On the propagation of analyticity of solutions of systems of linear differential equations with constant coefficients

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The purpose of this note is to prove a theorem on the propagation of analyticity of hyperfunction solutions of general systems of linear differential equations with constant coefficients. This theorem is proved in Kawai [1] Theorem 5.2.1 in a little more precise form when the parameter k used in the theorem *is equal to* 1. There we have used the theory of Fourier hyperfunctions and an inequality due to Malgrange and Hörmander on the division of holomorphic functions by a polynomial. The proof of the theorem given in this note is entirely different from the one given in our previous paper Kawai [1], though it will be possible to modify the proof given in Kawai [1] to prove the general case treated in this note. The present writer believes that the proof given in this note is the clearest one of this sort.

Note that this theorem is proved for distribution solutions by Palamodov [1] § 15.2° by the method of *a priori* estimate. We follow his reasonings as for the algebraic part, i.e., use his lemma in p. 412 of Palamodov [1], which is obvious intuitively. We use the same notations as in Palamodov [1] and refer the reader to the book about their definitions.

THEOREM. *Let a* $\mathcal{P}$*-module* M *satisfy* $\mathrm{Ext}^i(M, \mathcal{P})=0$ *for* $i=0, \cdots, k-1, 1\leqq k\leqq n$, *where* $\mathcal{P}$ *denotes the polynomial ring in* n *variables over* $\boldsymbol{C}$. *Denote by* Π *a compact polyhedron in* $\boldsymbol{R}^n$, *by* Γ_{n-k} *an open neighbourhood of the* $(n-k)$*-dimensional skeleton of* Π, *and by* Ω *an open neighbourhood of* Π. *Suppose that* $u(x)$ *belongs to* $\mathrm{Hom}(M, \mathcal{B}(\Omega))$, *where* $\mathcal{B}(\Omega)$ *denotes the space of hyperfunctions on* Ω, *which may be considered as a* (*left*)-$\mathcal{P}$*-module by the usual action of differentiation. Assume further that* $u(x)$ *is real analytic in* Γ_{n-k}. *Then* $u(x)$ *is real analytic in a neighbourhood of* Π.

PROOF. First assume that $k=1$. Then we can assume that there is a non-zero polynomial $q(\xi_1, \cdots, \xi_n)$ which vanishes on $N(M)$, the variety associated with the $\mathcal{P}$-module M. (See p. 138 of Palamodov [1] about the definition of $N(M)$.) Then applying the lemma in Palamodov [1] p. 412, we find a differential operator $Q(iD)$ such that $Q(iD)u=0$ holds, where Q is a diagonal matrix and its diagonal elements are sufficiently high powers of q. Therefore it is sufficient to prove that, in general, a hyperfunction $v(x)$ is real analytic in Ω if it is real analytic near the boundary of Ω, and if it satisfies $R(iD)v=0$ in Ω for some linear differential operator with constant coefficients $R(iD)$. Now let $E(x)$ be an elementary solution of $R(iD)$, i.e., $R(iD)E=\delta$, whose existence is well known.

Now consider the following trivial exact sequence (1) of sheaves, where $\mathcal{A}$ and $\mathcal{B}$ denote the sheaves of germs of real analytic functions and hyperfunctions respectively, and $\mathcal{B}/\mathcal{A}$ denotes the quotient sheaf.

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \xrightarrow{s} \mathcal{B}/\mathcal{A} \longrightarrow 0 . \tag{1}$$

The mapping s is by definition the natural surjection from $\mathcal{B}$ to $\mathcal{B}/\mathcal{A}$. By the assumption on v, $s(v)$ has compact support as an element of $H^0(\Omega, \mathcal{B}/\mathcal{A})$, hence $s(v)$ can be considered as an element of $H^0(\boldsymbol{R}^n, \mathcal{B}/\mathcal{A})$ if we define it as zero outside Ω. Since $H^1(\boldsymbol{R}^n, \mathcal{A})=0$, we can find a hyperfunction $w(x)$ on $\boldsymbol{R}^n$ such that $s(w)=s(v)$ holds. Then $s(R(iD)w)=0$ holds, because $w(x)$ is real analytic in $\boldsymbol{R}^n \backslash \operatorname{supp} s(v)$ and $R(iD)w=R(iD)v+f$ holds in Ω with some real analytic function $f(x)$ defined in Ω. Thus we have the following equality (2) between sections of sheaf $\mathcal{C}$. Note that the integrations which appear in (2) make sense as integrations along fiber of sections of sheaf $\mathcal{C}$. (Cf. Sato [1] Theorem 6.5.2.)

$$\begin{aligned} w(x) &= R(iD_x)\int E(x-y)w(y)dy = R(iD_x)\int E(y)w(x-y)dy \\ &= \int E(y)R(iD_x)w(x-y)dy = 0 . \end{aligned} \tag{2}$$

Therefore we conclude that $w(x)$ is real analytic on $\boldsymbol{R}^n$ and this proves that $v(x)$ is real analytic in Ω by the definition of $w(x)$.

Thus we have proved the theorem in the case $k=1$.

Let us now suppose that $k>1$ and use the induction on k.

Let κ be an arbitrary $(n-k+1)$-dimensional face of the polyhedron Π, and L be a linear variety of the same dimension containing κ. In $\boldsymbol{R}^n$ we choose a system of coordinates so that the axes $x_k, \cdots, x_n$ lie in L. By the assumptions of the theorem we may suppose without loss of generality that $N(M)$ is normally placed (in the sense of Palamodov [1]). Moreover we may assume without loss of generality that $L=\{x_1=\cdots=x_{k-1}=1\}$ for the sake of simplicity. Hence we can find a non-zero polynomial $q(\xi_k, \cdots, \xi_n)$ which vanishes on $N(M)$. Then applying the lemma in p. 412 of Palamodov [1] again, we can find a diagonal matrix of size $s\times s$, which contains in its diagonal a sufficiently high power of q and for which $Q(iD_{x''})u=0$ holds. Note that $Q(iD_{x''})$ contains only differentiations with respect to the variables $x''=(x_k, \cdots, x_n)$ by the definition. We also denote $(x_1, \cdots, x_{k-1})$ by x' in the sequel. By the assumption on Ω we may assume that $Q(iD_{x''})u(x)=0$ holds in

$$V=\{x' \mid |x_l-1|<\varepsilon,\ l=1, \cdots, k-1\}\times\{x'' \mid |x_j|<1+\varepsilon,\ j=k, \cdots, n\}\ ,$$

where ε is a positive constant. Moreover choosing the constant ε sufficiently small we can assume that $V\backslash W$ contains Γ_{n-k}, where W is by definition the set

$$\{x' \mid |x_l-1|<\varepsilon,\ l=1, \cdots, k-1\}\times\{x'' \mid |x_j|\leqq 1-\varepsilon,\ j=k, \cdots, n\}\ .$$

Now we use the flabbiness of sheaf $\mathcal{B}/\mathcal{A}$, which obviously follows from the flabbiness of sheaf $\mathcal{B}$ and the vanishing of cohomology group $H^1(\omega, \mathcal{A})$ for any open set ω in $\boldsymbol{R}^n$ (Sato's remark). (Note that the flabbiness of sheaf $\mathcal{B}/\mathcal{A}$ trivially follows from the flabbiness of sheaf $\mathcal{C}$ (Kashiwara [1].) Then we can find an s-tuple of hyperfunctions $w(x)$ which satisfies the following conditions:

(i) supp $s(w)$ is contained in the closure of V,

(ii) $s(w)=s(u)$ holds in V,

and

(iii) supp $s(Qw)$ is contained in the set

$$B=\{x' \mid |x_l-1|=\varepsilon,\ l=1, \cdots, k-1\}\times\{x'' \mid |x_j|\leqq 1+\varepsilon,\ j=k, \cdots, n\}\ .$$

Since the matrix Q is diagonal, it is sufficient to show that any component of $w(x)$, e.g., $w_1(x)$, the first component of $w(x)$, is real analytic in V. By the assumption of $w(x)$ we can assume that $q(iD_{x''})^p w_1(x)$ is real analytic outside B for some non-negative integer p. Let $E(x'')$ be an elementary solution of $q(iD_{x''})^p$ on $\boldsymbol{R}^{n-k+1}$, i.e., satisfy $q(iD_{x''})^p E(x'')=\delta(x'')$. Now just in the same way as in the proof of the case $k=1$, we have the following equality (3) among sections of sheaf $\mathcal{C}$. There $\delta(x')$ denotes the delta function in $(k-1)$-variables.

$$\begin{aligned}(3)\qquad w_1(x)&=q(iD_{x''})^p\int w_1(y', y'')\delta(x'-y')E(x''-y'')dy'dy''\\&=q(iD_{x''})^p\int w_1(x'-y', x''-y'')\delta(y')E(y'')dy'dy''\end{aligned}$$

$$=q(iD_{x''})^p\int w_1(x', x''-y'')E(y'')dy''$$

$$=\int q(iD_{x''})^p w_1(x', x''-y'')E(y'')dy'' .$$

By the condition (iii) on $w(x)$ mentioned above we conclude that the last term in the equality (3) vanishes as far as x belongs to V. Therefore $w_1(x)$ is real analytic in V, hence $u_1(x)$, the first component of $u(x)$, is real analytic in V. Thus we have proved that $u(x)$ is real analytic in a neighbourhood of $(n-(k-1))$-dimensional skeleton of Π. Then we can apply the induction hypothesis. This completes the proof of the theorem.

REMARK. As is obvious from the method of the proof given above, it is sufficient to assume that $s(u)$ belongs to Hom $(M, (\mathcal{B}/\mathcal{A})(\Omega))$.

References

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