Classical scattering for relativistic particles

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1. Introduction.

The Hamiltonian for a classical relativistic particle (mass m and charge e) in an external static electromagnetic field described by four vector $(\phi(x), A(x))=(A^0(x), \cdots, A^3(x))$ is given as

$$H(x, \xi)=((\xi-eA(x))^2+m^2)^{1/2}+e\phi(x),$$

where the velocity of light is normalized as $c=1$. The equation of motion is given as

$$dx/dt=\partial H/\partial\xi(x, \xi), \qquad d\xi/dt=-\partial H/\partial x(x, \xi). \tag{1.1}$$

Here x and $\xi\in \boldsymbol{R}^3$ are position and canonical momentum. If we introduce position-ordinary momentum variables (q, k) by $q=x$ and $k=\xi-eA(x)$, (1.1) can be rewritten in a familiar form

$$dq/dt=v(k), \qquad dk/dt=e(\boldsymbol{E}(q)+v(k)\wedge \boldsymbol{H}(q)), \tag{1.2}$$

where $v(k)=k/(m^2+k^2)^{1/2}$ is the velocity of the particle; $\boldsymbol{E}(x)$ and $\boldsymbol{H}(x)$ are the strengths of electric and magnetic field (Landau-Lifschitz [4]). The purpose of this paper is to study the scattering theory for the equation (1.1), or equivalently (1.2), under the following condition.

ASSUMPTION (A_n). (1) For any $j=0, 1, 2, 3$, $A^j(x)$ is a real-valued n-times continuously differentiable function on $\boldsymbol{R}^3$.

(2) There exist constants $\varepsilon>0$ and $C>0$ such that

$$|(\partial/\partial x)^\alpha A^j(x)|\leqq C(1+|x|)^{-1-\varepsilon-|\alpha|}, \qquad |\alpha|\leqq n.$$

Being stimulated by a success of mathematical scattering theory for non-relativistic quantum mechanics, Cook [1] and Hunziker [3] initiated the mathematical scattering theory for non-relativistic classical particles, and then it was developed by Simon [7] and Herbst [2]. The fundamental problems there were, as in quantum case, the construction of wave operators and the proof of their completeness. Cook and Hunziker treated the problem in the framework of Hilbert space over phase space $L^2(\boldsymbol{R}^6)$, and Simon and Herbst treated it directly

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in the phase space $\boldsymbol{R}^6$.

Here we examine the same problem in relativistic case, following Simon's framework [7]. In Sect. 2, we show the existence and the uniqueness of an asymptotically free solution parametrized suitably. In Sect. 3, we show the existence of the wave operators. In these sections, we discuss the problem in (q, k)-variables. In Sect. 4, we rewrite the results in Sect. 2 and Sect. 3 in terms of canonical variables (x, ξ) and then prove the completeness of wave operators.

We use the following notation and conventions. (q, k) and (a, b) are used to denote (position, ordinary momentum)-variables; (x, ξ) and (y, η) to denote (position, canonical momentum). These are related as $x=q$, $\xi=k+e\boldsymbol{A}(q)$. For two vectors $u, v\in\boldsymbol{R}^3$, $u\cdot v$ stands for their inner product; $u^2=u\cdot u$; $|u|=(u^2)^{1/2}$; $u\wedge v$ is vector product. For an $m\times m$-matrix A, $|A|$ denotes the norm of A regarded as a linear operator on the unitary space $\boldsymbol{C}^m$. For vector valued function $f(x)=(f_j(x))$, $\partial f/\partial x$ stands for the matrix $(\partial f_j/\partial x_k)$. For multi-index $\alpha=(\alpha_1, \alpha_2, \alpha_3)$, $|\alpha|=\alpha_1+\alpha_2+\alpha_3$ and $(\partial/\partial x)^\alpha=(\partial/\partial x_1)^{\alpha_1}(\partial/\partial x_2)^{\alpha_2}(\partial/\partial x_3)^{\alpha_3}$. If F is a map from a Banach space to another, derivatives of F means its Frécht derivatives. For two sets A and $B\subset\boldsymbol{R}^m$, $A\Subset B$ means that A is a precompact subset of $\boldsymbol{R}^m$ and its closure is contained in the interior of B. $\Gamma=\boldsymbol{R}^3\times(\boldsymbol{R}^3\setminus\{0\})$. If a formula contains $\pm$ or $\mp$ signs, it stands for two formulas, one for upper signs and the other for lower signs. Various constants appearing in inequalities are usually written as C and are not distinguished unless stated explicitly. By convention, "0-times continuously differentiable" means continuous, and C^0-diffeomorphism means homeomorphism.

2. Asymptotically free solutions.

In this and next sections, we describe a particle by $(q, k)\in\boldsymbol{R}^6$, its position and ordinary momentum. We write the Lorentz force as $\boldsymbol{F}(q, k)=e(\boldsymbol{E}(q)+v(k)\wedge\boldsymbol{H}(q))$. The main theorem in this section is the following theorem.

THEOREM 2.1. *Let Assumption* (A_n) $(n\geqq 2)$ *be satisfied. Let* $(a, b)\in\Gamma$. *Then there exists a unique solution* $(q_\pm(t, a, b), k_\pm(t, a, b))$ *of* (1.2) *such that as* $t\to\pm\infty$,

$$|q_\pm(t, a, b)-tv(b)-a|\longrightarrow 0, \qquad |k_\pm(t, a, b)-b|\longrightarrow 0. \tag{2.1}$$

Furthermore, for any $K\Subset\Gamma$, *there exists a constant* $C>0$ *independent of* $(a, b)\in K$ *such that*

$$|(\partial/\partial a)^\alpha(\partial/\partial b)^\beta(q_\pm(t, a, b)-tv(b)-a)|\leqq C(1+|t|)^{-\varepsilon} \tag{2.2}$$

$$|(\partial/\partial a)^\alpha(\partial/\partial b)^\beta(k_\pm(t, a, b)-b)|\leqq C(1+|t|)^{-1-\varepsilon} \tag{2.3}$$

for any multi-indices α, β *such that* $|\alpha|+|\beta|\leqq n-1$.

Thus there exists a unique asymptotically free solution of (1.2) parametrized by $(a, b)\in\Gamma$. To prove the theorem we need the following two lemmas.

LEMMA 2.2. *Let $(q(t), k(t))$ be a solution of (1.2) such that $(k(0)^2+m^2)^{1/2}+e\phi(q(0))>m$. Suppose that there exists a sequence $\{t_{n,\pm}\}_{n=1}^{\infty}\subset \boldsymbol{R}^1$ such that $\pm t_{n,\pm}\to\infty$ and $|q(t_{n,\pm})|\to\infty$ as $n\to\infty$. Then there exist constants $C>0$ and $T_0>0$ such that*

$$|q(t)|\geqq \pm Ct \qquad \text{for } \pm t>T_0 . \tag{2.4}$$

Furthermore the following statements hold.

(1) *The limits*

$$\lim_{t\to\pm\infty} k(t)=b_\pm , \qquad \lim_{t\to\pm\infty}(q(t)-tv(b_\pm))=a_\pm \tag{2.5}$$

exist.

(2) *There exists a constant $C>0$ such that*

$$|k(t)-b_\pm|\leqq C|t|^{-1-\varepsilon} \tag{2.6}$$

$$|q(t)-tv(b_\pm)-a_\pm|\leqq C|t|^{-\varepsilon} \tag{2.7}$$

for $|t|\geqq T_0$.

(3) *$q(t)$ and $k(t)$ satisfy the integral equation*

$$q(t)=a_\pm+tv(b_\pm)+\int_{\pm\infty}^{t}(v(k(s))-v(b_\pm))ds , \tag{2.8}$$

$$k(t)=b_\pm+\int_{\pm\infty}^{t}\boldsymbol{F}(q(s), k(s))ds . \tag{2.9}$$

PROOF. We prove "+" case only. The other case can be proved similarly. By equation (1.2) we have

$$(d/dt)^2(q(t)^2/2)=v(k(t))^2+q(t)\cdot(\partial v/\partial k)(k(t))\boldsymbol{F}(q(t), k(t)) . \tag{2.10}$$

Let us write as $E(t)=(k(t)^2+m^2)^{1/2}+e\phi(q(t))$. Then $E(t)$ is a constant of motion $E(t)=E(0)\equiv E>m$ and $v(k(t))^2=(E-e\phi(q(t)))^{-2}((E-e\phi(q(t)))^2-m^2)<1$. By Assumption (A_n), it is clear that there exist constants $R>0$ and $\delta>0$ such that

$$(d/dt)^2(q(t)^2/2)\geqq\delta>0 , \tag{2.11}$$

provided $|q(t)|\geqq R$. Since $|q(t_{n,+})|\to\infty$, (2.11) shows that $|q(t)|\to\infty$ and (2.4) holds. Then by equation (1.2), we have $|dk(t)/dt|\leqq C(1+t)^{-2-\varepsilon}$, which clearly shows the existence of the first limit of (2.5) and the inequality (2.6). By (2.6),

$$|v(k(t))-v(b_+)|\leqq C(1+t)^{-1-\varepsilon} . \tag{2.12}$$

Inequality (2.12) and equation (1.2) show the existence of the second limit of

(2.5) and (2.7). Now it is clear that the functions $q(t)$ and $k(t)$ satisfy integral equations (2.8) and (2.9). (Q.E.D.)

For $T\geqq 0$ and $(a, b)\in\Gamma$ we define a Banach space X^T and an integral operator $I^{a,b}$ on X^T as follows:

$$X^T=\Big\{(f, g): f, g\in C([T, \infty); \boldsymbol{R}^3),\ \sup_{t\geq T}(1+t)^{\varepsilon/4}|f(t)| + \sup_{t\geq T}(1+t)^{1+\varepsilon/2}|g(t)|\equiv\|(f, g)\|_{X^T}<\infty\Big\}; \tag{2.13}$$

$$I^{a,b}(f, g)=(I_1^{a,b}(f, g), I_2^{a,b}(f, g)), \tag{2.14}$$

$$I_1^{a,b}(f, g)(t)=\int_\infty^t\{v(g(s)+b)-v(b)\}\,ds, \tag{2.15}$$

$$I_2^{a,b}(f, g)(t)=\int_\infty^t \boldsymbol{F}(a+v(b)s+f(s), b+g(s))ds. \tag{2.16}$$

We write the unit ball of X^T as B^T. X^T is also regarded as

$$X^T=X_1^T\oplus X_2^T,$$

where

$$X_1^T=\Big\{f\in C([T, \infty); \boldsymbol{R}^3): \sup_{t\geq T}(1+t)^{\varepsilon/4}|f(t)|=\|f\|_{X_1^T}<\infty\Big\},$$

$$X_2^T=\Big\{f\in C([T, \infty); \boldsymbol{R}^3): \sup_{t\geq T}(1+t)^{1+\varepsilon/2}|f(t)|=\|f\|_{X_2^T}<\infty\Big\}.$$

LEMMA 2.3. i) *Let $K\Subset\Gamma$. Then there exist constants $T_0\geqq 0$ and $0<\delta<1$ such that for $(a, b)\in K$, $I^{a,b}$ maps B^{T_0} into B^{T_0} and for any (f_1, g_1) and (f_2, g_2) $\in B^{T_0}$,*

$$\|I^{a,b}(f_1, g_1)-I^{a,b}(f_2, g_2)\|_{X^{T_0}}\leqq\delta\|(f_1-f_2, g_1-g_2)\|_{X^{T_0}}. \tag{2.17}$$

ii) *For any $T\geqq 0$, the mapping $\Gamma\times X^T\ni(a, b, f, g)\to I^{a,b}(f, g)\in X^T$ is $(n-1)$-times continuously differentiable.*

PROOF. i) Let us take $T'>0$ such that for $(f_1, g_1), (f_2, g_2)\in B^{T'}$ and (a, b) $\in K$,

$$|f_1(t)+f_2(t)+a+tv(b)|>\frac{1}{2}|v(b)(t+1)|, \quad t>T'. \tag{2.18}$$

Let $(f, g)\in B^{T'}$. Then for $t\geqq T'$,

$$|I_1^{a,b}(f, g)(t)|\leqq\left|\int_\infty^t\int_0^1(\partial v/\partial k)(b+\theta g(s))g(s)d\theta ds\right| \leqq 2(m\varepsilon)^{-1}(1+t)^{-\varepsilon/2}\sup_{t\leq s}|(1+s)^{1+\varepsilon/2}g(s)| \tag{2.19}$$

and by Assumption (A_n),

$$(2.20)\qquad |I_2^{a,b}(f, g)(t)| \leqq C\int_t^\infty (1+s)^{-2-\varepsilon} ds = C(1+t)^{-1-\varepsilon}.$$

Here the constant $C>0$ in (2.20) depends only on K and the constants appearing in the Assumption (A_n). Let $(f_1, g_1), (f_2, g_2) \in B^T$, $T>T_1$ and $t>T$. Then as (2.19),

$$(2.21)\qquad \begin{aligned}&|I_1^{a,b}(f_2, g_2)(t) - I_1^{a,b}(f_1, g_1)(t)|\\ &\leqq 2(m\varepsilon)^{-1}(1+t)^{-\varepsilon/2} \sup_{s\geqq t}(1+s)^{1+\varepsilon/2}|(g_1(s)-g_2(s))|,\end{aligned}$$

and

$$(2.22)\qquad \begin{aligned}&|I_2^{a,b}(f_2, g_2) - I_2^{a,b}(f_1, g_1)(t)|\\ &\leqq \int_t^\infty\int_0^1 \{|(\partial \boldsymbol{F}/\partial x)(a+v(b)s+\theta(f_2-f_1)(s),\ k+\theta(g_2-g_1)(s))(f_2-f_1)(s)|\\ &\quad +|(\partial \boldsymbol{F}/\partial k)(a+v(b)s+\theta(f_2-f_1)(s),\ k+\theta(g_2-g_1)(s))(g_2-g_1)(s)|\}\, d\theta\, ds\\ &\leqq C\int_t^\infty \{(1+s)^{-3-\varepsilon}|(f_2-f_1)(s)| + (1+s)^{-2-\varepsilon}|(g_2-g_1)(s)|\}\, ds\\ &\leqq C\Big\{(1+t)^{-2-5\varepsilon/4} \sup_{s\geqq t}(1+s)^{\varepsilon/4}|f_2(s)-f_1(s)|\\ &\quad +(1+t)^{-2-3\varepsilon/2}\sup_{s\geqq t}(1+s)^{1+\varepsilon/2}|g_2(s)-g_1(s)|\Big\}.\end{aligned}$$

Thus by (2.19)—(2.22), we see that for $T\geqq T'$,

$$(2.23)\qquad \|I^{a,b}(f, g)\|_{X^T} \leqq C(1+T)^{-\varepsilon/4}\|(f, g)\|_{X^T},$$

$$(2.24)\qquad \|I^{a,b}(f_1, g_1) - I^{a,b}(f_2, g_2)\|_{X^T} \leqq C(1+T)^{-\varepsilon/4}\|(f_1, g_1)-(f_2, g_2)\|_{X^T},$$

provided (f, g), (f_1, g_1) and $(f_2, g_2) \in B^{T'}$. Here the constant in (2.23) and (2.24) depends only on $K \Subset \Gamma$ and the constants appearing in Assumption (A_n). This proves the first statement.

ii) By a direct calculation, using similar estimates as in the proof of i), we can easily see that $I^{a,b}(f, g)$ is $(n-1)$ times differentiable with respect to $(a, b, f, g) \in K\times B^T$. The derivative $(\partial/\partial a)^j(\partial/\partial b)^l(\partial/\partial f)^m(\partial/\partial g)^r I^{a,b}(f, g)$ is (j, l, m, r)-multi-linear form $(0\leqq j+l+m+r\leqq n-1)$ on $\boldsymbol{R}^3\times\boldsymbol{R}^3\times X_1^T\times X_2^T$ to $X_1^T\times X_2^T$ given as

$$(2.25)\qquad \begin{aligned}&((\partial/\partial a)^j(\partial/\partial b)^l(\partial/\partial f)^m(\partial/\partial g)^r I_1^{a,b}(f, g))(a^1, \cdots, a^j; b^1, \cdots, b^l;\\ &\qquad\qquad f^1, \cdots, f^m; g^1, \cdots, g^r)(t)\\ &=\begin{cases} \displaystyle\sum_{\alpha_i, \beta_i=1}^3 \int_\infty^t [(\partial^{l+r}/\partial k_{\alpha_1}\cdots\partial k_{\alpha_r}\partial b_{\beta_1}\cdots\partial b_{\beta_l})(v(k+b)-v(b))]_{k=g(s)}\\ \qquad\qquad \times g_{\alpha_1}^1(s)\cdots g_{\alpha_r}^r(s) b_{\beta_1}^1\cdots b_{\beta_l}^l\, ds, \qquad \text{if } j=m=0,\\ 0, \qquad \text{otherwise}\end{cases}\end{aligned}$$

$$(2.26)\qquad ((\partial/\partial a)^j(\partial/\partial b)^l(\partial/\partial f)^m(\partial/\partial g)^r I_2^{a,b}(f, g))(a^1, \cdots, a^j; b^1, \cdots, b^l; f^1, \cdots, f^m; g^1, \cdots, g^r)(t)$$
$$= \sum_{\alpha_i, \beta_i, \gamma_i, \delta_i=1}^{3} \int_{\infty}^{t} [(\partial^{j+l+m+r}/\partial a_{\alpha_1} \cdots \partial a_{\alpha_j} \partial b_{\beta_1} \cdots \partial b_{\beta_l} \partial q_{\gamma_1} \cdots \partial q_{\gamma_m} \partial k_{\delta_1} \cdots \partial k_{\delta_r})$$
$$\times \boldsymbol{F}(a+v(b)s+q, b+k)]_{q=f(s), k=g(s)} f_{\gamma_1}^1(s) \cdots$$
$$\times f_{\gamma_m}^m(s) g_{\delta_1}^1(s) \cdots g_{\delta_r}^r(s) a_{\alpha_1}^1 \cdots a_{\alpha_j}^j b_{\beta_1}^1 \cdots b_{\beta_l}^l ds .$$

Clearly $(\partial/\partial a)^j(\partial/\partial b)^l(\partial/\partial f)^m(\partial/\partial g)^r I^{a,b}(f, g)$ is a continuous function from $(a, b, f, g) \in K \times B^{T_0}$ to the space of (j, l, m, r)-multilinear forms on $\boldsymbol{R}^3 \times \boldsymbol{R}^3 \times X_1^{T_0} \times X_2^{T_0}$. We omit the details. (Q.E.D.)

PROOF OF THEOREM 2.1.

We prove the case where $t \to \infty$ only. The other case can be proved similarly. By Lemma 2.3 and a fixed point theorem for a contraction map (Loomis-Steinberg [5], Theorem 9.4, p. 231 and J. T. Schwartz [8], Corollary 1.21, p. 15), $I^{a,b}$ has a unique fixed point $(f^{a,b}, g^{a,b})$ in the unit ball B^{T_0} of X^{T_0} and $(f^{a,b}, g^{a,b})$ is an $(n-1)$-times continuously differentiable X^{T_0}-valued function of $(a, b) \in \Gamma$. Let us define as $q_+(t, a, b) = a + v(b)t + f^{a,b}(t)$ and $k_+(t, a, b) = b + g^{a,b}(t)$ for $t \geqq T$. Clearly $q_+(t, a, b)$ and $k_+(t, a, b)$ are the solutions of (2.8) and (2.9) (replace a_+, b_+ by a, b) and hence they are the desired unique solutions of (1.2) by Lemma 2.2. (2.2) and (2.3) are consequences of the differentiability of $(f^{a,b}, g^{a,b})$ as an X^{T_0} valued function of (a, b). (Q.E.D.)

3. Existence of wave operators.

Following Simon [7], we define classical wave operators $\widetilde{W}_{\pm}^{\mathrm{cl}}$ as follows:

$$(3.1)\qquad \widetilde{W}_{\pm}^{\mathrm{cl}}(a, b) = (q_{\pm}(0, a, b), k_{\pm}(0, a, b)) .$$

By Theorem 2.1, and Lemma 2.3 $\widetilde{W}_{\pm}^{\mathrm{cl}}$ is a C^{n-1} diffeomorphism, $n \geqq 2$. Comparing with the wave operators in quantum mechanics (see Reed-Simon [6]), we ask the following question: Let $(q_s(t, a, b), k_s(t, a, b))$ be the solution of (1.2) with the condition

$$(3.2)\qquad q_s(s, a, b) = a + sv(b), \qquad k_s(s, a, b) = b ,$$

and $\widetilde{U}(t)$, $\widetilde{U}_0(t)$ be the mappings defined as

$$(3.3)\qquad \widetilde{U}(t)(a, b) = (q_0(t, a, b), k_0(t, a, b)) ,$$

$$(3.4)\qquad \widetilde{U}_0(t)(a, b) = (a + tv(b), b) .$$

Now do $\lim_{t \to \pm\infty} \widetilde{U}(-t)\widetilde{U}_0(t)(a, b)$ exist and are equal to $\widetilde{W}^{\mathrm{cl}}(a, b)$?

THEOREM 3.1. *Let Assumption* (A_n) $(n\geqq 2)$ *be satisfied. Then for any multi-index* α *and* β *such that* $|\alpha|+|\beta|\leqq n-1$,

$$\lim_{t\to\pm\infty}(\partial/\partial a)^{\alpha}(\partial/\partial b)^{\beta}\tilde{U}(-t)\tilde{U}_0(t)(a,\ b)=(\partial/\partial a)^{\alpha}(\partial/\partial b)^{\beta}\tilde{W}_{\pm}^{cl}(a,\ b)\,, \tag{3.5}$$

for any $(a,\ b)\in\Gamma$, *where the convergence in* (3.5) *is uniform on every compact subset of* Γ.

We prove this theorem only for the case where $t\to\infty$. We omit the index "+" in what follows. To prove the theorem, we need the following lemma.

LEMMA 3.2. *Let* $K\Subset\Gamma$ *and let* $T_0>0$, $0<\delta<1$ *be the constants of Lemma* 2.3. *For* $(a,\ b)\in K$ *and* $s\geqq T$, *we define an integral operator* $I_s^{a,b}=(I_{s,1}^{a,b},\ I_{s,2}^{a,b})$ *as follows.*

$$I_{s,1}^{a,b}(f,\ g)(t)=Y(s-t)\int_{\infty}^{t}Y(s-u)(v(b+g(u))-v(b))du\,, \tag{3.6}$$

$$I_{s,2}^{a,b}(f,\ g)(t)=Y(s-t)\int_{\infty}^{t}Y(s-u)\boldsymbol{F}(a+v(b)u+f(u),\ b+g(u))du \tag{3.7}$$

where $Y(t)$ *is the Heaviside function*: $Y(t)=1$ *if* $t>0$ *and* $Y(t)=0$ *if* $t\leqq 0$. *Then* $I_s^{a,b}$ *maps* X^{T_0} *into* X^{T_0}, B^{T_0} *into* B^{T_0} *and*

$$\|I_s^{a,b}(f_2,\ g_2)-I_s^{a,b}(f_1,\ g_1)\|_{X^{T_0}}\leqq\delta\|(f_2,\ g_2)-(f_1,\ g_1)\|_{X^{T_0}} \tag{3.8}$$

for $(a,\ b)\in K$, $(f_2,\ g_2),\ (f_1,\ g_1)\in B^{T_0}$. *Furthermore* $I_s^{a,b}(f,\ g)$ *is an* $(n-1)$*-times continuously differentiable function of* $(a,\ b,\ f,\ g)\in K\times B^{T_0}$ *and* $I_s^{a,b}(f,\ g)$ *and its derivatives converge to* $I^{a,b}(f,\ g)$ *and its derivatives as* $s\to\infty$ *uniformly on* $(a,\ b,\ f,\ g)\in K\times B^{T_0}$.

PROOF. Since $Y(s-t)$ simply cuts off the tails of functions, the proof of Lemma 2.3 clearly shows that $I_s^{a,b}$ maps X^{T_0} into X^{T_0}; B^{T_0} into B^{T_0}; (3.8) holds for $(a,\ b)\in K$ and $(f_1,\ g_1),\ (f_2,\ g_2)\in B^{T_0}$; $I_s^{a,b}(f,\ g)$ is $(n-1)$-times continuously differentiable with respect to $(a,\ b,\ f,\ g)\in K\times B^{T_0}$. Let us prove that $I_s^{a,b}(f,\ g)$ converges to $I^{a,b}(f,\ g)$ as $s\to\infty$ uniformly in $(a,\ b)\in K$ and $(f,\ g)\in B^{T_0}$. Let $(f,\ g)\in B^{T_0}$. By a similar calculation as in the proof of Lemma 2.3, we have for $t\leqq s$

$$\begin{aligned}|I_{s,1}^{a,b}(f,\ g)(t)-I_1^{a,b}(f,\ g)(t)|&=\left|\int_{\infty}^{s}(v(b+g(u))-v(b))du\right|\\&\leqq 2(m\varepsilon)^{-1}(1+s)^{-\varepsilon/2}\,,\end{aligned} \tag{3.9}$$

$$\begin{aligned}|I_{s,2}^{a,b}(f,\ g)(t)-I_2^{a,b}(f,\ g)|&=\left|\int_{\infty}^{s}\boldsymbol{F}(a+v(b)u+f(u),\ b+g(u))du\right|\\&\leqq C(1+s)^{-1-\varepsilon}\,;\end{aligned} \tag{3.10}$$

for $t\geqq s$,

$$(3.11)\qquad |I_{s,1}^{a,b}(f, g)(t)-I_1^{a,b}(f, g)(t)|=|I_1^{a,b}(f, g)(t)|$$
$$=\left|\int_{\infty}^{t}(v(b+g(u))-v(b))du\right|\leqq 2(m\varepsilon)^{-1}(1+t)^{-\varepsilon/2},$$

$$(3.12)\qquad |I_{2,s}^{a,b}(f, g)(t)-I_2^{a,b}(f, g)(t)|=|I_2^{a,b}(f, g)(t)|$$
$$=\left|\int_{\infty}^{t}\boldsymbol{F}(a+v(b)u+f(u), b+g(u))du\right|\leqq C(1+t)^{-1-\varepsilon}.$$

Summarizing (3.9)-(3.12), we have

$$(3.13)\qquad \|I^{a,b}(f, g)-I_s^{a,b}(f, g)\|_{X^{T_0}}\leqq C(1+s)^{-\varepsilon/4},$$

for $(a, b)\in K$ and $(f, g)\in B^{T_0}$, where the constants in (3.9)-(3.13) are dependent only on K and the constants of Assumption (A_n). The convergence of the derivatives of $I_s^{a,b}(f, g)$ can be proved similarly. (Q. E. D)

PROOF OF THEOREM 3.1.

We prove the case $t\rightarrow\infty$ only. The other case can be proved similarly. Let $K\Subset\Gamma$. By Lemma 3.2, $I_s^{a,b}$ $(s\geqq T)$ has a unique fixed point $(f_s^{a,b}, g_s^{a,b})$ in B^{T_0} and $(\partial/\partial a)^{\alpha}(\partial/\partial b)^{\beta}(f_s^{a,b}, g_s^{a,b})$ converges to $(\partial/\partial a)^{\alpha}(\partial/\partial b)^{\beta}(f^{a,b}, g^{a,b})$ in X^{T_0} as $s\rightarrow\infty$ uniformly on $(a, b)\in K$ $(|\alpha|+|\beta|\leqq n-1)$. Let $q_s(t, a, b)=a+tv(b)+f_s^{a,b}(t)$ and $k_s(t, a, b)=b+g_s^{a,b}(t)$. Clearly $q_s(t, a, b)$ and $k_s(t, a, b)$ $(T_0\leqq t\leqq s)$ are the unique solutions of (1.2) with condition (3.2). By the unique existence theorem of the solution of the Cauchy problem for the ordinary differential equation (1.2), we see that

$$(3.14)\qquad \tilde{U}(-s)\tilde{U}_0(s)(a, b)=\tilde{U}(-T_0)\tilde{U}(T_0-s)\tilde{U}_0(s)(a, b)$$
$$=\tilde{U}(-T_0)(q_s(T_0, a, b), k_s(T_0, a, b)).$$

Since $\lim_{s\rightarrow\infty}(\partial/\partial a)^{\alpha}(\partial/\partial b)^{\beta}(q_s(T_0, a, b), k_s(T_0, a, b))=(\partial/\partial a)^{\alpha}(\partial/\partial b)^{\beta}(q_+(T_0, a, b), k_+(T_0, a, b))$ uniformly on $(a, b)\in K$, the continuous dependence of the solution of (1.2) on the initial condition shows the convergence of the limit (3.5). This proves the theorem. (Q. E. D.)

COROLLARY 3.3. *Let $K\Subset\Gamma$ and let T_0 be the constant in Lemma 2.3. Then there exists a constant C_n depending only on K such that*

$$(3.15)\qquad \sum_{|\alpha|+|\beta|\leqq n-1}|(\partial/\partial a)^{\alpha}(\partial/\partial b)^{\beta}(q_s(t, a, b)-a-tv(b))|\leqq C_n(1+|t|)^{-\varepsilon},$$

$$(3.16)\qquad \sum_{|\alpha|+|\beta|\leqq n-1}(\partial/\partial a)^{\alpha}(\partial/\partial b)^{\beta}(k_s(t, a, b)-b)|\leqq C_n(1+|t|)^{-1-\varepsilon},$$

for any $(a, b)\in K$ and $T_0\leqq\pm t\leqq\pm s$.

PROOF. By Lemma 3.2, we get (3.15) and (3.16) replacing ε by $\varepsilon/4$ in (3.15) and by $\varepsilon/2$ in (3.16). We note now $(f_s(t, a, b), g_s(t, a, b))=(q_s(t, a, b)-a-tv(b), k_s(t, a, b)-b)$ is a fixed point of $I_s^{a,b}$. Therefore $g_s(t, a, b)$ really satisfies (3.16) by (3.7) and using this with (3.6), we get (3.15). (Q. E. D.)

4. Hamilton formalism and completeness of wave operators.

In previous sections we described a particle by (q, k), position and ordinary momentum. Sometimes it is more convenient to describe the particle by (x, ξ), canonical variables. Especially this is the case when we discuss the relation between quantum mechanics and classical mechanics (Yajima [9]). So we rewrite here the results obtained in Sect. 2 and Sect. 3 in terms of (x, ξ). Thus we treat the equation (1.1) here.

We write the solution of (1.1) with condition

$$x(s)=sv(\eta)+y\,, \qquad \xi(s)=\eta \tag{4.1}$$

as $x_s(t, y, \eta)$ and $\xi_s(t, y, \eta)$. We define maps $U(t)$ and $U_0(t)$ as

$$U(t)(y, \eta)=(x_0(t, y, \eta), \xi_0(t, y, \eta))\,, \tag{4.2}$$

$$U_0(t)(y, \eta)=(tv(\eta)+y, \eta)\,. \tag{4.3}$$

The relation of (q, k)-variables and (x, ξ)-variables is given as

$$x=q\,, \qquad \xi=k+eA(q)\,. \tag{4.4}$$

Thus we have the following obvious relations:

(A) If $(q(t), k(t))$ is a solution of (1.2), $(x(t), \xi(t))=(q(t), k(t)+eA(q(t)))$ is a solution of (1.1).

(B) If (a, b) and (y, η) are related as

$$sv(b)+a=sv(\eta)+y\,, \qquad b+eA(sv(b)+a)=\eta\,, \tag{4.5}$$

$(x_s(t, y, \eta), \xi_s(t, y, \eta))$ and $(q_s(t, a, b), k_s(t, a, b))$ are related as

$$x_s(t, y, \eta)=q_s(t, a, b)\,, \quad \xi_s(t, y, \eta)=k_s(t, a, b)+eA(q_s(t, a, b))\,. \tag{4.6}$$

(C) The map $(x, \xi)\to(q, k)$ defined by (4.4) is a global diffeomorphism on $\boldsymbol{R}^3\times\boldsymbol{R}^3$ and its derivatives and the derivatives of its inverse map are uniformly bounded functions on $\boldsymbol{R}^3\times\boldsymbol{R}^3$ (global implicit function theorem, J. T. Schwartz [8], Theorem 1.22, p. 16).

Using these properties (A), (B) and (C), we prove the following theorems.

THEOREM 4.1. *Let Assumption* (A_n) *be satisfied* $(n\geqq 2)$. *Let* $(y, \eta)\in\Gamma$. *Then there exists a unique solution* $(x_\pm(t, y, \eta), \xi_\pm(t, y, \eta))$ *of* (1.1) *such that as* $t\to\pm\infty$,

$$|x_\pm(t, y, \eta)-tv(\eta)-y|\longrightarrow 0\,, \qquad |\xi_\pm(t, y, \eta)-\eta|\longrightarrow 0\,. \tag{4.7}$$

Furthermore for any $K \Subset \Gamma$ *and multi-index* α *and* β $(|\alpha|+|\beta| \leqq n-1)$, *there exists a constant* $C_{\alpha\beta}>0$ *such that*

$$(4.8) \qquad |(\partial/\partial y)^{\alpha}(\partial/\partial \eta)^{\beta}(x_{\pm}(t, y, \eta)-tv(\eta)-y)| \leqq C_{\alpha\beta}(1+|t|)^{-\varepsilon},$$

$$(4.9) \qquad |(\partial/\partial y)^{\alpha}(\partial/\partial \eta)^{\beta}(\xi_{\pm}(t, y, \eta)-\eta)| \leqq C_{\alpha\beta}(1+|t|)^{-1-\varepsilon},$$

for any $(y, \eta) \in K$ *and* $\pm t \geqq 0$.

PROOF. By (A), $(x_{\pm}(t), \xi_{\pm}(t))=(q_{\pm}(t, y, \eta), k_{\pm}(t, y, \eta)+eA(q_{\pm}(t, y, \eta)))$ is a solution of (1.1). Clearly this $(x_{\pm}(t), \xi_{\pm}(t))$ satisfies the condition (4.7). The uniqueness of $(x_{\pm}(t, y, \eta), \xi_{\pm}(t, y, \eta))$ follows from the uniqueness of $(q_{\pm}(t, a, b), k_{\pm}(t, a, b))$ of Theorem 2.1. The estimates (4.8) and (4.9) are obvious by (2.2) and (2.3). (Q. E. D.)

We define the wave operator $W_{\pm}^{cl}$ in canonical formalism as

$$(4.10) \qquad W_{\pm}^{cl}(y, \eta)=(x_{\pm}(0, y, \eta), \xi_{\pm}(0, y, \eta)).$$

Clearly $W_{\pm}^{cl}$ are C^{n-1}-diffeomorphisms on Γ.

THEOREM 4.2. *Let Assumption* (A_n) *be satisfied* $(n \geqq 2)$. *Then for any* $(y, \eta) \in \Gamma$ *and multi-index* α *and* β $(|\alpha|+|\beta| \leqq n-1)$,

$$(4.11) \qquad \lim_{t\to\pm\infty} (\partial/\partial y)^{\alpha}(\partial/\partial \eta)^{\beta} U(-t)U_0(t)(y, \eta)=(\partial/\partial y)^{\alpha}(\partial/\partial \eta)^{\beta} W_{\pm}^{cl}(y, \eta).$$

Furthermore the convergence in (4.11) *is uniform on every compact subset of* Γ.

To prove Theorem 4.2, we need the following lemma.

LEMMA 4.3. *Let* $K \Subset \Gamma$. *Then for sufficiently large* $|s|>0$, *equation* (4.5) *determines a unique map* $(a, b)=(a_s(y, \eta), b_s(y, \eta))$ *(and its inverse* $(y, \eta)=(y_s(a, b), \eta_s(a, b))$) *from* K *onto compact subset* $\tilde{K}_s \Subset \Gamma$ *(and* $K_s \Subset \Gamma$). *Furthermore for any multi-index* α *and* β $(|\alpha|+|\beta| \leqq n-1)$ *there exist a constant* $C_{\alpha\beta}$ *such that for* $(a, b) \in K$ *and* $(y, \eta) \in K$,

$$(4.12) \qquad |(\partial/\partial a)^{\alpha}(\partial/\partial b)^{\beta}(y_s(a, b)-a)| \leqq C_{\alpha\beta}(1+|s|)^{-\varepsilon},$$

$$(4.13) \qquad |(\partial/\partial a)^{\alpha}(\partial/\partial b)^{\beta}(\eta_s(a, b)-\eta)| \leqq C_{\alpha\beta}(1+|s|)^{-1-\varepsilon},$$

$$(4.14) \qquad |(\partial/\partial y)^{\alpha}(\partial/\partial \eta)^{\beta}(a_s(y, \eta)-y)| \leqq C_{\alpha\beta}(1+|s|)^{-\varepsilon},$$

$$(4.15) \qquad |(\partial/\partial y)^{\alpha}(\partial/\partial \eta)^{\beta}(b_s(y, \eta)-\eta)| \leqq C_{\alpha\beta}(1+|s|)^{-1-\varepsilon}.$$

PROOF. Equation (4.5) can be solved explicitly as

$$a=y+s(v(\eta)-v(\eta-eA(sv(y)+y))),$$

$$b=\eta-eA(sv(\eta)+y).$$

Therefore by mean-value theorem,

$$a_s(y,\eta)-y=e\Big(\int_0^1(\partial v/\partial k)(\eta-e\theta A(sv(\eta)+y))d\theta)sA(sv(\eta)+y))d\theta\Big)\,, \tag{4.16}$$

$$b_s(y,\eta)-\eta=-eA(sv(\eta)+y)\,. \tag{4.17}$$

The statements of the lemma are now clear by (4.16), (4.17), Assumption (A_n) and the global implicit function theorem (J. T. Schwartz [8]). (Q. E. D.)

PROOF OF THEOREM 4.2.
By property (B),

$$U(-s)U_0(s)(y,\eta)=(q_s(0,\ a_s(y,\eta),\ b_s(y,\eta))\,,\quad k_s(0,\ a_s(y,\eta),\ b_s(y,\eta))+eA(q_s(0,\ a_s(y,\eta),\ b_s(y,\eta))))\,. \tag{4.8}$$

The statement of Theorem 4.2 is a consequence of Theorem 3.1, and Lemma 4.2. (Q. E. D.)

COROLLARY 4.4. *Let $K\Subset\Gamma$. Then there exists a constant $T_0>0$ such that for any multi-index α and β ($|\alpha|+|\beta|\leqq n-1$) there exists a constant $C_{\alpha\beta}>0$ such that*

$$|(\partial/\partial y)^\alpha(\partial/\partial\eta)^\beta(x_s(t,y,\eta)-tv(\eta)-y)|\leqq C_{\alpha\beta}(1+|t|)^{-\varepsilon}\,, \tag{4.19}$$

$$|(\partial/\partial y)^\alpha(\partial/\partial\eta)^\beta(\xi_s(t,y,\eta)-\eta)|\leqq C_{\alpha\beta}(1+|t|)^{-1-\varepsilon}\,, \tag{4.20}$$

for any t, s such that $T_0\leqq\pm t\leqq\pm s$.

PROOF. By (4.6), we have

$$x_s(t,y,\eta)=a-v(\eta)t=(q_s(t,a_s,b_s)-a_s-v(b_s)t)+(1+t/s(a_s-y))\,, \tag{4.21}$$

$$\xi_s(t,y,\eta)-\eta=(k_s(t,a_s,b_s)-b_s)+e(A(q_s(t,a_s,b_s))-A(q_s(s,a_s,b_s)))\,, \tag{4.22}$$

where $a_s=a_s(y,\eta)$ and $b_s=b_s(y,\eta)$ are functions defined in Lemma 4.3. Thus by Lemma 4.3, Corollary 3.3 and Assumption (A_n) we get (4.19) and (4.20). (Q. E. D.)

Since $U(t)$ and $U_0(t)$ are canonical mappings, Hunziker-Siegel's argument [3] yields the following completeness theorem for wave operators $W_\pm^{cl}$ and $\widetilde{W}_\pm^{cl}$.

THEOREM 4.5. *Let Assumption (A_n) ($n\geqq 2$) be satisfied. Then there exist closed null set $e_\pm\subset\Gamma$ such that*

$$W_\pm^{cl}(\Gamma\backslash e_\pm)\subset W_\mp^{cl}(\Gamma)\,. \tag{4.23}$$

The same statement holds for $\widetilde{W}_\pm^{cl}$.

PROOF. By virtue of the argument of Siegel-Hunziker (Hunziker [3], Lemma 4, p. 290), we see that

$$e_{\pm}=\Gamma\cap(W_{\pm}^{cl})^{-1}(W_{\pm}^{cl}(\Gamma)\backslash W_{\mp}^{cl}(\Gamma))$$

has a Lebesgue measure zero. Since $W_{\pm}^{cl}$ are C^{n-3}-diffeomorphisms on Γ, $e_{\pm}$ is a closed set of Γ. Statement for $\widetilde{W}_{\pm}^{cl}$ clearly follows from that for $W_{\pm}^{cl}$.

(Q. E. D.)

The classical S-matrix is defined on $\Gamma\backslash e_{-}$ as

$$S^{cl}=(W_{+}^{cl})^{-1}(W_{-}^{cl})=(\widetilde{W}_{+}^{cl})^{-1}(\widetilde{W}_{-}^{cl})\,.$$

S^{cl} is a C^{n-1} canonical mapping.

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References

[1] Cook, J., Banach Algebras and Asymptotic Mechanics, Cargese Lectures in Theoretical Physics (F. Lurchit, ed.) Gordon and Breach, New York, 1967.
[2] Herbst, I., Classical scattering with long range forces, Comm. Math. Phys. **35** (1974), 193-214.
[3] Hunziker, W., The S-matrix in classical mechanics, Comm. Math. Phys. 8 (1968), 282-299.
[4] Landau, L. D. and E. M. Lifschitz, The Classical Theory of Fields, Pergamon Press, New York, 1971.
[5] Loomis, L. and S. Sternberg, Advanced Calculus, Addison-Wesley, Reading, Mass., 1968.
[6] Reed, M. and B. Simon, Methods of Modern Mathematical Physics, Vol. 3, Scattering Theory, Academic Press, New York, 1979.
[7] Simon, B., Wave operators for classical particle scattering, Comm. Math. Phys. **23** (1971), 37-48.
[8] Schwartz, J. T., Nonlinear Functional Analysis, Gordon and Breach, New York, 1969.
[9] Yajima, K., The quasi-classical approximation to Dirac equation, I, II, J. Fac. Sci. Univ. Tokyo Sect. IA **29** (1982), 161-194 and 371-386.

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