On maximal p-local subgroups of S_n and A_n

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§ 1. Introduction.

The object of this paper is to determine the structure and the conjugate classes of the maximal p-local subgroups of S_n and A_n. Here we understand by S_n and A_n, the symmetric and the alternating groups of degree n, respectively. If a subgroup H of a finite group G is of the shape $N_G(P)$ where P is a non-identity p-subgroup of G, we customarily say that H is a p-local subgroup of G. But in this paper we will adopt a little wider definition. Namely, H is a *p-local subgroup* of G if and only if H has a nonidentity normal p-subgroup. It is obvious that a maximal p-local subgroup in the narrow sense is also maximal in the wider sense, and conversely. We discuss the maximal p-local subgroups of S_n in § 3. As for the A_n, we must discuss the case $p=2$ separately, mainly because of the fact that $|S_n : A_n|=2$. We discuss the case $p=2$ in § 4, and the case $p\neq 2$ in § 5.

§ 2. Notation and necessary lemmas.

Let E_{p^e} denote an elementary abelian group of order $p^e \geq p$, and let H_{p^e} be the holomorph of E_{p^e}, i.e. the semidirect product of E_{p^e} by the full automorphism group A of E_{p^e}. We regard H_{p^e} as a permutation group on p^e letters. Namely, we let $ax \in H_{p^e}$ ($a \in A$, $x \in E_{p^e}$) act on $y \in E_{p^e}$ by $y^{(ax)}=y^a x$. Then E_{p^e} is regular on itself. Since A is the stabilizer of the identity element of E_{p^e}, and is transitive on remaining points, H_{p^e} is a doubly transitive group. Note that $H_n = S_n$ for $n=2, 3$ and 4. Since E_{p^e} is irreducible as an A-module, E_{p^e} is the largest normal p-subgroup of H_{p^e}.

Some further notations are required. For any finite group X, let $l_p(X)$ denote the number of conjugate classes of maximal p-local subgroups of X. Let $O_p(X)$ denote the largest normal p-subgroup of X. For any positive number n, let $[n]$ denote the largest integer not greater than n. If p^e is the largest power of p that divides n, we write $p^e \| n$.

Almost all of the notation concerning permutation groups are standard and may be found in Wielandt's book: Finite Permutation Groups, Academic Press, New York-London, 1964 (henceforth abbreviated by (W)). Particularly, we let S^{Ω} and A^{Ω} denote the symmetric and the alternating groups on a set Ω, and if G is a permutation group on Ω, G_{α} will denote the stabilizer of a point α in G. A *block* is a set of imprimitivity. It should be noted that if f is a mapping of a set X into a set Y, we denote by xf or x^{f} the image of x under f. If G is a permutation group on Ω, we often say that (G, Ω) is a permutation group. Let (G, Ω) be a permutation group. We will denote by $N(G)$ and $C(G)$, respectively, the normalizer and the centralizer of G in S^{Ω}. If (G, Ω) and (H, Δ) are isomorphic permutation groups, i.e. if there exists a group isomorphism $\lambda : G \to H$, and a bijection $\mu : \Omega \to \Delta$ such that $(\alpha g)\mu = (\alpha\mu)(g\lambda)$ for all $\alpha \in \Omega$ and $g \in G$, we will write $(G, \Omega) \cong (H, \Delta)$. If there is no fear of confusion, we will use the same letter for λ and μ, and say that $\lambda : (G, \Omega) \to (H, \Delta)$ is an isomorphism. Suppose G is a permutation group on $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_n$ (disjoint union) such that $\Omega_i^{G} = \Omega_i$ for $i = 1, 2, \ldots, n$. Then each $g \in G$ is uniquely expressed as the product $g = g_1 g_2 \cdot \cdots \cdot g_n$ where g_i are permutations on Ω such that $\Omega_i g_i = \Omega_i$ and that g_i fixes the points of $\Omega - \Omega_i$. We call g_i the *Ω_i-constituent* of g and denote it by g^{Ω_i}. If $\Delta \subseteq \Omega$, we denote by S^{Δ} the set of permutations on Ω which fix the points of $\Omega - \Delta$, since it is canonically isomorphic to the symmetric group on Δ.

Let (G, Δ) and (H, Σ) be permutation groups. Let $G^{(\Sigma)}$ denote the Cartesian product of G with the index set Σ. The *Wreath product* $G \int H$ of G by H is the semidirect product $G^{(\Sigma)} H$ where H acts on $G^{(\Sigma)}$ by:

$$(g_j)_{j \in \Sigma}^{h} = (g_{jh^{-1}})_{j \in \Sigma} \quad \text{for} \quad (g_j)_{j \in \Sigma} \in G^{(\Sigma)} \quad \text{and} \quad h \in H.$$

We let $G \int H$ act on $\Omega = \Delta \times \Sigma$ by:

$$(\alpha, i)\{(g_j)_j h\} = (\alpha g_i, ih) \quad \text{for} \quad (\alpha, i) \in \Omega \quad \text{and} \quad (g_j)_j h \in G \int H.$$

Then, $G \int H$ is transitive on Ω if and only if both (G, Δ) and (H, Σ) are transitive. For each $i \in \Sigma$ we set

$$\Delta^{(i)} = \{(\alpha, i);\ \alpha \in \Delta\} \quad \text{and}$$
$$G^{(i)} = \{(g_j)_j;\ (g_j)_j \in G^{(\Sigma)},\ g_j = 1 \text{ for each } j \neq i\}$$

Then, $G^{(i)}$ is a permutation group on $\Delta^{(i)}$, $(G^{(i)}, \Delta^{(i)}) \cong (G, \Delta)$ and $\{\Delta^{(i)}\}_i$ is a system of imprimitivity of $G \int H$.

(2.1) *Suppose* (G, Δ) *and* (H, Σ) *are permutation groups and* $\Omega = \Delta \times \Sigma$. *Then,*

(i) *If $K \leqq S^{\Omega}$ and $\{\Delta^{(i)}\}_i$ is a system of imprimitivity of K, then $K \leqq S^{\Delta} \int S^{\Sigma}$.*

(ii) *If (G, Δ) and (H, Σ) are primitive, then $\{\Delta^{(i)}\}_i$ is the unique nontrivial system of imprimitivity of $(G \int H, \Omega)$.*

PROOF. (i) Let $g \in K$. Then g induces a permutation $\bar{g}$ on Σ by $\Delta^{(i)} g = \Delta^{(i\bar{g})}$. Define a bijection $\varphi_i : \Delta \to \Delta^{(i)}$ by $\alpha\varphi_i = (\alpha, i)$. For each $i \in \Sigma$ set $g_i = \varphi_i g \varphi_{i\bar{g}}^{-1}$. Then clearly $g_i \in S^{\Delta}$ and $g = (g_j)_j \bar{g} \in S^{\Delta} \int S^{\Sigma}$.

(ii) Let Ψ be a nontrivial block of $(G \int H, \Omega)$. Suppose $\Psi \cap \Delta^{(k)} \neq 0 \neq \Psi \cap \Delta^{(l)}$ for $k \neq l$. Since $(G^{(i)}, \Delta^{(i)})$ is transitive, $\Psi \supseteq \Delta^{(k)} \cup \Delta^{(l)}$. Therefore Ψ is a union of more then one, but not all of $\Delta^{(i)}$'s, say $\Psi = \Delta^{(k)} \cup \Delta^{(l)} \cup \cdots \cup \Delta^{(m)}$. Since $\Delta^{(i)} h = \Delta^{(ih)}$ for each $h \in H$, $\{k, l, \ldots, m\}$ is a nontrivial block of (H, Σ), contrary to our hypothesis. Thus $\Psi \subseteq \Delta^{(i)}$ for some i. Since $(G^{(i)}, \Delta^{(i)})$ is primitive, $\Psi = \Delta^{(i)}$, q.e.d.

Let (G, Ω) be a permutation group, $\alpha \in \Omega$, and N be a regular abelian normal subgroup of G, Since $C(N) = N$ by (W) Prop. 4.4., G_α can be regarded as a permutation group on N acting by conjugation. It is easily seen that $(G_\alpha, \Omega) \simeq (G_\alpha, N)$.

(2.2) *Let A be a group of automorphisms of a finite group G, and C be the set of fixed points of A on G. Suppose A is primitive on $G-C$, regarded as a permutation group on $G-C$. Then one of the following possibilities occurs.*

(i) *G is an elementary abelian 2-group, $|C| = 1$.*

(ii) *G is a dihedral group of order $2p$ where p is a (not necessarily odd) prime, $|C| = p$.*

(iii) *G is a cyclic group of order 4, $|C| = 2$.*

(iv) *G is a cyclic group of order 3, $|C| = 1$.*

PROOF. For each subgroup D of C, a coset of D not contained in C is a block of A on $G-C$. Hence either (1) $C = 1$ or (2) $|G : C| = 2$ and $|C| = p$ is a prime. For each $g \in G-C$, $\{g, g^{-1}\}$ is a block of A on $G-C$. Hence either (3) all the elements of $G-C$ are of order 2 or (4) $G-C = \{g, g^{-1}\}$ with $g \neq g^{-1}$. If (1) and (3) hold, then G is an elementary abelian 2-group. Suppose (2) and (3) hold. Then for any $g, h \in G-C$, $(g \neq h)$, $gh \in C$ and gh is of order p. Thus (ii) holds. If (1) and (4) hold, then G is cyclic of order 3. If (2) and (4) hold, then $p = 2$ and (iii) holds, q.e.d.

(2.3) ((W) Th. 13.3) *A primitive group which contains a transposition is a symmetric group. A primitive group which contains a 3-cycle is either alternating or symmetric.*

§ 3. Maximal p-local subgroups of S_n.

Throughout this section let p denote a prime.

THEOREM 1. *Maximal p-local subgroups of S_n are $(H_{p^e} \int S_k) \times S_r$, with $n = p^e k + r$, $e \geq 1$, $k \geq 1$, $n > r \geq 0$.*

Exceptions are $(H_2 \int S_2) \times S_r$, $(H_{2^e} \int S_k) \times S_2$ with $2^e k \neq 4$, $(H_{2^e} \int S_k) \times S_4$ with $2^e k \neq 2$ and $(H_{3^e} \int S_k) \times S_3$.

Two maximal p-local subgroups $(H_{p^e} \int S_k) \times S_r$ and $(H_{p^f} \int S_l) \times S_s$ are conjugate in S_n if and only if one of the following conditions holds:

(i) $(e, k, r) = (f, l, s)$

(ii) $(p^e k, r) = (s, p^f l) = (4, 2)$ or $(2, 4)$.

REMARK. $(H_2 \times S_2) \times S_r \subsetneq (H_2 \int S_2) \times S_r \subsetneq S_4 \times S_r$, $(H_{2^e} \int S_k) \times S_2 \subsetneq S_{2^e k} \times S_2$ and $(H_{2^e} \int S_k) \times S_4 \subsetneq S_{2^e k} \times S_4$ if $2 \neq 2^e k \neq 4$, $H_4 \times S_4 \subsetneq H_4 \int S_2$, $(H_{3^e} \int S_k) \times S_3 \subsetneq S_{3^e k} \times S_3$ if $3^e k \neq 3$, and $H_3 \times S_3 \subsetneq H_3 \int S_2$.

As an illustration we list the maximal 2-local subgroups of S_n for $5 \leqq n \leqq 9$.

Table 1

S_5	$H_2 \times S_3$, $H_4 \times S_1$
S_6	$H_2 \int S_3$, $H_2 \times S_4 = H_4 \times S_2$
S_7	$H_2 \times S_5$, $H_4 \times S_3$, $(H_2 \int S_3) \times S_1$
S_8	$H_2 \int S_4$, $H_4 \int S_2$, H_8, $H_2 \times S_6$
S_9	$H_2 \times S_7$, $H_4 \times S_5$, $(H_2 \int S_3) \times S_3$, $(H_2 \int S_4) \times S_1$, $(H_4 \int S_2) \times S_1$, $H_8 \times S_1$

COROLLARY 1. *If $n \geqq 5$, $6 \neq n \neq 8$, $2^a \| n$, $2^b \| n-2$, and $2^c \| n-4$, then $l_2(S_n) = a - b - c - 1 + \sum_{i=1}^{\infty} [(n-1)/2^i]$. If $n \geqq 4$, $3^a \| n$ and $3^b \| n-3$, then $l_3(S_n) = a - b + \sum_{i=1}^{\infty} [(n-1)/3^i]$. If $p > 3$ and $p^a \| n$, then $l_p(S_n) = a + \sum_{i=1}^{\infty} [(n-1)/p^i]$.*

COROLLARY 2. *Maximal subgroups of S_n with nonidentity solvable normal subgroups are isomorphic to one of the following groups.*

(i) H_{p^e}, $n = p^e$, $p \neq 2$.

(ii) $S_i \int S_k$, $n = ik$, $i = 2, 3, 4$.

(iii) $S_i \times S_{n-i}$, $i = 2, 3, 4$.

(The author does not know whether H_{p^e}, $p \neq 2$, is actually maximal in S_{p^e} or not. It is easy to see that H_{2^e} is contained in A_{2^e} if $e \geqq 3$.)

The proof of Theorem 1 is divided into several steps. In the rest of this section, let Ω denote a finite set. We note that if a maximal p-local subgroup L of a finite group G normalizes a non-identity p-subgroup P of G, then $L = N_G(P)$.

(3.1) *Suppose $|\Omega|=p^e\geqq p$. Then, up to conjugacy, S^Ω has a unique regular elementary abelian subgroup. The normalizer of a regular elementary abelian subgroup of S^Ω is isomorphic to H_{p^e} as a permutation group.*

PROOF. Let E be a regular elementary abelian subgroup of S^Ω. Let $\alpha\in\Omega$ and define a bijection φ: $E\to\Omega$ by $x\varphi=\alpha x$ for each $x\in E$. Introduce group structure in Ω so that φ is an isomorphism, and denote this group by $\bar{\Omega}$. Taking the regular representation of $\bar{\Omega}$, we consider $\bar{\Omega}$ as a permutation group on Ω. Then clearly $(E,\Omega)\cong(\bar{\Omega},\bar{\Omega})$. Therefore if F is another regular elementary abelian subgroup of S^Ω, we have $(E,\Omega)\cong(F,\Omega)$, or equivalently E, F are conjugate in S^Ω.

Since E_{p^e} is a regular elementary abelian subgroup of H_{p^e}, there exists a permutation group H on Ω which is isomorphic to H_{p^e} as a permutation group and so has a regular elementary abelian normal subgroup E'. By the first paragraph, we may assume that $E=E'$. Since $C(E)=E$, $N(E)/E$ is embedded in $\mathrm{Aut}(E)$. In particular, $|N(E)|\leqq|H|$. Since $H\subseteq N(E)$, we conclude that $N(E)=H$, q.e.d.

(3.2) *Primitive maximal p-local subgroups of S^Ω are the normalizers of regular elementary abelian p-subgroups of S^Ω, and conversely. Therefore, the primitive maximal p-local subgroups of S^Ω constitute one conjugate class.*

PROOF. Let L be a primitive maximal p-local subgroup of S^Ω, and let E be a nonidentity normal elementary abelian p-subgroup of L. Since L is primitive, E is regular, and the first part of the assertion follows. Let H be the normlizer of a regular elementary abelian p-subgroup of S^Ω. By (3.1), H is primitive. Hence if L is a maximal p-local subgroup of S^Ω containing H, L is also primitive, whence $L=H$ by the first part and (3.1). The conjugacy part is a consequence of (3.1).

(3.3) *Suppose L is an imprimitive maximal p-local subgroup of S^Ω. Then we have $|\Omega|=p^ek$ for some $e\geqq1$ and $k>1$, and $L\cong H_{p^e}\int S_k$ as permutation groups.*

PROOF. Let E be maximal among elementary abelian p-subgroups of S^Ω normalized by L. If E is transitive, then E is regular and L is primitive by (3.1), contrary to our hypothesis. Hence E is intransitive. Let $\Delta=\Delta_1,\Delta_2,\dots,\Delta_k$ be the orbits of E. Since L normalizes E and is transitive on Ω, L permutes $\Delta_1,\Delta_2,\dots,\Delta_k$ transitively. In particular, $|\Delta|=|\Delta_i|$ for $1\leqq i\leqq k$. Suppose $|\Delta|=p^e$. Define N to be the subgroup of L that fixes all Δ_i. Then $L\trianglerighteq N\supseteq E$, and L/N, is a permutation group on $\Delta_1,\Delta_2,\dots,\Delta_k$. In particular, $|L:N|\leqq|S_k|$. Each element of N is represented as the product of its Δ_i-constituents (see §2). We will show that L permutes $E^{\Delta_1}, E^{\Delta_2}, \ldots, E^{\Delta_k}$. Suppose $x \in L$ and $\Delta_i x = \Delta_{i'}$ for $1 \leqq i, i' \leqq k$. If we denote the Δ_i-constituent of $a \in E$ by a_i, then $E \ni a^x = a_1^x a_2^x \cdots a_k^x$, $(\Delta_{i'}) a_i^x = \Delta_i a_i x = \Delta_i x = \Delta_{i'}$, and for any $\delta \in \Delta_{j'}$, $i \neq j$, $\delta a_i^x = (\delta x^{-1}) a_i x = (\delta x^{-1}) x = \delta$. Thus a_i^x is the $\Delta_{i'}$-constituent of a^x. This implies that $(E^{\Delta_i})^x = E^{\Delta_{i'}}$. Since L permutes $E^{\Delta_1}, E^{\Delta_2}, \ldots, E^{\Delta_k}$, L normalizes the elementary abelian p-group $B = E^{\Delta_1} E^{\Delta_2} \cdots E^{\Delta_k}$. By the maximality of E, $E = E^{\Delta_1} E^{\Delta_2} \cdots E^{\Delta_k}$. By the definition of N, each E^{Δ_i} is normalized by N, whence by N^{Δ_j}, for each N^{Δ_j}, $j \neq i$, centralizes E^{Δ_i}. Thus $N^{\Delta_i} \subseteq N_{S^{\Delta_i}}(E^{\Delta_i})$. Set $N_i = N_{S^{\Delta_i}}(E^{\Delta_i})$. Since N_i normalizes each E^{Δ_j}, $1 \leqq j \leqq k$, $N_i \subseteq N(B) = L$. Since N_i fixes each Δ_j, $N_i \subseteq N$ whence $N_i \subseteq N^{\Delta_i}$. We conclude that $N^{\Delta_i} = N_i$ and $N = N_1 N_2 \cdots N_k$.

Since E^{Δ_i} is a regular permutation group on Δ_i, (N_i, Δ_i) is isomorphic to a primitive maximal p-local subgroup (H, Δ) of S^{Δ}. Let λ_i: $(H, \Delta) \to (N_i, \Delta_i)$ be an isomorphism. Set $\Sigma = \{1, 2, \ldots, k\}$ and define a bijection μ: $\Delta \times \Sigma \to \Omega$ by $(\alpha, i)\mu = \alpha\lambda_i$. Define a mapping λ: $G = H \int S_k \to S^{\Omega}$ by:

$$\{(\alpha, i)\mu\}(g\lambda) = (\alpha h_i)\lambda_{is} \quad \text{for} \quad g = (h_j)_j s \in G \quad \text{and} \quad (\alpha, i) \in \Delta \times \Sigma.$$

Certainly $g\lambda$ is a permutation on Ω, and λ is a monomorphism. Furthermore, λ, μ induce an isomorphism $(G, \Delta \times \Sigma) \cong (G\lambda, \Omega)$. Let F be the regular elementary abelian normal subgroup of H, and set $C = \{(h_j)_j \in G;\ h_j \in F\}$. We will show that $C\lambda = E$. Let $c = (h_j)_j \in C$. Then $\{(\alpha, i)\mu\}(c\lambda) = (\alpha h_i)\lambda_i = (\alpha\lambda_i)(h_i\lambda_i) = \{(\alpha, i)\mu\}(h_i\lambda_i)$, for λ_i is a permutation isomorphism. This implies that $c\lambda = (h_1\lambda_1)(h_2\lambda_2) \cdots (h_k\lambda_k)$. Since $F = O_p(H)$ and $E^{\Delta_i} = O_p(N_i)$, we have $F\lambda_i = E^{\Delta_i}$, thus $C\lambda = E^{\Delta_1} E^{\Delta_2} \cdots E^{\Delta_k} = E$. Since $C \lhd G$, $E = C\lambda \lhd G\lambda$, whence $G\lambda \subseteq N(E) = L$. Since $|L| = |L : N| \cdot |N| = |L : N| \cdot |H|^k \leqq |S_k| \cdot |H|^k = |G|$, $G\lambda = L$. Therefore $(G, \Delta \times \Sigma) \cong (L, \Omega)$. The proof is complete.

(3.4) *Suppose H is a primitive maximal p-local subgroup of S^{Δ}, and $\Sigma = \{1, 2, \ldots, k\}$, $k > 1$. Then $G = H \int S^{\Sigma}$ is a maximal p-local subgroup of $S^{\Delta \times \Sigma}$ unless $p = |\Delta| = k = 2$.*

PROOF. Set $\Omega = \Delta \times \Sigma$. Let L be a maximal p-local subgroup containing G. Since G is transitive, so is L. Assume first that L is imprimitive. By (2.1), G has a unique nontrivial system of imprimitivity. By (3.3), $|G| = |L|$. Thus $G = L$. Assume that L is primitive. Let E be the regular elementary abelian normal p-subgroup of L, and $\alpha \in \Delta^{(2)}$ (see §2 for the definition of $\Delta^{(2)}$). As noted in §2, if we regard L_α as a permutation group on E, then $(L_\alpha, \Omega) \cong (L_\alpha, E)$. Therefore we may assume that Ω is an elementary abelian p-group (with an identity element α) and L_α acts on Ω as a group of automorphisms of Ω. Since $H^{(1)} \subseteq L_\alpha$

and $H^{(1)}$ is primitive on $\Delta^{(1)}$, and fixes the points of $\Omega-\Delta^{(1)}$, we can apply Proposition (2.2) with $H^{(1)}, \Omega$ and $\Omega-\Delta^{(1)}$ in place of A, G and C, respectively. Since Ω is an elementary abelian p-group and $|\Omega-\Delta^{(1)}|\neq 1$, (ii) of (2.2) must holds. We conclude that $p=|\Delta|=k=2$.

(3.5) *An imprimitive maximal p-local subgroup of S^Ω has a unique nontrivial system of imprimitivity. Two imprimitive maximal p-local subgroups of S^Ω are conjugate in S^Ω if and only if their nontrivial blocks contain the same number of points.*

PROOF. Trivial.

(3.6) *Suppose L is an intransitive maximal p-local subgroup of S^Ω. Then $L=M\times S^{\Omega-\Delta}$ for a transitive maximal p-local subgroup M of S^Δ, $\Delta\subsetneqq\Omega$.*

PROOF. Let $\Delta_1, \dots, \Delta_k$ be the orbits of L, and let P be a nonidentity normal p-subgroup of L. As in the proof of (3.3), it is easily proved that L normalizes each P^{Δ_i}. Since $P\neq 1$, $P^{\Delta_i}\neq 1$ for some i, say for $i=1$. Set $M=N_{S^\Delta}(P^\Delta)$, where $\Delta=\Delta_1$, Then $L^\Delta\subseteq M$. In particular M is transitive on Δ. Since $L\subseteq L^{\Delta_1}L^{\Delta_2}\cdots L^{\Delta_k}\subseteq MS^{\Omega-\Delta}$ and $MS^{\Omega-\Delta}$ is p-local, we have $L=MS^{\Omega-\Delta}$, q.e.d.

(3.7) *Suppose M is a transitive maximal p-local subgroup of S^Δ, $\Delta\subsetneqq\Omega$, and $G=M\times S^{\Omega-\Delta}$. Then G is not a maximal p-local subgroup of S^Ω if and only if one of the following conditions holds.*

(i) $p=2$, $|\Omega-\Delta|=2$, $|\Delta|\neq 4$

(ii) $p=2$, $|\Omega-\Delta|=4$, $|\Delta|\neq 2$

(iii) $p=3$, $|\Omega-\Delta|=3$.

PROOF. If Ψ is a nontrivial block of G on Ω, then either $\Psi\subseteq\Delta$ or $\Psi=\Omega-\Delta$. Let L be a maximal p-local subgroup of S^Ω containing G. Assume that L is intransitive. Then by (3.6), we have $L=N\times S^{\Omega-\Sigma}$ for some transitive maximal p-local subgroup N of S^Σ, $\Sigma\subsetneqq\Omega$. Since Σ is a fixed block of G, either $\Sigma=\Delta$ or $\Sigma=\Omega-\Delta$. If $\Sigma=\Delta$, then $M=N$ by the maximality of M, whence $G=L$. If $\Sigma=\Omega-\Delta$, then $S^\Sigma=N$. Therefore the following condition holds.

(*) $\quad p=2,\ |\Omega-\Delta|=2\ $ or $\ p=2,\ |\Omega-\Delta|=4\ $ or $\ p=3,\ |\Omega-\Delta|=3.$

Assume that L is imprimitive. Then L has a unique nontrivial system of imprimitivity $\Delta_1, \Delta_2, \dots, \Delta_k$. We may assume that $\Delta=\Delta_2\cup\cdots\cup\Delta_k$ and $\Omega-\Delta=\Delta_1$. Let L_i be the subgroup of L which fixes the points of $\Omega-\Delta_i$. Then by (3.3), L_i is a primitive maximal p-local subgroup of S^{Δ_i}. Since $S^{\Delta_i}=S^{\Omega-\Delta}\subseteq L_1$, (*) holds. Assume that L is primitive. Since p divides both $|\Omega|$ and $|\Delta|$, $|\Omega-\Delta|\geqq p\geqq 2$. Hence L contains a transposition. By (2.3), $L=S^\Omega$. Again (*) holds. Finally assume that (*) holds. Suppose $p=2$ and $2\neq|\Delta|\neq 4$ or $p=3$ and $|\Delta|\neq 3$. Then G is a proper subgroup of $S^{\Delta}\times S^{\Omega-\Delta}$ which is p-local. The $S_2\times S_4$ is a maximal 2-local subgroup of S_6. For the candidates for the maximal 2-local subgrougs of S_6 are $S_2\times S_4$ and $S_2\int S_3$, and $|S_2\times S_4|=|S_2\int S_3|$. Since $H_n\times S_n=S_n\times S_n\subsetneqq H_n\int S_2$, for $n=2$, 3 and 4, $H_n\times S_n$ is not maximal for $n=2$, 3 and 4.

(3.8) *Suppose $\Delta, \Sigma\subsetneqq\Omega$. Let M and N be transitive maximal p-local subgroups of S^{Δ} and S^{Σ}, respectively. Then $M\times S^{\Omega-\Delta}$ is conjugate to $N\times S^{\Omega-\Sigma}$ in S^{Ω} if and only if one of the following conditions holds.*

(i) $(M, \Delta)\cong(N, \Sigma)$

(ii) $(|\Delta|, |\Omega-\Delta|)=(|\Omega-\Sigma|, |\Sigma|)=(2, 4)$ *or* $(4, 2)$, $p=2$.

PROOF. Suppose $M\times S^{\Omega-\Delta}$ is conjugate to $N\times S^{\Omega-\Sigma}$. We may assume $M\times S^{\Omega-\Delta}=N\times S^{\Omega-\Sigma}$. Then either $\Sigma=\Delta$ or $\Sigma=\Omega-\Delta$. If $\Sigma=\Delta$, then $M=N$, whence $(M, \Delta)\cong(N, \Sigma)$. If $\Sigma=\Omega-\Delta$, then $S^{\Omega-\Delta}=N$ and $S^{\Omega-\Sigma}=M$. Therefore (i) or (ii) holds, q.e.d.

Proposition (3.8) concludes the proof of Theorem 1. We now prove Corollary 1. Suppose $p\neq 2$. Then there exists a one to one correspondence between the conjugate classes of transitive maximal p-local subgroups of S_n and the p-powers dividing n. If $p>3$, then there exists a one to one correspondence between the conjugate classes of intransitive maximal p-local subgroups of S_n and the pairs (e, k) of positive integers such that $p^e k<n$. Thus, if $p^a\| n$,

$$l_p(S_n)=a+\sum_i\left\{\left[\frac{n-1}{p^i}\right]-\left[\frac{n-1}{p^{i+1}}\right]\right\}i=a+\sum_i\left[\frac{n-1}{p^i}\right].$$

In case $p=2$ or 3, modify this formula according to (3.4) and (3.7). Corollary 2 is an immediate consequence of Theorem 1.

§4. Maximal 2-local subgroups of A_n.

THEOREM 2. *Maximal 2-local subgroups of A_n, $n\geqq 5$, are $L\cap A_n$ where L is a subgroup of S_n which satisfies the following condition.*

(*) *L is a maximal 2-local subgroup of S_n and $L\not\cong S_2\times S_{n-2}$, or $L=M\times S_2$ where M is a transitive maximal 2-local subgroup of S_{n-2}.*

Exceptions are $(H_2\int S_4)\cap A_8$, $((H_2\int S_4)\times S_1)\cap A_9$ and $((H_2\int S_{k-1})\times S_2)\cap A_{2k}$, $k\geqq 4$.

Let L_1 and L_2 satisfy (*). If $L_1\cap A_n$ is conjugate to $L_2\cap A_n$ in A_n, then L_1 is conjugate to L_2 in A_n. If L satisfies (*), then $N(L\cap A_n)=N(L)=L$.

REMARK. $(H_2\int S_4)\cap A_8\subsetneqq H_8\subsetneqq A_8$, $((H_2\int S_4)\times S_1)\cap A_9\subsetneqq H_8\times S_1\subsetneqq A_9$, $(H_2\int S_{k-1})\times S_2\subsetneqq H_2\int S_k$.

Theorem 2, together with Theorem 1 and Proposition (3.8), provides enough information about maximal 2-local subgroups of A_n. Let L satisfy (*), and let C be the conjugate class of L in S_n. If $L\subseteq A_n$, then, C decomposes into two conjugate classes in A_n. If $L\nsubseteq A_n$, $C'=\{K\cap A_n;\ K\in C\}$ is a conjugate class in A_n. We will see that $L\subseteq A_n$ if and only if L is one of the following groups.

$H_{2^e}\int S_k,\ n=2^e k,\ e\geqq 3,\ k\geqq 1$

$(H_{2^e}\int S_k)\times S_1,\ n=2^e k+1,\ e\geqq 3,\ k\geqq 1.$

As an illustration we list the maximal 2-local subgroups of A_n for $5\leqq n\leqq 9$.

Table 2

A_5	$(H_4\times S_1)\cap A_5=A_4\times S_1$
A_6	$(H_2\int S_3)\cap A_6$, $(H_4\times S_2)\cap A_6$
A_7	$(H_4\times S_3)\cap A_7$, $(H_2\int S_3\times S_1)\cap A_7$
A_8	$(H_4\int S_2)\cap A_8$, H_8 (two classes)
A_9	$(H_4\times S_5)\cap A_9$, $((H_2\int S_3)\times S_3)\cap A_9$, $((H_4\int S_2)\times S_1)\cap A_9$, $(H_8\times S_1)\cap A_9$ (two classes)

COROLLARY 3. *If* $n\geqq 5$, $6\neq n\neq 9$, $2^a\| n$, $2^d\| n-1$ *and* $2^c\| n-4$, *then*

$$l_2(A_n)=\begin{Bmatrix}2a-2 & (a\geqq 3)\\ a & (a\leqq 2)\end{Bmatrix}+\begin{Bmatrix}d-2 & (d\geqq 3)\\ 0 & (d\leqq 2)\end{Bmatrix}+\begin{Bmatrix}-c-3 & (c\geqq 1)\\ -2 & (c=0)\end{Bmatrix}+\sum_i\left[\frac{n-1}{2^i}\right].$$

The proof is divided into five steps.

(4.1) *Suppose G is a maximal 2-local subgroup of A^Ω ($|\Omega|=n\geqq 5$). Then $G=L\cap A^\Omega$ where L satisfies the following condition.*

(*) *L is a maximal 2-local subgroup of S^Ω and $L\neq S_2\times S_{n-2}$, or $L=M\times S^{\Omega-\Delta}$ where M is a transitive maximal 2-local subgroup of S^Δ, $\Delta\subseteq\Omega$, $|\Omega-\Delta|=2$.*

PROOF. Let L be a maximal 2-local subgroup of S^Ω containing G, and set $P=O_2(L)$. Since $L\cap A^\Omega\trianglerighteq P\cap A^\Omega$ and $L\cap A^\Omega\supseteq G$, $G=L\cap A^\Omega$ unless $P\cap A^\Omega=1$. If $P\cap A^\Omega=1$, then clearly $|P|=2$. Therefore, by a result of §3, $L=S^\Delta\times S^{\Omega-\Delta}$, $\Delta\subseteq\Omega$ and $|\Omega-\Delta|=2$. Since G is a maximal 2-local subgroup of $L\cap A^\Omega$, and the mapping $L\cap A^\Omega\ni x\mapsto x^\Delta\in L^\Delta$ is an isomorphism, $M=G^\Delta$ is a maximal 2-local subgroup of $L^\Delta=S^\Delta$. In particular, $|O_2(M\times S^{\Omega-\Delta})|>2$. Since $G\subseteq(M\times S^{\Omega-\Delta})\cap A^\Omega$, we have $G=(M\times S^{\Omega-\Delta})\cap A^\Omega$. Assume that M is intransitive on Δ. Then by (3.6), we have $M=K\times S^{\Delta-\Sigma}$ for a transitive maximal 2-local subgroup K of S^Σ, $\Sigma\subsetneq\Delta$. Thus $G\subseteq X=K\times S^{\Delta-\Sigma}\times S^{\Omega-\Delta}$. If $|\Sigma|=2$, $X\subsetneq S^{(\Omega-\Delta)\cup\Sigma}\times S^{\Delta-\Sigma}=Y$ and $|O_2(Y)|>2$. Thus $G=X\cap A^\Omega\subsetneq Y\cap A^\Omega$ and $Y\cap A^\Omega$ is 2-local, contrary to the maximality of G. If $|\Sigma|>2$, then $X\subsetneq K\times S^{\Omega-\Sigma}=Z$ and $|O_2(Z)|>2$, again a contradiction. Therefore M

is transitive on Δ, q.e.d.

(4.2) *Let K be a transitive maximal 2-local subgroup of S^{Ω}, $|\Omega|=n\geqq 5$, and let L satisfy* (*) *in* (4.1). *Suppose $K\cap A^{\Omega}\subseteq L$. Then one of the following statements holds.*

(i) $K=L$

(ii) $|\Omega|=8$, $K\cong H_2\int S_4$ *and* $L\cong H_8$.

PROOF. Assume first that K is primitive. Then $|\Omega|=2^e$ for some integer e. The stabilizer K_α of $\alpha\in\Omega$ is isomorphic to $GL(e,2)$ and is simple as $e\geqq 3$. Therefore $K\subseteq A^{\Omega}$, and (i) holds. Assume that K is imprimitive. We may assume that $K=H\int S^{\Sigma}$ where H is a primitive maximal 2-local subgroup of S^{Δ} and $\Omega=\Delta\times\Sigma$. The S^{Σ} is contained in A^{Ω}. For if the cycle decomposition of $\sigma\in S^{\Sigma}$ is $\sigma=(i,j,\cdots,k)(i',j',\cdots,k')\cdots$, the cycle decomposition of σ in S^{Ω} is

$$\sigma=\prod_{\alpha\in\Delta}((\alpha,i),(\alpha,j),\cdots,(\alpha,k))((\alpha,i'),(\alpha,j'),\cdots,(\alpha,k'))\cdots$$

Since $|\Delta|$ is even, $\sigma\in A^{\Omega}$. We have $|\Delta|=2^e$ and $|\Sigma|=k$ for some e and k. Assume $e\geqq 3$. Then $H^{(i)}\subseteq A^{\Delta^{(i)}}$ for each i, whence $K\subseteq A^{\Omega}$ and (i) holds. Suppose $e\leqq 2$. Then $H=S^{\Delta}$. Since $K\cap A^{\Omega}$ is transitive, L is a transitive maximal 2-local subgroup of S^{Ω}. Suppose $e=2$. Then $A^{\Delta}\int S^{\Sigma}\subseteq K\cap A^{\Omega}$, whence $\Delta^{(1)},\Delta^{(2)},\dots,\Delta^{(k)}$ is a unique nontrivial system of imprimitivity for $K\cap A^{\Omega}$. Therefore if L is imprimitive, we have $L=S^{\Delta}\int S^{\Sigma}=K$ by (2.1) and the maximality of L. Assume that L is primitive. Then $L=A^{\Omega}$ or S^{Ω} by (2.3), since $A^{\Delta}\int S^{\Sigma}$ contains a 3-cycle. But this is not the case since we are assuming $n\geqq 5$. Assume $e=1$. Then $k\geqq 3$ by our hypothesis. We can prove in exactly the same way as in the proof of (2.1) that $\Delta^{(1)},\Delta^{(2)},\dots,\Delta^{(k)}$ is the unique nontrivial system of imprimitivity for $K\cap A^{\Omega}$. Therefore if L is imprimitive, we have $K=L$.

Assume that L is primitive. Then $k\geqq 4$ by our hypothesis. We will show that (ii) holds. Let $a\in\Delta^{(1)}$. Identifying Ω with the regular elementary abelian normal subgroup of L, we may consider that L_a acts on Ω as a group of automorphisms. Let us adopt the additive notation, and rewrite a by 0. Let $0\neq\eta\in\Delta^{(1)}$ and let $\Delta^{(2)}=\{\alpha,x\}$. Clearly $\alpha+\eta\neq 0$, η or α. Suppose $\alpha+\eta\neq x$. Since $k\geqq 4$, there exists an $f\in(K\cap A^{\Omega})_0$ such that $\alpha f=x$ and $(\alpha+\eta)f=(\alpha+\eta)$. It follows that $\eta f=(\alpha+(\alpha+\eta))f=x+\alpha+\eta\neq\eta$. This is a contradiction since $\Delta^{(1)}$ is a block of K. Therefore we may assume $\Delta^{(2)}=\{\alpha,\alpha+\eta\}$ and $\Delta^{(3)}=\{\beta,\beta+\eta\}$. Since $\alpha+\beta\notin\Delta^{(1)}\cup\Delta^{(2)}\cup\Delta^{(3)}$, we may assume $\Delta^{(4)}=\{\alpha+\beta,\alpha+\beta+\eta\}$. If $k\geqq 5$, then there exists an $f\in(K\cap A^{\Omega})_0$ such that $\alpha f=\alpha$, $\beta f=\beta$, $(\alpha+\beta)f=\alpha+\beta+\eta$. This yields a contradiction. Thus (ii) holds.

REMARK. $(H_2 \int S_4) \cap A_8$ is actually contained in H_8. Let Ω be an elementary abelian (additive) group of order 8, $\Delta=\{0, \gamma\}$ a subgroup of Ω of order 2, and $\Sigma=\{0, \alpha, \beta, \alpha+\beta\}$ a subgroup of Ω of order 4 not containing γ. Define a bijection $\lambda: \Delta \times \Sigma \to \Omega$ by $(x, y)\lambda = x+y$ and extend it to an embedding $\lambda: S^{\Delta} \int S^{\Sigma} \to S^{\Omega}$. We will show that $G=(S^{\Delta} \int S^{\Sigma})\lambda \cap A^{\Omega}$ is contained in the holomorph H^{Ω} of Ω. Suppose $f=(0, \alpha)(\gamma, \alpha+\gamma)$ and $g=(0, \gamma)(\alpha, \alpha+\gamma)$. Both f and g are typical generators of G. Let a, b be elements of Aut Ω such that $\gamma a=\gamma$, $\alpha a=\alpha$, $\beta a=\alpha+\beta$, and $\gamma b=\gamma$, $\alpha b=\alpha$, $\beta b=\beta+\gamma$. Then $f=a\alpha \in H^{\Omega}$ and $g=b\gamma \in H^{\Omega}$.

(4.3) *Suppose $K=N\times S^{\Omega-\Lambda}$ where N is a transitive maximal 2-local subgroup of S^{Λ}, $\Lambda \subsetneqq \Omega$ and $4 \leqq |\Lambda| < |\Omega|$. Then,*

(i) *$N\cap A^{\Lambda}$ is transitive on Λ, and has at most one nontrivial system of imprimitivity on Λ which coincides with that of N.*

(ii) *Suppose Ψ is a nontrivial block of $K\cap A^{\Omega}$ on Ω. Then one of the following possibilities occurs.*

(1) $\Psi \subseteq \Lambda$.

(2) $\Psi = \Omega - \Lambda$

(3) $\Lambda \subsetneqq \Psi \subsetneqq \Omega$, $|\Omega-\Lambda|=2$.

PROOF. (i) We may assume that $N=H \int S^{\Sigma}$ where H is a primitive maximal 2-local subgroup of S^{Γ}, $\Lambda=\Gamma\times\Sigma$, $|\Gamma|=2^e$ and $|\Sigma|=m$. If $e \geqq 3$, then $N \subseteq A^{\Lambda}$ and the assertion follows from (2.1). If $e=2$, then $A^{\Gamma} \int S^{\Sigma} \subseteq N\cap A^{\Lambda}$ and we can apply (2.1). If $e=1$, we have $m \geqq 3$ since $|\Lambda| \geqq 4$ and N is a maximal 2-local subgroup of S^{Λ}. Thus the proof of (2.1) can apply.

(ii) We first note that $(N\cap A^{\Lambda})\times A^{\Omega-\Lambda} \subseteq K\cap A^{\Omega}$. Assume $\Psi\cap\Lambda \neq \phi \neq \Psi\cap(\Omega-\Lambda)$. Then by (i), $\Lambda \subseteq \Psi$. Therefore if $|\Omega-\Lambda| \leqq 2$, (1) or (3) holds. Suppose $|\Omega-\Lambda| \geqq 3$. Then $\Psi=\Lambda$ since $A^{\Omega-\Lambda}$ is transitive on $\Omega-\Lambda$. If $\Psi \subsetneqq \Omega-\Lambda$, then $|\Omega-\Lambda| \geqq 3$. But this is a contradiction since $A^{\Omega-\Lambda}$ is primitive on $\Omega-\Lambda$.

(4.4) *Let K be an intransitive maximal 2-local subgroup of S^{Ω}, $|\Omega|=n \geqq 5$, such that $K \neq S_2\times S_{n-2}$. Assume $K=N\times S^{\Omega-\Lambda}$ where N is a transitive maximal 2-local subgroup of S^{Λ}, $\Lambda \subsetneqq \Omega$. If L satisfies (*) in (4.1) and $K\cap A^{\Omega} \subseteq L$, then one of the following statements holds.*

(i) $K=L$.

(ii) $|\Lambda|=8$, $|\Omega|=9$, $N \cong H_2 \int S_4$, $L \cong H_8 \times S_1$.

PROOF. By our hypothesis $|\Lambda| \geqq 4$, and so (4.3) can apply. Furthermore it follows from (3.7) that $2 \neq |\Omega-\Lambda| \neq 4$.

Suppose $L=M\times S^{\Omega-\Delta}$ where M is a transitive maximal 2-local subgroup of S^{Δ},

$\Delta \subsetneq \Omega$, $|\Omega-\Delta|=2$. Since both Δ and $\Omega-\Delta$ are fixed nontrivial blocks of $K\cap A^{\Omega}$, we have $\Delta=\Lambda$. Thus $|\Omega-\Lambda|=2$, a contradiction.

Suppose L is an intransitive maximal 2-local subgroup of S^{Ω} such that $L\neq S_2\times S_{n-2}$. Let $L=M\times S^{\Omega-\Delta}$ where M is a transitive maximal 2-local subgroup of S^{Δ}, $\Delta\subsetneq\Omega$. Since both Δ and $\Omega-\Delta$ are fixed nontrivial blocks of $K\cap A^{\Omega}$, either $\Delta=\Lambda$ or $\Delta=\Omega-\Lambda$. Assume $\Delta=\Lambda$. Since $L=M\times S^{\Omega-\Delta}\supseteq K\cap A^{\Omega}\supseteq(N\cap A^{\Lambda})\times A^{\Omega-\Lambda}$, $M\supseteq N\cap A^{\Lambda}$. If $|\Lambda|=4$, then $M=S^{\Lambda}=N$ and (i) holds. If $|\Lambda|\geqq 5$, it follows from (4.2) that either $N=M$ or $|\Lambda|=8$, $N\cong H_2\int S_4$ and $M\cong H_8$. In the latter case, $L\cap A^{\Omega}=M\times A^{\Omega-\Lambda}$ as $M\subsetneq A^{\Lambda}$. Since $L\cap A^{\Omega}\supseteq(N\times S^{\Omega-\Lambda})\cap A^{\Omega}$ and $N\nsubseteq A^{\Lambda}$, we have $|\Omega-\Lambda|=1$. Thus (ii) holds. Assume $\Delta=\Omega-\Lambda$. Since $L=S^{\Lambda}\times M\supseteq(N\cap A^{\Lambda})\times A^{\Omega-\Lambda}$, $M\supseteq A^{\Omega-\Lambda}$. But this is not the case since $|\Omega-\Lambda|=|\Delta|>4$.

Suppose L is an imprimitive maximal 2-local subgroup of S^{Ω}. Let $\Sigma_1, \Sigma_2, \dots, \Sigma_m$ be the canonical system of imprimitivity of L. By (4.3), we may assume that $\Omega-\Lambda=\Sigma_1$ and $\Lambda=\Sigma_2\cup\cdots\cup\Sigma_m$. Let L_1 be the set of elements of L that fix the points of Λ. Then L_1 is a primitive maximal 2-local subgroup of $S^{\Omega-\Lambda}$. Since $L\supseteq(N\cap A^{\Lambda})\times A^{\Omega-\Lambda}$ $L_1\supseteq A^{\Omega-\Lambda}$. But this is not the case since $|\Omega-\Lambda|\geqq 8$.

Suppose L is a primitive maximal 2-local subgroup of S^{Ω}. Since both $|\Omega|$ and $|\Lambda|$ are even, and $L\supseteq(N\cap A^{\Lambda})\times A^{\Omega-\Lambda}$, L contains a 3-cycle. By (2.3), this yields a contradiction.

(4.5) *Let $K=N\times S^{\Omega-\Lambda}$ where N is a transitive maximal 2-local subgroup of S^{Λ}, $\Lambda\subseteq\Omega$, $|\Omega-\Lambda|=2$ and $|\Omega|=n\geqq 5$. If L satisfies (*) in (4.1) and $K\cap A^{\Omega}\subseteq L$, then one of the following statements holds.*

(i) $K=L$.

(ii) $N\cong H_2\int S_{m-1}$. $L\cong H_2\int S_m$, $n=2m$.

(iii) $|\Omega|=8$, $N\cong H_2\int S_3$, $L\cong H_8$.

PROOF. Since $|\Lambda|$ is even, $|\Lambda|\geqq 4$, whence (4.3) can apply. Suppose $L=M\times S^{\Omega-\Delta}$ where M is a transitive maximal 2-local subgroup of S^{Δ}, $\Delta\subseteq\Omega$ and $|\Omega-\Delta|=2$. Since both Δ and $\Omega-\Delta$ are fixed nontrivial blocks of $K\cap A^{\Omega}$, we have $\Delta=\Lambda$. Hence $M\supseteq N\cap A^{\Lambda}$. By (4.2), either $N=M$ or $|\Lambda|=8$, $N\cong H_2\int S_4$ and $M\cong H_8$. In the latter case, $L\cap A^{\Omega}=M\times A^{\Omega-\Lambda}\supseteq(N\times S^{\Omega-\Lambda})\cap A^{\Omega}$. This is a contradiction since $|\Omega-\Lambda|=2$ and $N\nsubseteq A^{\Lambda}$.

Suppose L is an intransitive maximal 2-local subgroup of S^{Ω} such that $L\ncong S_2\times S_{n-2}$. Assume $L=M\times S^{\Omega-\Delta}$ where M is a transitive maximal 2-local subgroup of S^{Δ}, $\Delta\subsetneq\Omega$. Then $|\Delta|\geqq 4$ and $2\neq|\Omega-\Delta|\neq 4$. Therefore by (4.3), $\Lambda\subsetneq\Delta\subsetneq\Omega$. But this is not the case since both $|\Lambda|$ and $|\Delta|$ are even.

Suppose L is an imprimitive maximal 2-local subgroup of S^{Ω}. Let $\Sigma_1, \Sigma_2, \ldots, \Sigma_m$ be the canonical system of imprimitivity of L. By (4.3), we may assume that $\Lambda = \Sigma_1 \cup \cdots \cup \Sigma_{m-1}$ and $\Omega - \Lambda = \Sigma_m$. In particular $|\Sigma_i| = 2$ for each i. Whence $m \geqq 3$. Therefore by (4.3), N is imprimitive with the canonical system of imprimitivity $\Sigma_1, \ldots, \Sigma_{m-1}$. Thus (ii) holds.

Suppose L is a primitive maximal 2-local subgroup of S^{Ω}. As before, we identify Ω with the regular elementary abelian normal subgroup of L. We will assume that $0 \in \Omega - \Lambda$ (0 is the identity element of Ω). We may assume that $N = H \int S^{\Sigma}$ where H is a primitive maximal 2-local subgroup of S^{Γ}, $\Lambda = \Gamma \times \Sigma$, $|\Gamma| = 2^e \geqq 2$ and $|\Sigma| = k \geqq 1$. If $e \geqq 3$, then $N \subsetneqq A^{\Lambda}$, and therefore $H^{(i)} \subsetneqq N \cap A^{\Lambda} \subsetneqq L$ (see § 2 for the definition of $H^{(i)}$). Since $H^{(i)}$ is primitive on $\Gamma^{(i)}$ and fixes the points of $\Omega - \Gamma^{(i)}$, we have $|\Omega| = 4$ by (2.2), contrary to our hypothesis. Hence $e \leqq 2$. If $e = 2$, then $A^{\Gamma} \int S^{\Sigma} \subsetneqq N \cap A^{\Lambda} \subsetneqq L$. Again this is a contradiction. Thus $e = 1$, and therefore $k \geqq 3$ since $|\Omega| = 2(k+1)$ is a power of 2. Since $k \geqq 3$, we can prove as in the last paragraph of (4.2) that $k = 3$. Thus (iii) holds. This completes the proof of (4.5).

Finally we show that $N(L) = L$ if L satisfies (*). If L is transitive, the assertion is clear by (2.1) and the uniqueness of the nontrivial system of imprimitivity of L. Suppose L is intransitive and $\Delta, \Omega - \Delta$ are its fixed blocks. If $|\Delta| = |\Omega - \Delta|$, then $L^{\Delta} \not\cong L^{\Omega - \Delta}$ by the maximality of L. In any case $N(L)$ fixes Δ and $\Omega - \Delta$. Therefore $N(L) = L$.

§ 5. Maximal p-local subgroups of A_n, $p \neq 2$.

Throughout this section let p be an odd prime.

THEOREM 3. *Maximal p-local subgroups of A_n are $L \cap A_n$ where L are maximal p-local subgroups of S_n. Let L_1, L_2 be maximal p-local subgroups of S_n. Then $L_1 \cap A_n$ is conjugate to $L_2 \cap A_n$ in A_n if and only if L_1 is conjugate to L_2 in S_n.*

The proof is divided into three steps. Let Ω be a finite set.

(5.1) *Suppose G is a maximal p-local subgroup of A^{Ω}. Then there exists a maximal p-local subgroup L of S^{Ω} such that $G = L \cap A^{\Omega}$.*

PROOF. Trivial.

(5.2) *Suppose L is a maximal p-local subgroup of S^{Ω}. Then*

(i) $L \not\subseteq A^{\Omega}$.

(ii) *If L is primitive, so is $L \cap A^{\Omega}$.*

(iii) *If L is imprimitive, $L \cap A^{\Omega}$ is transitive and has a unique nontrivial* *system of imprimitivity.*

(iv) *Suppose L is intransitive and $L=M\times S^{\Omega-\Delta}$ where M is a transitive maximal p-local subgroup of $S^{\Delta}, \Delta\subsetneq\Omega$. If ϕ is a nontrivial block of $L\cap A^{\Omega}$ on Ω, then either $\phi\subseteq\Delta$ or $\phi=\Omega-\Delta$.*

PROOF. (i) By Th. 1, we may assume that L is primitive. Identifying Ω with the regular elementary abelian normal subgroup E of L, we choose a basis $\omega_1, \omega_2, \ldots, \omega_r$ of Ω. We regarded Ω as a vector space over $GF(p)$. Let a be a generator of the multiplicative group of $GF(p)$ and let g be an element of L_0 such that $\omega_i g=\omega_i$ for $i<r$ and $\omega_r g=a\omega_r$. Then g is the product of p^{r-1} $(p-1)$-cycles, and so is an odd permutation in L.

(ii) Let E be the regular elementary abelian normal subgroup of L. Since $E\subseteq L\cap A^{\Omega}$, $L\cap A^{\Omega}$ is transitive. Suppose $|\Omega|>p$. Let $\alpha\in\Omega$. If we identify Ω with E and L_{α} with $GL(\Omega)$, respectively, then $(L\cap A^{\Omega})_{\alpha}\supseteq SL(\Omega)$. Since $SL(\Omega)$ is transitive on $\Omega-\{0\}$, $L\cap A^{\Omega}$ is doubly transitive.

(iii) Let $\Delta_1, \ldots, \Delta_m$ be the canonical system of imprimitivity of L. To prove that $L\cap A^{\Omega}$ is transitive, it is sufficient to show that $L\cap A^{\Omega}$ permutes $\Delta_1, \ldots, \Delta_m$ transitively. But it is clear by (i). The second assertion is an immediate consequence of (2.1) and (ii).

(iv) is an immediate consequence of (i) (ii) and (iii).

(5.3) *Let K, L be maximal p-local subgroups of S^{Ω}. If $K\cap A^{\Omega}\subseteq L$, then $K=L$.*

PROOF. Suppose K is primitive. By (5.2), L is primitive. Let E, F be regular elementary abelian normal subgroups of K, L, respectively. Since $K\cap A^{\Omega}=L\cap A^{\Omega}$, E, F are minimal normal subgroups of $K\cap A^{\Omega}$. If $E\neq F$, then $[E, F]\subseteq E\cap F=1$, contrary to the fact that $C(E)=E$. Thus $E=F$ and $K=L$.

Suppose K is imprimitive. By (5.2), L is transitive. We may assume that $K=H\int S^{\Sigma}$ where H is a primitive maximal p-local subgroup of $S^{\Delta}, \Delta\times\Sigma=\Omega$. Assume that L is primitive. Since $p\neq 2$ and $H^{(i)}\cap A^{\Delta^{(i)}}$ is primitive on $\Delta^{(i)}$ by (5.2), (2.2) yields a contradiction. Thus L is imprimitive. Let $L^{(i)}$ be the set of elements of L that fix the points of $\Omega-\Delta^{(i)}$. Since $H^{(i)}\cap A^{\Delta^{(i)}}\subseteq L^{(i)}$, we have, by the first paragraph, $H^{(i)}=L^{(i)}$. Since $O_p(H^{(1)})\cdots O_p(H^{(m)})\trianglelefteq K$ and $O_p(L^{(1)})\cdots O_p(L^{(m)})\trianglelefteq L$, we have $K=L$.

Suppose K is intransitive and $K=N\times S^{\Omega-\Delta}$ where N is a transitive maximal p-local subgroup of $S^{\Delta}, \Delta\subsetneq\Omega$. Then $(p, |\Omega-\Delta|)\neq(3,3)$. If L is primitive, then, by (2.3), $L=A^{\Omega}$ or S^{Ω} since $L\supseteq(N\cap A^{\Delta})\times A^{\Omega-\Delta}$ and p divides $|\Omega-\Delta|$. But this is not the case. If L is imprimitive, then, by (5.2), $\Omega-\Lambda$ is a nontrivial block of L. Since $L \supsetneqq (N \times A^{\Lambda}) \times A^{\Omega-\Lambda}$, $A^{\Omega-\Lambda}$ is contained in a primitive maximal p-local subgroup of $S^{\Omega-\Lambda}$, a contradiction. Thus L is intransitive. Then we have $L = M \times S^{\Omega-\Delta}$ where M is a transitive maximal p-local subgroup of S^{Δ}, $\Delta \subseteq \Omega$. Then $\Delta=\Lambda$ or $\Delta=\Omega-\Lambda$. Assume $\Delta=\Lambda$. Then $M \supseteq N \cap A^{\Lambda}$. Therefore $M=N$ by the preceding paragraph, whence $K=L$. If $\Delta=\Omega-\Lambda$, then $M \supseteq A^{\Omega-\Lambda}$, a contradiction. The proof is complete.

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