z\subset G\}=\{z\in B\,;\ \inf\{\|F(z+h)-h\|_H\,;\ h\in H\backslash G\}>0\}.$$

Suppose that $\inf\{\|F(z+h)-h\|_H\,;h\in H\backslash G\}>0$, then $F(z+h)\neq h$ for any $h\in H\backslash G$, and so $K_z\subset G$. Conversely suppose that $\inf\{\|F(z+h)-h\|_H\,;h\in H\subset G\}=0$. Then there exists a sequence $\{h_n\}_{n=1}^{\infty}\subset H\backslash G$ such that $\|F(z+h_n)-h_n\|_H\to 0$, $n\to\infty$. According to our assumption (2), $\{\|h_n\|_H\}$ must be bounded. So taking a subsequence if necessary, we may assume that $h_n\to h_0$, $n\to\infty$, weakly in H for some $h_0\in H$. Since $F(z+h_n)-h_n\to 0$ strongly, $n\to\infty$, we obtain that $h_n\to h_0$, $n\to\infty$, strongly and $F(z+h_0)=h_0$. By the closedness of $H\backslash G$, we get $h_0\in H\backslash G$, which shows that $K_z\not\subset H\backslash G$. This completes the proof.

Let $\{h_n\}_{n=1}^{\infty}$ be a dense countable subset of $H\backslash G$, then

$$\inf\{\|F(z+h)-h\|_H;\ h\in H\backslash G\}=\inf\{\|F(z+h_n)-h_n\|_H;\ n=1, 2, \cdots\}.$$

This shows that $\{z\in B;\ K_z\in G\}$ is measurable. According to D. W. Stroock and S. R. S. Varadhan [16] Chapter 12, we see that $K\colon B\to\mathrm{comp}(H)$ is measurable.

Step 4. By the measurable selection theorem in D. W. Stroock and S. R. S. Varadhan [16] Chapter 12, we see that there exists a measurable map $G\colon B\to H$ such that $G(z)\in K_z$ for each $z\in B$. Then $(I_B-F)(I_B+G)z=z+Gz-F(z+Gz)=z$ for each $z\in B$. Let $\nu=(I_B+G)\mu$ and $\nu=\nu_1+\nu_2$ be the Lebesgue decomposition of ν relative to μ, i.e. ν_1 is absolutely continuous and ν_2 is singular relative to μ. Then $\mu=(I_B-F)\nu=(I_B-F)\nu_1+(I_B-F)\nu_2$. But $(I_B-F)\nu_2$ and μ must be mutually singular in view of Theorem 8.1. So we have got $\nu_2=0$, and accordingly ν is absolutely continuous relative to μ. Thus $\mu=(I_B-F)\nu$ is absolutely continuous relative to $(I_B-F)\mu$. This completes the proof.

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