On trace of Hecke operators for discontinuous groups operating on the product of the upper half planes

By Hirofumi ISHIKAWA

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Introduction.

Let H_n be the direct product of n upper half planes and G the connected component of the identity of the group of all $g=(g^{(1)},\cdots,g^{(n)})$ with $g^{(i)}\in GL_2(\boldsymbol{R})$. We consider G as a group of transformations in H_n. Let Γ be a subgroup of G operating on H_n discontinuously with a compact fundamental domain, and satisfying (G.1) and (G.2) (§1). We fix once for all an element α in G such that Γ and $\alpha\Gamma\alpha^{-1}$ are commensurable, and denote by Γ' the subgroup of G generated by Γ and α. Let χ be a representation of Γ' by unitary matrices. Let $\{k_i\}_{i=1}^n$ be the set of positive integers. We assume that χ satisfies the conditions (R.1) and (R.2) in §1. Let $T=T(\Gamma\alpha\Gamma)$ be the Hecke operator acting on the space of automorphic forms of type $(\Gamma,\{k_i\},\chi)$ (see below). For the case of all $k_i>2$, the trace of T has been explicitly calculated in Shimizu [8][1]. Also for the case of $n=1$, $k=2$, the trace has been calculated in our previous paper [5][1]. To calculate the trace for the remaining case where at least one of k_i is 2, in this note we shall regard all k_i being equal to 2 as $2+s$ $(s>0)$; we carry T over to a subspace of $L^2(\tilde{H}_n,\Gamma)$ isomorphic to the space of automorphic forms (§1), and extend its action to $L^2(\tilde{H}_n,\Gamma)$. We consider an integral operator in $L^2(\tilde{H}_n,\Gamma)$ whose kernel function depends on k_i and s (§3), and join it with T. We follow the method of Selberg [6], and get the trace formula for our case, letting s tend towards zero.

§§1-2 are concerned with preliminary statements. In §§3-4 an explicit formula for the trace of T will be given (Theorem 1). In §5 we shall apply Theorem 1 to some discrete groups which are defined arithmetically and give the trace of the operator $T(\mathfrak{q})$ defined in Shimura [10] (Theorem 2). In §6, as an application of Theorem 2 we shall see some equations between zeta functions of quaternion algebras over a totally real algebraic number field, which are obtained in Shimizu [9] for the case that all $k_i>2$ (Theorems 3, 4).

[1] These works treated the problem under the condition that Γ has a fundamental domain of finite volume, more generally.

Notation. $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ denote the ring of rational integers, the field of rational numbers, the field of real numbers, the field of complex numbers and the division quaternion over $\boldsymbol{R}$, respectively. R being a ring, R^* denotes the group of all invertible elements in R.

§ 1. The Hecke operators.

Let H_n be the direct product of n upper half planes and consider $\tilde{H}_n = H_n \times (\boldsymbol{R}/2\pi\boldsymbol{Z})^n$ with elements (z, ϕ), where $z = (z^{(1)}, \cdots, z^{(n)}) \in H_n$, $\phi = (\phi^{(1)}, \cdots, \phi^{(n)}) \in \boldsymbol{R}^n$, $\phi^{(i)}$ being identified with $\phi^{(i)} + 2\pi$. Let G be the connected component of the identity of the group of all $g = (g^{(1)}, \cdots, g^{(n)})$ with $g^{(i)} \in GL_2(\boldsymbol{R})$, and $\tilde{G} = G \times (\boldsymbol{R}/2\pi\boldsymbol{Z})^n$ with elements (g, θ) $(\theta = (\theta^{(1)}, \cdots, \theta^{(n)}))$, and it acts on the space (z, ϕ) as

$$(g, \theta)(z, \theta) = (g^{(1)}z^{(1)}, \cdots, g^{(n)}z^{(n)}, \phi^{(1)} + \arg(c^{(1)}z^{(1)} + d^{(1)}) - \theta^{(1)}, \cdots,$$
$$\phi^{(n)} + \arg(c^{(n)}z^{(n)} + d^{(n)}) - \theta^{(n)})$$

$$g^{(i)}z^{(i)} = \frac{a^{(i)}z^{(i)} + b^{(i)}}{c^{(i)}z^{(i)} + d^{(i)}}, \quad \left(g^{(i)} = \begin{pmatrix} a^{(i)}, & b^{(i)} \\ c^{(i)}, & d^{(i)} \end{pmatrix}\right).$$

We denote by $Z(G)$ the center of G and by ι the canonical homomorphism of G onto $G/Z(G)$. Let Γ be a subgroup of G operating on H_n discontinuously with a compact fundamental domain. It is assumed through out this paper that

(G.1) $\iota(\Gamma)$ is an irreducible[2] subgroup of $\iota(G)$,

(G.2) Γ contains all elements of the intersection of $Z(G)$ and the direct product of $SL_2(R)^n$.

Then the subgroup $\tilde{\Gamma} = \Gamma \times \{0\}$ of $\tilde{G}$ acts on $\tilde{H}_n$ discontinuously with a compact fundamental domain. We identify Γ with $\tilde{\Gamma}$; so we shall write Γ instead of $\tilde{\Gamma}$. We fix once and for all an element α in G such that $\alpha\Gamma\alpha^{-1}$ is commensurable with Γ, and denote by Γ' the subgroup of G generated by Γ and α. Let χ be a representation of Γ' by unitary matrices of degree ν and denote by $V = \boldsymbol{C}^\nu$ the representation space of χ. We assume that

(R.1) the kernel Γ_χ of χ in Γ is of finite index in Γ,

(R.2) $\chi(\varepsilon) = \prod_{i=1}^{n} (\operatorname{sgn} \varepsilon^{(i)})^{k_i}$, for $\varepsilon \in Z(\Gamma) = Z(G) \cap \Gamma$.

By an automorphic form of type $(\Gamma, \{k_i\}, \chi)$, we understand a function $F(z)$ on H_n

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[2] Cf. [8].

taking values in the representation space of χ, which satisfies the following conditions:

(A.1) $\quad F(z)$ is holomorphic on H_n,

(A.2) $\quad F(\gamma z)=\chi(\gamma)j(\gamma, z)^{-1}F(z)$, for $\gamma\in\Gamma$.

Here, $j(g,z)$ denotes $\prod_{i=1}^{n}(c^{(i)}z^{(i)}+d^{(i)})^{-k_i}(\det g^{(i)})^{k_i/2}$, for $g\in G$ and $z\in H_n$. The linear space consisting of all $F(z)$ is denoted by $\mathscr{A}(\Gamma,\{k_i\},\chi)$. By the definition, $F(z)$ is invariant under the action of the elements of $Z(\Gamma)$. We now define the Hecke operator $T(\Gamma\alpha\Gamma)$ in $\mathscr{A}(\Gamma,\{k_i\},\chi)$. Let $\Gamma\alpha\Gamma=\bigcup_{\mu=1}^{d}\alpha_\mu\Gamma$ be the right Γ-cosets decomposition of $\Gamma\alpha\Gamma$. For $F\in\mathscr{A}(\Gamma,\{k_i\},\chi)$, we set

$$T(\Gamma\alpha\Gamma)F(z)=\sum_{\mu=1}^{d}\chi(\alpha_\mu)j(\alpha_\mu^{-1},z)F(\alpha_\mu^{-1}z)\,. \tag{1.1}$$

Let $L^2(\tilde{H}_n,\Gamma)$ be the space of functions $F(z,\phi)$ on $\tilde{H}_n$ taking values in $\boldsymbol{C}^\nu$ and satisfying the following conditions:

(i) $F(z,\phi)=\begin{pmatrix}f_1(z,\phi)\\ \vdots\\ f_\nu(z,\phi)\end{pmatrix}$, each $f_i(z,\phi)$ is a measurable function on $\tilde{H}_n$ taking values in $\boldsymbol{C}$,

(ii) $F(\gamma(z,\phi))=\chi(\gamma)F(z,\phi)$, for $\gamma\in\Gamma$,

(iii) $\int_{\Gamma\backslash\tilde{H}_n}{}^tF(z,\phi)\overline{F(z,\phi)}dz\,d\phi<\infty,\left(dz=\prod_{i=1}^{n}\frac{dx^{(i)}dy^{(i)}}{y^{(i)2}},\ d\phi=\prod_{i=1}^{n}d\phi^{(i)}\right).$

Put $G_1=SL_2(\boldsymbol{R})\times\cdots\times SL_2(\boldsymbol{R})$ (n-times) and $\tilde{G}_1=G_1\times(\boldsymbol{R}/2\pi\boldsymbol{Z})^n$. Let $C^\infty(\tilde{H}_n)$ be the space of C^∞-class functions on $\tilde{H}_n$ taking values in $\boldsymbol{C}$. It is well known that the ring of all $\tilde{G}_1$-invariant differential operators on $C^\infty(\tilde{H}_n)$ is generated by

$$\frac{\partial}{\partial\phi^{(i)}},\ \tilde{\Delta}^{(i)}=y^{(i)2}\left(\frac{\partial^2}{\partial x^{(i)2}}+\frac{\partial^2}{\partial y^{(i)2}}\right)+y^{(i)}\frac{\partial}{\partial x^{(i)}}\frac{\partial}{\partial\phi^{(i)}}\qquad(1\leq i\leq n)\,. \tag{1.2}$$

Generally, for a $\tilde{G}_1$-invariant differential-integral operator L in $C^\infty(\tilde{H}_n)$ and $F\in L^2(\tilde{H}_n,\Gamma)$, we define LF simply by

$$LF\in\begin{pmatrix}Lf_1\\ \vdots\\ Lf_\nu\end{pmatrix},$$

if $Lf_1,\cdots,Lf_\nu$ are well-defined and if $LF\in L^2(\tilde{H}_n,\Gamma)$. In such a case, we regard L as an operator in $L^2(\tilde{H}_n,\Gamma)$. Thus $\frac{\partial}{\partial\phi^{(i)}}$, $\Delta^{(i)}$ and etc. will also be considered as operators in $L^2(\tilde{H}_n,\Gamma)$.

§2. The decomposition of $L^2(\tilde{H}_n, \Gamma)$.

We shall give the classification of eigen spaces of the differential operators given by (1.2) in $L^2(\tilde{H}_n, \Gamma)$, with the aid of the representation theory of groups. Since Bargmann has given the classification of irreducible unitary representations of $SL_2(\boldsymbol{R})$, we get the classification of those of G_1 as tenser product representations.

Now, we make each element $g \in G_1$ correspond a unitary operator T_g in $L^2(\tilde{H}_n, \Gamma)$ of the following kind:

$$T_g \varphi(\tilde{\omega} g') = \varphi(\tilde{\omega} g' g) ,$$

where

$$\tilde{\omega}: g' = \left(\begin{pmatrix} 1, x^{(i)} \\ 0, \ 1 \end{pmatrix} \begin{pmatrix} y^{(i)1/2}, & 0 \\ 0 & , y^{(i)-1/2} \end{pmatrix} \begin{pmatrix} \cos\phi^{(i)}, & -\sin\phi^{(i)} \\ \sin\phi^{(i)}, & \cos\phi^{(i)} \end{pmatrix} \right) \longrightarrow ((z^{(i)} = x^{(i)} + \sqrt{-1}\, y^{(i)}), (\phi^{(i)}))$$

is the canonical isomorphism of G_1 onto $\tilde{H}_n$. Put $K = SO_2(\boldsymbol{R}) \times \cdots \times SO_2(\boldsymbol{R})$ (n-times) and denote a one-dimensional representation of K by

$$\sigma_m: \kappa = \left(\begin{pmatrix} \cos\theta^{(i)}, & -\sin\theta^{(i)} \\ \sin\theta^{(i)}, & \cos\theta^{(i)} \end{pmatrix} \right) \longrightarrow \exp\left(-\sqrt{-1} \sum_{i=1}^{n} m^{(i)} \theta^{(i)}\right), \quad (m = (m^{(1)}, \cdots, m^{(n)}) \in \boldsymbol{Z}^n).$$

As is well known, $L^2(\tilde{H}_n, \Gamma)$ decomposes into the sum of a countable number of irreducible representations and each irreducible representation enters into $L^2(\tilde{H}_n, \Gamma)$ with finite multiplicity. Let $(T, \mathscr{H})$ be an irreducible representation of G_1. Put

$$\mathscr{H}(\sigma_m) = \{\varphi \in \mathscr{H} \mid T(\kappa)\varphi = \sigma_m(\kappa)\varphi , \quad \text{for all } \kappa \in K\} ,$$

$$\mathscr{H}_0 = \sum_m \mathscr{H}(\sigma_m) .$$

Then $\mathscr{H}_0$ is a dense set of analytic vectors in $\mathscr{H}$. Denote by $\mathfrak{G}$, $U(\mathfrak{G})$ and by $U(\mathfrak{G})_C$ the Lie algebra of G_1, the universal envelopping algebra of $\mathfrak{G}$ and $U(\mathfrak{G}) \otimes \boldsymbol{C}$, respectively. Then as is known, we can give the differential representation of T in $\mathscr{H}_0$ by

$$T(X)\varphi = \left(\frac{d}{dt} T_{\exp(tX)} \varphi \right)_{t=0} , \quad \text{for} \quad X \in \mathfrak{G} .$$

It is well known that this representation is able to be extended to the representation of $U(\mathfrak{G})$ and then to that of $U(\mathfrak{G})_C$, $\boldsymbol{C}$-linearly. Choose a basis of $sl(2, \boldsymbol{R})$ as follows:

$$X_1 = \begin{pmatrix} 1, & 0 \\ 0, & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0, 1 \\ 1, 0 \end{pmatrix} \quad \text{and} \quad X_3 = \begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix}.$$

Put

$$V^{+}=X_1+\sqrt{-1}\,X_2,\quad V^{-}=X_1-\sqrt{-1}\,X_2\quad\text{and}\quad D=X_1^2+X_2^2-X_3^2.$$

For each i, define elements of $U(\mathfrak{G})_C$ by:

$$X_{3,i}=(\overbrace{0,\cdots,0,X_3}^{i},0,\cdots,0),$$

$$V_i^{\pm}=(0,\cdots,0,V^{\pm},0,\cdots,0),$$

$$D_i=(0,\cdots,0,D,0,\cdots,0).$$

If φ is an element of $\mathscr{H}(\sigma_m)$, we get

$$T(X_{3,i})\varphi=\sqrt{-1}\,m^{(i)}\varphi,$$

$$T(V_i^{\pm})\varphi\in\mathscr{H}(\sigma_{m\pm\underbrace{(0,\cdots,0,2}_{i},0,\cdots,0)}).$$

Now, we define the subspace $M(m,\lambda)$ of $L^2(\tilde{H}_n,\Gamma)$ consisting of φ satisfying the following conditions:

(i) $T_\kappa\varphi=\sigma_m(\kappa)\varphi$, for $\kappa\in K$,

(ii) $T(D_i)\varphi=4\lambda^{(i)}\varphi$, $(1\leq i\leq n)$.

We carry out the same argument as [5, §2.2] substituting $X_{3,i}, V_i^{\pm}, D_i$ for $X_3, V^{\pm}$, D, respectively. In our case of $n>1$, we take into consideration the following fact. Suppose there exists a function φ in $M(m,\lambda)$ with $m^{(i)}=0$ and $\lambda^{(i)}=0$ for all $i\in I$ which is a subset of $[1,n]$, then φ depends only on $(z^{(j)},\phi^{(j)})$ for $j\in J=[1,n]-I$. It follows from Proposition 1.1 in Matsushima and Shimura[3)] that the projection of Γ_χ into the partial product of G corresponding to $j\in J$ is dense in the partial product. As φ is an eigenfunction of $\tilde{\Delta}^{(j)}(j\in J)$, φ is constant and $m^{(j)}=0$ for $j\in J$. Now we get the following proposition.

PROPOSITION 1. *If Γ and χ satisfy the conditions given in* §1, *the classification of the eigen spaces in $L^2(\tilde{H}_n,\Gamma)$ for each eigenvalue-set $(\{-\sqrt{-1}\,m^{(i)}\},\{\lambda^{(i)}\})$ of $\left(\left(\frac{\partial}{\partial\phi^{(i)}}\right),(\tilde{\Delta}^{(i)})\right)$ which is restricted within non-negative $m^{(i)}$'s is given by the Table* I. *In the series A, $\lambda_\rho^{(i)}$ ranges over all eigenvalues of $\tilde{\Delta}^{(i)}$ satisfying $M(\{\delta^{(i)}\},\{\lambda_\rho^i\})\neq\{0\}$, $(\delta^{(i)}=0,1)$, except $\lambda_\rho^{(i)}=0,-\frac{1}{2}$. In the series C, $\lambda_\rho'^{(i)}$ ranges over all eigenvalues of $\tilde{\Delta}^{(i)}$ satisfying $M(\{\delta^{(i)}\},\{\lambda_\rho'^{(i)}\})\neq\{0\}$, except $\lambda_\rho'^{(i)}=0,-\frac{1}{2}$.*

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3) Matsushima, Y. and G. Shimura, On the cohomology groups attached to certain vector valued differential forms on the product of the upper half planes, Ann. of Math. **78** (1963), 417-449.

Table I.

Series[4]	$\{m^{(i)}\}$	$\{\lambda^{(i)}\}$	Isomorphic space to $M(\{m^{(i)}\}, \{\lambda^{(i)}\})$
A	$m^{(i)}\equiv 0(2)$, $(k_i=\text{even})$ $m^{(i)}\equiv 1(2)$, $(k_i=\text{odd})$	$\lambda_\rho^{(i)}$	$M(\{\delta^{(i)}\}, \{\lambda_\rho^{(i)}\})$ $(\delta^{(i)}=0$ or $1)$
B	$m^{(i)}=k_i+2r^{(i)}$ $(r^{(i)}=0,1,2,\cdots)$	$\lambda^{(i)}=\frac{1}{4}k_i^2-\frac{1}{2}k_i$	$\mathscr{A}(\Gamma, \{k_i\}, \chi)$
C	$m^{(i)}\equiv 0(2)$, $(k_i=\text{even})$ $m^{(i)}\equiv 1(2)$, $(k_i=\text{odd})$ $m^{(j)}=k_j+2r^{(j)}$	$\lambda_\rho'^{(i)}$ $\lambda^{(j)}=\frac{1}{4}k_j^2-\frac{1}{2}k_j$	$M(\{\delta^{(i)}, k_j\}, \{\lambda_\rho'^{(i)}, \lambda^{(j)}\})$
D	0	0	V^Γ

Here, V^Γ is the subspace of V of which elements are stable under the action of Γ; the isomorphism of $M\left(\{k_i\}, \left\{\frac{1}{4}k_i^2-\frac{1}{2}k_i\right\}\right)$ onto $\mathscr{A}(\Gamma, \{k_i\}, \chi)$ is given by

$$F(z, \phi)\longrightarrow \prod_{i=1}^{n}(\exp(-\sqrt{-1}\,k_i\phi^{(i)})y^{(i)k_i/2})^{-1}F(z, \phi)\,, \tag{2.1}$$

for $F(z, \phi)\in M\left(\{k_i\}, \left\{\frac{1}{4}k_i^2-\frac{1}{2}k_i\right\}\right)$.

§3. An operator K_s.

In order to calculate the trace of Hecke operators acting on $\mathscr{A}(\Gamma, \{k_i\}, \chi)$, we shall write down the action of the Hecke operators carried over to the space

$$M\left(\{k_i\}, \left\{\frac{1}{4}k_i^2-\frac{1}{2}k_i\right\}\right)$$

by the canonical isomorphism (2.1), and extend it to the space $L^2(\tilde{H}_n, \Gamma)$. Thus

$$T(\Gamma\alpha\Gamma)F(z, \phi)=\sum_{\mu=1}^{d}\chi(\alpha_\mu)F(\alpha_\mu^{-1}(z, \phi)) \tag{3.1}$$

for $F\in L^2(\tilde{H}_n, \Gamma)$. If necessary exchanging the indexes of $\{k_i\}$, we can assume, from now on, that $k_1=\cdots=k_{n_0}=2, k_j>2$ $(n_0<j\leq n)$. For the calculation of its trace we consider a $\tilde{G}$-invariant integral operator k_s in $C^\infty(\tilde{H}_n)$ defined by a point pair invariant kernel: for $s>0$,

[4] Series A, B and C correspond to the tenser product representations of principal and/or supplementary series, discrete series, and discrete and principal and/or supplementary series, respectively.

$$k_s(z,\phi,z',\phi')=\prod_{i=1}^{n_0}(k_s^{(2)}(z^{(i)},\phi^{(i)},z'^{(i)},\phi'^{(i)})-\hat{c}(s)\hat{k}_s^{(2)}(z^{(i)},z'^{(i)}))$$
$$\times\prod_{j=n_0+1}^{n}k^{(k_j)}(z^{(j)},\phi^{(j)},z'^{(j)},\phi'^{(j)}),$$

$$k_s^{(2)}(z,\phi,z',\phi')=\exp(-2\sqrt{-1}(\phi-\phi'))\left[\frac{(yy')^{1/2}}{(z-\bar{z}')/2\sqrt{-1}}\right]^2\frac{(yy')^{s/2}}{|(z-\bar{z}')/2\sqrt{-1}|^s}, \tag{3.2}$$

$$\hat{k}_s^{(2)}(z,z')=\frac{(yy')^{(2+s)/2}}{|(z-\bar{z}')/2\sqrt{-1}|^{2+s}},$$

$$k^{(k_j)}(z,\phi,z',\phi')=\exp(-k_j\sqrt{-1}(\phi-\phi'))\left[\frac{(yy')^{1/2}}{(z-\bar{z}')/2\sqrt{-1}}\right]^{k_j},$$

$$\hat{c}(s)=\frac{s}{2+s}.$$

By the general theory, every element in $M(\{m^{(i)}\},\{\lambda^{(i)}\})$ is an eigenfunction of k_s and its eigenvalue depends only on $(\{m^{(i)}\},\{\lambda^{(i)}\})$; so we write the eigenvalue of k_s with $h_s(\{m^{(i)}\},\{\lambda^{(i)}\})$. We can express the operator $T(\Gamma\alpha\Gamma)$, restricted to $M(\{m^{(i)}\},\{\lambda^{(i)}\})$, by k_s in the following way.

$$T(\Gamma\alpha\Gamma)F(z,\phi)=h_s(\{m^{(i)}\},\{\lambda^{(i)}\})^{-1}\times\sum_{\mu=1}^{d}\int_{\tilde{H}_n}\chi(\alpha_\mu)k_s(\alpha_\mu^{-1}(z,\phi),z',\phi')F(z',\phi')dz'd\phi'. \tag{3.3}$$

But for $s>0$, the kernel $k_s(z,\phi,z',\phi')$ is of (a)-(b) type in the sense of Selberg [6], therefore

$$\sum_{\gamma\in\Gamma}\chi(\gamma)k_s(z,\phi,\gamma(z',\phi'))$$

is absolutely convergent for all $(z,\phi),(z',\phi')\in\tilde{H}_n$, and uniformly if (z,ϕ) and (z',ϕ') are contained in some compact subregion of $\tilde{H}_n$. Now we have

$$\sum_{\mu=1}^{d}\int_{\tilde{H}_n}\chi(\alpha_\mu)k_s(\alpha_\mu^{-1}(z,\phi),z',\phi')dz'd\phi'=\int_{\Gamma\backslash\tilde{H}_n}K_s(z,\phi,z',\phi')dz'd\phi',$$

where

$$K_s(z,\phi,z',\phi')=\sum_{g\in\Gamma\alpha\Gamma}\chi(g)k_s(z,\phi,g(z',\phi')).$$

As the fundamental domain of Γ is compact, the operator K_s with the kernel $K_s(z,\phi,z',\phi')$ is completely continuous. The next proposition is proved by the direct calculations.

PROPOSITION 2. *The eigenvalue $h_s(\{m^{(i)}\},\{\lambda^{(i)}\})$ for $M(\{m^{(i)}\},\{\lambda^{(i)}\})$ in which k_s does not vanish is given in 'Table II'. Here, the series A and D appear only if all $k_i=2$.*

Table II.

Series	$\{m^{(i)}\}$	$\{\lambda^{(i)}\}$	Eigenvalue $h_s(\{m^{(i)}\},\{\lambda^{(i)}\})$ of k_s	Trace of T in $M(\{m^{(i)}\},\{\lambda^{(i)}\})$
A	$m^{(i)}=0$ or 2 $(1\leq i\leq n)$	$\lambda^{(i)}=\lambda_\rho^{(i)}$	$\left(-8\pi 2^s\dfrac{c(s)}{\Gamma(1+s)}\right)^n \times\prod_{i=1}^{n}\Gamma\left(\dfrac{s}{2}+\delta_\rho^{(i)}\right) \times\Gamma\left(\dfrac{s+2}{2}-\delta_\rho^{(i)}\right)$	$t_{m,\rho}$
B	$m^{(i)}=k_i$ $(1\leq i\leq n)$	$\lambda^{(i)}=\dfrac{1}{4}k_i^2-\dfrac{1}{2}k_i$	$a(\{k_i\})^{-1}\left(2\cdot 2^s\dfrac{\Gamma\left(1+\frac{s}{2}\right)^2\Gamma\left(\frac{1}{2}\right)}{\Gamma(1+s)} \times\dfrac{\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(2+\frac{s}{2}\right)}\right)^{n_0}(2\pi)^{n-n_0}$	t_0
C	$m^{(i)}=0$ or 2 $(i\in I)$ $m^{(j)}=k_j$ $(j\in J)$	$\lambda^{(i)}=\lambda_{\rho'}^{(i)}$ $\lambda^{(j)}=\dfrac{1}{4}k_j^2-\dfrac{1}{2}k_j$	$a(\{k_i\})^{-1}\prod_{i\in I}\left(-2\cdot 2^s\dfrac{c(s)}{\Gamma(1+s)} \times\Gamma\left(\dfrac{s}{2}+\delta_{\rho'}^{(i)}\right)\Gamma\left(\dfrac{s+2}{2}-\delta_{\rho'}^{(i)}\right)\right) \times\prod_{j\in J,\, j\leqq n_0}\left(2\cdot 2^s\dfrac{\Gamma\left(1+\frac{s}{2}\right)^2\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1+s}{2}\right)}{\Gamma(1+s)\Gamma\left(2+\frac{s}{2}\right)}\right) \times\prod_{j\in J,\, j>n_0}(2\pi)$	$t_{m,\rho'}$
D	$m^{(i)}=0$	$\lambda^{(i)}=0$	$\left(-8\pi 2^s\dfrac{\Gamma\left(1+\frac{s}{2}\right)^2\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1+s}{2}\right)}{\Gamma(1+s)\Gamma\left(2+\frac{s}{2}\right)}\right)^n$	t_1

The notations used in this table is defined as follows.

$$c(s)=\frac{s}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(2+\frac{s}{2}\right)},$$

$$\lambda^{(i)}=\delta^{(i)}(\delta^{(i)}-1),$$

$$a(\{k_i\})=(4\pi)^{-n}\prod_{i=1}^{n}(k_i-1),$$

$\lambda_\rho^{(i)}, \lambda_{\rho'}^{(i)}$ are as same as defined in Table I. In the series C, I denotes a subset of $\{1,\cdots,n_0\}$ and $I\cup J=\{1,\cdots,n\}$.

From now on, for simplicity, we regard that the series A is concluded in the series C. Considering the trace of K_s in $L^2(\tilde{H}_n,\Gamma)$, we obtain

$$\text{(3.4)}\qquad \operatorname{tr} K_s=h_s\left(\{k_i\},\left\{\frac{1}{4}k_i^2-\frac{1}{2}k_i\right\}\right)t_0+\sum_{(m,\rho')}h_s(\{m^{(i)}\},\{\lambda_{\rho'}^{(i)}\})t_{m,\rho'}+\delta_2 h_s(\{0\},\{0\})t_1,$$

where

$$\delta_2=\begin{cases}1\cdots & \text{if}\quad k_1=\cdots=k_n=2\\ 0\cdots & \text{otherwise}.\end{cases}$$

By Selberg [6], the second term on the right hand side of (3.4) is absolutely convergent. We fix a positive number s_0 which is small enough. Let N be a positive number which is large enough, and put

$$U=\left\{u+\sqrt{-1}\,v;\quad 0\leqq u\leqq\frac{1}{2}+s_0,\ |v|\leqq N\right\}.$$

In U, $|\Gamma|$ has the maximal value c_1 and the minimal value $c_2\neq 0$. For $0<s<s_0$, we have

$$B\left(\frac{1+s}{2}+\sqrt{-1}\,v,\frac{1+s}{2}-\sqrt{-1}\,v\right)\leqq\left(\frac{c_1}{c_2}\right)^2 B\left(\frac{1+s_0}{2}+\sqrt{-1}\,v,\frac{1+s_0}{2}-\sqrt{-1}\,v\right),\quad (|v|\leqq N).$$

On the other hand, when $|v|>N$, Stirling's formula for Γ-function gives

$$B\left(\frac{1+s}{2}+\sqrt{-1}\,v,\frac{1+s}{2}-\sqrt{-1}\,v\right)<B\left(\frac{1+s_0}{2}+\sqrt{-1}\,v,\frac{1+s_0}{2}-\sqrt{-1}\,v\right).$$

Then, as $\lim_{s\to 0}c(s)=0$, we have $\lim_{s\to 0}\sum_{(m,\rho')}h_s(\{m^{(i)}\},\{\lambda_{\rho'}^{(i)}\})t_{m,\rho'}=0$. Now we obtain

$$\text{(3.5)}\qquad \lim_{s\to 0}\operatorname{tr} K_s=(8\pi^2)^n t_0+\delta_2(-8\pi^2)^n t_1.$$

§4. An explicit formula for $\operatorname{tr} T(\Gamma\alpha\Gamma)$.

In this section, we shall calculate the trace of $T(\Gamma\alpha\Gamma)$ in $L^2(\tilde{H}_n, \Gamma)$. Define the equivalence relation of elements of $\Gamma\alpha\Gamma$ by

$$g\sim g' \Longleftrightarrow g'=\varepsilon\gamma g\gamma^{-1} \quad \text{for} \quad \gamma\in\Gamma, \varepsilon\in Z(\Gamma). \tag{4.1}$$

Let $[g]$ denote an equivalence class in $\Gamma\alpha\Gamma$ containing g, and put

$$\Gamma(g)=\{\gamma\in\Gamma \mid g=\varepsilon\gamma g\gamma^{-1}, \quad \text{for some } \varepsilon\in Z(\Gamma)\}.$$

In these notations, we can write

$$\operatorname{tr}\int_{\Gamma\backslash\tilde{H}_n} K_s(z,\phi,z,\phi)dzd\phi=\sum_{[g],g\in\Gamma\alpha\Gamma}(2\pi)^n \operatorname{tr}\chi(g)\int_{\Gamma\backslash H_n} k_s(z,0,g(z,0))dz.$$

For simplicity, we put

$$A(g,s)=(2\pi)^n \operatorname{tr}\chi(g)\int_{\Gamma(g)\backslash H_n} k_s(z,0,g(z,0))dz.$$

On account of the assumption of Γ and of [7, Lemma 1.1], we can classify an element g_0 in $\Gamma\alpha\Gamma$ to one of the following types:

i) $g_0\in Z(G)$,

ii) g_0 is elliptic,[5)]

iii) g_0 is hyperbolic,[5)]

iv) g_0 is mixed.

Case i). Suppose $Z(\Gamma\alpha\Gamma)\equiv Z(G)\cap\Gamma\alpha\Gamma\neq\phi$ and let g_0 be an element of $Z(\Gamma\alpha\Gamma)$. Then $\Gamma\alpha\Gamma=g_0\Gamma$ and $Z(\Gamma\alpha\Gamma)=g_0Z(\Gamma)$. Consequently $Z(\Gamma\alpha\Gamma)$ consists of a single equivalence class. We have

$$\begin{aligned} A(g_0,s)&=(2\pi)^n\prod_{i=1}^n(\operatorname{sgn} g_0^{(i)})^{k_i}\int_{\Gamma(g_0)\backslash H_n} dz \operatorname{tr}\chi(g_0)\\ &=(2\pi)^n\prod_{i=1}^n(\operatorname{sgn} g_0^{(i)})^{k_i}v(\Gamma\backslash H_n)\operatorname{tr}\chi(g_0), \end{aligned}\tag{4.2}$$

where $v(\Gamma\backslash H_n)$ denotes the volume of a fundamental domain of Γ in H_n relative to dz.

Case ii) g_0 is elliptic. Consider a linear transformation that maps H_n into the product of the n unit circles, and a fixed point of g_0 to its origin. Let $\zeta^{(i)}$

---

5) We say that an element g of G is elliptic or hyperbolic according as all the $g^{(i)}$ are of corresponding types in the usual sense, where the identify is excluded from all of these types. If g is not the identity and belongs to none of the above types, we say that g is mixed.

and $\eta^{(i)}$ be the eigenvalues of $g_0^{(i)}$ and suppose that

$$\frac{g_0^{(i)}z^{(i)}-z_0^{(i)}}{g_0^{(i)}z^{(i)}-\bar{z}_0^{(i)}}=\zeta^{(i)}\eta^{(i)-1}\frac{z^{(i)}-z_0^{(i)}}{z^{(i)}-\bar{z}_0^{(i)}}, \tag{4.3}$$

($z_0\in H_n$ is the fixed point of g_0). By a simple calculation, we get

$$\lim_{s\to 0}A(g_0,s)=a(\{k_i\})^{-1}\frac{(2\pi)^n}{[\Gamma(g_0):Z(\Gamma)]}\prod_{i=1}^{n}\frac{\eta^{(i)k_i-1}}{\zeta^{(i)}-\eta^{(i)}}(\det g_0)^{1-k_i/2}. \tag{4.4}$$

Case iii) g_0 is hyperbolic. We may assume that 0 and ∞ are fixed point of g_0. As is well known that $\Gamma(g_0)$ has n independent generators; let $\{\gamma_1,\cdots,\gamma_n\}$ be a system of such generators; $(\gamma_j z)^{(i)}=\lambda_j^{(i)}z^{(i)}$. Put $l_j^{(i)}=\log\lambda_j^{(i)}$. Writing $z^{(i)}=\rho^{(i)}\exp(\sqrt{-1}\,\theta^{(i)})$ and $\log\rho^{(i)}=u_1l_1^{(i)}+\cdots+u_nl_n^{(i)}$ with $u_i\in \boldsymbol{R}$, the set of z such that $0<u_i<1$ $(1\leqq i\leqq n)$ and $0<\theta^{(i)}<\pi$ $(1\leqq i\leqq n)$ forms a fundamental domain of $\Gamma(g_0)$. It follows from the previous paper [5] that $A(g_0,s)=0$.

Case iv) g_0 is mixed. At least one of $g_0^{(i)}$'s is of hyperbolic in the usual sense. We assume $g_0^{(i)}$ is of hyperbolic. If $k_i=2$, it also follows from [5] that $A(g_0,s)=0$. If $k_i>2$, $A(g_0,s)$ vanishes following from the equation:

$$\int_0^{\pi}\frac{(\sin\theta)^{k_i-2}}{(\exp(\sqrt{-1}\,\theta)-\lambda\exp(-\sqrt{-1}\,\theta))^{k_i}}d\theta=0.$$

Noting that $t_1=\sum_{\mu=1}^{d}\mathrm{tr}_{V\Gamma}\,\chi(\alpha_\mu)$, we get Theorem 1.

THEOREM 1. *If $k_i\geqq 2$, the trace of $T(\Gamma\alpha\Gamma)$ is given by the following formula:*

$$\begin{aligned}\mathrm{Tr}\,T(\Gamma\alpha\Gamma)&=\delta_1 a(\{k_i\})v(\Gamma\backslash H_n)\prod_{i=1}^{n}(\mathrm{sgn}\,g_0^{(i)})^{k_i}\,\mathrm{tr}\,\chi(g_0)\\&+\sum_{[g]\in\mathfrak{Z}}\frac{\mathrm{tr}\,\chi(g)}{[\Gamma(g):Z(\Gamma)]}\prod_{i=1}^{n}\frac{\eta^{(i)k_i-1}}{\zeta^{(i)}-\eta^{(i)}}(\det g_0^{(i)})^{1-k_i/2}-\delta_2(-1)^n\sum_{\mu=1}^{d}\mathrm{tr}_{V\Gamma}\,\chi(\alpha_\mu).\end{aligned}$$

The notations used in this formula are defined as follows;

$$\delta_1=\begin{cases}1\cdots & \text{if } \Gamma\alpha\Gamma\cap Z(G)\neq\phi\\ 0\cdots & \text{otherwise}\end{cases},$$

$$\delta_2=\begin{cases}1\cdots & \text{if } k_1=\cdots=k_n=2\\ 0\cdots & \text{otherwise}\end{cases},$$

$g_0\in\Gamma\alpha\Gamma\cap Z(G)$,

$v(\Gamma\backslash H_n)$; the volume of fundamental domain of Γ in H_n relative to the invariant measure $dz=\prod_{i=1}^{n}\frac{dx^{(i)}dy^{(i)}}{y^{(i)2}}$,

$a(\{k_i\})=(4\pi)^{-n}\prod_{i=1}^{n}(k_i-1)$,

$\mathfrak{S}$; a complete system of inequivalent elliptic elements in $\Gamma\alpha\Gamma$ with respect to the equivalence relation (4.1).

$\zeta^{(i)}, \eta^{(i)}$; the eigenvalues of an elliptic element $g^{(i)}$ satisfying the formula (4.3),

$V^{\Gamma}=\{v\in V\,|\,\chi(\gamma)v=v\,, \quad \text{for} \quad \gamma\in\Gamma\}$.

Let $\mathfrak{F}_n$ be the set of all $z=(z^{(1)},\cdots,z^{(n)})$ with $z^{(i)}\in \boldsymbol{C}$, $\operatorname{Im} z^{(i)}\neq 0$, and G' the group of all $g=(g^{(1)},\cdots,g^{(n)})$ with $g^{(i)}\in GL_2(\boldsymbol{R})$, anew. We consider that G' operates on $\mathfrak{F}_n$ as the usual way. Now we shall give the trace of Hecke operators operating on the space of automorphic forms defined on $\mathfrak{F}_n$. Let Γ be a subgroup of G' operating on $\mathfrak{F}_n$ discontinuously with a compact fundamental domain and satisfying (G.1) and (G.2) in §1 substituting G' for G. Let Γ^+ be the group of all $\gamma\in\Gamma$ such that $\det\gamma^{(i)}>0$ $(1\leqq i\leqq n)$. We shall choose $\alpha\in G$ and a representation χ as same as in §1. By an automorphic form on $\mathfrak{F}_n$ of type $(\Gamma,\{k_i\},\chi)$, we understand a function $F(z)$ on $\mathfrak{F}_n$ taking values in the representation space of χ satisfying (A.2) in §1 and (A.1′):

(A.1′) $F(z)$ is holomorphic on each connected component of $\mathfrak{F}_n$.

Again we denote by $\mathscr{A}(\Gamma,\{k_i\},\chi)$ the linear space consisting of all $F(z)$. Also we define the Hecke operator $T(\Gamma\alpha\Gamma)$ by (1.1).

THEOREM 1′. *If $k_i\geqq 2$, the trace of $T(\Gamma\alpha\Gamma)$ operating on the space of automorphic forms on $\mathfrak{F}_n$ of type $(\Gamma,\{k_i\},\chi)$ is given in the following formula:*

$$
\begin{aligned}
\operatorname{Tr} T(\Gamma\alpha\Gamma)=\;&\delta_1 a(\{k_i\})v(\Gamma\backslash\mathfrak{F}_n)\prod_{i=1}^{n}(\operatorname{sgn} g_0^{(i)})^{k_i}\operatorname{tr}\chi(g_0)\\
&+\sum_{[g]\in\mathfrak{S}}\frac{(-1)^n\operatorname{tr}\chi(g)}{[\Gamma(g):Z(\Gamma)]}\prod_{i=1}^{n}\frac{\zeta^{(i)k_i-1}-\eta^{(i)k_i-1}}{\zeta^{(i)}-\eta^{(i)}}(\det g^{(i)})^{1-k_i/2}\\
&-\delta_2(-1)^n\frac{2^n}{[\Gamma:\Gamma^+]}\sum_{\mu=1}^{d}\operatorname{tr}_{V^{\Gamma^+}}\chi(\alpha_\mu)\,.
\end{aligned}
\tag{4.6}
$$

The notations used in this formula are the same as in Theorem 1.

PROOF. Let $\mathfrak{H}_1=H_n,\cdots,\mathfrak{H}_{2^n}$ be the connected components of $\mathfrak{F}_n$. Each $\gamma\in\Gamma$ induces a permutation of $\{\mathfrak{H}_j\}_{j=1}^{2^n}$ and this permutation is identity if and only if $\gamma\in\Gamma^+$. Therefore the quotient group Γ/Γ^+ is identified with a subgroup of permutations of $\{\mathfrak{H}_j\}$. Let $\mathscr{A}(\Gamma^+,\mathfrak{H}_j)$ be the set of holomorphic vector functions defined on each $\mathfrak{H}_j$ satisfying (A, 2) for Γ^+. If, for some i and j, there exists an element δ of Γ such that $\delta(\mathfrak{H}_i)=\mathfrak{H}_j$, then the mapping $F(z)\to\chi(\delta)^{-1}j(\delta,z)F(\delta(z))$ gives an isomorphism of $\mathscr{A}(\Gamma^+, \mathfrak{H}_j)$ onto $\mathscr{A}(\Gamma^+, \mathfrak{H}_i)$. When there is not such a δ, we say $\mathfrak{H}_i$ and $\mathfrak{H}_j$ are independent. We observe that there exist exactly $2^n/[\Gamma:\Gamma^+]$ independent $\mathfrak{H}_j$, we fix a subset, say $\{\mathfrak{H}_1, \cdots, \mathfrak{H}_f\}$ $(f=2^n/[\Gamma:\Gamma^+])$; and $\mathscr{A}(\Gamma, \{k_i\}, \chi)$ is canonically isomorphic to the direct sum of the f vector spaces $\mathscr{A}(\Gamma^+, \mathfrak{H}_j)$ for such $\mathfrak{H}_j$. If F_j is a fundamental domain of Γ^+ in $\mathfrak{H}_j$, the union of $F=\bigcup_{j=1}^{f} F_j$ is obviously a fundamental domain of Γ in $\mathfrak{H}_n$. Note that, for an elliptic element g of $\Gamma\alpha\Gamma$,

$$\int_{\Gamma^+(g)\backslash\mathfrak{H}_j} k_s(z, 0, g(z, 0))dz = a(\{k_i\})(-1)^{n-p}\prod_{i=1}^{p}\frac{\eta^{(i)k_i-1}}{\zeta^{(i)}-\eta^{(i)}}(\det g^{(i)})^{1-k_i/2}$$
$$\times\prod_{i'=p+1}^{n}\frac{\zeta^{(i')k_{i'}-1}}{\zeta^{(i')}-\eta^{(i')}}(\det g^{(i')})^{1-k_{i'}/2},$$

if $\mathfrak{H}_j$ is defined by $\operatorname{Im} z^{(i)}>0$ $(1\leqq i\leqq p)$, $\operatorname{Im} z^{(i')}<0$ $(p+1\leqq i'\leqq n)$. Then the trace formula (4.6) follows from Theorem 1.

§5. Discrete groups defined arithmetically.

5.1. Let Φ be a totally real algebraic number field of degree m over $\boldsymbol{Q}$. Let $\mathfrak{g}$ be the ring of integers in Φ and E_0 the group of units in $\mathfrak{g}$. We denote by $\Phi^{(i)}$ the completion of Φ with respect to the infinite valuations $\mathfrak{p}_{\infty,i}$ of Φ. Let A be an indefinite quaternion algebra of discriminant δ over Φ and assume once for all that $A^{(i)}=M_2(\boldsymbol{R})$ for $1\leqq i\leqq n$ and $A^{(i)}=\boldsymbol{K}$ for $n<i\leqq m$ $(A^{(i)}=A\otimes\Phi^{(i)})$. Let $\mathfrak{J}$ be the idèle group of A. The $\mathfrak{p}$-component of an element x in $\mathfrak{J}$ is denoted by $x_{\mathfrak{p}}$. Let $\mathfrak{O}$ be an order in A of the level $\delta\delta'$ and U the group of all idèles x such that $x_{\mathfrak{p}}$ is a unit of $\mathfrak{O}_{\mathfrak{p}}$ for all finite prime $\mathfrak{p}$. We limite ourselves to orders $\mathfrak{O}$ such that the level $\delta\delta'$ is square-free. Put $\Gamma=U\cap A^*$. Every x in $\mathfrak{J}$ is made to act on $\mathfrak{H}_n$ by putting

$$x(z)=(x^{(1)}(z^{(1)}), \cdots, x^{(n)}(z^{(n)}))\,.$$

Then Γ satisfies the assumption (G.1) and (G.2). If and only if A is a division algebra, $\Gamma\backslash\mathfrak{H}_n$ is compact. From now on, we assume that A is a division algebra.

Let $\mathfrak{A}$ be an integral two-sided $\mathfrak{O}$-ideal of norm $\mathfrak{a}$, and ρ a representation of $(\mathfrak{O}/\mathfrak{A})^*$ which we consider as a representation of $V_{\mathfrak{a}}=\{x\in\mathfrak{J}\,|\,x_{\mathfrak{p}}\in U_{\mathfrak{p}}$ for all $\mathfrak{p}$ dividing $\mathfrak{a}\}$ by means of a natural homomorphism of $V_{\mathfrak{a}}$ onto $(\mathfrak{O}/\mathfrak{A})^*$. Let φ_i $(n+1\leqq i\leqq m)$ be an irreducible unitary representation of $\boldsymbol{K}^*$. We put

$$\chi(x)=\rho(x)\otimes\varphi_{n+1}(x^{(n+1)})\otimes\cdots\otimes\varphi_m(x^{(m)})\,. \tag{5.1}$$

Then χ satisfies (R.1). We assume that χ satisfies also (R.2).

$\mathfrak{J}$ is a finite union of double cosets of U and A^* in the following way;

$$\mathfrak{J}=\bigcup_{\lambda=1}^{h} Ux_\lambda A^*\,,\quad (x_\lambda\in V_{\mathfrak{a}},\ h \text{ is the class number of } \mathfrak{O})\,. \tag{5.2}$$

Put $\mathfrak{O}_\lambda=\bigcap_{\mathfrak{p}} x_{\lambda\mathfrak{p}}^{-1}\mathfrak{O}_{\mathfrak{p}}x_{\lambda\mathfrak{p}}$, $U_\lambda=x_\lambda^{-1}Ux_\lambda$ and $\Gamma_\lambda=A^*\cap U_\lambda$ for $\lambda=1,\cdots,h$. Let $\mathscr{A}_\lambda$ be the space of all automorphic forms on $\mathfrak{F}_n$ of type $(\Gamma_\lambda,\{k_i\},\chi)$ and $\mathscr{A}$ the direct product of $\mathscr{A}_1,\cdots,\mathscr{A}_h$. For $x\in V_{\mathfrak{a}}$ and an integral ideal $\mathfrak{q}$ in $\mathfrak{g}$, we denote by $T(UxU)$ and $T(\mathfrak{q})$ the linear operators in $\mathscr{A}$ defined as [10, §3.4]. $T(\mathfrak{q})\neq 0$ only if $\mathfrak{q}$ is a principal ideal and only if we can write $\mathfrak{q}=q\mathfrak{g}$ with a totally positive element q in $\mathfrak{g}$. Hence we limit ourselves that $\mathfrak{q}=q\mathfrak{g}$ and that q is totally positive.

Now, in the case of $T(\mathfrak{q})$ we shall consider the part corresponding to the last term in (4.6), (which is denoted by t_1). It follows from Theorem 1′ that if $k_i\geqq 2$ $(1\leqq i\leqq n)$, we have

$$t_1=-\delta_2(-1)^n\sum_{\lambda=1}^{h}\frac{2^n}{[\Gamma:\Gamma^+]}\sum_{\substack{\alpha\in B_\lambda^+(\mathfrak{q})\\ \alpha \bmod \Gamma_\lambda^+}}\mathrm{tr}_{V^{\Gamma_\lambda^+}}\chi(\alpha)\,,$$

where $B_\lambda^+(\mathfrak{q})=\{\alpha\in\mathfrak{O}_\lambda \mid N(\alpha)\mathfrak{g}=\mathfrak{q},\ N(\alpha) \text{ is totally positive}\}$; $V^{\Gamma_\lambda^+}$ is defined as same as in Table I substituting Γ_λ^+ for Γ. Firstly we assume that there exists some j $(n<j\leqq m)$ such that φ_j is not identity. It follows from Proposition 1.1 in Matsushima and Shimura[3)] that the projection of Γ_λ^+ into $A^{(j)}$ is dense in $A^{(j)}$. Then we have $t_1=0$. Secondly we assume that all φ_j are the identities $(n<j\leqq m)$. We consider ρ as a representation of $U_{\mathfrak{a}}=\prod_{\mathfrak{p}|\mathfrak{a}}\mathfrak{O}_{\mathfrak{p}}^*$. Γ_λ^1, $U_{\mathfrak{a}}^1$ denote the subgroups of Γ_λ, $U_{\mathfrak{a}}$ consisting of elements γ, u such that $N(\gamma)=1$, $N(u_{\mathfrak{p}})=1$ (for all $\mathfrak{p}|\mathfrak{a}$), respectively. Let E_0^+ and E_1 be the subgroup of E_0 consisting of all totally positive units and the subgroup of all elements ε such that $\varepsilon^{(i)}>0$ for all $i>n$. It follows from Eichler's approximation theorem that, for $u\in U_{\mathfrak{a}}^1$, there exists $\gamma\in\Gamma_\lambda^1$ such that $\gamma\equiv u_{\mathfrak{p}} (\bmod\ \mathfrak{A}_{\mathfrak{p}})$ for all $\mathfrak{p}$ dividing $\mathfrak{a}$. Then we have $V^{\Gamma_\lambda^1}=V^{U_{\mathfrak{a}}^1}$ for all λ. Let ρ' be the restriction of ρ to $V^{U_{\mathfrak{a}}^1}$. As $U_{\mathfrak{a}}/U_{\mathfrak{a}}^1\cong\prod_{\mathfrak{p}|\mathfrak{a}}\mathfrak{g}_{\mathfrak{p}}^*$, there exists representation ρ_0 of $\prod_{\mathfrak{p}|\mathfrak{a}}\mathfrak{g}_{\mathfrak{p}}^*$ such that $\rho'(u)=\rho_0(N(u))$. If $\varepsilon\in E_0^+$, $\rho_0(\varepsilon)$ is trivial on $V^{\Gamma_\lambda^+}$. For $\alpha\in B_\lambda^+(\mathfrak{q})$, we can write $N(\alpha)=q\varepsilon$ $(\varepsilon\in E_0^+)$. Again it follows from the approximation theorem that the number of the representatives of $B_\lambda^+(\mathfrak{q}) \bmod \Gamma_\lambda^+$ is equal to the number of integral $\mathfrak{O}_\lambda$-ideals having norm $\mathfrak{q}$. Then we have

$$\sum_{\substack{\alpha\in B_\lambda^+(\mathfrak{q})\\ \alpha \bmod \Gamma_\lambda^+}}\mathrm{tr}_{V^{\Gamma_\lambda^+}}\rho(\alpha)=\mathrm{tr}_{V^{E_0^+}}\rho_0(q)\sum_{\substack{\mathfrak{n}|\mathfrak{q}\\ (\mathfrak{n},\delta\delta')=1}}N(\mathfrak{n})\prod_{\mathfrak{p}|(\delta',\mathfrak{q})}(2(N(\mathfrak{p}^{a(\mathfrak{p})})-1)(N(\mathfrak{p})-1)^{-1}-1)\,,$$

$$(\mathfrak{q}=\prod_{\mathfrak{p}|\mathfrak{q}}\mathfrak{p}^{a(\mathfrak{p})})\,,$$

where

$$V^{E_0^+}=\{v\in V;\ \rho_0(\varepsilon)v=v \quad \text{for all} \quad \varepsilon\in E_0^+\},$$

LEMMA 1. $[\Gamma:\Gamma^+]=[E_1:E_0^+]$.

PROOF. It is obvious that $[\Gamma:\Gamma^+]\geqq[E_1:E_0^+]$. We fix a maximal order $\mathfrak{O}_0$ containing $\mathfrak{O}$ and denote by $\mathfrak{f}$ the conducter of $\mathfrak{O}$. Let b be an element of E_1 and α an element of $\mathfrak{O}$ such that $N(\alpha)\equiv b \bmod^*\mathfrak{f}\cap\mathfrak{g}$ (here mod* means the multiplicative congruence). It follows from the approximation theorem that there is an element β in $\mathfrak{O}_0$ such that $\beta\equiv\alpha \bmod \mathfrak{f}$ and $N(\beta)=b$. Our lemma is thereby proved.

On the other hand, it follows from [9, Lemma 2.4] that $h=h_0 2^{m-n}/[E_0:E_1]$, h_0 being the class number of Φ.

Combining above things with [9,§§3.2-3.3 and Appendix], we obtain

THEOREM 2. *Let $\mathfrak{q}=q\mathfrak{g}$ be a principal ideal in Φ with a totally positive element q in $\mathfrak{g}$. If $k_i\geqq 2$, the trace $T(\mathfrak{q})$ is given by the following formula:*

$$\begin{aligned}
\operatorname{Tr} T(\mathfrak{q}) = {} & \delta(\mathfrak{q})(2\pi)^{-2m}2h_0D_0^{3/2}\zeta_0(2)\operatorname{tr}\chi'(q_0)\prod_{\mathfrak{p}|\delta}(N(\mathfrak{p})-1)\prod_{\mathfrak{p}|\delta'}(N(\mathfrak{p})+1)\prod_{i=1}^{n}(k_i-1)(\operatorname{sgn} q_0^{(i)})^{k_i} \\
& -\delta(\{k_i\},\{\varphi_j\})(-1)^n\frac{2^m}{[E_0:E_0^+]}h_0\operatorname{tr}_{V^{E_0^+}}\rho_0(q) \\
& \times\Big(\sum_{\substack{\mathfrak{n}|\mathfrak{q}\\(\mathfrak{n},\delta\delta')=1}}N(\mathfrak{n})\Big)\prod_{\mathfrak{p}|(\delta',\mathfrak{q})}(2(N(\mathfrak{p}^{a(\mathfrak{p})})-1)(N(\mathfrak{p})-1)^{-1}-1) \\
& +(-1)^n2^{-\nu-1}\sum_{\mathfrak{o}\in\Omega}\frac{h(\mathfrak{o})}{w(\mathfrak{o})}\prod_{\mathfrak{p}|\delta}\left(1-\left\{\frac{\mathfrak{o}}{\mathfrak{p}}\right\}\right)\prod_{\mathfrak{p}|\delta'}\left(1+\left\{\frac{\mathfrak{o}}{\mathfrak{p}}\right\}\right)\sum_{\mathfrak{r}|\delta\delta'}\sum_{\substack{\alpha\in I(\mathfrak{q},\mathfrak{o})\\ \alpha \bmod E_0}}\operatorname{tr}\chi'(\pi_{\mathfrak{r}}\psi(\alpha)\pi_{\mathfrak{r}}^{-1}).
\end{aligned}\tag{5.3}$$

The notations are as follows. h_0, D_0 are ζ_0 are the class number of Φ, the discriminant of Φ over $\boldsymbol{Q}$ and the zeta function of Φ, respectively. $\delta(\mathfrak{q})=1$ if $\mathfrak{q}=q_0^2\mathfrak{g}$ for some $q_0\in\mathfrak{g}$ and otherwise $\delta(\mathfrak{q})=0$. $\delta(\{k_i\},\{\varphi_j\})=1$ if all $k_i=2$ and all φ_j's are trivial and otherwise $\delta(\{k_i\},\{\varphi_j\})=0$. $\mathfrak{n}$ runs over all divisors of $\mathfrak{q}$, prime to $\delta\delta'$. $\mathfrak{q}=\prod_{\mathfrak{p}|\mathfrak{q}}\mathfrak{p}^{a(\mathfrak{p})}$. ρ_0 is defined as above. Ω is the set of all orders $\mathfrak{o}$ (taken up to isomorphism) in totally imaginary quadratic extensions of Φ. $h(\mathfrak{o})$ is the class number of $\mathfrak{o}$, and $w(\mathfrak{o})$ is the index of E_0 in the group of units in $\mathfrak{o}$. Let K be the quadratic extension of Φ containing $\mathfrak{o}$; then ψ is an embedding of the adèle of K into the adèle of A such that $\psi(\mathfrak{o}_\mathfrak{p})=\psi(K_\mathfrak{p})\cap\mathfrak{O}_\mathfrak{p}$ for all $\mathfrak{p}$; $I(\mathfrak{q},\mathfrak{o})$ is the set of all $\alpha\in K-\Phi$ such that $\psi(\alpha)$ is contained in the union of double cosets UxU appearing in $T(\mathfrak{q})$. Put

$$\chi'(x)=\prod|N(x^{(i)})|^{1-k_i/2}\chi(x)\otimes\Phi_{k_1-2}(x^{(1)})\otimes\cdots\otimes\Phi_{k_n-2}(x^{(n)}).$$

Here, Φ_k denotes the symmetric tensor representation of $GL_2(\boldsymbol{C})$ of degree k. $\pi_\mathfrak{r}$ is an element in $\mathfrak{J}$ such that $\bigcap_\mathfrak{p}\mathfrak{O}_\mathfrak{p}(\pi_\mathfrak{r})_\mathfrak{p}$ is a two-sided $\mathfrak{O}$-ideal and of norm $\mathfrak{r}$. ν is the number of prime divisor of $\delta\delta'$.

5.2. Here we shall give some numerical examples. Let us take $\Phi=\boldsymbol{Q}(\sqrt{p})$ $(p>0)$. We assume that the class number of Φ is one and that Φ contains a unit element of norm -1. Let A be a quaternion algebra over Φ and assume that A is unramified at all archimedean prime of Φ. Let $\mathfrak{O}$ be the maximal order in A. In this section, we limit ourselves to the case all $k_i=2$ and $\mathfrak{q}=\mathfrak{g}$. In this situation, it follows from Theorem 2 that

$$\dim \mathscr{A}=\frac{D_0^{3/2}}{8\pi^4}\zeta_0(2)\prod_{\mathfrak{p}|\delta}(N(\mathfrak{p})-1)-1 \tag{5.4}$$
$$+\frac{1}{2}\sum\frac{h(\mathfrak{o}(s,\mathfrak{f}))}{w(\mathfrak{o}(s,\mathfrak{f}))}(w(\mathfrak{o}(s,\mathfrak{f}))-1)\prod_{\mathfrak{p}|\delta}\left(1-\left\{\frac{\mathfrak{o}(s,\mathfrak{f})}{\mathfrak{p}}\right\}\right).$$

Here,

s : integers in Φ, satisfying $4-s^2$ is totally positive,

$\mathfrak{f}$: integral ideal in Φ, satisfying $(s^2-4)\mathfrak{f}^{-2}$ is an integral ideal in Φ,

$\mathfrak{o}(s,\mathfrak{f})$: the order in $K(s)=\Phi(\sqrt{s^2-4})$ with the discriminant $(s^2-4)\mathfrak{f}^{-2}$ over Φ.

The summation runs over all $\mathfrak{o}(s,\mathfrak{f})$ (taken up to isomorphism).

By the functional equation of $\zeta_0(s)$, the first term in right hand side of (5.4) is equal to $(1/2)\zeta_0(-1)$. If p is a square-free, we can use the explicit formula for $\zeta_0(-1)$ given by Siegel. Note that in the summation of the right hand side of (5.4) s which satisfies the condition is only $0, \pm1, \pm\sqrt{2}$ or $\pm\dfrac{1\pm\sqrt{5}}{2}$. Considering the numbers of residue classes for the conductor $\mathfrak{F}$ of $\mathfrak{o}(s,\mathfrak{f})$ in the principal order $\mathfrak{o}_0(s)$ which are prime to $\mathfrak{F}$ in $\mathfrak{o}_0(s)$ and in $\mathfrak{o}(s,\mathfrak{f})$, we get

$$\frac{h(\mathfrak{o}(s,\mathfrak{f}))}{w(\mathfrak{o}(s,\mathfrak{f}))}=\frac{h(\mathfrak{o}_0(s))}{w(\mathfrak{o}_0(s))}N(\mathfrak{f})\prod_{\mathfrak{p}|\mathfrak{f}}\left(1-\left(\frac{K(s)}{\mathfrak{p}}\right)N(\mathfrak{p})^{-1}\right).$$

The consideration of the zeta function of $K(s)$ gives the following lemma.

LEMMA 2. *Suppose $K(s)=\boldsymbol{Q}(\sqrt{p},\sqrt{-D})$ with a square-free positive number D. Then we get*

$$h(\mathfrak{o}_0(s))=(2)\frac{1}{2}h(-D)h(-pD),$$

where $h(D_1)$ $(D_1\in\boldsymbol{Z})$ denotes the class number of $\boldsymbol{Q}(\sqrt{D_1})$; the factor 2 appears only if $p=2$, $D=1$.

Summing up above things, we obtain the following formulae.

i) $\Phi=\boldsymbol{Q}(\sqrt{2})$, $\dim\mathscr{A}=-1+\dfrac{1}{24}\prod_{\mathfrak{p}|\delta}(N(\mathfrak{p})-1)+\dfrac{1}{3}\prod_{\mathfrak{p}|\delta}\left(1-\left(\dfrac{\Phi(\sqrt{-3})}{\mathfrak{p}}\right)\right)$
$$+\frac{\xi(\delta)}{8}\prod_{\mathfrak{p}|\delta}\left(1-\left(\frac{\Phi(\sqrt{-1})}{\mathfrak{p}}\right)\right),$$

where $\xi(\delta)=3$ if δ is divisible by $\sqrt{2}$ and otherwise $\xi(\delta)=5$.

ii) $\Phi=\boldsymbol{Q}(\sqrt{5}), \dim \mathscr{A}=-1+\frac{1}{60}\prod_{\mathfrak{p}\mid\delta}(N(\mathfrak{p})-1)+\frac{1}{3}\prod_{\mathfrak{p}\mid\delta}\left(1-\left(\frac{\Phi(\sqrt{-3})}{\mathfrak{p}}\right)\right)$

$$+\frac{1}{4}\prod_{\mathfrak{p}\mid\delta}\left(1-\left(\frac{\Phi(\sqrt{-1})}{\mathfrak{p}}\right)\right)+\frac{2}{5}\prod_{\mathfrak{p}\mid\delta}\left(1-\left(\frac{\Phi(\zeta_{10})}{\mathfrak{p}}\right)\right),$$

where ζ_{10} is a primitive 10-th root of 1.

iii) $\Phi=\boldsymbol{Q}(\sqrt{p})$, ($p>5$; suppose Φ satisfies the conditions as above),

$$\dim \mathscr{A}=-1+\frac{1}{2}\zeta_0(-1)\prod_{\mathfrak{p}\mid\delta}(N(\mathfrak{p})-1)+\frac{h(-3p)}{6}\prod_{\mathfrak{p}\mid\delta}\left(1-\left(\frac{\Phi(\sqrt{-3})}{\mathfrak{p}}\right)\right)$$
$$+\frac{h(-p)}{8}\prod_{\mathfrak{p}\mid\delta}\left(1-\left(\frac{\Phi(\sqrt{-1})}{\mathfrak{p}}\right)\right).$$

§ 6. Zeta functions of quaternion algebras.

6.1. In § 6, we admit a definite quaternion as A. Let k_i' ($1\leqq i\leqq m$) be non-negative integers. Let $\mathfrak{i}_i$ be an injection of $A^{(i)}$ into $M_2(\boldsymbol{C})$ ($n<i\leqq m$). For $i>n$, put $\varphi_i(x^{(i)})=|N(x^{(i)})|^{-k_i'/2}\Phi_{k_i'}(\mathfrak{i}_i(x^{(i)}))$ in (5.1).

Firstly, we assume A is a definite quaternion algebra. Let $M(\rho, \{k_i'\})$ be the space of all continuous functions $f(x)$ on $\mathfrak{J}$, taking values in the representation space of χ', which satisfy

$$f(ux\alpha)=\chi'(u)f(x), \tag{6.1}$$

for all $u\in U, \alpha\in A^*$. Define an endomorphism $T(UyU)$ on $M(\rho, \{k_i'\})$ by

$$T(UyU)f(x)=\sum_{\mu}\chi'(z_\mu)f(z_\mu^{-1}x), \tag{6.2}$$

for $UyU=\bigcup_{\mu} z_\mu U$.

Let us assume all $k_i'=0$. Denote by $\Phi_A, \Phi_{A\infty}$ the adèle of Φ, the infinite part of Φ_A, respectively. Let ψ be a character of $\Phi_A^*/\Phi^*(\prod_{\mathfrak{p}}\mathfrak{g}_{\mathfrak{p}}^*)(\Phi_{A\infty}^*)_+$ and ρ_0 the representation of $\prod_{\mathfrak{p}\mid\mathfrak{a}}\mathfrak{g}_{\mathfrak{p}}^*$ defined in § 5.1. For $v\in V^{E_0^+}$, $u\in U$, $\alpha\in A^*$, put

$$f_{\psi,v}(ux_\lambda\alpha)=\psi(N(x_\lambda))\rho_0(N(u))v \qquad (1\leqq\lambda\leqq h),$$

and $M_1(\rho)$ denotes the subspace of $M(\rho, \{0\})$ consisting of such $f_{\psi,v}$ for all ψ and v. As is well known, $[\Phi_A^*: \Phi^*(\prod_{\mathfrak{p}}\mathfrak{g}_{\mathfrak{p}}^*)(\Phi_{A\infty}^*)_+]=h_0 2^m/[E_0: E_0^+]$. If q is a totally positive element in $\mathfrak{g}$ the trace t_1' of $T(q\mathfrak{g})$ restricted on $M_1(\rho)$ is given by

$$t_1'=\frac{2^m}{[E_0: E_0^+]}h_0\,\mathrm{tr}_{V^{E_0^+}}\rho_0(q)\sum_{\substack{\mathfrak{n}\mid q\mathfrak{g}\\(\mathfrak{n},\delta\delta')=1}}N(\mathfrak{n})\prod_{\mathfrak{p}\mid(\delta',q\mathfrak{g})}(2(N(\mathfrak{p}^{a(\mathfrak{p})})-1)(N(\mathfrak{p})-1)^{-1}-1), \tag{6.3}$$

$$(q\mathfrak{g}=\prod_{\mathfrak{p}\mid q\mathfrak{g}}\mathfrak{p}^{a(\mathfrak{p})}).$$

Note that, unless an ideal $\mathfrak{q}$ can be written in the form $q\mathfrak{g}$ with a totally positive element q of $\mathfrak{g}$, t_1' vanishes and the trace of $T(\mathfrak{q})$ on $M(\rho, \{0\})$ also vanishes. From now on, if all $k_i'=0$ we consider the trace of $T(\mathfrak{q})$ on $M'(\rho, \{0\})$ which is the orthogonal complement of $M_1(\rho)$ in $M(\rho, \{0\})$ instead of $M(\rho, \{0\})$, and we also express the restriction of $T(\mathfrak{q})$ to $M'(\rho, \{0\})$ by $T(\mathfrak{q})$.

Secondly, we assume A is an indefinite division algebra. Let $\mathscr{A}_\lambda$ be the space of all automorphic forms on $\mathfrak{H}_n$ of type $(\Gamma_\lambda, \{k_1'+2, \cdots, k_n'+2\}, \chi)$ and $\mathscr{A}(\rho, \{k_i'\})$ the direct product of $\mathscr{A}_1, \cdots, \mathscr{A}_h$. Note that, for $T(\mathfrak{q})$, we have

$$t_1 = -\delta(\{k_i'\})(-1)^n \frac{2^m}{[E_0 : E_0^+]} h_0 \operatorname{tr}_{V E_0^+} \rho_0(q)$$
$$\times \sum_{\substack{\mathfrak{n}|\mathfrak{q} \\ (\mathfrak{n}, \delta\delta')=1}} N(\mathfrak{n}) \prod_{\mathfrak{p}|(\delta', \mathfrak{q})} (2(N(\mathfrak{p}^{a(\mathfrak{p})})-1)(N(\mathfrak{p})-1)^{-1}-1),$$

where $\delta(\{k_i'\})=1$ if all $k_i'=0$ and otherwise $\delta(\{k_i'\})=0$.

6.2. In this section A may be definite or indifinite. Let $\mathfrak{O}, \rho, \mathfrak{a}, \{k_i'\}$ be in § 5 and § 6.1. In the same way as in [9], we define a Dirichlet series[6] of a complex variable s by

$$\zeta(s) = \sum_{\mathfrak{q}} T(\mathfrak{q}) N(\mathfrak{q})^{-s}, \tag{6.4}$$

where the sum runs over all integral ideals $\mathfrak{q}$ in $\mathfrak{g}$ prime to $\mathfrak{a}$. We regard $\zeta(s)$ as a matrix-valued function fixing a basis in $M(\rho, \{k_i'\})$ or in $\mathscr{A}(\rho, \{k_i'\})$. We write $\zeta(A, \mathfrak{O}; \rho, \{k_i'\}; s)$ for $\zeta(s)$.

6.3. Let δ, δ' be integral square-free ideals in Φ prime to each other and $\mathfrak{c}$ a prime ideal in Φ prime to $\delta\delta'$. Let $\mathfrak{O}, \mathfrak{O}', \mathfrak{O}''$ be orders of level $\delta\delta', \delta\delta'\mathfrak{c}, \delta\mathfrak{c}\delta'$ in division quaternion algebras A, A', A'' of discriminant δ, δ, $\delta\mathfrak{c}$, respectively. Let $\mathfrak{A}, \mathfrak{A}', \mathfrak{A}''$ be integral two-sided ideals in $\mathfrak{O}, \mathfrak{O}', \mathfrak{O}''$ with the properties indicated in [9, § 4.1]. We choose representations ρ, η, Ξ and Ξ' of $(\mathfrak{O}'/\mathfrak{A}')^*$, $(\mathfrak{O}''/\mathfrak{A}'')^*$, $(\mathfrak{O}/\mathfrak{A})^*$, and of $(\mathfrak{O}/\mathfrak{A})^*$, respectively as in [9, (23)-(29)]. Let $U_\mathfrak{c}, U_\mathfrak{c}', U_\mathfrak{c}''$ be the unit groups of $\mathfrak{O}_\mathfrak{c}, \mathfrak{O}_\mathfrak{c}', \mathfrak{O}_\mathfrak{c}''$, respectively. Denote by $\rho_{\mathfrak{c},0}, \eta_{\mathfrak{c},0}, \Xi_{\mathfrak{c},0}$ and by $\Xi_{\mathfrak{c},0}'$ the representations of $\mathfrak{g}_\mathfrak{c}^*$ defined as the same way in § 5.1. We assume that $\rho_{\mathfrak{c},0}$, $\eta_{\mathfrak{c},0}, \Xi_{\mathfrak{c},0}$ and $\Xi_{\mathfrak{c},0}'$ satisfy the following conditions; for $\varepsilon \in \mathfrak{g}_\mathfrak{c}^*$,

$$\operatorname{tr}_{V E_0^+} \rho_{\mathfrak{c},0}(\varepsilon) = \operatorname{tr}_{V E_0^+} \Xi_{\mathfrak{c},0}(\varepsilon) - \operatorname{tr}_{V E_0^+} \Xi_{\mathfrak{c},0}'(\varepsilon)$$
$$\operatorname{tr}_{V E_0^+} \eta_{\mathfrak{c},0}(\varepsilon) = \operatorname{tr}_{V E_0^+} \Xi_{\mathfrak{c},0}(\varepsilon) - \operatorname{tr}_{V E_0^+} \Xi_{\mathfrak{c},0}'(\varepsilon) .$$

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[6] Cf. Shimura [10] and T. Tamagawa, On the ζ-functions of a division algebra, Ann. of Math. **77** (1963), 387-405.

We regard that two matrix-valued function $f(s)$, $g(s)$ are identified if there exists a non-singular matrix M such that $Mf(s)M^{-1}=g(s)$. By the same argument as in [9, § 4], we have

THEOREM 3. *Let δ, δ' be integral square-free ideals in Φ prime to each other and $\mathfrak{c}$ a prime ideal in Φ prime to $\delta\delta'$. Let $A, A', A'', \mathfrak{O}, \mathfrak{O}', \mathfrak{O}'', \rho, \eta, \Xi, \Xi'$ be as above. If $k_i'\geqq 0$, we have*

$$\begin{aligned}\zeta(A', \mathfrak{O}'; \rho, \{k_i'\}; s)&\oplus 2\cdot\zeta(A, \mathfrak{O}; \Xi', \{k_i'\}; s)\\ =\zeta(A'', \mathfrak{O}''; \eta, \{k_i'\}; s)&\oplus 2\cdot\zeta(A, \mathfrak{O}; \Xi, \{k_i'\}; s)^{7)}\,.\end{aligned} \tag{6.5}$$

REMARK. In the case $\Phi=\boldsymbol{Q}$, the above theorem is also valid without the limitation that A, A' and A'' are all division, by [5, Theorem].

6.4. Let δ, δ' be the same as § 6.3 and let $\mathfrak{O}$, $\mathfrak{O}'$ be orders of level $\delta\delta'$ in division algebras A, A' of discriminant δ, respectively. Let $\mathfrak{A}, \mathfrak{A}'$ be integral two-sided ideals of $\mathfrak{O}, \mathfrak{O}'$ such that $\mathfrak{A}_{\mathfrak{p}}=\mathfrak{A}'_{\mathfrak{p}}$ for all $\mathfrak{p}$, and put $\mathfrak{a}=N(\mathfrak{A})$. Let ρ be a common representation of $(\mathfrak{O}/\mathfrak{A})^*$ and of $(\mathfrak{O}'/\mathfrak{A}')^*$. We assume ρ satisfies (R.2). Again with the same argument in [9, § 4], we obtain the following theorem.

THEOREM 4. *The notations are the same as above. If $k_i'\geqq 0$, we have*

$$\zeta(A, \mathfrak{O}; \rho, \{k_i'\}; s)=\zeta(A', \mathfrak{O}'; \rho, \{k_i'\}; s)\,. \tag{6.6}$$

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7) X, Y being matrices, $X\oplus 2\cdot Y$ denotes the direct sum of X and two copies of Y.

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Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan