Spectral and scattering theory for Schrödinger operators with Stark-effect, II

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§ 1. Introduction, assumption and theorems.

The purpose of this article is to make some remarks on the author's previous paper [18], which we refer to as [I] hereafter. In [I] we studied some spectral properties and scattering theory for Schrödinger operator H^E of the form

$$H^E=-\frac{1}{2m}\Delta+eE\cdot x+V(x), \tag{1.1}$$

which is a model Hamiltonian for the Stark-effect. Here $\Delta=\left(\frac{\partial}{\partial x_1}\right)^2+\cdots+\left(\frac{\partial}{\partial x_n}\right)^2$, m (mass) and e (charge) are positive constants, $\boldsymbol{E}\in\boldsymbol{R}^n$ (electric field), and $e\boldsymbol{E}\cdot x$, $V(x)$ are multiplication operators ($V(x)$ is a real-valued function). We choose the units and coordinates such that $m=\frac{1}{2}$, $e=1$ and $\boldsymbol{E}=(\varepsilon, 0, \cdots, 0)$, $\varepsilon>0$, and we write as $H^E=H^\varepsilon=-\Delta+\varepsilon x_1+V(x)$. In [I], we considered the operator H^ε as a perturbed operator of $H_0^\varepsilon=-\Delta+\varepsilon x_1$ in the Hilbert space $\mathcal{H}=L^2(\boldsymbol{R}^n)$, and proved that, roughly speaking, if $V(x)=o(|x_1|^{-1/2-\sigma})$, $\sigma>0$, as $x_1\to-\infty$ and $V(x)=o(x_1)$ as $x_1\to\infty$, then i) the spectrum $\sigma(H^\varepsilon)$ of H^ε fills up $\boldsymbol{R}^1$ and consists of absolutely continuous part $\sigma_{ac}(H^\varepsilon)$ and point spectrum $\sigma_p(H^\varepsilon)$; ii) wave operators $W_\pm^\varepsilon=\operatorname*{s-lim}_{t\to\pm\infty} e^{itH^\varepsilon}e^{-itH_0^\varepsilon}$ exist and are complete: $R(W_\pm^\varepsilon)=\mathcal{H}_{ac}(H^\varepsilon)=$the absolutely continuous subspace of $\mathcal{H}$ w.r.t. H^ε; iii) the absolutely continuous part H_{ac}^ε of H^ε is unitarily equivalent to H_0^ε via the wave operators. Therefore the scattering operator $S^\varepsilon=W_+^{\varepsilon*}W_-^\varepsilon$ is a unitary operator on $\mathcal{H}$ and commutes with H_0^ε. Then by a spectral representation theorem, H_0^ε and S^ε are simultaneously diagonalizable: $H_0^\varepsilon=\{\lambda I\,;\,\lambda\in\boldsymbol{R}^1\}$ and $S^\varepsilon=\{S^\varepsilon(\lambda)\,;\,\lambda\in\boldsymbol{R}^1\}$, where I is the identity operator and $S^\varepsilon(\lambda)$ is an operator on an accessory space $\boldsymbol{h}=L^2(\boldsymbol{R}^{n-1})$ (see (2.3) and (3.1)). $S^\varepsilon(\lambda)$ is called scattering matrix.

Here, continuing the study of the operator H^ε, we shall discuss the following two problems. (We assume $n\geqq 3$ hereafter).

a) The analyticity of the scattering matrix $S^\varepsilon(\lambda)$ w.r.t. λ (holding $\varepsilon>0$

*) Partially supported by Fûjû-kai and Sakko-kai Foundations.

fixed).

b) The asymptotic behaviour of scattering operator S^ε as the coupling constant $\varepsilon>0$ approaches zero.

These two problems will be treated separately, under different conditions. However, the asymptotic behaviour of the poles of $S^\varepsilon(\lambda)$ as $\varepsilon\to 0$ connects these two problems. To state our results we prepare the notation and conventions. $\boldsymbol{C}$ is the complex plane and $\boldsymbol{C}_\pm=\{z\in\boldsymbol{C}: \operatorname{Im} z\gtrless 0\}$. For $1\leqq p\leqq\infty$, $L^p(\boldsymbol{R}^n)$ is the Banach space of all p-summable functions equipped with the norm $\|\ \|_{L^p}$:

$$\|f\|_{L^p}=\left(\int |f(x)|^p dx\right)^{1/p}.$$

$\mathcal{S}(\boldsymbol{R}^n)$ is the space of all rapidly decreasing functions and $\mathcal{S}'(\boldsymbol{R}^n)$ tempered distributions. For $f\in\mathcal{S}'$, $\mathcal{F}f=\hat{f}$ stands for its Fourier transform. For s, $\rho\in\boldsymbol{R}^1$, $H^s_\rho(\boldsymbol{R}^n)$ is the weighted Sobolev-space:

$$H^s_\rho(\boldsymbol{R}^n)=\{f\in\mathcal{S}': \|(1+\xi^2)^{s/2}\widehat{(1+x^2)^{\rho/2}f}(\xi)\|_{L^2}=\|f\|_{H^s_\rho}<\infty\}.$$

$H^s(\boldsymbol{R}^n)=H^s_0(\boldsymbol{R}^n)$ and $L^2_\rho(\boldsymbol{R}^n)=H^0_\rho(\boldsymbol{R}^n)$. $H^s_\rho(\boldsymbol{R}^n)$ is a Hilbert space and the dual space of $H^s_\rho(\boldsymbol{R}^n)$ is $H^{-s}_{-\rho}(\boldsymbol{R}^n)$. Under our conditions H^ε is selfadjoint and we write as

$$R^\varepsilon(z)=(H^\varepsilon-z)^{-1},\quad R^\varepsilon_0(z)=(H^\varepsilon_0-z)^{-1},\quad R_0(z)=R^0_0(z)\quad\text{and}\quad R(z)=R^0(z).$$

Spectral measures of H^ε, H^ε_0, $H=H^0$, and $H_0=H^0_0$ are denoted respectively by $E^\varepsilon(d\lambda)$, $E^\varepsilon_0(d\lambda)$, $E(d\lambda)$, and $E_0(d\lambda)$. $\sigma_p(H)$ is the point spectrum of H. For Banach spaces X and Y, $\boldsymbol{B}(X, Y)$ is the space of all bounded linear operators from X to Y, and $\boldsymbol{B}_\infty(X, Y)$ all compact operators. $\boldsymbol{B}(X)=\boldsymbol{B}(X, X)$ and $\boldsymbol{B}_\infty(X)=\boldsymbol{B}_\infty(X, X)$. X^*, Y^* are the dual spaces of X and Y, and for $T\in\boldsymbol{B}(X, Y)$, $T^*\in\boldsymbol{B}(Y^*, X^*)$ is the adjoint operator of T. In addition to the usual weighted L^2-spaces $L^2_s(\boldsymbol{R}^n)$, we use one-sided exponentially weighted L^2-spaces $L^2_{e(b)}(\boldsymbol{R}^n)$, $b\in\boldsymbol{R}^1$:

$$L^2_{e(b)}(\boldsymbol{R}^n)=\{f\in L^2_{\mathrm{loc}}(\boldsymbol{R}^n): \|(\chi(x_1)+e^{-bx_1}\chi(-x_1))f\|_{L^2}=\|f\|_{L^2_{e(b)}}<\infty\},$$

where $\chi(x_1)\in C^\infty(\boldsymbol{R}^1)$ is such that $\chi(x_1)=1$ for $x_1>-1$, and $\chi(x_1)=0$ for $x_1<-2$. Theorems, formulas and etc, of [I] are referred to as Theorem 1.1.I, Lemma 2.2.I and etc.

Our Theorems read as follows.

THEOREM 1. *Suppose that* $V(x)$ *satisfies the following condition* (EX): *There exist a constant* $a>0$ *and two real-valued functions* $V_1(x)$ *and* $V_2(x)$ *such that*

a) $V(x)=(e^{ax_1}\chi(-x_1)+\chi(x_1))(V_1(x)+V_2(x))$,

b) $V_1(x)\in L^\infty(\boldsymbol{R}^n)$ *and* $\lim_{|x|\to\infty} V_1(x)=0$,

c) $(1+|x|)^{\gamma}V_2(x)\in L^p(\boldsymbol{R}^n)$ *for some* $p>\frac{n}{2}$ ($p\geqq 2$ *if* $n=3$) *and* $\gamma>1$.

Then the following statements hold.

i) *For any b satisfying* $0<b<a/2$, $R^{\varepsilon}(z)$ ($z\in \boldsymbol{C}_{\pm}$) *can be extended to* $\boldsymbol{C}$ *as a* $\boldsymbol{B}(L^2_{e(b)}(\boldsymbol{R}^n), L^2_{e(-b)}(\boldsymbol{R}^n))$*-valued meromorphic function of* $z\in\boldsymbol{C}$. *We write the function extended from* $\boldsymbol{C}_{\pm}$ *as* $R^{\varepsilon}_{\pm}(z)$.

ii) $S^{\varepsilon}(\lambda)$ *and* $S^{\varepsilon}(\lambda)^{-1}$ *can be expressed as a* $\boldsymbol{B}(L^2(\boldsymbol{R}^{n-1}))$*-valued function of* $\lambda\in\boldsymbol{R}^1$ *and they can be extended to* $\boldsymbol{C}$ *as* $\boldsymbol{B}(L^2(\boldsymbol{R}^{n-1}))$*-valued meromorphic functions* $S^{\varepsilon}(z)$ *and* $S^{\varepsilon}(z)^{-1}$ *of* $z\in\boldsymbol{C}$.

iii) *The non-real poles of* $S^{\varepsilon}(z)$ *and* $S^{\varepsilon}(z)^{-1}$ *are the same as those of* $R^{\varepsilon}_{+}(z)$ *and* $R^{\varepsilon}_{-}(z)$, *respectively.*

THEOREM 2. *Suppose that* $V(x)$ *satisfies the following condition* (VSR): *There exist a constant* $\delta>1$ *and two real-valued functions* $V_1(x)$ *and* $V_2(x)$ *such that*

a) $V(x)=(1+x^2)^{-\delta}(V_1(x)+V_2(x))$,

b) $V_1(x)\in L^{\infty}(\boldsymbol{R}^n)$ *and* $V_2(x)\in L^p(\boldsymbol{R}^n)$ *for some* $p>\frac{n}{2}$ ($p\geqq 2$ *if* $n=3$).

Then the scattering operator S^{ε} *converges strongly to the scattering operator* S *associated with* H *and* H_0 *as* ε *approaches zero*: $\operatorname{s-lim}_{\varepsilon\downarrow 0} S^{\varepsilon}=S$.

THEOREM 3. *Suppose that* $V(x)$ *satisfy both conditions* (EX) *and* (VSR) *and that* $\sigma_p(H^{\varepsilon})=\emptyset$ *for all* $\varepsilon>0$. *Then the following statements hold.*

i) *For any negative eigenvalue* μ *of* H, *there exists a neighborhood* U *of* μ *such that for* ε *sufficiently small there are exactly (counting multiplicities)* $m(\mu)$ *poles of* $S^{\varepsilon}(z)$ (*or* $S^{\varepsilon}(z)^{-1}$) *in* U, *and all of these poles converge to* μ *as* ε *tends to zero, where* $m(\mu)$ *is the multiplicity of the eigenvalue* μ.

ii) *There occurs the spectral concentration at* μ. (For the concept of spectral concentration, see Howland [6], Theorem 2.1.)

iii) *If* μ *is a simple eigenvalue, then the location of the pole of* $S^{\varepsilon}(z)$ *or* $S^{\varepsilon}(z)^{-1}$ *which approaches* μ *as* ε *tends to zero is asymptotically described by the Rayleigh-Schrödinger series.*

REMARK 1.1. The assumptions (EX) and (VSR) are stronger than the assumption (A) of [I]. Hence $V(x)$ is H^{ε}_0-compact; the wave operators $W^{\varepsilon}_{\pm}$ exist and are complete.

REMARK 1.2. Under an additional smoothness condition to (VSR), Avron and Herbst [1] proved $\sigma_p(H^{\varepsilon})=\emptyset$.

Topics related to these results have been studied by several authors. For one-dimensional case, Titchmarsh [15] proved the analyticity of the resolvent $R^{\varepsilon}(z)$ under fairly general conditions. For more than 2-dimensional case, Herbst [4] studied the analyticity of $R^{\varepsilon}(z)$, using dilation-analyticity machinaries (see also Herbst-Simon [5]). Topics related to Theorem 3 have been studied in conjunction with the phenomena of spectral concentration by Conley and Rejto [2] or Kato [10], and an abstract theory for it has been developed by Howland [6], [7] and [8]. However, unfortunately, the analyticity of the scattering matrix associated with this concrete operator, Stark-effect Hamiltonian, has not been proved so far and therefore the relation of the poles of $S^{\varepsilon}(z)$ and the so-called "resonance" energies has not been completely settled. The problem about the convergence of S^{ε} to the "switched off" scattering operator S has been completely open. So, in spite of the fact that our conditions on potential are too restrictive to accommodate the Coulomb potential, we hope the present note still has a little interest.

This work was done while the author was on leave from the University of Tokyo and visiting the University of Virginia. The author would like to express his sincere thanks to the Department of Mathematics of both universities. He thanks Dr. A. Jensen and the referee who pointed out errors in the original manuscript.

§ 2. Lemmas.

In this section we present several preliminary lemmas which are necessary to prove the theorems. We write $G(p)=(1/3)p_1^3+p_1(p_2^2+\cdots+p_n^2)$ for $p=(p_1, \cdots, p_n)\in \boldsymbol{R}^n$. We define integral operators U^{ε} and $U^{\varepsilon*}$, $\varepsilon>0$ as follows:

$$(U^{\varepsilon}f)(x)=(2\pi\varepsilon)^{-n/2}\int_{R^n} e^{i(x\cdot p-G(p))/\varepsilon}\hat{f}(p)dp\,, \tag{2.1}$$

$$(U^{\varepsilon*}f)(x)=(2\pi\varepsilon)^{-n/2}\int_{R^n} e^{i(x\cdot p+G(p))/\varepsilon}\hat{f}(p)dp\,. \tag{2.2}$$

As we showed in Theorem 2.4.I, U^{ε} is a unitary operator in $\mathcal{H}$ with $U^{\varepsilon*}=(U^{\varepsilon})^{-1}$ and

$$H_0^{\varepsilon}=(U^{\varepsilon})^{*}x_1U^{\varepsilon}\,. \tag{2.3}$$

We define an operator-valued function $T_0^{\varepsilon}(\lambda)$ from $\mathcal{H}$ to $\boldsymbol{h}=L^2(\boldsymbol{R}^{n-1})$ of $\lambda\in\boldsymbol{R}^1$ as

$$T_0^{\varepsilon}(\lambda)f(y)=(2\pi\varepsilon)^{-n/2}\int_{R^n} e^{i(\lambda\cdot p_1+y\cdot p'-G(p))/\varepsilon}\hat{f}(p)dp\,,$$

where $p'=(p_2, \cdots, p_n)$ and $y\cdot p'=\sum_{j=1}^{n-1} y_jp_{j+1}$. By the unitarity of U^{ε}'s and the relation (2.3), it is obvious that $\{T_0^\varepsilon(\lambda), \boldsymbol{h}, -\infty<\lambda<\infty\}$ gives a spectral representation of H_0^ε, that is, i) for any $f\in\mathcal{H}$, $T_0^\varepsilon(\lambda)f$ is an $\boldsymbol{h}$-valued square integrable function on $\boldsymbol{R}^1$, and ii) for any Borel set $I\subset\boldsymbol{R}^1$, $T_0^\varepsilon(\lambda)[E_0^\varepsilon(I)f]=\chi_I(\lambda)T_0^\varepsilon(\lambda)f$ for a.e. $\lambda\in\boldsymbol{R}^1$, where $\chi_I(\lambda)$ is the characteristic function of I.

We first assume $V(x)$ to satisfy the condition (EX). We write as $A(x)=|V(x)|^{1/2}$ and $B(x)=|V(x)|^{1/2}\cdot\operatorname{sgn} V(x)$. A and B are the multiplication operator by $A(x)$ and $B(x)$.

LEMMA 2.1. *Suppose $V(x)$ satisfies* (EX). *For each $\lambda\in\boldsymbol{R}^1$, $T_0^\varepsilon(\lambda, A)=T_0^\varepsilon(\lambda)A$ is a bounded operator from $\mathcal{H}$ to $\boldsymbol{h}$. Moreover $\boldsymbol{B}(\mathcal{H}, \boldsymbol{h})$-valued function $T_0^\varepsilon(\lambda, A)$ of $\lambda\in\boldsymbol{R}^1$ can be extended to $\boldsymbol{C}$ as an entire analytic function. The same statement holds true for $T_0^\varepsilon(\lambda, B)$.*

PROOF. This lemmas is essentially proved in [1]. First of all by Sobolev's embedding theorem, $A(x)f(x)$ can be written as

$$A(x)f(x)=(e^{x_1\cdot a/2}\chi(-x_1)+\chi(x_1))g(x)\equiv g_1(x) \tag{2.4}$$

where $g\in H^{-1}(R^n)$ and $\|g\|_{H^{-1}}\leqq C\|f\|$, C is independent of $f\in\mathcal{H}$. For $g\in C_0^\infty(\boldsymbol{R}^n)$,

$$\hat{g}_1(z)=(2\pi)^{-n/2}\int e^{-iz\cdot x}(e^{x_1\cdot a/2}\chi(-x_1)+\chi(x_1))g(x)dx$$

is an entire function of z decaying rapidly to all real direction. If $e_1=(1, 0, \cdots, 0)\in\boldsymbol{R}^1$ and $-a/2<q_1<0$,

$$\begin{aligned}\|(1+|p|)^{-1}\hat{g}_1(p+iq_1e_1)\|_{L^2(R_p^n)}&\leqq\|(1+|p|)^{-1}\mathcal{F}(e^{x_1(a+2q_1)/2}\chi(-x_1)g)(p)\|\\&+\|(1+|p|)^{-1}\mathcal{F}(e^{q_1\cdot x_1}\chi(x_1)g)(p)\|\leqq C\|g\|_{-1}\end{aligned}$$

since the multiplication by a C^∞-function with bounded derivatives is a bounded operator in $H^s(\boldsymbol{R}^n)$ for any $s\in\boldsymbol{R}^1$. Hence for $g\in C_0^\infty(\boldsymbol{R}^n)$, we get

$$\begin{aligned}(T_0^\varepsilon(\lambda)g_1)(y)&=e^{-\varepsilon^{-1}(q_1^3/3+\lambda\cdot q_1)}(2\pi\varepsilon)^{-n/2}\\&\times\int e^{i\varepsilon^{-1}(\lambda p_1+y\cdot p'-(p_1^3-3p_1p_1^2+3p_1p'^2)/3)+\varepsilon^{-1}q_1\cdot p_1^2}\hat{g}_1(p+iq_1e_1)dp\,,\end{aligned} \tag{2.5$_1$}$$

by changing the region of integration $\boldsymbol{R}^n$ to $\boldsymbol{R}^n+iq_1e_1$, $-a/2<q_1<0$. Therefore it is obvious that $T_0^\varepsilon(\lambda)g_1$ is the restriction onto the real line of $\boldsymbol{h}$-valued entire analytic function $T_0^\varepsilon(z)g_1$. Furthermore, by Minkowski's inequality, Parseval relation and Schwarz' inequality we get

$$\begin{aligned}\|T_0^\varepsilon(z)g_1\|&\leqq\exp(-\varepsilon^{-1}(q_1^3/3+\operatorname{Re} z\cdot q_1))\cdot(2\pi\varepsilon)^{-1/2}\int_{-\infty}^{\infty}dp_1e^{\varepsilon^{-1}(q_1p_1^2-\operatorname{Im} z\cdot p_1)}\\&\times(1+|p_1|)\end{aligned} \tag{2.5$_2$}$$

$$\times\|(2\pi\varepsilon)^{-(n-1)/2}\int_{R^{n-1}} e^{i\varepsilon^{-1}y\cdot p'} e^{-i\varepsilon^{-1}(p_1^3-3p_1q_1^2+3p_1p'^2)/3+\varepsilon^{-1}q_1\cdot p'^2}$$

$$\times(1+|p_1|)^{-1}\hat{g}_1(p+iq_1e_1)dp'\|$$

$$\leqq \exp(-\varepsilon^{-1}(q_1^3/3+\mathrm{Re}\, z\cdot q_1))(2\pi\varepsilon)^{-1/2}\Big(\int_{-\infty}^{\infty}(1+|p_1|)^2 e^{2\varepsilon^{-1}(q_1p_1^2-\mathrm{Im}\, z\cdot p_1)}dp_1\Big)^{1/2}$$

$$\times\Big(\int e^{2\varepsilon^{-1}q_1\cdot p'^2}(1+|p_1|)^{-2}|\hat{g}_1(p+iq_1e_1)|^2dp\Big)^{1/2}$$

$$\leqq C(\varepsilon,\, q_1,\, \zeta)\|g\|_{H^{-1}},$$

where $C(\varepsilon, q_1, \zeta)$ is a constant depending only on ε, q_1, ζ and this constant can be taken uniformly when ζ varies on a compact subset of the complex plane $\boldsymbol{C}$. Therefore the standard limiting procedure and Weierstass' theorem imply the statement of the lemma. (Q.E.D.)

REMARK 2.2. The proof of the lemma shows actually $T_0^\varepsilon(z)(e^{bx_1}\chi(-x_1)+\chi(x_1))\in \boldsymbol{B}(H^{-s}(\boldsymbol{R}^n),\, L^2(\boldsymbol{R}^{n-1}))$ for any $b>0$ and $s\in\boldsymbol{R}^1$ and $z\in\boldsymbol{C}$.

COROLLARY 2.3. *If $V(x)$ satisfies condition* (EX), *$AR_0^\varepsilon(z)B$, $z\in\boldsymbol{C}_\pm$ can be extended to the whole complex plane as a $B_\infty(\mathscr{H})$-valued entire analytic function. We write the function extended from $\boldsymbol{C}_\pm$ as $AR_{0,\pm}^\varepsilon(z)B$. Then*

$$AR_{0,\pm}^\varepsilon(z)B=AR_0^\varepsilon(z)B\pm 2\pi i[T_0^\varepsilon(\bar{z})A]^*T_0^\varepsilon(z)B\,,\qquad z\in\boldsymbol{C}_\mp\,. \tag{2.6}$$

PROOF. We only prove that $AR_0^\varepsilon(z)B$ can be extended from $\boldsymbol{C}_+$ to $\boldsymbol{C}$. The other case can be proved similarly. Let us take an open precompact interval I of the real axis. We first note that $AR_0^\varepsilon(z)E_0^\varepsilon(I^c)B$ can be analytically continued across the interval I and for $z\in\boldsymbol{C}_-$ the continued function is equal to $AR_0^\varepsilon(z)E_0^\varepsilon(I^c)B$. This can be seen from the following facts:

a) $AR_0^\varepsilon(z)E_0^\varepsilon(I^c)B=[AU^{\varepsilon *}](x_1+i)^{-1/2}\cdot(x_1+i)(x_1-z)^{-1}\chi_{I^c}(x_1)\cdot(x_1+i)^{-1/2}U^\varepsilon B$;

b) $(x+i)\chi_{I^c}(x_1)(x_1-z)^{-1}\in\boldsymbol{B}(\mathscr{H})$ can be analytically continued across the interval I;

c) $[AU^{\varepsilon *}](x_1+i)^{-1/2}$ and $(x_1+i)^{-1/2}U^\varepsilon B\in\boldsymbol{B}(\mathscr{H})$, by statements 4) and 5) of Lemma 2.1.I, interpolation theorem and Sobolev's embedding theorem.

On the other hand, considering as an operator in $\mathscr{H}$, we have

$$AR_0^\varepsilon(z)E_0^\varepsilon(I)B=\int_I \frac{[T_0^\varepsilon(\lambda)A]^*T_0^\varepsilon(\lambda)B}{\lambda-z}\,d\lambda\,,\qquad z\in\boldsymbol{C}_\pm$$

(see Kuroda [7]). Since $[T_0^\varepsilon(\lambda)A]^*T_0^\varepsilon(\lambda)B$ is the restriction of the $\boldsymbol{B}(\mathscr{H})$-valued entire function $[T_0^\varepsilon(\bar{z})A]^*T_0^\varepsilon(z)B$, Cauchy's integral formula shows that $AR_0^\varepsilon(z)E_0^\varepsilon(I)B$ can be analytically continued across I and the extended function can be written as $AR_0^\varepsilon(z)E_0^\varepsilon(I)B+2\pi i[T_0^\varepsilon(\bar{z})A]^*[T_0^\varepsilon(z)B]$ for $z\in \boldsymbol{C}_-$. Thus for completing the proof, it suffices to prove $AR_0^\varepsilon(i)B\in \boldsymbol{B}_\infty(\mathcal{H})$. However, this is obvious (see the proof of Theorem 2.5.I). (Q.E.D.)

Now we want to discuss the asymptotic behaviour of $R_0^\varepsilon(z)$ as $\varepsilon\downarrow 0$ for z in the closed cut plane $\boldsymbol{C}_\pm\cup \boldsymbol{R}^1$. We first prove the following lemma.

LEMMA 2.4. *For $\sigma\in \boldsymbol{R}^1$ there exists a constant $C>0$ independent of $0<\varepsilon\leqq 1$ such that*

$$\|(1+x^2)^{\sigma/2}R_0^\varepsilon(\pm i)(1+x^2)^{-\sigma/2}\|_{B(\mathcal{H})}\leqq C. \tag{2.8}$$

Moreover, if $1\leqq\sigma$,

$$\|(1-\Delta)R_0^\varepsilon(\pm i)(1+x^2)^{-\sigma/2}\|_{B(\mathcal{H})}\leqq C. \tag{2.9}$$

PROOF. We give a proof only for $R_0^\varepsilon(i)$ with $\sigma\geqq 0$. $R_0^\varepsilon(-i)$ with $\sigma\geqq 0$ can be proved similarly and the case $\sigma<0$ can be proved by duality argument. For proving (2.8), it is sufficient to prove the case where $\sigma=0, 1, 2, \cdots$, by virtue of the interpolation theorem. For (2.9), clearly it suffices to prove only the case $\sigma=1$. The inequality (2.8) for $\sigma=0$ is obvious. Let us prove (2.8) and (2.9) for $\sigma=1$. If $f\in C_0^\infty(R^n)$, it is obvious that

$$\mathcal{F}R_0^\varepsilon(i)\mathcal{F}^*f(p)=e^{ip_1^3/3\varepsilon}\Big(p'^2+i\varepsilon\frac{\partial}{\partial p_1}-i\Big)^{-1}e^{-ip_1^3/3\varepsilon}\hat{f}(p). \tag{2.10}$$

Hence writing as $K(q, p')=\dfrac{q^3}{3}+qp'^2-iq$ ($q\in \boldsymbol{R}^1$, $p\in \boldsymbol{R}^{n-1}$), we get

$$\mathcal{F}R_0^\varepsilon(i)\mathcal{F}^*f(p)=\frac{i}{\varepsilon}\int_{p_1}^\infty e^{i(K(p_1,p')-K(q,p'))/\varepsilon}\hat{f}(q, p')dq. \tag{2.11}$$

By partial integration, the right hand side of (2.11) is equal to i times

$$(1+ip^2)^{-1}\hat{f}(p)+\int_{p_1}^\infty e^{i(K(p_1,p')-K(q,p'))/\varepsilon}\frac{\partial}{\partial q}((1+iq^2+ip'^2)^{-1}\times\hat{f}(q, p'))dq. \tag{2.12}$$

Taking derivatives of (2.11) and using partial integration, we get

$$\begin{aligned}\frac{\partial}{\partial p_1}(\mathcal{F}R_0^\varepsilon(i)\mathcal{F}^*f)(p)=\frac{i}{\varepsilon}\int_{p_1}^\infty e^{i(K(p_1,p')-K(q,p'))/\varepsilon}(1+ip^2)\Big\{&(1+iq^2+ip'^2)^{-1}\\ &\times\frac{\partial\hat{f}}{\partial q}-2iq(1+iq^2+ip'^2)^{-2}\hat{f}(q, p')\Big\}dq,\end{aligned} \tag{2.13$_1$}$$

$$\frac{\partial}{\partial p_j}(\mathcal{F}R_0^\varepsilon(i)\mathcal{F}^*f)(p)=\frac{1}{\varepsilon}\int_{p_1}^\infty e^{i(K(p_1,p')-K(q,p'))/\varepsilon}\frac{\partial\hat{f}}{\partial p_j}(q, p')dq \tag{2.13$_j$}$$

$$+\frac{i}{\varepsilon}\int_{p_1}^{\infty} e^{i(K(p_1,p')-K(q,p'))/\varepsilon}\frac{\partial}{\partial p}\Big(\frac{2i(q-p_1)p_j}{1+iq^2+ip'^2}\hat{f}(q,\ p')\Big)dq\,, \qquad j\geqq 2\,.$$

In any case, $j=1,\ 2,\ \cdots,\ n$, we get

$$\Big\|\frac{\partial}{\partial p_j}(\mathscr{F}R_0^{\varepsilon}(i)\mathscr{F}^*f)\Big\| \tag{2.14}$$

$$\leqq\Big\|\frac{4}{\varepsilon}\int_{p_1}^{\infty}e^{(p_1-q)/\varepsilon}\Big(1+\frac{1+p_1^2+p'^2}{1+q^2+p'^2}\Big)\Big(|\hat{f}(q,\ p')|+\sum_{j=1}^{n}\Big|\frac{\partial\hat{f}}{\partial p_j}(q,\ p')\Big|\Big)dq\Big\|.$$

By (2.12), we also have

$$\|(1+p^2)(\mathscr{F}R_0^{\varepsilon}(i)\mathscr{F}^*f)(p)\| \tag{2.15}$$

$$\leqq 2\|\hat{f}\|+2\Big\|\int_{p_1}^{\infty}e^{(p_1-q)/\varepsilon}\Big(1+\frac{1+p_1^2+p'^2}{1+q^2+p'^2}\Big)\Big(|\hat{f}(q,\ p')|+\sum_{j=1}^{n}\Big|\frac{\partial\hat{f}}{\partial p_j}(q,\ p')\Big|\Big)dq\Big\|.$$

Now we apply an elementary inequality that for $0<\varepsilon\leqq 1$, $a\geqq 1$,

$$\int_{-\infty}^{\infty}\Big\{\frac{1}{\varepsilon}\int_{x}^{\infty}e^{(x-y)/\varepsilon}\Big(\frac{x^2+a^2}{y^2+a^2}\Big)^k f(y)dy\Big\}^2dx\leqq C_k\int_{-\infty}^{\infty}f(x)^2dx\,, \tag{2.16}$$

where C_k is a constant depending only on $k\in\boldsymbol{R}^1$. Thus relations (2.8) and (2.9) for $\sigma=1$ are obtained by combing (2.14) and (2.15) with (2.16). The rest to prove is the relation (2.8) for $\sigma=2,\ 3,\ \cdots$. For such σ, (2.8) can be proved by repeating the above procedure, so details are omitted here. (Q.E.D.)

COROLLARY 2.5. *If* $\rho>\sigma>1$, $(1+x^2)^{\sigma/2}R_0^{\varepsilon}(i)(1+x^2)^{-\rho/2}\in\boldsymbol{B}_{\infty}(\mathscr{H})$ *and*

$$\lim_{\varepsilon\downarrow 0}\|(1+x^2)^{\sigma/2}R_0^{\varepsilon}(i)(1+x^2)^{-\rho/2}-(1+x^2)^{\sigma/2}R_0(i)(1+x^2)^{-\rho/2}\|=0\,. \tag{2.17}$$

PROOF. We denote as K the Hilbert space $H^2\cap L_\rho^2$ equipped with the norm $\|f\|_K=(\|f\|_{H^2}^2+\|f\|_{L_\rho^2}^2)^{1/2}$ and as K^* its dual space. Clearly $K^*=H^{-2}+L_{-\rho}^2$ with the norm $\|f\|_{K^*}=\inf\{(\|g\|_{H^{-2}}^2+\|h\|_{L_{-\rho}^2}^2)^{1/2}:\ f=g+h,\ g\in H^{-2},\ h\in L_{-\rho}^2\}$, the injection map $L_{-\sigma}^2\subset K^*$ is compact and $C_0^{\infty}(\boldsymbol{R}^n)$ is dense in K^*. By resolvent equation, we have

$$\|R_0^{\varepsilon}(-i)f-R_0(-i)f\|_{L_{-\rho}^2}\leqq\|R_0^{\varepsilon}(-i)f-R_0(-i)f\|_{\mathscr{H}} \tag{2.18}$$

$$\leqq\varepsilon\|R_0^{\varepsilon}(-i)x_1R_0(-i)f\|\leqq\varepsilon\|x_1R_0(-i)f\|$$

for $f\in C_0^{\infty}(\boldsymbol{R}^n)$. Since $\|R_0^{\varepsilon}(-i)\|_{B(K^*,L_{-\rho}^2)}$ is uniformly bounded by the dual statement of Lemma 2.4, (2.18) implies that $R_0^{\varepsilon}(-i)$ converges to $R_0(-i)$ strongly in $B(K^*,\ L_{-\rho}^2)$, hence in norm in $B(L_{-\sigma}^2,\ L_{-\rho}^2)$ which implies (2.17) by duality.

(Q.E.D.)

LEMMA 2.6. *Suppose* $V(x)$ *satisfies the conditions* (VSR), $A(x)=|V(x)|^{1/2}$ *and*

$B(x)=|V(x)|^{1/2}\operatorname{sgn} V(x)$. *Then there exists a constant* C *independent of* ε *and* $z\in \boldsymbol{C}_{\pm}$ *such that*

$$\|AR_0^{\varepsilon}(z)B\|_{B(\mathcal{H})}\leqq C\max(\|V\|_{(n/2)-\gamma}, \|v\|_{(n/2)+\gamma}) \tag{2.19}$$

where $\gamma>0$ *is a constant such that* $v\in L^{(n/2)-\gamma}(\boldsymbol{R}^n)\cap L^{(n/2)+\gamma}(\boldsymbol{R}^n)$. *Moreover* $AR_0^{\varepsilon}(z)B$ *can be extended to the closed cut plane* $\boldsymbol{C}_{\pm}\cup \boldsymbol{R}^1$ *as a* $B(\mathcal{H})$*-valued Hölder continuous function. We write the boundary values as* $AR_0^{\varepsilon}(\lambda\pm i0)B$.

PROOF. By the relation of Avron-Herbst [1]:

$$e^{itH_0^{\varepsilon}}f(x)=e^{-i\varepsilon tx_1-i\varepsilon t^3/3}(e^{-itH_0}f)(x_1-\varepsilon t^2, x'), \tag{2.20}$$

we get easily,

$$\|Ae^{-itH_0^{\varepsilon}}Bf\|=\|Ae^{-itH_0}(Bf)_t\|, \quad (Bf)_t(x)=(Bf)(x_1-\varepsilon t^2, x'). \tag{2.21}$$

Hence usual technique for proving $-\Delta$-smoothness of potentials (see Kato [11] or Ginibre-Moulin [3], and the proof of the following lemma) implies (2.19).

(Q.E.D.)

LEMMA 2.7. *Suppose* $V(x)$ *satisfies* (VSR). *Then for any* $z\in \boldsymbol{C}_{\pm}\cup \boldsymbol{R}^1$, $\lim_{\varepsilon\downarrow 0}\|AR_0^{\varepsilon}(z)B-AR_0(z)B\|=0$. *Moreover, this convergence is locally uniform in* $z\in \boldsymbol{C}_{\pm}\cup \boldsymbol{R}^1$.

PROOF. We prove the case $\operatorname{Im} z\geqq 0$ only. The other case can be proved similarly. By virtue of the inequality (2.19) it suffices to prove the lemma in the case where $V_1(x)$ and $V_2(x)$ in the expression (VSR) (a) for $V(x)$ are smooth bounded functions (three ε-argument). By the resolvent equation, we have

$$AR_0^{\varepsilon}(z)B=AR_0^{\varepsilon}(i)B+(i-z)AR_0^{\varepsilon}(z)(1+x^2)^{-\sigma/2}\cdot(1+x^2)^{\sigma/2}R_0^{\varepsilon}(i)B, \tag{2.22}$$

where $1<\sigma<\delta$. In the right hand side of (2.22) the first summand $AR_0^{\varepsilon}(i)B$ obviously converges to $AR_0(i)B$ in $\boldsymbol{B}(\mathcal{H})$ by Corollary 2.5. In the second summand, $(1+x^2)^{\sigma/2}R_0^{\varepsilon}(i)B\in \boldsymbol{B}_{\infty}(\mathcal{H})$ and converges to $(1+x^2)^{\sigma/2}R_0(i)B$ in $\boldsymbol{B}_{\infty}(\mathcal{H})$. Therefore it is sufficient to prove that $AR_0^{\varepsilon}(z)(1+x^2)^{-\sigma/2}$ converges strongly to $AR_0(z)(1+x^2)^{-\sigma/2}$. Let $f\in\mathcal{H}$ and write $(1+x^2)^{-\sigma/2}=C(x)$. Then

$$\begin{aligned}
&AR_0^{\varepsilon}(z)Cf(x)-AR_0(z)Cf(x)\\
&\qquad=i\int_0^{\infty}e^{itz}Ae^{-itH_0^{\varepsilon}}Cf\,dt-i\int_0^{\infty}e^{itz}Ae^{-itH_0}Cf\,dt\\
&\qquad=i\int_0^{\infty}e^{itz}A(e^{-i\varepsilon tx_1-i\varepsilon t3/3}-1)e^{-itH_0}(Cf)_t(x)dt\\
&\qquad\quad+i\int_0^{\infty}e^{itz}Ae^{-itH_0}[(Cf)_t-Cf](x)dt.
\end{aligned}$$

Let us take p and q such that $1\leqq p<n<q<\infty$ and $A,\ C\in L^{p}(\boldsymbol{R}^{n})\cap L^{q}(\boldsymbol{R}^{n})$. Then, writing as $q'=2q/q+2$, we have

$$\text{(2.23)}\qquad \int_{0}^{\infty}\|A(e^{-i\varepsilon t x_{1}-i\varepsilon t^{3}/3}-1)e^{-itH_{0}}(Cf)_{t}\|dt$$

$$\leqq\left\{\int_{0}^{1}\|A(e^{-i\varepsilon tx_{1}-i\varepsilon t^{3}/3}-1)\|_{q}(4\pi t)^{-n/q}\|C\|_{q}dt\right.$$

$$\left.+\int_{1}^{\infty}\|A(e^{-i\varepsilon tx_{1}-i\varepsilon t^{3}/3}-1)\|_{p}(4\pi t)^{-n/p}\|C\|_{p}dt\right\}\|f\|$$

and

$$\text{(2.24)}\qquad \int_{0}^{\infty}\|Ae^{-itH_{0}}[(Cf)_{t}-(Cf)]\|dt$$

$$\leqq\int_{0}^{1}\|A\|_{q}(4\pi t)^{-n/q}\|(Cf)_{t}-Cf\|_{q'}dt+\int_{1}^{\infty}\|A\|_{p}(4\pi t)^{-n/p}\|(Cf)_{t}-Cf\|_{p'}dt\,.$$

Obviously, for $r=p$ or q, and $r'=2r/r+2$, we have

$$\text{(2.25)}_{1}\qquad \|A(e^{-i\varepsilon tx_{1}-i\varepsilon t^{3}/3}-1)\|_{r}\leqq 2\|A\|_{r}\,;$$

$$\text{(2.25)}_{2}\qquad \lim_{\varepsilon\downarrow 0}\|A(e^{-i\varepsilon tx_{1}-i\varepsilon t^{3}/3}-1)\|_{r}=0\qquad\text{for any } t>0\,;$$

$$\text{(2.26)}_{1}\qquad \|(Cf)_{t}-Cf\|_{r'}=\|(Cf)(x_{1}-\varepsilon t^{2},\ x')-(Cf)(x)\|_{r'}\leqq 2\|C\|_{r}\|f\|\,;$$

$$\text{(2.26)}_{2}\qquad \lim_{\varepsilon\downarrow 0}\|(Cf)_{t}-Cf\|_{r'}=0\qquad\text{for any } t>0\,.$$

Hence by Lebesgue's dominated convergence theorem, $\lim_{\varepsilon\downarrow 0}\|AR_{0}^{\varepsilon}(z)Cf-AR_{0}(z)Cf\|=0$, which proves the lemma. (Q.E.D.)

§ 3. Proof of Theorem 1.

Here we prove Theorem 1, assuming the condition (EX). We first remark that if we define $A(x)=|V(x)|^{1/2}$ and $B(x)=|V(x)|^{1/2}\operatorname{sgn}V(x)$, all assumption of Kuroda's abstract theory [12] are satisfied, and therefore the existence and the completeness of wave operators holds, although this fact is proved in [I] under weaker assumptions. Furthermore by Theorem 6.3 of [12] (see also Yajima [17]), scattering matrix $S(\lambda)$ can be represented as an operator in $\boldsymbol{h}=L^{2}(\boldsymbol{R}^{n-1})$ as

$$\text{(3.1)}_{1}\qquad S^{\varepsilon}(\lambda)=1-2\pi iT_{0}^{\varepsilon}(\lambda,\ B)(1+AR_{0}^{\varepsilon}(\lambda+i0)B)^{-1}T_{0}^{\varepsilon}(\lambda,\ A)^{*}$$

and

$$\text{(3.1)}_{2}\qquad S^{\varepsilon}(\lambda)^{-1}=1+2\pi iT_{0}^{\varepsilon}(\lambda,\ B)(1+AR_{0}^{\varepsilon}(\lambda-i0)B)^{-1}T_{0}^{\varepsilon}(\lambda,\ A)^{*}$$

except for $\lambda\in\sigma_d(H^\varepsilon)$ where $(1+AR_0^\varepsilon(\lambda\pm i0)B)^{-1}$ fails to exist.

For $\operatorname{Im} z\neq 0$, we can easily see (see Kato [11]),

$$R^\varepsilon(z)=R_0^\varepsilon(z)+[BR_0^\varepsilon(\bar z)]^*(1+AR_0^\varepsilon(z)B)^{-1}AR_0^\varepsilon(z)\,. \tag{3.2}$$

For $z\in\boldsymbol{C}_\pm$ and $0<b<a/2$, $AR_0^\varepsilon(z)$, $BR_0^\varepsilon(z)\in\boldsymbol{B}(L^2_{e(b)}, \mathcal{H})$ and they can be extended to $\boldsymbol{C}$ as a $\boldsymbol{B}(L^2_{e(b)}, \mathcal{H})$-valued entire analytic functions by Corollary 2.3. Moreover Corollary 2.3 tells us that $AR_0^\varepsilon(z)B\in\boldsymbol{B}_\infty(\mathcal{H})$ for $z\in\boldsymbol{C}_\pm$ and it can be extended to $\boldsymbol{C}$ as a $\boldsymbol{B}_\infty(\mathcal{H})$-valued entire function $AR^\varepsilon_{0,\pm}(z)B$ from $\boldsymbol{C}_+$ and/or $\boldsymbol{C}_-$, respectively. Therefore by the well-known theorem for operator-valued analytic functions (Steinberg [14]), $(1+AR^\varepsilon_{0,\pm}(z)B)^{-1}$ is a $\boldsymbol{B}(\mathcal{H})$-valued meromorphic functions and at each of its pole the principal part of $(1+AR^\varepsilon_{0,\pm}(z)B)^{-1}$ is of finite rank. Thus $R^\varepsilon(z)$, $z\in\boldsymbol{C}_\pm$ can be extended to $\boldsymbol{C}$ as a $\boldsymbol{B}(L^2_{e(b)}, L^2_{e(-b)})$-valued meromorphic function $R^\varepsilon_\pm(z)$ and for $z\in\boldsymbol{C}_\mp$,

$$R^\varepsilon_\pm(z)=R^\varepsilon_{0,\pm}(z)+[BR^\varepsilon_{0,\mp}(\bar z)]^*(1+AR^\varepsilon_{0,\pm}(z)B)^{-1}AR^\varepsilon_{0,\pm}(z)\,. \tag{3.3}$$

At each of its poles the principal part is of finite rank. This proves the first statement i). To prove ii) we define as

$$S^\varepsilon(z)=1-2\pi i T_0^\varepsilon(z,\ B)(1+AR^\varepsilon_{0,+}(z)B)^{-1}T_0^\varepsilon(\bar z,\ A)^* \tag*{(3.4)$_1$}$$

$$\tilde S^\varepsilon(z)=1+2\pi i T_0^\varepsilon(z,\ B)(1+AR^\varepsilon_{0,-}(z)B)^{-1}T_0^\varepsilon(\bar z,\ A)^*\,. \tag*{(3.4)$_2$}$$

By Lemma 2.1, $(3.1)_1$, $(3.1)_2$ and the argument for the proof of i), it is obvious that (a) $S^\varepsilon(z)$ and $\tilde S^\varepsilon(z)$ are $\boldsymbol{B}(\boldsymbol{h})$-valued meromorphic functions and at each of their poles the principal part is of finite rank; (b) the restriction of $S^\varepsilon(z)$ and $\tilde S^\varepsilon(z)$ to the real axis are $S^\varepsilon(\lambda)$ and $S^\varepsilon(\lambda)^{-1}$. This proves ii) and the equation

$$S^\varepsilon(z)\tilde S^\varepsilon(z)=\tilde S^\varepsilon(z)S^\varepsilon(z)=I\,. \tag{3.5}$$

Now we prove iii). Suppose first that $z=z_0\in\boldsymbol{C}_\mp$ is a pole of $R^\varepsilon_\pm(z)$. It is clear by (3.3) that there exists a vector $0\neq f\in\mathcal{H}$ such that $(1+AR^\varepsilon_{0,\pm}(z_0)B)f=(1+AR^\varepsilon_0(z_0)B)f\pm 2\pi i T_0^\varepsilon(\bar z_0,\ A)^*T_0^\varepsilon(z_0,\ B)f=0$. Since $1+AR_0^\varepsilon(z_0)B$ is invertible, it is easy to see that $0\neq g\equiv T_0^\varepsilon(z_0,\ B)f\in\boldsymbol{h}$ and

$$g\pm 2\pi i T_0^\varepsilon(z_0,\ B)(1+AR_0^\varepsilon(z_0)B)^{-1}T_0^\varepsilon(\bar z_0,\ A)^*g=0\,. \tag{3.6}$$

Since $AR^\varepsilon_{0,\mp}(z_0)B=AR^\varepsilon_0(z_0)B$ for $z_0\in\boldsymbol{C}_\mp$, (3.4), (3.5) and (3.6) clearly imply that $z=z_0$ is a pole of $S(z)^{\pm 1}$. Let us assume now conversely that $z=z_0\in\boldsymbol{C}_\mp$ is a pole of $S(z)^{\pm 1}$. Then there exists $0\neq g\in\boldsymbol{h}$ such that (3.6) is satisfied. We set $f=(1+AR_0^\varepsilon(z_0)B)^{-1}T_0^\varepsilon(\bar z_0,\ A)^*g$. It is easy to see that $f\neq 0$ and

$$(1+AR^\varepsilon_{0,\pm}(z_0)B)f=0\,, \tag{3.7}$$

$$f=AT_0^\varepsilon(\bar z_0)^*g+AR^\varepsilon(z_0)VT_0(\bar z)^*g\,. \tag{3.8}$$

By Theorem 2.5 [I] and Remark 2.2, (3.8) implies $Bf\in L^2_{e(b)}$. Thus (3.7) implies $f\in R(AR^\varepsilon_{0,\pm}(z_0))$, and $f\notin N([BR^\varepsilon_{0,\mp}(\bar z)]^*)$. Therefore (3.3) and (3.7) show that $z=z_0$ is a pole of $R^\varepsilon_\pm(z)$. (Q.E.D.)

§4. Proof of Theorem 2.

We prove here Theorem 2. The way of the proof is similar to that used by the author in a different context [16]. Here we assume $V(x)$ to satisfy the condition (VSR).

LEMMA 4.1. *For any compact interval* $I=[a, b]$,

$$\text{s-}\lim_{\varepsilon\downarrow 0} E^\varepsilon_0(I)=E_0(I)\,. \tag{4.1}$$

PROOF. If f, $g\in L^2_\rho(\boldsymbol{R}^n)$ with $\rho>1$, we have by Stone's theorem and Lemma 2.7,

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}(E^\varepsilon_0(I)f,\ g)&=\lim_{\varepsilon\downarrow 0}(2\pi i)^{-1}\int_a^b <(R^\varepsilon_0(\lambda+i0)-R^\varepsilon_0(\lambda-i0))f,\ g>d\lambda\\ &=(2\pi i)^{-1}\int_a^b <(R_0(\lambda+i0)-R_0(\lambda-i0)f,\ g>d\lambda\\ &=(E_0(I)f,\ g)\,.\end{aligned} \tag{4.2}$$

Since $\|E^\varepsilon_0(I)\|\leqq 1$, (4.2) implies $\text{w-}\lim_{\varepsilon\downarrow 0} E^\varepsilon_0(I)=E_0(I)$. On the other hand, setting as $g=f$ in (4.2), we get $\lim_{\varepsilon\downarrow 0}\|E^\varepsilon_0(I)f\|=\|E_0(I)f\|$ for $f\in L^2_\rho(\boldsymbol{R}^n)$. Thus by Banach-Steinhaus' theorem, $E^\varepsilon_0(I)$ converges strongly to $E_0(I)$ as $\varepsilon\downarrow 0$. (Q.E.D.)

LEMMA 4.2. $\text{s-}\lim_{\varepsilon\downarrow 0} W^\varepsilon_\pm=W_\pm$.

PROOF. Let us take a compact interval $I=[a, b]$ such that $I\cap\sigma_p(H)=\emptyset$. Then it is well-known that $AR_0(\lambda\pm i0)B$ is norm-continuous on I and $(1+AR_0(\lambda\pm i0)B)^{-1}$ exists for every $\lambda\in I$. Hence by Lemma 2.7, we can see easily that for sufficiently small $\varepsilon>0$, $(1+AR^\varepsilon_0(\lambda\pm i0)B)^{-1}$ exists and converges to $(1+AR_0(\lambda\pm i0)B)^{-1}$ in norm as $\varepsilon\downarrow 0$ uniformly on $\lambda\in I$ (Neumann series expansion). Thus by the above argument and Lemma 2.7 again, if $\rho>1$, $R^\varepsilon(\lambda\pm i0)=R^\varepsilon_0(\lambda\pm i0)-R^\varepsilon_0(\lambda\pm i0)B\cdot(1+AR^\varepsilon_0(\lambda\pm i0)B)^{-1}AR^\varepsilon_0(\lambda\pm i0)$ converges to $R(\lambda\pm i0)$ in $\boldsymbol{B}(L^2_\rho,\ L^2_{-\rho})$-norm as $\varepsilon\downarrow 0$ uniformly on I. Now using the expression of the wave operators in terms of the resolvents (see Kuroda [12] or Kako-Yajima [9]), we have for f, $g\in L^2_\rho$,

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}(W^\varepsilon_\pm E^\varepsilon_0(I)f,\ g)\\ &\quad=\lim_{\varepsilon\downarrow 0}(E^\varepsilon_0(I)f,\ g)+\end{aligned} \tag{4.3}$$

$$+\lim_{\varepsilon\downarrow 0}(2\pi i)^{-1}\int_a^b (AR_0^\varepsilon(\lambda\pm i0)f,\ B(R^\varepsilon(\lambda+i0)-R^\varepsilon(\lambda-i0))f)d\lambda$$

$$=(E_0(I)f,\ g)+(2\pi i)^{-1}\int_a^b (AR_0(\lambda\pm i0)f,\ B(R(\lambda+i0)-R(\lambda-i0))f)d\lambda$$

$$=(W_\pm E_0(I)f,\ g)\,.$$

Since $\|W_\pm^\varepsilon E_0^\varepsilon(I)\|\leqq 1$, (4.3) implies $\operatorname{w-lim}_{\varepsilon\downarrow 0} W_\pm^\varepsilon E_0^\varepsilon(I)=W_\pm E_0(I)$. Then

$$\operatorname{s-lim}_{\varepsilon\downarrow 0} W_\pm^\varepsilon E_0^\varepsilon(I)=W_\pm E_0(I)\,, \tag{4.4}$$

since $\lim_{\varepsilon\downarrow 0}\|W_\pm^\varepsilon E_0^\varepsilon(I)f\|=\lim_{\varepsilon\downarrow 0}\|E_0^\varepsilon(I)f\|=\|E_0(I)f\|=\|W_\pm E_0(I)f\|$ for any $f\in\mathcal{H}$ by the isometry property of $W_\pm^\varepsilon$, $W_\pm$ and Lemma 4.1. Let us take $f\in\mathcal{H}$ such that $E_0(I)f=f$. Then, $\lim_{\varepsilon\downarrow 0}\|W_\pm^\varepsilon(E_0^\varepsilon(I)-E_0(I))f\|=0$ by Lemma 4.1 and (4.4). This fact proves the lemma, because i) the linear hull of $\{E_0(I)f: f\in\mathcal{H}, I\subset\boldsymbol{R}^1$ is a compact interval such that $I\cap\sigma_p(H)=\varnothing\}$ is a dense set of $\mathcal{H}$ since $\sigma_p(H)$ is discrete and $H_0=-\Delta$ is absolutely continuous; ii) operators $W_\pm^\varepsilon$ are isometries on $\mathcal{H}$.

(Q.E.D.)

PROOF OF THEOREM 2. Since $S^\varepsilon=(W_+^\varepsilon)^*W_-^\varepsilon$ by definition, we immediately have $\operatorname{w-lim}_{\varepsilon\downarrow 0} S^\varepsilon=S$ by Lemma 4.2. On the other hand $\|S^\varepsilon f\|=\|Sf\|=\|f\|$ for any $\varepsilon>0$, since the wave operators are complete and hence S^ε are unitary. Thus $\operatorname{s-lim}_{\varepsilon\downarrow 0} S^\varepsilon=S$. (Q.E.D.)

§5. Proof of Theorem 3.

In this section we assume $V(x)$ to satisfy both (EX) and (VSR). Let us take an eigenvalue $\mu<0$ of H. We take a small complex neighborhood $U\subset\boldsymbol{C}$ of μ such that μ is the only eigenvalue of H in U. Since we assume that H^ε has no eigenvalues, the poles of $R_\pm^\varepsilon(z)$ near μ appear only in $\boldsymbol{C}_\mp$. We prove statements (i) and (ii) by exploiting Theorem 1.5 and Theorem 2.1 of Howland [5]. Setting $\varepsilon=n\geqq 0$, $A_\varepsilon=A(x)$, $B_\varepsilon=B(x)$ and $T_\varepsilon=-\Delta+\varepsilon x_1$, we check his hypothesis. Hypothesis I is obvious; Hypothesis II is proved by Corollary 2.3. In Hypothesis III, (b), (c) and (e) are obvious; (d) is easy since $\|AR_0^\varepsilon(z)f-AR_0(z)f\|\leqq\varepsilon\|AR_0(z)x_1\|\,\|R_0^\varepsilon(z)f\|\leqq\varepsilon|\operatorname{Im} z|^{-1}\|AR_0(z)x_1\|_{B(\mathcal{H})}$ and $\|AR_0(z)x_1\|<\infty$. To prove Hypothesis III, (a) we need the following

LEMMA 5.1. *If $z\in U$, and $\operatorname{Re} z=\rho<0$, $\|T_0^\varepsilon(z, A)\|_{B(\mathcal{H},\boldsymbol{h})}\leqq Ce^{-d/\varepsilon}$ for some $d>0$, where d and C are determined only by ρ and U.*

PROOF. Calculating the integral appearing $(2.5)_2$ explicitly we have for any sufficiently small $q_1<0$,

$$\|T_0^\varepsilon(\zeta, A)\| \leqq C\exp\left(-\varepsilon^{-1}\left(\frac{q_1^3}{3}+(\mathrm{Re}\,\zeta-\mathrm{Im}\,\zeta)q_1\right)\right)(2\pi\varepsilon)^{-1/2}|q_1|^{-3/2}F(q_1, \mathrm{Im}\,\zeta), \tag{5.1}$$

where $F(q_1, \mathrm{Im}\,\zeta)$ is a polynomial of $|q_1|^{1/2}$ and $\mathrm{Im}\,\zeta$ of order 5. By (5.1) the statement of the lemma is obvious. (Q.E.D.)

By (2.6), Lemma 2.7 and Lemma 5.1, as $\varepsilon \downarrow 0$, $AR_{0,\pm}^\varepsilon(z)B$ converges to $AR_0(z)B$ uniformly on U, in operator norm, which proves Hypothesis III, (a) of [5]. Thus by Theorem 1.5 [5] and Theorem 1, we get i). ii) is the consequence of Theorem 2.1 [5]. Statement iii) can be proved by a standard method (cf. [4]) and we omit the proof. (Q.E.D.)

Concluding Remark. By a similar argument employed by Steinberg [14], we can easily prove that the poles $\{z_n\}$ of $S^\varepsilon(z)$ and $S^\varepsilon(z)^{-1}$ (or $R_\pm^\varepsilon(z)$) are actually simple; z_n's are the points where the equation $(-\Delta+\varepsilon x_1+V(x))u_n(x)=z_n u_n(x)$ has a solution $u_n(x)\in L^2_{e(-b)}$ for some $0<b<a/2$; the residue of $R_\pm^\varepsilon(z)$ at the pole z_n can be written as $\sum_{n_j}<\cdot, u_{n_j}>u_{n_j}$, where u_{n_j}'s are solutions of the above equation belonging to $L^2_{e(-b)}$. However, we shall not discuss the details here.

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(Received November 16, 1978)

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