Comparison of the Brauer group with the Tate-Šafarevič group

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To the memory of Takuro Shintani

1. Statements.

Let K be a number field or a function field in one variable with finite field of constants, and let V be respectively the spectrum of the ring of integers in K or the complete smooth curve with function field K. We write V° for the set of finite primes of V (identical to the set of closed points of V), V^{∞} for the set of infinite primes, and $\hat{V}$ for $V^{\circ} \cup V^{\infty}$. The completion of K at $v \in \hat{V}$ is denoted by K_v.

Let X be a regular connected scheme of dimension 2, and let $\pi : X \to V$ be a proper morphism such that

(1.1a) the generic fibre of π, $X_K \overset{\mathrm{df}}{=} X \times_V \operatorname{spec} K$, is a smooth, geometrically connected curve over K, and

(1.1b) for all $v \in \hat{V}$, the curve $X_{K_v} \overset{\mathrm{df}}{=} X \times_V \operatorname{spec} K_v$ has index 1.

Recall that the index of a curve C over a field k is the greatest common divisor of the degrees of the fields k' over k such that C has a k'-rational point. Equivalently, it can be described as the least positive degree of a divisor on C.

Let A be the Jacobian variety of X_K. When π has a section, so that in particular X_K has index 1, M. Artin has shown (see [12, § 3]) that the Tate-Šafarevič group $Ш(A)$ of A is isomorphic to the group of elements in the Brauer group $\mathrm{Br}(X)$ of X becoming trivial on X_{K_v} for all $v \in V^{\infty}$; i. e., if we set

$$\mathrm{Br}(X)' = \mathrm{Ker}\,(\mathrm{Br}(X) \longrightarrow \bigoplus_{v \in V^{\infty}} \mathrm{Br}(X_{K_v}))$$

$$Ш(A) = \mathrm{Ker}\,(H^1(K, A) \longrightarrow \bigoplus_{v \in \hat{V}} H^1(K_v, A))$$

then $\mathrm{Br}(X)' \xrightarrow{\approx} Ш(A)$. The purpose of this paper is to extend this result to the case that X_K has index $\delta > 1$, and to draw some conclusions concerning the conjectures of Artin and Tate and of Birch and Swinnerton-Dyer.

THEOREM 1.2. *Assume, with the above notations, that* $Ш(A)$ *has no nonzero*

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infinitely divisible elements; then there is an exact sequence

$$0 \longrightarrow \mathrm{Br}\,(X)' \longrightarrow Ш(A)/T_1 \longrightarrow T_2 \longrightarrow 0$$

in which T_1 and T_2 are finite groups of orders

$$[T_1]=\delta$$

$$[T_2]=\delta/p^r,\ r\geqq 0,\ p=\text{characteristic exponent of } K.$$

COROLLARY 1.3. *If one of $Ш(A)$ or $\mathrm{Br}\,(X)'$ is finite, so is the other, and their orders are related by*

$$\delta^2[\mathrm{Br}\,(X)']=[Ш(A)] \qquad (\textit{number field case})$$

$$\delta^2[\mathrm{Br}\,(X)]=p^r[Ш(A)] \qquad (\textit{function field case}).$$

REMARK 1.4. Everything except the information on the order of T_2 can be found in [4, § 4]. Our proof of that part of the theorem uses in an essential way Tate's duality theorem for abelian varieties over global fields, and it is because the p-torsion part of Tate's theorem has not been proved in characteristic p (although the techniques seem to be available to do so) that we are unable to show that $[T_2]=\delta$ in this case.

REMARK 1.5. Under the hypotheses of the theorem, W. Gordon [3] has shown (in the function field case) that the conjecture of Artin and Tate for X [12, Conj. C] is equivalent to the conjecture of Birch and Swinnerton-Dyer for A [12, Conj. B] if and only if $\delta^2[\mathrm{Br}\,(X)]=[Ш(A)]$. Thus our theorem demonstrates the equivalence of the conjectures when $p \nmid \delta$ (and without restriction on δ if Tate's theorem is assumed). Cf. Conjecture (d) of [12].

REMARK 1.6. Let K be a function field and let A be an abelian variety over K arising, as above, from a map $\pi: X \to V$ for which $p \nmid \delta$ (or else assume Tate's theorem). On combining (1.5) with the main theorem of [7], we find that the following statements are equivalent:

(a) The L-series $L(A, s)$ of A has a zero $s=1$ of order equal to the rank of $A(K)$;

(b) for some prime l ($l=p$ is allowed), the l-primary component of $Ш(A)$ is finite;

(c) $Ш(A)$ is finite, and the conjecture of Birch and Swinnerton-Dyer is true for A.

REMARK 1.7. Let X and V be as above, and assume $\pi: X \to V$ is a proper map satisfying (1.1a).

(a) The conditions imply that π is surjective, and it follows that the fibres $X_v \overset{\mathrm{df}}{=} \pi^{-1}(v)$ of π all have dimension 1 and that π is flat [6, III Ex. 10.9]. Consequently $\pi_* O_X$ is a torsion-free O_V-module which (1.1a) shows to have rank 1; therefore $O_V \xrightarrow{\approx} \pi_* O_X$, and all X_v are connected [6, III, 11.3].

(b) For $v \in V^\circ$, let $X_v = \sum m_i C_i$ (as a divisor on X), and let $\Gamma(C_i, O_{C_i})$ have degree n_i over the residue field $k(v)$ at v. Then the index of X_{K_v} is g.c.d. $(m_i n_i)$ (see [3, 6.2]), and so (1.1b) implies that X has no multiple fibres and (because of [9, 7.2.1]) is cohomologically flat in dimension zero. Thus (in the function field case) the conditions (1.1) imply that (X, V, π) is a "fibration" in the sense of [3, 2.1].

§2. Proofs.

For the rest of the paper K and V will be as in the first paragraph, and $\pi: X \to V$ will satisfy the conditions (1.1). We shall also use the following notations: The separable algebraic closure of a field k is denoted by $\bar{k}$. For any regular connected scheme Y, $R(Y)$ denotes the field of rational functions on Y, and $\mathcal{R}$ the sheaf $U \mapsto R(U)$ for the étale topology on Y; also $\mathrm{Div}(Y)$ denotes the group of Weil divisors of Y, and $\mathcal{D}iv$ the sheaf $U \mapsto \mathrm{Div}(U)$. The set of points y of Y on dimension i (i.e., such that $\dim \overline{\{y\}} = i$) is denoted Y^i. We use Γ_v to denote $\mathrm{Gal}(\bar{K}_v/K_v)$ and Γ to denote $\mathrm{Gal}(\bar{K}/K)$ (although not exclusively). All cohomology groups will be with respect to the étale topology or will be Galois cohomology groups.

We begin by listing some exact sequences, several of which are well-known.

LEMMA 2.1. *Let Y be a regular scheme; then $H^1(Y, \mathcal{R}^\times) = 0$ and there is an exact sequence*

$$0 \longrightarrow H^2(Y, O_Y^\times) \longrightarrow H^2(Y, \mathcal{R}^\times) \longrightarrow \bigoplus_y H^2(y, \boldsymbol{Z})$$

where the sum is over the points y of Y of codimension 1.

PROOF. This follows from studying the cohomology sequence of

$$0 \longrightarrow O_Y^\times \longrightarrow \mathcal{R}^\times \longrightarrow \mathcal{D}iv \longrightarrow 0\ ;$$

see [8, III 2.22].

LEMMA 2.2. *Let C be a complete smooth curve over a field k, and let $\bar{C} = C \otimes_k \bar{k}$ and $\Gamma = \mathrm{Gal}(\bar{k}/k)$; then there are exact sequences*

(a) $0 \longrightarrow \mathrm{Br}(C) \longrightarrow H^2(\Gamma, R(\bar{C})^\times) \longrightarrow H^2(\Gamma, \mathrm{Div}(\bar{C}))$, *and*

(b) $0 \longrightarrow \mathrm{Pic}(C) \longrightarrow \mathrm{Pic}(\bar{C})^{\Gamma} \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(C) \longrightarrow H^1(\Gamma, \mathrm{Pic}(\bar{C}))$
$\longrightarrow H^3(\Gamma, \bar{k}^{\times})$.

PROOF. (a) On applying (2.1) to C and $\bar{C}$, we get the rows of the following diagram,

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Br}(C) \arrow[r] \arrow[d, "\alpha"] & H^2(C, \mathcal{R}^{\times}) \arrow[r] \arrow[d, "\beta"] & \bigoplus_{x \in C^0} H^2(x, \boldsymbol{Z}) \arrow[d, "\gamma"] \\ 0 \arrow[r] & \mathrm{Br}(\bar{C}) \arrow[r] & H^2(\bar{C}, \mathcal{R}^{\times}) \arrow[r] & \bigoplus_{x \in \bar{C}^0} H^2(x, \boldsymbol{Z}). \end{tikzcd}$$

As $\mathrm{Br}(\bar{C})=0$ [4, Cor. 5.8] and $H^2(x, \boldsymbol{Z})=0$ for $x \in \bar{C}^{\circ}$, we see that $\mathrm{Ker}(\alpha) =\mathrm{Br}(C)$ and $\mathrm{Ker}(\gamma)=\bigoplus H^2(x, \boldsymbol{Z})=\bigoplus H^2(\mathrm{Gal}(\overline{k(x)}/k(x)), \boldsymbol{Z})$. Shapiro's lemma shows that this last group equals $H^2(\Gamma, \mathrm{Div}(\bar{C}))$, and so it remains to show that $\mathrm{Ker}(\beta)=H^2(\Gamma, R(\bar{C})^{\times})$. But, as $H^1(\bar{C}, \mathcal{R}^{\times})=0$ the Hochschild-Serre spectral sequence for $\bar{C}/C$,

$$H^r(\Gamma, H^s(\bar{C}, \mathcal{R}^{\times})) \Rightarrow H^{r+s}(C, \mathcal{R}^{\times})$$

gives rise to an exact sequence,

$$0 \longrightarrow H^2(\Gamma, R(\bar{C})^{\times}) \longrightarrow H^2(C, \mathcal{R}^{\times}) \longrightarrow H^2(\bar{C}, \mathcal{R}^{\times}).$$

(b) Since $H^s(\bar{C}, O_{\bar{C}}^{\times})=0$ for $s \geqq 2$, the Hochschild-Serre spectral sequence

$$H^r(\Gamma, H^s(\bar{C}, O_{\bar{C}}^{\times})) \Rightarrow H^{r+s}(C, O_C^{\times})$$

immediately yields the required sequence.

REMARK 2.3. One can also construct the sequence in (2.2b) using the exact sequence of Γ-modules

$$0 \longrightarrow \bar{k}^{\times} \longrightarrow R(\bar{C})^{\times} \longrightarrow \mathrm{Div}(\bar{C}) \longrightarrow \mathrm{Pic}(\bar{C}) \longrightarrow 0.$$

On splitting this sequence into two short exact sequences and forming their cohomology sequences, we obtain the following diagram:

$$\begin{tikzcd} & & H^2(\Gamma, \mathrm{Div}(\bar{C})) & \\ \mathrm{Br}(k) \arrow[r] & H^2(\Gamma, R(\bar{C})^{\times}) \arrow[r] \arrow[ur] & H^2(\Gamma, R(\bar{C})^{\times}/\bar{k}^{\times}) \arrow[r] \arrow[u] & H^3(\Gamma, \bar{k}^{\times}) \\ & & H^1(\Gamma, \mathrm{Pic}(\bar{C})) \arrow[u] & \\ & & 0=H^1(\Gamma, \mathrm{Div}(\bar{C})) \arrow[u] & \end{tikzcd}$$

From this diagram and (2.2a) it is easy to derive most of (2.2b). This approach has the advantage that it gives us an explicit description of the map $\psi:\mathrm{Br}(C)\to H^1(\Gamma, \mathrm{Pic}(\bar{C}))$: let $\beta\in\mathrm{Br}(C)$, and let the image of β in $H^2(\Gamma, R(\bar{C})^\times)$ be represented by the 2-cocycle $b\in Z^2(\Gamma, R(\bar{C})^\times)$; the image (b) of b in $Z^2(\Gamma, \mathrm{Div}(\bar{C}))$ is a coboundary, say $(b)=\delta\mathfrak{b}$, $\mathfrak{b}\in C^1(\Gamma, \mathrm{Div}(\bar{C}))$; the image of $\mathfrak{b}$ in $C^1(\Gamma, \mathrm{Pic}(\bar{C}))$ is a cocycle, and represents $\psi(\beta)$.

LEMMA 2.4. *There is an exact sequence*

$$0 \longrightarrow \mathrm{Br}(K) \longrightarrow \bigoplus_{v\in\hat{V}} \mathrm{Br}(K_v) \longrightarrow \boldsymbol{Q}/\boldsymbol{Z} \longrightarrow 0$$

$$(\beta_v) \longmapsto \sum \mathrm{inv}_v(\beta_v)$$

PROOF. This statement is a major part of class field theory; see [2, VII. 11].

We now write P for $\underline{\mathrm{Pic}}_{X_K/K}$, the Picard scheme of X_K/K. The degree map $\deg: P(K)\to\boldsymbol{Z}$ has kernel A, and the image of $\deg: P(K)\to\boldsymbol{Z}$ is, by definition, $\delta'\boldsymbol{Z}$ where δ' is the period of X_K.

PROPOSITION 2.5. *The period of X_K equals its index, $\delta'=\delta$.*

PROOF. The degree map on $\mathrm{Div}(X_K)$ factors into

$$\mathrm{Div}(X_K) \longrightarrow P(K) \xrightarrow{\deg} \boldsymbol{Z},$$

and so $\delta\boldsymbol{Z}\subset\delta'\boldsymbol{Z}$; therefore $\delta'|\delta$.

As $\mathrm{Div}(X_K)\to\mathrm{Pic}(X_K)$ is surjective and $\mathrm{Pic}(X_{\bar{K}})^\Gamma=P(\bar{K})^\Gamma=P(K)$, we can extract from (2.2b) the following diagram

$$\begin{array}{ccccccc}
\mathrm{Div}(X_K) & \longrightarrow & P(K) & \longrightarrow & \mathrm{Br}(K) & & \\
\downarrow & & \downarrow & & \downarrow\cap & & \\
\bigoplus_v \mathrm{Div}(X_{K_v}) & \longrightarrow & \bigoplus_v P(K_v) & \xrightarrow{0} & \bigoplus_v \mathrm{Br}(K_v) & \longrightarrow & \bigoplus_v \mathrm{Br}(X_{K_v})
\end{array} \qquad (v\in\hat{V}).$$

The map $\mathrm{Br}(K_v)\to\mathrm{Br}(X_{K_v})$ is injective because X_{K_v} has index 1, and therefore $P(K_v)\to\mathrm{Br}(K_v)$ is the zero map. Moreover, (2.4) shows that $\mathrm{Br}(K)\to\bigoplus\mathrm{Br}(K_v)$ is injective, and an easy diagram chase shows that $\mathrm{Div}(X_K)\to P(K)$ is surjective; therefore $\delta=\delta'$.

LEMMA 2.6. *There is an exact sequence,*

$$0 \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(X_K) \longrightarrow \bigoplus_{v\in V^\circ} \mathrm{Br}(X_{K_v}).$$

PROOF. Consider $X_K\overset{h}{\hookrightarrow}X$. I claim that $R^r h_*\boldsymbol{G}_m=0$ for $r>0$. For this it suffices to show that, for any geometric point $\bar{x}\to X$ with image $x\notin X_K$, the stalk $(R^r h_* \boldsymbol{G}_m)_{\bar{x}}=0$, $r>0$. Let B be the strictly local ring at $\bar{x}$ and let $t\in B$ be a local uniformizing parameter at $v=\pi(x)$. Then $\operatorname{spec} B\times_X X_K=\operatorname{spec} B\otimes_{O_v}K$ $=\operatorname{spec} B[t^{-1}]$, and so

$$(R^r h_* \boldsymbol{G}_m)_{\bar{x}}=H^r(\operatorname{spec} B[t^{-1}], \boldsymbol{G}_m).$$

The groups $H^r(\operatorname{spec} B[t^{-1}], \boldsymbol{G}_m)$ are torsion for $r>0$, and the purity theorem in étale cohomology shows that they are zero except possibly for p-torsion in characteristic p [4, §6]. The possibility of p-torsion can be eliminated by making a direct calculation using the second exact sequence of [8, p 129]. (The reader should ignore this p case, since it plays no role in the final result.)

It follows that $H^r(X, h_*\boldsymbol{G}_m)\xrightarrow{\approx}H^r(X_K, \boldsymbol{G}_m)$ for all r, and so the exact sequence

$$0\longrightarrow \boldsymbol{G}_m\longrightarrow h_*\boldsymbol{G}_m\longrightarrow \bigoplus_{x\in X^1,\ \pi(x)\in V^0}\boldsymbol{Z}\longrightarrow 0$$

gives rise to the first row of

$$\begin{array}{ccccccc}
0\longrightarrow \operatorname{Br}(X) & \longrightarrow & \operatorname{Br}(X_K) & \longrightarrow & \bigoplus_{\pi(x)\in V^0} H^2(x, \boldsymbol{Z}) & & (x\in X^1)\\
\downarrow & & \downarrow & & \downarrow & & \\
0\longrightarrow \operatorname{Br}(X_{R_v}) & \longrightarrow & \operatorname{Br}(X_{K_v}) & \longrightarrow & \bigoplus_{\pi(x)=v} H^2(x, \boldsymbol{Z}) & & (x\in X^1).
\end{array}$$

The second row is constructed the same way as the first starting from $X_{R_v}\to\operatorname{spec} R_v$, where R_v is the ring of integers in K_v for some $v\in V^\circ$. Artin's theorem [4, 3.1] shows that $\operatorname{Br}(X_{R_v})=\operatorname{Br}(C)$ where C is the reduced curve over $k(v)$ associated with X_v. When C is smooth, it is well-known that $\operatorname{Br}(C)=0$. Otherwise we consider its normalization $f:\tilde{C}\to C$. There is an exact sequence of sheaves on C_{et}

$$0\longrightarrow O_C^\times\longrightarrow f_*O_{\tilde{C}}^\times\longrightarrow \bigoplus i_{c*}G_c\longrightarrow 0$$

in which the sum is over the singular points of C and G_c is a connected algebraic group on c. Since $H^r(c, G_c)=0$ for $r>0$, we see that again

$$\operatorname{Br}(C)\xrightarrow{\approx}H^2(C, f_*O_{\tilde{C}}^\times)\xrightarrow{\approx}\operatorname{Br}(\tilde{C})=0.$$

Thus $\operatorname{Br}(X_{R_v})=0$, and so an element $\beta\in\operatorname{Br}(X)$ maps to zero in $\operatorname{Br}(X_{K_v})$ if and only if it maps to zero in $H^2(x, \boldsymbol{Z})$ for all $x\in X^1$ with $\pi(x)=v$; we conclude that $\operatorname{Br}(X)=\operatorname{Ker}(\operatorname{Br}(X_K)\to\bigoplus\operatorname{Br}(X_{K_v}))$.

Define

$$Ш(P)=\operatorname{Ker}(H^1(K, P)\longrightarrow \bigoplus_{v\in V} H^1(K_v, P))$$

PROPOSITION 2.7. *There is an exact sequence*

$$0 \longrightarrow \mathrm{Br}(X)' \longrightarrow Ш(P) \xrightarrow{\phi} \boldsymbol{Q}/\boldsymbol{Z}\,.$$

PROOF. On applying (2.2b) to X_K and X_{K_v}, we obtain a diagram

$$\begin{array}{ccccccc} \mathrm{Br}(K) & \longrightarrow & \mathrm{Br}(X_K) & \longrightarrow & H^1(\Gamma, P(\bar{K})) & \longrightarrow 0 & \\ \downarrow & & \downarrow & & \downarrow & & (2.7.1) \\ 0 \longrightarrow \bigoplus_v \mathrm{Br}(K_v) & \longrightarrow & \bigoplus_v \mathrm{Br}(X_{K_v}) & \longrightarrow & \bigoplus_v H^1(\Gamma_v, P(\bar{K}_v)) & & (v \in \hat{V})\,. \end{array}$$

The zero at upper-right comes from the fact that $H^3(\Gamma, \bar{K}^\times)=0$ ([2, VII. 11.4], [10, II. 4]), and the zero at lower-left is a consequence of our assumption (1.1b). On applying the serpent lemma, we get an exact sequence

$$0 \longrightarrow \mathrm{Br}(X)' \longrightarrow Ш(P) \longrightarrow Q$$

where $Q \overset{\mathrm{df}}{=} \mathrm{Coker}\,(\mathrm{Br}(K) \to \bigoplus \mathrm{Br}(K_v)) \approx \boldsymbol{Q}/\boldsymbol{Z}$.

REMARK 2.8. A section to π splits the rows in (2.7.1), and so in this case $\mathrm{Br}(X)' \xrightarrow{\approx} Ш(P)$; moreover $P=A\oplus \boldsymbol{Z}$, and so $Ш(A)=Ш(P)$: we have recovered Artin's theorem that $\mathrm{Br}(X)' \xrightarrow{\approx} Ш(A)$.

REMARK 2.9. Using (2.3), we can give an explicit description of the map ϕ. Write S for the map $\mathrm{Div}(\bar{X}) \to P(\bar{K})$. Represent $\alpha \in Ш(P)$ by a cocycle $a \in Z^1(\Gamma, P(\bar{K}))$, and let $\mathfrak{a} \in C^1(\Gamma, \mathrm{Div}(X_{\bar{K}}))$ be such that $S(\mathfrak{a})=a$. Then $\delta(\mathfrak{a}) \in Z^2(\Gamma, R(X_{\bar{K}})^\times/\bar{K}^\times)$ and, because $H^3(\Gamma, \bar{K}^\times)=0$, it can be lifted to an element $f \in Z^2(\Gamma, R(X_{\bar{K}})^\times)$. On the other hand, $a=\delta(a_v)$ with $a_v \in C^0(\Gamma_v, P(\bar{K}_v))$; let $\mathfrak{a}_v$ be such that $S(\mathfrak{a}_v)=a_v$; then $\mathfrak{a}=\delta\mathfrak{a}_v+(f_v)$ with $f_v \in C^1(\Gamma, R(X_{\bar{K}_v})^\times)$, and $f/\delta f_v \in Z^2(\Gamma, \bar{K}_v^\times)$. Let γ_v be the class of $f/\delta f_v$ in $\mathrm{Br}(K_v)$; then $\phi(\alpha)=\sum \mathrm{inv}_v(\gamma_v)$.

Note that if $\mathfrak{c}$ is any divisor of degree 1 on X_{K_v} such that neither f nor δf_v has a zero or a pole in the support of $\mathfrak{c}$, then $f(\mathfrak{c})/\delta f_v(\mathfrak{c})=f/\delta f_v$. As $\delta f_v(\mathfrak{c}) =\delta(f_v(\mathfrak{c}))$ with $f_v(\mathfrak{c}) \in C^1(\Gamma_v, \bar{K}_v^\times)$, we see that γ_v is represented by $f(\mathfrak{c})$.

We next define a canonical pairing

$$\langle\ ,\ \rangle : Ш(A) \times Ш(A) \longrightarrow \boldsymbol{Q}/\boldsymbol{Z}\,.$$

Let $\alpha \in Ш(A)$ be represented by $a \in Z^1(\Gamma, A(\bar{K}))$, and let $a=\delta a_v$ with $a_v \in Z^0(\Gamma, A(\bar{K}_v))$. Write

$$a=S(\mathfrak{a})\,, \qquad \mathfrak{a} \in C^1(\Gamma, \mathrm{Div}^0(X_{\bar{K}}))$$

$$a_v=S(\mathfrak{a}_v)\,, \qquad \mathfrak{a}_v \in C^0(\Gamma_v, \mathrm{Div}^0(X_{\bar{K}_v}))\,.$$

Then $\mathfrak{a}=\delta\mathfrak{a}_v+(f_v)$ in $C^1(\Gamma_v, \mathrm{Div}^0(X_{\bar{K}_v}))$ with $f_v \in C^1(\Gamma_v, R(X_{\bar{K}_v})^\times)$. Moreover, $\delta\mathfrak{a} =(f)$, $f \in Z^2(\Gamma, R(X_{\bar{K}})^\times)$. Let β be a second element of $Ш(A)$ and define $\mathfrak{b}$, $\mathfrak{b}_v$, g_v, g as for α. Set

$$\langle \alpha, \beta \rangle = \sum \mathrm{inv}_v(\gamma_v), \qquad \gamma_v = \text{class of } g_v \cup \mathfrak{a} - \mathfrak{b}_v \cup f,$$

where $\cup$ denotes the cup-product pairing induced by $(f, \mathfrak{a}) \mapsto f(\mathfrak{a})$, $f \in R(X_{\bar{K}})^{\times}$, $\mathfrak{a} \in \mathrm{Div}(X_{\bar{K}})$. One shows without serious difficulty that f, $\mathfrak{b}_v$, g_v, and $\mathfrak{a}$ can be chosen so that $f(\mathfrak{b}_v)$ and $g_v(\mathfrak{a})$ are defined, that $\langle \alpha, \beta \rangle$ is independent of the choices, and that $\langle \beta, \alpha \rangle = -\langle \alpha, \beta \rangle$.

THEOREM 2.10. *The pairing*

$$\langle\ ,\ \rangle : Ш(A) \times Ш(A) \longrightarrow \boldsymbol{Q}/\boldsymbol{Z}$$

annihilates only the divisible part of $Ш(A)$ (*except possibly on the p-primary component in characteristic p*).

PROOF. A proof in the one-dimensional case can be found in [1]. As Tate explains in [11], the general case can be proved similarly, once one knows [11, 3.1]; a proof of this last result can be found in [5].

Consider the diagram

$$\begin{array}{ccccccccc}
P(K) & \xrightarrow{\deg} & \boldsymbol{Z} & \longrightarrow & H^1(K, A) & \longrightarrow & H^1(K, P) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\oplus_v P(K_v) & \xrightarrow{\deg} & \oplus_v \boldsymbol{Z} & \longrightarrow & \oplus_v H^1(K_v, A) & \longrightarrow & \oplus_v H^1(K_v, P) & \longrightarrow & 0 \qquad (v \in \hat{V})
\end{array}$$

in which the rows are the cohomology sequences of

$$0 \longrightarrow A \longrightarrow P \longrightarrow \boldsymbol{Z} \longrightarrow 0$$

over K and K_v. By assumption $P(K_v) \to \boldsymbol{Z}$ is surjective, and so the serpent lemma provides us with an exact sequence

$$0 \longrightarrow \boldsymbol{Z}/\delta'\boldsymbol{Z} \longrightarrow Ш(A) \xrightarrow{\rho} Ш(P) \longrightarrow 0.$$

Let T_1 be the image of $\boldsymbol{Z}/\delta'\boldsymbol{Z}$ in $Ш(A)$, and let T_2 be the image of the map $\phi : Ш(P) \to \boldsymbol{Q}/\boldsymbol{Z}$ in (2.7). Then we have an exact sequence,

$$0 \longrightarrow \mathrm{Br}(X)' \longrightarrow Ш(A)/T_1 \longrightarrow T_2 \longrightarrow 0.$$

According to (2.5), T_1 has order δ, and the next lemma (together with (2.10)) shows that T_2 has the order asserted by theorem.

LEMMA 2.11. *Let β be a generator of $T_1 \subset Ш(A)$. Then the composite*

$$Ш(A) \overset{\rho}{\twoheadrightarrow} Ш(P) \xrightarrow{\phi} \boldsymbol{Q}/\boldsymbol{Z}$$

is $\alpha \mapsto \langle \alpha, \beta \rangle$.

PROOF. Let $\alpha \in Ш(A)$ and define $\mathfrak{a}$, $\mathfrak{a}_v$, f_v, and f as in the discussion preceding (2.10); then (2.9) shows that $\phi(\rho(\alpha)) = \sum \mathrm{inv}_v(\gamma_v)$ where γ_v is represented by $f(\mathfrak{c}_v)$ for any divisor $\mathfrak{c}_v$ of degree 1 on X_{K_v}.

On the other hand, we can choose $\beta \in Ш(A)$ to be represented by $b = S(\mathfrak{b})$ where $\mathfrak{b} = \delta P$, P any point (prime divisor) on $X_{\bar{K}}$. Moreover, we can choose $\mathfrak{b}_v = P - \mathfrak{c}_v$ with $\mathfrak{c}_v$ as above; this forces $g_v = 0$. Therefore $\langle \alpha, \beta \rangle = -\sum \mathrm{inv}_v(\gamma'_v)$ where γ'_v is represented by $f(P - \mathfrak{c}_v) = f(P)/f(\mathfrak{c}_v)$. Let γ be the class of $f(P)$ in $\mathrm{Br}(K)$. Then $\langle \alpha, \beta \rangle = -\sum \mathrm{inv}_v(\gamma'_v) = -\sum \mathrm{inv}_v(\gamma/\gamma_v) = \sum \mathrm{inv}_v(\gamma_v) - \sum \mathrm{inv}_v(\gamma) = \phi(\rho(\alpha))$ because $\sum \mathrm{inv}_v(\gamma) = 0$.

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