Finiteness property of weakly proper birational maps

Dedicated to Professor S. Furuya on his 60th birthday

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§ 1. Introduction.

Let k be an algebraically closed field of characteristic zero. We shall work in the category of schemes over k.

For any n-dimensional algebraic variety V, we have defined the space $T_M(V)$ of logarithmic M-forms of V where M is an n-tuple of non-negative integers $m_1, \cdots, m_n$ ([1], [3]). If there is a proper birational morphism $f: W \to V$, then $T_M(V) \simeq T_M(W)$ via the linear map f^* induced from f. Hence the logarithmic M-genus $\bar{P}_M(V) = \dim T_M(V)$ is invariant under proper birational maps (Proposition 1 in [1]). We have shown that if V is of hyperbolic type, i.e., $\bar{\kappa}(V) = n$, then the group of proper birational maps of V into itself is a finite group (Theorem 6 in [1]).

We note that if Z is a closed subset of codimension $\geqq 2$ of a non-singular algebraic variety V, then

$$T_M(V-Z) = T_M(V)$$

by a Hartogs-type theorem. We say that a rational map $\varphi: W \to V$ is a weakly proper birational map if there are algebraic varieties $V_0 = V, V_1, \cdots, V_l = W$ and rational maps $\varphi_j: V_j \to V_{j-1}$ such that each φ_j is one of the following four types: 1) φ_j is a proper birational morphism, 2) φ_j^{-1} is a proper birational morphism, 3) V_{j-1} is non-singular and $\varphi_j: V_j = V_{j-1} - Z_{j-1} \subset V_{j-1}$ is an open immersion where Z_{j-1} is a closed subset of codimension $\geqq 2$, 4) φ_j^{-1} is the open immersion of the type 3), and such that $\varphi = \varphi_1 \cdots \varphi_l$. Then φ induces the isomorphisms $\varphi^* = \varphi_l^* \cdots \varphi_1^*: T_M(V) \simeq T_M(V_1) \simeq \cdots \simeq T_M(W)$. Hence $\bar{P}_M(V) = \bar{P}_M(V_1) = \cdots = \bar{P}_M(W)$. When such a φ exists between W and V, we say that W is *weakly proper birationally equivalent* (or WPB-equivalent) to V. Our purpose here is to prove the following

THEOREM 1. *If V is an algebraic variety of hyperbolic type, then the group $WPB(V)$ consisting of weakly proper birational maps of V into itself is a finite group.*

A weakly proper birational map is abbreviated by *WPB-map*. A rational map $f: W \to V$ is called a *WSR-map* if there is a *WPB-map* $\varphi: Y \to W$ such that

$g=f\cdot\varphi$ is a morphism of Y into V. Then for any $M=(m_1,\cdots,m_n)$, n being $\dim V$, we have

$$T_M(V) \xrightarrow{g^*} T_M(Y) \underset{\sim}{\overset{\varphi^*}{\longleftarrow}} T_M(W)\,.$$

Hence the linear map $f^*=\varphi^{*-1}\cdot g^*: T_M(V)\to T_M(W)$ is defined. Moreover, if f is dominant, f^* is injective. Therefore, letting $\mathrm{WSB}(V)$ be the group generated by birational WSR-maps (, which are abbreviated to WSB-maps) of V into itself, we have the representation

$$\rho_M:\ \mathrm{WSB}(V) \longrightarrow \mathrm{GL}(T_M(V))\,.$$

By this representation, we shall prove

THEOREM 2. *If V is of hyperbolic type, then $WSB(V)$ is a finite group. Hence $WSB(V)=WPB(V)$.*

§ 2. Linear system $\Lambda_m(V)$.

Let V be a non-singular algebraic variety of hyperbolic type. Take a completion $\bar{V}$ of V with smooth boundary D. Then by definition there exists $m>0$ such that $\Phi_m:\ \bar{V}\to\Phi_m(\bar{V})$ is the birational map, Φ_m being the rational map associated with $|m(K(\bar{V})+\bar{D})|$. Fix such an m. We eliminate the points of indeterminacy of Φ_m by a proper birational morphism $\mu:\ \bar{V}^*\to\bar{V}$ such that $V^*=\mu^{-1}(V)$ is non-singular and $\bar{V}^*$ is a completion of V^* with smooth boundary $D^*=\mu^{-1}(D)$. Then $\Phi_m^*=\Phi_m\cdot\mu$ is the rational map associated with $|m(K(\bar{V}^*)+D^*)|$. Consider the normalization of $\Phi_m^*:\ \bar{V}^*\to\Phi_m(\bar{V})$, which we denote by $f:\ \bar{V}^*\to\bar{W}$. Hence, $\bar{W}$ is the normalization of $\Phi_m(\bar{V})$.

We have the following linear system on $\bar{W}$:

$$\Lambda_m(V)=f_*|m(K(\bar{V}^*)+D^*)|=\{f_*\Delta\,;\ \Delta\in|m(K(\bar{V}^*)+D^*)|\}\,.$$

Note that $\Lambda_m(V)$ may be incomplete.

PROPOSITION 1. *$\Lambda_m(V)$ depends only upon V.*

PROOF. First, consider a fixed $\bar{V}$ and another $\bar{V}^\#$ and a proper birational

$$\begin{tikzcd}
& \bar{V}^* \arrow[dr, "\mu"] & \\
\bar{V}^\star \arrow[ur, "\sigma"] \arrow[rr, "\tau"] \arrow[dr, "\sigma_1"'] & & \bar{V} \\
& \bar{V}^\# \arrow[ur, "\rho"'] &
\end{tikzcd}$$

Fig. 1.

morphism $\rho: \bar{V}^{\#} \to \bar{V}$ satisfying the same condition as $\bar{V}^*$ and μ. Then there exist $\bar{V}^{\star}$ and $\tau: \bar{V}^{\star} \to \bar{V}$ satisfying the same condition as $\bar{V}^*$ and μ such that there exist birational morphisms $\sigma: \bar{V}^{\star} \to \bar{V}^*$ and $\sigma_1: \bar{V}^{\star} \to \bar{V}^{\#}$ in which $\tau = \mu \cdot \sigma = \rho \cdot \sigma_1$ (Fig. 1). Write $D^{\star} = \sigma^{-1}(D^*) = \tau^{-1}(D)$ and $D^{\star} = \sigma_1^{-1}(D^{\#}) = \tau^{-1}(D)$. We have by the logarithmic ramification formula ([1]),

$$K(\bar{V}^{\star}) + D^{\star} \sim \sigma^*(K(\bar{V}^*) + D^*) + \bar{R}_\sigma,$$

$\bar{R}_\sigma$ being the logarithmic ramification divisor of $\sigma | V^{\star}$. Hence,

$$|m(K(\bar{V}^{\star}) + D^{\star})| = \sigma^* |m(K(\bar{V}^*) + D^*)| + m\bar{R}_\sigma.$$

$f \cdot \sigma$ is a proper morphism which is induced from the rational map associated with $|m(K(\bar{V}^{\star}) + D^{\star})|$. Thus

$$(f \cdot \sigma)_* |m(K(\bar{V}^{\star}) + D^{\star})| = f_* \cdot \sigma_* \sigma^* |m(K(\bar{V}^*) + D^*)| + m f_* \sigma_* \bar{R}_\sigma = f_* |m(K(\bar{V}^*) + D^*)|.$$

Similarly, we get

$$(f_1 \cdot \sigma_1)_* |m(K(\bar{V}^{\star}) + D^{\star})| = f_{1*} |m(K(\bar{V}^{\#}) + D^{\#})|,$$

where f_1 is a proper morphism which is derived from the rational map associated with $|m(K(V^{\#}) + D^{\#})|$.

Next, we consider another arbitrary completion $\bar{V}_1$ of V with smooth boundary D_1. We may assume that there exists a proper birational morphism $\lambda: \bar{V}_1 \to \bar{V}$. Then $D_1 = \lambda^{-1} D$. Take a complete non-singular algebraic variety $\bar{V}_1^*$ and a proper birational morphism $\mu_1: \bar{V}_1^* \to \bar{V}_1$ satisfying the same condition as $\bar{V}^*$ and μ. By choosing a suitable $\bar{V}_1^*$ we can assume that $\lambda_1 = \mu^{-1} \cdot \lambda \cdot \mu_1$ is a morphism. Then $f \cdot \lambda_1$ is a morphism derived from the rational map associated with $|m(K(\bar{V}_1^*) + D_1^*)|$. By the same argument as above, we get

$$(f \cdot \lambda_1)_* |m(K(\bar{V}_1^*) + D_1^*)| = f_* |m(K(\bar{V}^*) + D^*)|.$$

Q.E.D.

In general, Φ_m is not a morphism. $\Phi'_m: \bar{V} \to \bar{W}$ is induced from Φ_m and $\Phi'_m(D) = f(\mu^{-1}(D)) = f(D^*)$ is a closed set of $\bar{W}$. We are interested in the behaviors of $A_m(V)$ and $\Phi'_m(D)$ under proper birational morphisms and open immersions of type 3) in § 1.

Let V_2 be a non-singular algebraic variety and $\varphi: V_2 \to V$ a proper birational morphism. Let $\bar{V}_2$ denote a completion of V_2 with smooth boundary D_2. φ defines a rational map $\bar{\varphi}: \bar{V}_2 \to \bar{V}$. Consider $\bar{V}_2^*$ and μ_2 as in the previous argument. $\bar{\varphi}$ defines a rational map $\tilde{\varphi}$, which may be assumed to be a morphism (see Fig. 2). Since φ is proper and birational, we have the isomorphism:

$$\varphi^*: T_m(V) \xrightarrow{\sim} T_m(V_2),$$

which defines the isomorphism $\psi: \bar{W} \to \bar{W}$. By the logarithmic ramification formula,

$$\begin{tikzcd}
\bar{V}_2^* \arrow[rrr, bend left, "\tilde{\varphi}"] \arrow[dr, "\mu_2"] \arrow[ddr, "f_2"'] & & & V^* \arrow[dl, "\mu"'] \arrow[ddl, "f"] \\
& \bar{V}_2 \supset V_2 \arrow[r, "\varphi"] \arrow[r, bend left, "\bar{\varphi}"] & V \subset \bar{V} & \\
& \overline{W} \arrow[r, "\psi"] & \overline{W} &
\end{tikzcd}$$

Fig. 2.

$$|m(K(\bar{V}_2^*)+D_2^*)|=\tilde{\varphi}^*|m(K(\bar{V}^*)+D^*)|+m\bar{R}_\varphi\,.$$

Then

$$\begin{aligned}
\psi_*\Lambda_m(V_2)&=\psi_*f_{2*}|m(K(\bar{V}_2^*)+D_2^*)| \\
&=\psi_*f_{2*}\tilde{\varphi}^*|m(K(\bar{V}^*)+D^*)|+m\psi_*f_{2*}\bar{R}_\varphi \\
&=f_*|m(K(\bar{V}^*)+D^*)|+mf_*\tilde{\varphi}^*\bar{R}_\varphi \\
&=f_*\Lambda_m(V)\,.
\end{aligned}$$

Moreover, $\psi f_2(D_2^*)=f\tilde{\varphi}\tilde{\varphi}^{-1}(D^*)=f(D^*)$. Thus we obtain

PROPOSITION 2. *For a proper birational morphism* φ: $V_2\to V$, *we have the isomorphism* ψ: $\overline{W}\to\overline{W}$, *which satisfies*

$$\psi_*(\Lambda_m(V_2))=\Lambda_m(V)\,,$$
$$\psi\Phi'_m(D_2)=\Phi'_m(D)\,.$$

Let Z be a closed subset of V of codimension $\geqq 2$. By $\bar{Z}$ we denote the closure of Z in $\bar{V}$. Perform a finite succession of monoidal transformations on $\bar{V}$ with non-singular centers in $\bar{Z}$. Then we have a proper birational morphism μ: $\bar{V}^*\to\bar{V}$ such that $\bar{V}^*$ is non-singular and $\mu^{-1}(D)+Z^*$, Z^* being the closure of $\mu^{-1}(Z)$ in $\bar{V}^*$, has only simple normal crossings. In other words, $\bar{V}^*$ is a completion of $V^*=V-Z$ with smooth boundary $\mu^{-1}(D)+Z^*$. Then we have

PROPOSITION 3.

$$\mathrm{Supp}(\Lambda_m(V-Z)_{\mathrm{fix}})=\mathrm{Supp}(\Lambda_m(V)_{\mathrm{fix}})\geqq f_*(Z^*)$$
$$\Phi_m(\bar{Z}\cup D)=\Phi_m(D)\cup\Phi_m(\bar{Z})\,.$$

Here, the symbol Λ_{fix} indicates the fixed part of a linear system Λ.

PROOF. We may assume that $\Phi_m'\cdot\mu$ is a morphism f. Then by definition, we get

$$\begin{aligned}\Lambda_m(V)&=f_*|m(K(\bar V^*)+\mu^{-1}(D))|\\&=f_*\mu^*|m(K(\bar V)+D)|+f_*m\bar R_\mu,\end{aligned}$$

and

$$\begin{aligned}\Lambda_m(V-Z)&=f_*|m(K(\bar V^*)+\mu^{-1}(D)+Z^*)|\\&=f_*(|m(K(\bar V^*)+\mu^{-1}(D))|+mZ^*)\\&=\Lambda_m(V)+mf_*(Z^*).\end{aligned}$$

Hence $\mathrm{Supp}\,(\Lambda_m(V-Z)_{\mathrm{fix}})=\mathrm{Supp}(\Lambda_m(V)_{\mathrm{fix}})\cup f_*(Z^*)$. On the other hand, since Z^* is the closure of $\mu^{-1}(Z)$, we have $Z^*\leqq\bar R_\mu$. Thus

$$\Lambda_m(V)_{\mathrm{fix}}\geqq mf_*(Z^*)\quad\text{and so}\quad \mathrm{Supp}\,\Lambda_m(V)_{\mathrm{fix}}\supset f_*(Z^*).$$

Accordingly, $\mathrm{Supp}(\Lambda_m(V-Z)_{\mathrm{fix}})=\mathrm{Supp}(\Lambda_m(V)_{\mathrm{fix}})$. The latter part of Proposition 3 is obvious. Q.E.D.

§ 3. Proof of Theorem 2.

Let $\varphi: V\to W$ be a WPB-map. Then by definition there are algebraic varieties $V_0=V, V_1, \cdots, V_l=Y$ and rational maps $\varphi_j: V_j\to V_{j-1}$ each of which is one of the four types 1), 2), 3), 4) in § 1 such that $\varphi=\varphi_1\cdots\varphi_l$. Considering non-singular models, we may assume that, all V_j are non-singular. Let $\bar V$ be a completion of V with smooth boundary D. Choose $m\gg 0$ such that $\Phi_m=\Phi_{m,V}: \bar V\to\Phi_m(\bar V)$ is birational, since $\bar\kappa(V)=n$. By $\Phi_m': \bar V\to X=\Phi_m(\bar V)'$, we denote the normalization of $\Phi_m: \bar V\to\Phi_m(\bar V)$. φ_j induces the linear isomorphism $T_m(V_{j-1})\simeq T_m(V_j)$, which defines the isomorphism $\psi_j: \Phi_m(\bar V_j)'\to\Phi_m(\bar V_{j-1})'$. Here $\bar V_j$ is a completion of V_j with smooth boundary D_j and $T_m(V_j)=H^0(\mathcal{O}(m(K(\bar V_j)+D_j)))$. Then by Propositions 1 and 2,

$$\psi_j\,(\mathrm{Supp}\,\Lambda_m(V_j)_{\mathrm{fix}})=\mathrm{Supp}\,\Lambda_m(V_{j-1})_{\mathrm{fix}}$$

and if φ_j is a proper birational morphism, then

$$\psi_j(\Phi_m^{(j)}(D_j))=\Phi_m^{(j-1)}(D_{j-1}),$$

where $\Phi_m^{(j)}$ is the normalization of the rational map associated to $|m(K(\bar V_j)+D_j)|$. If φ_j is the inverse of a proper birational morphism, then

$$\psi_j^{-1}(\Phi_m^{(j-1)}(D_{j-1}))=(\Phi_m^{(j)}(D_j)).$$

On the other hand, if $\varphi_j: V_j=V_{j-1}-Z_{j-1}\subset V_{j-1}$ is the open immersion with codimension $Z_{j-1}\geqq 2$, then $\Phi_m^{(j)}=\Phi_m^{(j-1)}$, $\psi_j=\mathrm{id}$ and

$$\Phi_m^{(j)}(D_j)=\Phi_m^{(j-1)}(\bar{Z}_{j-1}\cup D_{j-1})=\Phi_m^{(j-1)}(D_{j-1})\cup\Phi_m^{(j-1)}(\bar{Z}_{j-1}).$$

Therefore, letting $X=\Phi_m(\bar{V})'$, $S=\Phi_m(\bar{Y})'$ and $\psi=\psi_l\cdots\psi_1\colon S\to X$ we have

$$(*)\qquad \begin{cases} \psi\,(\mathrm{Supp}\,\Lambda_m(Y)_{\mathrm{fix}})=\mathrm{Supp}\,(\Lambda_m(V)_{\mathrm{fix}}) \\ \text{and}\quad \psi(\Phi_m^{(l)}(D_l))+\Delta=\Phi_m(D)+\Delta' \end{cases}$$

where Δ and Δ' are sums of closures of certain $\Phi_m^{(j)}(Z_j)$. Since $g=f\cdot\mu\colon Y\to V$ is a morphism, we may assume that $\bar{g}\colon \bar{Y}\to\bar{V}$ is a morphism after changing $\bar{Y}$ if necessary. g induces the linear isomorphism $T_m(V)\simeq T_m(Y)$ which defines the isomorphism $h\colon S\to X$. Then writing

$$\Sigma_Y=\mathrm{Supp}\,\Lambda_m(Y)_{\mathrm{fix}}\,,\qquad \Sigma_V=\mathrm{Supp}\,\Lambda_m(V)_{\mathrm{fix}}\quad\text{and}\quad D_Y=D_l\,,$$

we get

$$h(\Sigma_Y)=\Sigma_V\quad\text{and}\quad D\subset\bar{g}(D_Y)\,.$$

Hence

$$\Phi'_m(D)\subset\Phi'_m\bar{g}(D_Y)=h\Phi_m^{(l)}(D_Y)\,.$$

Put $\qquad F_Y=$ the purely 1-codimensional part of $\Phi_m^{(l)}(D_Y)$

and $\qquad F_V=$ the purely 1-codimensional part of $\Phi'(D)$.

Then,

$$h(\Sigma_Y\cup F_Y)\supset\Sigma_V\cup F_V\,.$$

Moreover, by $(*)$ we obtain

$$\psi(\Sigma_Y\cup F_Y)=\Sigma_V\cup F_V\,.$$

Since h and ψ are isomorphisms, we have

$$h\psi^{-1}(\Sigma_V\cup F_V)=h(\Sigma_Y\cup F_Y)\supset\Sigma_V\cup F_V\,.$$

The isomorphism $\eta=h\psi^{-1}\colon X\to X$ satisfies $\eta(\Sigma_V\cup F_V)=\Sigma_V\cup F_V$ by virtue of the following

LEMMA. *Let X be a noetherian space and F a closed subset of X. If an isomorphism $\eta\colon X\to X$ satisfies $\eta(F)\supset F$, then $\eta(F)=F$.*

PROOF. We have a descending chain of closed subsets: $F\supset\eta^{-1}F\supset\eta^{-2}F\supset\cdots$. Since X is noetherian, there is an $r>0$ such that $\eta^{-r}(F)=\eta^{-r-1}(F)$. Hence $\eta(F)=F$. Q.E.D.

Therefore, any WSB-map f of V into itself induces $\eta\in\mathcal{L}=\{\alpha\in PGL(\bar{P}_m(V),k)\,;\ \hat{\alpha}\Phi_m(V)'=\Phi_m(V)',\ \hat{\alpha}(\Sigma_V\cup F_V)=\Sigma_V\cup F_V,\ \hat{\alpha}$ being the automorphism of $X=\Phi_m(\bar{V})'$ induced from $\alpha\}$. Thus we have the group homomorphism $\beta_V\colon \mathrm{WSB}(V)\to\mathcal{L}$

such that $\beta_V(f)=\eta$. Since Φ'_m: $\bar{V}\to X=\Phi_m(V)'$ is birational, β_V is injective. Hence if WSB(V) were not a finite group, the affine algebraic group $\mathcal{L}$ would have a non-trivial connected component $\mathcal{G}$. Thus

$$\mathcal{G}\subset \mathrm{Aut}\,(X-\Sigma_V\cup F_V)\,.$$

Thanks to $\mathrm{Aut}\,(X-\Sigma_V\cup F_V)\subset \mathrm{Aut}\,(\mathrm{Reg}\,(X)-\Sigma_V\cup F_V)$, we have $\mathcal{G}\subset$ $\mathrm{Aut}\,(\mathrm{Reg}\,X-\Sigma_V\cup F_V)$. Hence

$$\bar{\kappa}(\mathrm{Reg}\,X-\Sigma_V\cup F_V)<n\,.$$

Since X is normal, $\mathrm{codim}\,(\Phi'_m(D)-F_V)\geqq 2$. We have

$$\bar{\kappa}(\mathrm{Reg}\,X-\Sigma_V\cup\Phi'_m(D))=\bar{\kappa}(\mathrm{Reg}\,X-\Sigma_V\cup F_V)<n\,.$$

Moreover,

$$\bar{\kappa}(X-\Phi'_m(D))\leqq\bar{\kappa}(X-\Sigma_V\cup\Phi'_m(D))\leqq\bar{\kappa}(\mathrm{Reg}\,X-\Sigma_V\cup\Phi'_m(D))<n\,.$$

On the other hand, setting $V_0=\bar{V}-\Phi'^{-1}_m(\Phi'_m(D))\subset V$, we have a proper birational morphism $\Phi'_m|V_0$: $V_0\to X-\Phi'_m(D)$. Hence

$$n=\bar{\kappa}(V)\leqq\bar{\kappa}(V_0)=\bar{\kappa}(X-\Phi'_m(D))<n\,,$$

which is a contradiction.

§ 4. Minimality of affine algebraic varieties and quasi-abelian varieties.

THEOREM 3. *A WSR-map φ of a normal variety V (resp. non-singular variety V) into an affine algebraic variety A (resp. a quasi-abelian variety) is a morphism.*

PROOF. By definition, there is a WPB-map ψ: $Y\to V$ such that $\varphi\cdot\psi=g$ is a morphism $Y\to A$. By the definition of WPB-map, there are algebraic varieties and rational maps as follows:

$$Y=V_l\xrightarrow{\psi_l}V_{l-1}\longrightarrow\cdots\longrightarrow V_1\xrightarrow{\psi_1}V_0=V\,,$$
$$\psi=\psi_1\cdots\psi_l\,.$$

We use the induction on l. Hence a rational map $\varphi_1=\varphi\cdot\psi_1$: $V_1\to A$ is a morphism. If ψ_1^{-1} is a morphism, φ is of course a morphism. If ψ_1 is a proper birational morphism, then φ is strictly rational and hence φ is a morphism. If ψ_1: $V_1=V-Z\subset V$, Z being a closed subset with codimension $\geqq 2$, is an open immersion, φ_1: $V-Z\to A$ extends to a morphism $V\to A$, which is φ. In the case where A is a quasi-abelian variety, we assume V to be non-singular. Then it is clear that $Y, V_{l-1},\cdots, V_1$ could be assumed to be non-singular. Hence by the similar argument as above, we complete the proof.

COROLLARY. *Let V_1 and V_2 be affine algebraic varieties. Then V_1 is WPB-equivalent to V_2 if and only if the normalization of V_1 is isomorphic to that of V_2.*

Example. Let V be an affine normal algebraic variety of dimension $\geqq 2$. Take a few points $p_1, \cdots, p_r \in V$. Then

$$\mathrm{Aut}(V-(p_1, \cdots, p_r)) = \mathrm{P\,Bir}(V-(p_1, \cdots, p_r)) \subset \mathrm{WPB}(V-(p_1, \cdots, p_r)) = \mathrm{Aut}(V).$$

§ 5. Fixed part of $\Lambda_m(V, \bar{V})$.

Let V be a normal algebraic variety with $\bar{P}_m(V) \geqq 1$. Take a normal completion $\bar{V}$ of V and define a linear system $\Lambda_m(V, \bar{V})$ on $\bar{V}$ as follows: Let $\mu: \bar{V}^* \to \bar{V}$ be a non-singular model of $\bar{V}$ such that $\bar{V}^*$ is a completion of $V^* = \mu^{-1}(V)$ with smooth boundary $D = \bar{V}^* - V^*$. Then a linear system $\mu_* |m(K(\bar{V}^*) + D)|$ does not depend on the choice of $\bar{V}^*$, which we call the *logarithmic m-canonical linear system* of V. And we write this as $\Lambda_m(V, \bar{V})$.

PROPOSITION 3. *Let V be a normal variety with $\bar{P}_m(V) \geqq 1$ and φ a proper birational map of W into V, W being a non-singular algebraic variety. Suppose that there is a closed subset Z of W with codimension $\geqq 2$ such that $\varphi(Z)$ contains an effective divisor H. Then the closure $\bar{H}$ in $\bar{V}$ is a component of $(1/m)(\Lambda_m(V, \bar{V})_{\mathrm{fix}})$.*

PROOF. Let $\bar{W}$ be a completion of W with smooth boundary D_W. Since $\varphi^{-1}(\mu | V^*): V^* \to W$ is a proper birational map, we may assume that $\psi = \varphi^{-1}(\mu | V^*)$ and $\bar{\psi}: \bar{V}^* \to \bar{W}$ are morphisms after changing μ suitably. From the logarithmic ramification formula, it follows that

$$m(K(\bar{V}^*) + D) \sim \psi^* m(K(\bar{W}) + D_W) + m\bar{R}_{\psi}.$$

Then

$$\Lambda_m(V, \bar{V}) = \mu_* \psi^* |m(K(\bar{W}) + D_W)| + m\mu_* \bar{R}_{\psi}.$$

Since $(\mu | V^*)^{-1}(\varphi(Z)) = \psi^{-1}(\bar{Z}) \cap V^*$ is an effective divisor, we have

$$\bar{R}_{\psi} \geqq \text{the closure of } \psi^{-1}(\bar{Z}) \cap V^* \quad \text{in} \quad \bar{V}^*.$$

Hence

$$\Lambda_m(V, \bar{V})_{\mathrm{fix}} \geqq m\mu_* \bar{R}_{\psi} = m\bar{H}.$$

Q.E.D.

COROLLARY. *Let V be a non-singular algebraic surface with $\bar{P}_m(V) \geqq 1$ and suppoe $\Lambda_m(V, \bar{V})_{\mathrm{fix}} = \phi$, $\bar{V}$ being a normal completion of V. A WPB-map $\varphi: W \to V$, W being non-singular, turns out to be a morphism. Hence, $WPB(V) = \mathrm{Aut}(V)$.*

§ 6. Saturated equivalence and W$^\infty$PB-equivalence.

Let $\tilde{R}$ be a subset of the set of birational maps which defines an equivalence relation R of algebraic varieties. For simplicity, we write $\tilde{R}=R$. We say that R is saturated if R satisfies the following property (S): Let V and W be algebraic varieties and $f: V\to W$ a morphism. If there exist an algebraic variety U and a morphism $g: U\to V$ or $h: W\to U$ such that $f\cdot g\in R$ or $h\cdot f\in U$, respectively, then f and g or h belong to R.

For example, the proper birational equivalence is saturated. However, WPB is not so. Hence, put

$$\mathcal{W}=\{f: V\longrightarrow W;\ \text{there exist } U \text{ and } g \text{ or } h \text{ such that}\\ f\cdot g\in \mathrm{WPB} \text{ or } h\cdot f\in \mathrm{WPB}, \text{ respectively}\}.$$

Note that if $f\in\mathcal{W}$, then we have, for any $M=(m_1,\cdots,m_n)$, n-being dim V,

$$\bar{P}_M(U)\geqq\bar{P}_M(V)\geqq\bar{P}_M(W)=\bar{P}_M(U)$$

or

$$\bar{P}_M(V)\geqq\bar{P}_M(W)\geqq\bar{P}_M(U)=\bar{P}_M(V)\,.$$

Hence $\bar{P}_M(V)=\bar{P}_M(W)$. Define

$$\mathrm{WWPB}=\{f_1\cdot\cdots\cdot f_m;\ f_j^{\pm1}\in\mathcal{W}\},$$

which defines a WWPB-equivalence. Needless to say, $\bar{P}_M$ is a WWPB-invariant. In the same way as in §§ 1, 2, 3, we can define WWPB-map, WWSR-map, WWSB-map; WWSB(V), WWPB(V). We have the following

THEOREM 4. *If V is of hyperbolic type, then $WWSB(V)$ is a finite group and hence $WWSB(V)=WWPB(V)$.*

Proof is easy and omitted.

THEOREM 5. *Let V be an algebraic variety of dimension $n=2$ or 3. Suppose that $\bar{\kappa}(V)=\kappa(V)=0$ and $q(V)=n$. Then V is WWPB-equivalent to an abelian variety.*

PROOF. We may assume that V is non-singular. By $\bar{V}$ we denote a completion of V with smooth boundary $\bar{D}$. The Albanese map α: $\bar{V}\to\mathcal{A}$ is a birational morphism by Ueno's Theorem [5]. Put $F=\alpha(\bar{D})$. Then by Theorem 5 [3], codim $F\geqq2$. Therefore

$$\bar{V}-\alpha^{-1}(F)\subset V\xrightarrow{\alpha^\circ}\mathcal{A}-F,\quad \alpha^\circ \text{ being } \alpha|V\,.$$

Since $(\bar{V}-\alpha^{-1}(F)\to\mathcal{A}-F)\in\mathrm{PB}$ and $(\mathcal{A}-F\hookrightarrow\mathcal{A})\in\mathrm{WPB}$, α° belongs to $\mathcal{W}$. Hence the quasi-Albanese map α_V of V is WWPB-map. Q.E.D.

WWPB may not be saturated. We introduce a sequence of birational equivalences of certain kinds:

$$\mathrm{W}^r\mathrm{PB}=\mathrm{WW}^{(r-1)}\mathrm{PB}.$$

Moreover, we have the notion of W^rPB-map, W^rSB-map, $\mathrm{W}^r\mathrm{PB}(V)$, $\mathrm{W}^r\mathrm{SB}(V)$. Finally, $\mathrm{W}^\infty\mathrm{PB}=\bigcup_r \mathrm{W}^r\mathrm{PB}$ is introduced. Then it is clear that W^∞PB is saturated.

$\bar{P}_M(V)$ is a W^∞PB-invariant. We have the following final

THEOREM 6. *If V is of hyperbolic type, then $W^rSB(V)$ is a finite group. Hence $W^\infty SB(V)=W^\infty PB(V)$.*

We can prove the minimality of affine varieties and quasi-abelian varieties.

PROPOSITION 4. *A $W^\infty SR$-map φ of a normal algebraic variety V (resp. non-singular variety) into an affine variety A (resp. a quasi-abelian variety) turns out to be a morphism.*

PROOF. For simplicity, we assume that φ is a W^2SR-map. We use the same notation and proof as in the proof of Theorem 3. It suffices to prove that φ is a morphism under the assumption that $\varphi_1=\varphi\cdot\phi_1$: $V_1\to A$ is a morphism. If ϕ_1^{-1} is a morphism, φ is of course a morphism. Assume $\phi_1\in\mathcal{W}$. We have an algebraic variety U and a morphism g: $U\to V_1$ such that $\phi_1\cdot g\in$WPB or a morphism h: $V\to U$ such that $h\cdot\phi_1\in$WPB. $\varphi_1\cdot g$ is a morphism $U\to A$ such that $g\cdot\phi_1\in$WPB. Hence by Theorem 3, φ turns out to be a morphism. Next, we consider h. Then take the normalization μ: $U'\to U$ of U. We have a

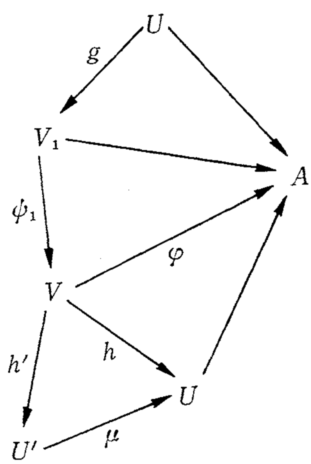

Fig. 3.

morphism h': $V \to U'$ derived from h. Then $h' \cdot \psi_1 \in$ WPB and so $\varphi \cdot h'^{-1}: U' \to A$ is a morphism by Theorem 3. Hence φ turns out to a morphism. The similar argument will do for quasi-abelian varieties. Q.E.D.

REMARK. It is expected that the following statements for an algebraic variety V of dimension n are equivalent to each other:

a) V is W^2PB-equivalent to a quasi-abelian variety,

a)′ V is W^∞PB-equivalent to a quasi-abelian variety,

b) $\bar{\kappa}(V) \geqq 0$ and $\bar{q}(V) \geqq n$,

c) $\dim W^\infty PB(V) \geqq n$ and $\bar{\kappa}(V) \geqq 0$.

We have seen that a)⇒a)′, a)′⇒b) and c).

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