On the second cohomology groups (Schur multipliers) of finite reflection groups

By Shin-ichiro IHARA and Takeo YOKONUMA

§ 0. Introduction.

The theory of projective representations of finite groups was founded by I. Schur [3], [4]. Schur has associated to every finite group G a finite abelian group M, called the multiplier of G, consisting of all the equivalence classes of factor sets of projective representations of G (cf. § 1). In the terminology of cohomology theory of groups, the multiplier M of G is nothing but the second cohomology group $\mathrm{H}^2(G, \boldsymbol{C}^*)$ for the coefficient group $\boldsymbol{C}^*$(= the multiplicative group of the complex number field $\boldsymbol{C}$) under the trivial action of G on $\boldsymbol{C}^*$. Schur has shown the existence of a representation-group $\tilde{G}$ of a given finite group G and thus reduced the problem of the determination of all projective representations of G to the determination of the ordinary representations of $\tilde{G}$. Namely, there exists a central extension

$$1 \to M \to \tilde{G} \to G \to 1 \qquad (M = \mathrm{H}^2(G, \boldsymbol{C}^*))$$

of G with the following property: for any projective representation $\varphi: G \to PGL(n, \boldsymbol{C}) = GL(n, \boldsymbol{C})/\boldsymbol{C}^* \cdot I_n$, there exists an ordinary representation $\tilde{\varphi}: \tilde{G} \to GL(n, \boldsymbol{C})$ such that $\tilde{\varphi}(M) \subset \boldsymbol{C}^* \cdot I_n$ and $\pi \circ \tilde{\varphi} = \varphi$ where $\pi: GL(n, \boldsymbol{C}) \to PGL(n, \boldsymbol{C})$ is the natural projection. (We refer to Schur [3], [4] or Curtis-Reiner [2] or K. Yamazaki [7] for the details.)

In [5] Schur has determined the group $\mathrm{H}^2(G, \boldsymbol{C}^*)$ for $G = \mathfrak{S}_n$ (the symmetric group of degree n) or $G = \mathfrak{A}_n$ (the alternating group of degree n) and has determined the projective representations of the groups $\mathfrak{S}_n, \mathfrak{A}_n$.

The purpose of this note is to determine the cohomology group $\mathrm{H}^2(G, \boldsymbol{C}^*)$ for the case where G is a finite reflection group on a Euclidean space. Our main result is the following.

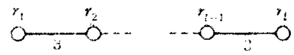
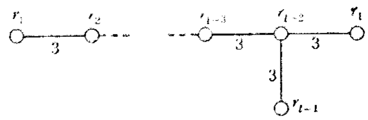
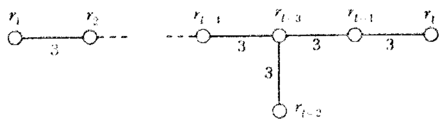
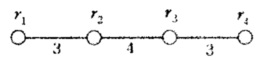
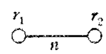
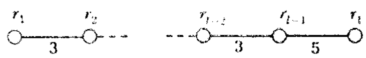
THEOREM. *Let G be a finite reflection group on a Euclidean space E. Then $\mathrm{H}^2(G, \boldsymbol{C}^*)$ is given as follows:*

$$\mathrm{H}^2(G, \boldsymbol{C}^*) \cong \underbrace{\boldsymbol{Z}_2 \times \cdots \times \boldsymbol{Z}_2}_{\kappa}$$

where $\boldsymbol{Z}_2$ means the cyclic group of order 2 and κ is a non-negative integer.

i) *The case where G is irreducible on E.*

Table I

type of G		Diagram	κ
A_l	$l \leqq 2$		0
A_l	$l \geqq 3$		1
B_2			1
B_3			2
B_l	$l \geqq 4$		3
D_4			3
D_l	$l \geqq 5$		2
E_l	$l = 6, 7, 8$		1
F_4			2
$G_2^{(n)}$	n: odd		0
$G_2^{(n)}$	n: even		1
H_l	$l = 3, 4$		1

ii) *Let $G \cong G_1 \times \cdots \times G_r$ be a decomposition of G into the irreducible components. Then*

$$\mathrm{H}^2(G, \boldsymbol{C}^*) \cong \prod_i \mathrm{H}^2(G_i, \boldsymbol{C}^*) \times \prod_{i<j} \mathrm{P}(G_i, G_j),$$

where $\mathrm{P}(G_i, G_j)$ is the group of all pairings of G_i and G_j in $\boldsymbol{C}^*$, i.e. the set of all mappings $\varphi: G_i \times G_j \to \boldsymbol{C}^*$ which satisfy the conditions

$$\varphi(a_1 b_1, c_2) = \varphi(a_1, c_2)\varphi(b_1, c_2)$$
$$\varphi(a_1, b_2 c_2) = \varphi(a_1, b_2)\varphi(a_1, c_2)$$

for all $a_1, b_1, c_1 \in G_i$, and *$a_2, b_2, c_2 \in G_j$.*

COROLLARY. *Let K be an algebraically closed field of characteristic p. Then*

$$H^2(G, K^*) \cong \begin{cases} 1 & \text{if } p=2 \\ \underbrace{Z_2 \times \cdots \times Z_2}_{\kappa} & \text{otherwise} \end{cases}$$

Our method is similar to that of Schur [5] for the case of symmetric groups. Also if G is a dihedral group, $H^2(G, C^*)$ was given in Schur [3]. We shall review in §1, for the convenience of the reader, the concepts of projective representations, associated factor sets and cohomology classes etc. In §2, we also review the classification of finite reflection groups due to Coxeter [1] and Witt [6]. In §3 we shall show that for a finite reflection group G, $H^2(G, C^*)$ is isomorphic to the direct product of several copies of Z_2. Thus $H^2(G, C^*)$ will be determined by its order. We shall also give an upper bound 2^{κ_0} for the order of $H^2(G, C^*)$ by constructing an injective homomorphism: $H^2(G, C^*) \to \mathfrak{N} \cong Z_2 \times \cdots \times Z_2$ (κ_0 times), where $\mathfrak{N}$ is a group consisting of "normalized" factor sets on G. In §4, we shall show that above injective homomorphism $H^2(G, C^*) \to \mathfrak{N}$ is also surjective by constructing actually the projective representations of G with the corresponding factor sets for each element of $\mathfrak{N}$. This will lead to the main theorem.

Finally we should like to thank Prof. N. Iwahori for many suggestions. Among other things, at first our result was only concerning to Weyl groups of complex simple Lie algebras and he pointed out that the method in this paper is available to the finite reflection groups.

§1. Projective representations, factor sets and the second cohomology groups with the coefficient group C^*.

Let G be a finite group. A *projective representation* of G over C is a homomorphism $\varphi: G \to PGL(n, C)$, n being called the degree of φ. A mapping $T: G \to GL(n, C)$ is called a *section* of φ if $\varphi = \pi \circ T$, where $\pi: GL(n, C) \to PGL(n, C)$ is the natural projection. If T is a section of φ, we have

$$T(a)T(b) = \alpha(a, b)T(ab)$$

for all $a, b \in G$, where α is a function: $G \times G \to C^*$. The function α is called the factor set associated to the section T of the representation φ. Two projective representations $\varphi: G \to PGL(n, C)$ and $\psi: G \to PGL(m, C)$ are called to be equivalent if $n=m$ and there exists a linear isomorphism $f \in GL(n, C)$ such that $\tilde{f} \circ \varphi = \psi$, where $\tilde{f}$ is the isomorphism $PGL(n, C) \to PGL(n, C)$ which is induced by the isomorphism $GL(n, \boldsymbol{C}) \to GL(n, \boldsymbol{C})$: $h \to fhf^{-1}(h \in GL(n, \boldsymbol{C}))$.

From the associativity of the product in $GL(n, \boldsymbol{C})$ one obtains the relations

$$\alpha(a, bc)\alpha(b, c) = \alpha(a, b)\alpha(ab, c) \tag{1}$$

for all $a, b, c \in G$.

A function $\alpha: G \times G \to \boldsymbol{C}^*$ which satisfies (1) is called a *factor set* or a $\boldsymbol{C}^*$-*valued* 2-*cocycle on* G (under the trivial action of G on $\boldsymbol{C}^*$). Let α, β be factor sets. One defines the factor sets $\alpha\beta, \alpha^{-1}$ by

$$(\alpha\beta)(a, b) = \alpha(a, b)\beta(a, b)$$
$$\alpha^{-1}(a, b) = (\alpha(a, b))^{-1} \qquad a, b \in G$$

Then the set $\mathrm{Z}^2(G, \boldsymbol{C}^*)$ of all factor sets forms an abelian group with respect to this operation.

Two factor sets α and β are called equivalent (or cohomologous) if there exists a function $\rho: G \to \boldsymbol{C}^*$ such that

$$\beta(a, b) = \alpha(a, b)\rho(ab)\rho(a)^{-1}\rho(b)^{-1}$$

for all $a, b \in G$. This is an equivalence relation, and the equivalence class containing a factor set α will be denoted by $\{\alpha\}$. One can define a product of equivalence classes by $\{\alpha\} \cdot \{\beta\} = \{\alpha\beta\}$, then the set M of all equivalence classes forms an abelian group. The identity element of M is $\{1\}$ where 1 is the factor set $1(a, b) = 1$ for all $a, b \in G$, and for any $\{\alpha\} \in M$ one has $\{\alpha\}^{-1} = \{\alpha^{-1}\}$. This group M is called the *multiplier* of G or the *second cohomology group* $\mathrm{H}^2(G, \boldsymbol{C}^*)$ for the coefficient group $\boldsymbol{C}^*$.

LEMMA 1. *Let* α, β *be equivalent factor sets of* G. *Then* $\alpha(a, b)\alpha(b, a)^{-1} = \beta(a, b)\beta(b, a)^{-1}$ *for any* $a, b \in G$ *such that* $ab = ba$.

PROOF. Since α and β are equivalent, there exists a function $\rho: G \to \boldsymbol{C}^*$ such that

$$\beta(a, b) = \alpha(a, b)\rho(ab)\rho(a)^{-1}\rho(b)^{-1}$$
$$\beta(b, a) = \alpha(b, a)\rho(ba)\rho(b)^{-1}\rho(a)^{-1}.$$

Hence we have $\beta(a, b)\beta(b, a)^{-1} = \alpha(a, b)\alpha(b, a)^{-1}$, Q.E.D.

It is clear that, if T and S are two sections of a projective representation φ, and α (resp. β) the factor set of φ associated to T(resp. S), they are equivalent. Also, equivalent projective representations have equivalent factor sets. Thus we have a mapping

$$\mathfrak{P}(G) \to M$$

where $\mathfrak{P}(G)$ is the set of all equivalent classes of projective representations of G over C. It is known that this mapping is surjective (Schur [3]). We note here that, if $\{\alpha\}$ (resp. $\{\beta\}$) is the class of factor sets corresponding to a projective representation φ (resp. ψ), $\{\alpha\beta\}$ (resp. $\{\alpha\}^{-1}$) corresponds to the projective representation $\varphi\otimes\psi$(resp. ${}^t\varphi^{-1}$), where $\varphi\otimes\psi$ is the tensor product of φ and ψ and ${}^t\varphi^{-1}$ is the contragredient representation of φ. (Note that $\varphi\otimes\psi$ or ${}^t\varphi^{-1}$ is defined up to the equivalence.)

§ 2. Finite reflection groups.

For a hyperplane P in an l-dimensional Euclidean space $E^{(l)}$ there corresponds a linear isometric transformation R called *reflection*. Let p be the normal unit vector of P and (x, y) the inner product of $x, y \in E^{(l)}$. Then R is expressed as $R\cdot x = x - 2(x, p)p$ $(x \in E^{(l)})$.

Let $P_1, P_2, \cdots, P_l$ be l different hyperplanes in $E^{(l)}$, and $R_1, R_2, \cdots, R_l$ the corresponding reflections respectively. Assume that the angle between P_i and $P_j (i \neq j)$ is equal to π/m_{ij} where $m_{ij} = 2, 3, \cdots, \infty$, and put $m_{ii} = 1$. Then it can be easily seen that R_iR_j has the order m_{ij} for $i, j = 1, 2, \cdots, l$;

$$(2) \qquad (R_iR_j)^{m_{ij}} = 1 \ (i, j = 1, 2, \cdots, l).$$

Under this assumption the group Γ of linear transformations generated by $R_1, R_2, \cdots, R_l$ will be called a (*discrete*) *refrection group* (see e.g. [6]). Reflection groups were studied and classified for the first by Coxeter [1], and later by Witt [6] more systematically. We shall resume here their results only for the case where Γ is finite according to [6].

Let D be the closure of a fixed connected component of $E^{(l)} - \bigcup_{i=1}^{l} P_i$. Take the normal unit vectors p_i of P_i for all $i = 1, 2, \cdots, l$ as $\widehat{p_ip_j} = \pi - \pi/m_{ij}$, put $a_{ij} = -\cos(\pi/m_{ij}) = (p_i, p_j)$, and define a quadratic form f by

$$(3) \qquad f = \sum_{i,j=1}^{l} a_{ij}\xi_i\xi_j.$$

Since $f = (\sum \xi_i p_i, \sum \xi_j p_j)$, f is positive definite or positive semi-definite according as $p_1, p_2, \cdots, p_l$ are linearly independent or not. If f is positive definite, the domain D is an angular domain in $E^{(l)}$, for we may assume that D is the set of $x \in E^{(l)}$ such that $(x, p_i) \leqq 0$ $(i = 1, 2, \cdots, l)$.

If p_i and p_j are mutually orthogonal, so $m_{ij} = 2$, R_i and R_j are commutative by (2). Therefore, if any one of the hyperplanes $P_1, \cdots, P_\nu$ is orthogonal to any of the hyperplanes $P_{\nu+1}, \cdots, P_l$, the reflection group Γ is the direct product of the two subgroups Γ_1 and Γ_2 generated by $\{R_1, \cdots, R_\nu\}$ and by $\{R_{\nu+1}, \cdots, R_l\}$ respectively. Γ_1 is considered as a reflection group in a ν-dimensional subspace $E^{(\nu)}$ containing $p_1, \cdots, p_\nu$. Also Γ_2 may be regarded as a reflection group in the same way. In this case the quadratic form f is decomposed as f_1+f_2 where $f_1=\sum_{1\leq i,j\leq\nu} a_{ij}\xi_i\xi_j$ and $f_2=\sum_{\nu+1\leq i,j\leq l} a_{ij}\xi_i\xi_j$, and f_i is associated to Γ_i. Conversely, if $f=\sum_{1\leq i,j\leq\nu} a_{ij}\xi_i\xi_j+\sum_{\nu+1\leq i,j\leq l} a_{ij}\xi_i\xi_j$, then $a_{ij}=(p_i, p_j)=0$ for $1\leq i\leq\nu$ and $\nu+1\leq j\leq l$, and hence $P_1, \cdots, P_\nu$ are orthogonal to $P_{\nu+1}, \cdots, P_l$.

If $\{P_1, \cdots, P_l\}$ can not be decomposed into two subsets such that any element in one of them is orthogonal to that of another, or that is to say, if $\{R_1, \cdots, R_l\}$ can not be a union of two subsets A and B such that $R_iR_j=R_jR_i$ for any $R_i\in A$ and any $R_j\in B$, then the reflection group Γ, the set of generators $\{R_1, \cdots, R_l\}$ of Γ and the quadratic form f will be called *irreducible*. Thus one can restrict the consideration to the irreducible case.

Now, let the quadratic form f be irreducible and of rank r. If f is positive definite, $r=l$ and D is an angular domain in $E^{(l)}$. For the case where f is positive semi-definite, we shall show that $r=l-1$, and D is a direct product of a simplex in a $(l-1)$-dimensional subspace of $E^{(l)}$ and a 1-dimensional subspace of $E^{(l)}$. Let V be the subspace of $E^{(l)}$ spanned by $p_1, \cdots, p_l$ which are linearly dependent in E. Let $\xi_1p_1+\cdots+\xi_lp_l=0$, and put $x_1=\sum_{i=1}^{l}|\xi_i|p_i$. Since $(p_i, p_j)\leqq 0$ $(i\neq j)$, it is easy to show that $(x_1, x_1)\leqq(\sum\xi_ip_i, \sum\xi_ip_i)=0$, and so $x_1=0$ which is another relation between $p_1, \cdots, p_l$. If for instance $\xi_1\cdot\cdots\cdot\xi_\nu\neq 0$ and $\xi_{\nu+1}=\cdots=\xi_l=0(\nu<l)$, then $\left(x_1, \sum_{j=\nu+1}^{l} p_j\right)=\sum_{\substack{1\leqq i\leqq\nu\\ \nu+1\leqq j\leqq l}}|\xi_i|(p_i, p_j)=0$ and so $(p_i, p_j)=a_{ij}=0$ for all $1\leq i\leq\nu$ and $\nu+1\leq j\leq l$, but this contradicts to the assumption that f is irreducible. Hence $|\xi_i|$'s are altogether positive, or equal to zero simultaneously, and so, if $|\xi_1|=\xi_1$, one can conclude that $|\xi_i|=\xi_i$ for all $i=1, \cdots, l$. If $\sum_{i=1}^{l}\xi_ip_i=0$ and $\sum_{i=1}^{l}\eta_ip_i=0$, it can also be seen that $\xi_i=\lambda\eta_i$ for all $i=1, \cdots, l$ taking λ as $\xi_1=\lambda\eta_1$. Hence there is essentially one and only one linear relation between $p_1, \cdots, p_l$ with all positive coefficients. Therefore V is an $(l-1)$-dimensional subspace $E^{(l-1)}$ and so $r=l-1$, and D is a simplex in $E^{(l-1)}$ as we claimed.

The following Proposition 1 and Proposition 2 are proved in [6] as Satz 2 and Satz 3 respectively.

PROPOSITION 1. i) *The relations* (2) *give the fundamental relations between the generators* $R_1, \cdots, R_l$ *of* Γ.

ii) *D gives a fundamental set of Γ, i.e. $E^{(l)}=\Gamma\cdot D$ and $S\cdot D=D$ $(S\in\Gamma)$ if and only if S is the identity of Γ.*

iii) *If f is positive definite, Γ is finite, otherwise Γ is infinite.*

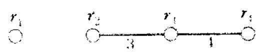
PROPOSITION 2. *Let I be a subset of $\{1, 2, \cdots, l\}$ and H a subgroup generated by $\{R_i; i \in I\}$ in Γ. Then the fundamental relations between $\{R_i; i \in I\}$ in H are given by $(R_iR_j)^{m_{ij}}=1 (i, j \in I)$.*

Now let G be an abstract group generated by $r_1, \cdots, r_l$ with the defining relations

$$(r_ir_j)^{m_{ij}}=1;\ m_{ii}=1,\ m_{ij}=m_{ji}=2, 3, \cdots, \infty \text{ for } i, j=1, \cdots, l, i \neq j. \tag{4}$$

To such a group G, a diagram $\Pi(G)$ is associated as follows: $\Pi(G)$ contains l points corresponding to the generators $r_i (i=1, 2, \cdots, l)$ in one-to-one way, and if $m_{ij} \neq 2 (i \neq j)$, two points corresponding to r_i and r_j respectively are connected by a segment together with the number m_{ij}. $\Pi(G)$ is called sometimes the Coxeter diagram of G.

EXAMPLE. If G is the abstract group generated by $\{r_1, r_2, r_3, r_4\}$ with the relations $r_i^2=1$ $(i=1, 2, 3, 4)$, $(r_1r_2)^2=(r_1r_3)^2=(r_1r_4)^2=(r_2r_4)^2=1$, $(r_2r_3)^3=1$ and $(r_3r_4)^4=1$, then $\Pi(G)$ is given as follows:



$(m_{23}=3,\ m_{34}=4)$.

Assume that $\Pi(G)$ is a union of two subdiagrams Π_1 and Π_2 consisting of the points corresponding for instance to $\{r_1, \cdots, r_\nu\}$ and to $\{r_{\nu+1}, \cdots, r_l\}$ respectively, and assume that any point of Π_1 is not connected to that of Π_2. Then $G=G_1 \times G_2$ (direct product) where G_1 is generated by $\{r_1, \cdots, r_\nu\}$ and G_2 is generated by $\{r_{\nu+1}, \cdots, r_l\}$, since $m_{ij}=2$ $(1 \leq i \leq \nu, \nu+1 \leq j \leq l)$. If G is finite, we shall say that $\Pi(G)$ is *of finite type*.

Put $a_{ij}=-\cos(\pi/m_{ij})$ and define a quadratic form f by (3). Then we have the following Proposition which is proved in [6] as Satz 6.

PROPOSITION 3. *The abstract group G is finite if and only if the quadratic form f is positive definite.*

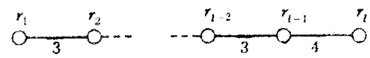
In the following we consider only the case where G is finite and so f is positive definite. Let $\{v_1, \cdots, v_l\}$ be a basis of a real l-dimensional vector space V. For any two vectors $\sum \xi_i v_i$ and $\sum \eta_j v_j$ of V, define an inner product formally by $(\sum \xi_i v_i, \sum \eta_j v_j)=\sum a_{ij}\xi_i\eta_j$, which is positive, symmetric since f is positive definite. Then, if $i \neq j$, the angle between v_i and v_j is equal to $\pi-\pi/m_{ij}$. Now we can define a linear representation Φ of G on V by

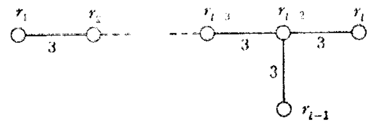
$$R_i=\Phi(r_i), \quad R_i \cdot x=x-2(x, v_i)v_i \ (i=1, \cdots, l).$$

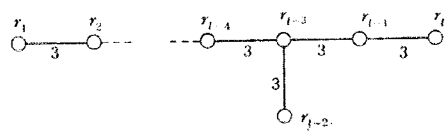
R_i is a reflection defined by the hyperplane $P_i=\{x \in V; (x, v_i)=0\}$. Thus we have a finite reflection group Γ generated by $R_1, \cdots, R_l$, and $\Gamma=\Phi(G)$. On the other hand we have $(R_iR_j)^{m_{ij}}=1(i, j=1, \cdots, l)$ by (4), but these equations are fundamental relations between the generators $R_1, \cdots, R_l$ of Γ (Proposition 1, i)). Therefore Φ is an isomorphism. So, in the following of this paper, we shall say that G is a finite reflection group if it is an abstract group generated by $r_1, \cdots, r_l$ with the defining relations (4) and if G is finite. It is easy to show that G is irreducible, i.e. $\Gamma=\Phi(G)$ is irreducible, if and only if the diagram $\Pi=\Pi(G)$ of G is connected. Thus the classification of the irreducible finite reflection groups G is equivalent to the classification of the connected diagrams Π of finite types, which is given by the following Lemma (see Satz 8 in [6]).

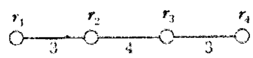
LEMMA 2. *There is no other connected diagram of finite type than the following types (the numbers under the segments give m_{ij}'s in* (4)):

$A_l(l\geqq 1)$;  (*the Weyl group of* $SL(l+1, \boldsymbol{C})$)

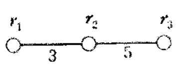
$B_l(l\geqq 2)$;  (*the Weyl group of* $SO(2l+1, \boldsymbol{C}), Sp(l, \boldsymbol{C})$)

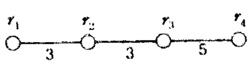
$D_l(l\geqq 4)$;  (*the Weyl group of* $SO(2l, \boldsymbol{C})$)

$E_l(l=6, 7, 8)$;  (*the Weyl group of complex Lie group of type* (E_l))

F_4 ;  (*the Weyl group of complex Lie group of type* (F_4))

$G_2^{(n)}(n\geqq 5)$;  (*the dihedral group of order* $2n$)

H_3 ;  (*the finite group* $\mathfrak{A}_5\times \boldsymbol{Z}_2$)

H_4 ;  (*the group of order* 120^2)

If a diagram is of finite type, it is a disjoint union of some finite numbers of those in Lemma 2.

Now, let us refer to the structure of the group $\mathrm{P}(G_1, G_2; C^*)$ of pairings (see § 0) of two finite reflection groups G_1 and G_2, that will be tacitly used in the following sections. Let G be a finite reflection group generated by $r_1, \cdots, r_l$ with the relations (4), $[G, G]$ be the commutator subgroup of G, and $\bar{a}$ be the image of $a \in G$ by the canonical homomorphism from G onto the abelian group $\bar{G}=G/[G, G]$. $\bar{G}$ is generated by $\bar{r}_1, \cdots, \bar{r}_l$; $\bar{r}_i^2=1$, hence

$$\mathrm{Hom}(G, \boldsymbol{C}^*)\cong\bar{G}\cong \boldsymbol{Z}_2\times\cdots\times \boldsymbol{Z}_2 \ (\lambda \text{ times}) \tag{5}$$

for a certain integer $\lambda\leqq l$, where $\boldsymbol{Z}_2$ is the group of order 2. For the determination of λ, it is sufficient to examine the case where G is irreducible. For, if $G\cong G_1\times G_2$ the direct product of two finite reflection groups, then $[G, G]\cong [G_1, G_1]\times[G_2, G_2]$, hence $\bar{G}\cong\bar{G}_1\times\bar{G}_2$. Since $[G, G]$ is generated by $(r_ir_j)^2=r_ir_jr_i^{-1}r_j^{-1}$ $(i, j=1, 2, \cdots, l)$, we have $\bar{r}_i\bar{r}_j=1$ and so $\bar{r}_i=\bar{r}_j^{-1}=\bar{r}_j$ if m_{ij} is odd, on the other hand $\bar{r}_i\bar{r}_j=\bar{r}_j\bar{r}_i$ for all $i, j=1, \cdots, l$. Thus we obtain the following values of λ for the irreducible finite reflection groups:

$$\begin{aligned} \lambda=1 &\quad \text{for} \quad A_l, D_l, E_l, G_2^{(n)}(n\text{: odd}), H_l. \\ 2 &\quad \text{for} \quad B_l, F_4, G_2^{(n)}(n\text{: even}). \end{aligned}$$

LEMMA 3. *Let G_1 and G_2 be finite reflection groups. Then*

$$\mathrm{P}(G_1, G_2; \boldsymbol{C}^*)\cong \boldsymbol{Z}_2\times\cdots\times \boldsymbol{Z}_2 \ (\lambda_1\lambda_2 \textit{ times})$$

where λ_1, λ_2 are the numbers of $\boldsymbol{Z}_2$ in (5) *for $\bar{G}_1, \bar{G}_2$ respectively.*

PROOF. Since $\mathrm{P}(G_1, G_2; \boldsymbol{C}^*)\cong\mathrm{Hom}(G_1, \bar{G}_2)\cong\mathrm{Hom}(G_2, \bar{G}_1)$, it is clear, Q.E.D.

§ 3. Normalization of factor set.

In this paper, we are going to determine the structure of the multiplier $\mathrm{H}^2(G, \boldsymbol{C}^*)$ in the case where G is the finite reflection group described in § 2.

In the following we may assume that G is irreducible on the Euclidean space E. For, in the case where $G\cong G_1\times\cdots\times G_r$, G_i being irreducible components,

$$\mathrm{H}^2(G, \boldsymbol{C}^*)\cong \prod_i \mathrm{H}^2(G_i, \boldsymbol{C}^*)\times \prod_{i<j}\mathrm{P}(G_i, G_j)$$

by theorem 2.1 in Yamazaki [7]. This shows the second part of the theorem.

In order to determine the group $\mathrm{H}^2(G, \boldsymbol{C}^*)$, in this section we shall construct an injective homomorphism θ from $\mathrm{H}^2(G, \boldsymbol{C}^*)$ into $\mathfrak{N}\cong(\boldsymbol{Z}_2)^{\kappa_0}$, where $\mathfrak{N}$ is a group consisting of "normalized" factor sets on G (in the sense specified below) and $(\boldsymbol{Z}_2)^{\kappa_0}$ a direct product of κ_0 copies of the cyclic group $\boldsymbol{Z}_2$. The number κ_0 is determined respectively for each type of a reflection group.

Let $\{\alpha\}$ be an element of $\mathrm{H}^2(G, \boldsymbol{C}^*)$. There exist (see [3]) projective representations whose factor sets belong to $\{\alpha\}$. Denote one of them by φ and let

T be a section of φ. Denote $T(r_i)$ by T_i. Then the following relations are valid according to (4):

$$(6) \qquad T_i^2 = \varepsilon_i I \qquad i=1, \cdots, l$$

$$(7) \qquad (T_i T_j)^{n_{ij}} = \alpha_{ij}(T_j T_i)^{n_{ij}} \qquad \text{if } m_{ij} \text{ is even}: \quad m_{ij}=2n_{ij}$$

$$(8) \qquad (T_i T_j)^{n_{ij}} T_i = \beta_{ij}(T_j T_i)^{n_{ij}} T_j \quad \text{if } m_{ij} \text{ is odd}: \quad m_{ij}=2n_{ij}+1,$$

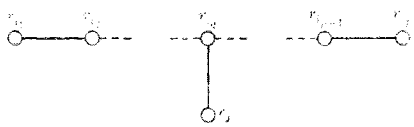
where I denote the identity matrix and ε_i, α_{ij} and β_{ij} belong to $\boldsymbol{C}^*$.

Now, let us normalize the numbers ε_i, α_{ij}, β_{ij}. Replacing T_i by $\xi_i T_i$ where $\xi_i^2 = \varepsilon_i^{-1}$, i.e. changing the section T of φ, we may assume $\varepsilon_i = 1$ without impropriety. If we multiply T_i to the left and $T_j^{-1}(=T_j)$ to the right of both sides of (7), we have easily $\alpha_{ji} = \alpha_{ij}^{-1} = \alpha_{ij} = \pm 1$. As for the coefficients β_{ij}, squaring both sides of (8), we have $\beta_{ij}^2 = 1$ and consequently $\beta_{ji} = \beta_{ij}^{-1} = \beta_{ij} = \pm 1$. Moreover we may assume that all of them are equal to 1. For, if

is a connected chain in which all m_{ij}'s are odd in the diagram of G, we can replace as

$$T_{i_1} \to T_{i_1},\ T_{i_2} \to \beta_{i_1 i_2} T_{i_2},\ T_{i_3} \to \beta_{i_1 i_2}\beta_{i_2 i_3} T_{i_3},\ \cdots,$$
$$T_{i_p} \to \beta_{i_1 i_2}\beta_{i_2 i_3} \cdots \beta_{i_{p-1} i_p} T_{i_p},$$

keeping the other equations (6) and (7) invariant. All $\beta_{i_t i_{t+1}}$ are then equal to 1. When in the diagram of G there exists a point which is connected directly to three other points, i.e. when a connected diagram of type



in which all m_{ij}'s are odd is contained in the diagram of G, we may assume $\beta_{i_t i_{t+1}} = 1$ as before, and replacing T_j by $\beta_{i_q j} T_j$ we may assume $\beta_{i_q j} = 1$. Thus, normalizing all the $\beta_{ij} = 1$, we can associate to each φ the system of numbers $\{\alpha_{ij}\}$.

We have then by definitions and (7)

$$(9) \qquad \alpha_{ij} = \frac{\alpha(r, s)}{\alpha(s, r)},$$

where $r = r_i$, $s = (r_j r_i)^{n_{ij}-1} r_j$, $m_{ij} = 2n_{ij}$, and α is the factor set of T.

By Lemma 1 and (9), the system $\{\alpha_{ij}\}$ is independent of the choice of the section T of φ and determined only by $\{\alpha\}$. Thus we have defined a map

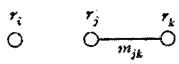
$$\theta: \mathrm{H}^2(G, \boldsymbol{C}^*) \to \boldsymbol{C}^* \times \cdots \times \boldsymbol{C}^*$$

by $\{\alpha\} \to \{\alpha_{ij}\}$. By (9) this map θ is a homomorphism.

Now let us show that the homomorphism θ is injective. In fact, let φ and ψ be projective representations of G and T and S sections of φ and ψ respectively. Assume the systems $\{\alpha_{ij}\}$ coincide for φ and ψ. To each element of G, we fix once for all an expression by the generators and construct mappings $T_1: G \to GL(n, \boldsymbol{C})$ and $S_1: G \to GL(m, \boldsymbol{C})$ (n = the degree of φ, m = the degree of ψ) as follows: If $G \ni a = r_{i_1} \cdots r_{i_p}$ is thus fixed expression and $T(r_i) = T_i$, $S(r_i) = S_i$, $T_1(a) = T_{i_1} \cdots T_{i_p}$ and $S_1(a) = S_{i_1} \cdots S_{i_p}$. Let α (resp. β) be the factor set of φ (resp. ψ) determined by the section T_1 (resp. S_1). It is clear that $\alpha(a, b) = \beta(a, b)$ for all $a, b \in G$, and $\{\alpha\} = \{\beta\}$.

Many of the α_{ij} may be equal. We shall now consider more explicitly the "degree of freedom" of the α_{ij}, i.e. the situations where we have $\alpha_{ij} = \alpha_{kl}$. To begin with, let r_i, r_j, r_k and r_l be generators of G such that $m_{ij} = m_{kl} = 2$. Assume that there is an element r of G such that $r_k = r r_i r^{-1}$, $r_l = r r_j r^{-1}$ at the same time. Then $\alpha_{kl} = \alpha_{ij}$. In fact we have $T_k = \lambda T_0 T_i T_0^{-1}$ and $T = \mu T_0 T_j T_0^{-1}$, where $\lambda, \mu \in \boldsymbol{C}^*$ and $T_0 = T(r)$. Therefore, $T_k T_l = \lambda T_0 T_i T_j T_0^{-1} = \alpha_{ij} T_l T_k$ and we have $\alpha_{kl} = \alpha_{ij}$. On the other hand, we have the following lemma.

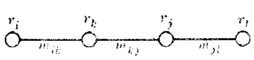
LEMMA 4. 1) *Let r_i, r_j and r_k be generators of G such that they are arranged in the diagram as*



$$m_{jk} = 2n_{jk} + 1.$$

There exists an element r in G such that $r r_i r^{-1} = r_i$, $r r_j r^{-1} = r_k$. Therefore we have $\alpha_{ij} = \alpha_{ik}$ in this case.

2) *Let r_i, r_j, r_k and r_l be generators of G such that they are arranged in the diagram as*



$$m_{ik} = 2n_{ik} + 1, \quad m_{jl} = 2n_{jl} + 1.$$

We can find an element r in G such that $r r_i r^{-1} = r_k$, $r r_j r^{-1} = r_l$. Therefore we have $\alpha_{ij} = \alpha_{kl}$ in this case.

PROOF. It is sufficient only to take r as $(r_j r_k)^{n_{jk}}$ for 1) and as $(r_i r_k)^{n_{ik}}(r_j r_l)^{n_{jl}}$ for 2), Q.E.D.

Using Lemma 4 we easily see that, if r_i and r_j runs throughout a certain connected chain in the diagram of G, keeping the condition that $m_{ij} = 2$ all the

α_{ij} attached to (7) are mutually equal.

Consequently we can conclude without difficulties that when G is of type $A_l(l\geqq 3)$, B_2, E_l, $G_2^{(n)}$ (n : even) and H_l, all α_{ij} are equal. We shall henceforth denote this common value of α_{ij} by α_1. When G is of type B_3, $D_l(l\geqq 5)$ and F_4 all α_{ij} are divided into at most two distinct values α_1, α_2 equal to ± 1. In the case of $B_l(l\geqq 4)$ and D_4, there are three sorts of α_{ij} that may be independently equal to ± 1. We denote them by $\alpha_1, \alpha_2, \alpha_3$.

Table II.

type of G	normalized relations (6)~(8)	κ_0
A_1	$T_1^2=I$.	0
A_2	$T_i^2=I$ $(i=1,2)$, $T_1T_2T_1=T_2T_1T_2$.	0
A_l $(l\geqq 3)$	$T_i^2=I$ $(i=1,\cdots,l)$, $T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}$ $(i=1,\cdots,l-1)$, $T_iT_j=\alpha_1T_jT_i$ if $m_{ij}=2$.	1
B_2	$T_i^2=I$ $(i=1,2)$, $(T_1T_2)^2=\alpha_1(T_2T_1)^2$.	1
B_3	$T_i^2=I$ $(i=1,2,3)$, $T_1T_2T_1=T_2T_1T_2$, $T_1T_3=\alpha_1T_3T_1$, $(T_2T_3)^2=\alpha_2(T_3T_2)^2$.	2
B_l $(l\geqq 4)$	$T_i^2=I$ $(i=1,\cdots,l)$, $T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}$ $(i=1,\cdots,l-2)$, $T_iT_j=\alpha_1T_jT_i$ $(1\leqq i<j\leqq l-1,\ m_{ij}=2)$, $T_iT_l=\alpha_2T_lT_i$ $(1\leqq i\leqq l-2)$, $(T_{l-1}T_l)^2=\alpha_3(T_lT_{l-1})^2$.	3
D_4	$T_i^2=I$ $(i=1,2,3,4)$, $T_iT_jT_i=T_jT_iT_j$ if $m_{ij}=3$, $T_1T_3=\alpha_1T_3T_1$, $T_1T_4=\alpha_2T_4T_1$, $T_3T_4=\alpha_3T_4T_3$.	3
D_l $(l\geqq 5)$	$T_i^2=I$ $(i=1,\cdots,l)$, $T_iT_jT_i=T_jT_iT_j$ if $m_{ij}=3$, $T_iT_j=\alpha_1T_jT_i$ $(1\leqq i<j\leqq l,\ m_{ij}=2,\ i\neq l-1)$, $T_{l-1}T_l=\alpha_2T_lT_{l-1}$.	2
E_l $(l=6,7,8)$	$T_i^2=I$ $(i=1,\cdots,l)$, $T_iT_jT_i=T_jT_iT_j$ if $m_{ij}=3$, $T_iT_j=\alpha_1T_jT_i$ if $m_{ij}=2$.	1
F_4	$T_i^2=I$ $(i=1,\cdots,4)$, $T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}$ $(i=1,3)$, $T_iT_j=\alpha_1T_jT_i$ $(1\leqq i<j\leqq 4,\ m_{ij}=2)$, $(T_2T_3)^2=\alpha_2(T_3T_2)^2$.	2
$G_2^{(n)}$ n: even n: odd	$T_i^2=I$ $(i=1,2)$, $(T_1T_2)^{\frac{n}{2}}=\alpha_1(T_2T_1)^{\frac{n}{2}}$. $T_i^2=I$ $(i=1,2)$, $(T_1T_2)^{\frac{n-1}{2}}T_1=(T_2T_1)^{\frac{n-1}{2}}T_2$.	1 0
H_l $(l=3,4)$	$T_i^2=I$ $(i=1,\cdots,l)$, $T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}$ $(i=1,\cdots,l-2)$, $(T_{l-1}T_l)^2T_{l-1}=(T_lT_{l-1})^2T_l$, $T_iT_j=\alpha_1T_jT_i$ if $m_{ij}=2$.	1

Accordingly, the numbers κ_0 of α_i which may be independently equal to ± 1 are given in the Table II.

The correspondence $\theta: \{\alpha\} \to \{\alpha_{ij}\}$ then can be considered as a homomorphism from $H^2(G, C^*)$ into $Z_2 \times \cdots \times Z_2$ (κ_0 times). We denote this group by $\mathfrak{N}$ in the following. We have thus the following lemma.

LEMMA 5. *$H^2(G, C^*)$ is isomorphic to a direct product of several copies of the cyclic group Z_2, or otherwise to $\{1\}$.*

§4. Construction of projective representations

We can now prove the theorem, by showing that the homomorphism $\theta: H^2(G, C^*) \to \mathfrak{N}$ given in §3 is surjective.

For this purpose it is enough to construct projective representations whose factor sets correspond to κ_0 generators of $\mathfrak{N}$; namely we have only to find the system of matrices which satisfy the normalized relations in the table II for the generating system of numbers ± 1 given to α_i.

Let A, B be two matrices of degree n. We denote $AB+BA$ by $\{A, B\}$. $\{\ ,\ \}$ is a bilinear mapping: $\mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$ where $\mathfrak{M}$ is the complex vector space of all matrices of degree n.

LEMMA 6. (Schur [5]) *Let m be a positive integer. Then there exist $2m+1$ matrices $M_1, \cdots, M_{2m+1}$ in $GL(2^m, C)$ which satisfy the following relations.*

$$\text{(10)} \qquad \begin{cases} M_{2k}^2 = -I,\ M_{2k+1}^2 = I & (k=0, 1, \cdots, m) \\ \{M_k, M_l\} = 0 & (1 \leqq k \neq l \leqq 2m+1) \\ M_{2m+1} \cdots M_1 = I \end{cases}$$

Put $N_{2k} = iM_{2k}$, where $i^2 = -1$ and $N_{2k+1} = M_{2k+1}$. Then from (10) $2m+1$ matrices $N_1, \cdots, N_{2m+1}$ satisfy the following relations.

$$\text{(11)} \qquad \begin{cases} N_k^2 = I & (k=1, \cdots, 2m+1) \\ \{N_k, N_l\} = 0 & (1 \leqq k \neq l \leqq 2m+1) \\ N_{2m+1} \cdots N_1 = (i)^m I \end{cases}$$

In the following a system of matrices $\{N_1, \cdots, N_{2m+1}\}$ which satisfy (11) will be denoted by $\sum(m)$.

Using Lemma 6, Schur proved that for type $A_l (l \geqq 3)$ there exists a system of matrices satisfying the relations in the table II with $\alpha_1 = -1$.

Namely,

LEMMA 7. (Schur [5]) *Let l be a positive integer and $\left[\dfrac{l}{2}\right] = m$. Then there exist l matrices $A_1, \cdots, A_l$ in $GL(2^m, C)$ which satisfy the following relations.*

$$(12)\qquad \begin{cases} A_j^2=I & j=1,\cdots,l \\ \{A_j, A_{j+1}\}=-I & j=1,\cdots,l-1 \\ \{A_j, A_k\}=0 & 1\leqq j<k\leqq l,\ |j-k|\geqq 2. \end{cases}$$

PROOF. Let us take $\sum(m)=\{N_1,\cdots,N_{2m+1}\}$, and put $A_1=-N_1$, $A_j=a_{j-1}N_{j-1}+b_jN_j$ $(j=2,\cdots,l)$ where

$$a_{2k}=-\frac{\sqrt{k}}{\sqrt{2k+1}} \qquad b_{2k+1}=-\frac{\sqrt{k+1}}{\sqrt{2k+1}}$$
$$a_{2k+1}=\frac{\sqrt{2k+1}}{2\sqrt{k+1}} \qquad b_{2k+2}=\frac{\sqrt{2k+3}}{2\sqrt{k+1}}$$

Then these numbers satisfy the relations

$$a_{j-1}^2+b_j^2=1 \qquad 2a_jb_j=-1$$

and the relations (12) are easily verified by straight forward calculations,

Q.E.D.

In the following a system of matrices $\{A_1,\cdots,A_l\}$ satisfying (12) will be denoted by $\Lambda(l)$.

Let us observe the following elementary facts which will facilitate our computations.

Let A, B be two matrices of the same degree.

1) If $A^2=B^2=I$ and $\{A, B\}=-I$, then $(AB)^3=I$.

2) If $A^2=B^2=I$ and $\{A, B\}=0$, then $CAC=-B$, $C^2=I$, and $(AC)^4=-I$ where $C=\frac{1}{\sqrt{2}}(A-B)$.

In general we shall show the follwing lemma.

LEMMA 8. *Let l be a given positive integer and A, B two matrices of degree n such that $A^2=B^2=I$ and A and B or AB and I are linearly independent in $\mathfrak{M}$. Then there exist complex numbers α such that A, B satisfy the following relation* (13) *if* $\{A, B\}=\alpha I$.

$$(13)\qquad (AB)^l=(-1)^{l-1}I$$

PROOF. If $A^2=B^2=I$ and $\{A, B\}=\alpha I$, we can write

$$(AB)^k=p_kAB+q_kI$$
$$(BA)^k=p'_kAB+q'_kI$$

where p_k, q_k, p'_k, q'_k, are complex numbers determined by α and k. Then

$$(AB)^{k+1}=(p_k\alpha+q_k)AB-p_kI$$
$$(BA)^{k+1}=-q'_kAB+(p'_k+\alpha q'_k)I$$

Assume $\alpha^2\neq 4$. We put

$$f_k=\frac{1}{\sqrt{\alpha^2-4}}\left\{\left(\frac{\alpha+\sqrt{\alpha^2-4}}{2}\right)^k-\left(\frac{\alpha-\sqrt{\alpha^2-4}}{2}\right)^k\right\}$$

Then f_k is a polynomial of α and $p_k=f_k$, $q_k=-f_{k-1}$, $q'_k=f_{k+1}$, $p'_k=-f_k$.

Suppose that l is even and put $l=2m$. For $(AB)^l=-I$ it is necessary and sufficient $(AB)^m=-(BA)^m$. This is equivalent to $f_{m+1}=f_{m-1}$. This is an algebraic equation for α. ± 2 are not its solutions.

Suppose that l is odd and put $l=2m+1$. For $(AB)^l=I$ it is equivalent to $(AB)^mA=(BA)^mB$. This is equivalent to $f_{m+1}+f_m=0$. This is also an algebraic equation for α and has not ± 2 as its solutions, Q.E.D.

Now let us construct a system of matrices which satisfy the normalized relations in the table II for the pre-assigned value for α_i. We shall only write down how to take T_i's. It will be easily verified that they satisfy the desired relations.

I-1 B_l $l\geqq 4$

i) $(\alpha_1, \alpha_2, \alpha_3)=(-1, 1, 1)$

Let us take $\Lambda(l-1)=\{A_1, \cdots, A_{l-1}\}$, and put $T_i=A_i$ $i=1, \cdots, l-1$, $T_l=I$.

ii) $(\alpha_1, \alpha_2, \alpha_3)=(1, -1, 1)$

Let us take $\Sigma(1)=\{N_1, N_2, N_3\}$, and put $T_1=\cdots=T_{l-1}=N_1$, $T_l=N_2$.

iii) $(\alpha_1, \alpha_2, \alpha_3)=(-1, -1, -1)$

Suppose $m=\left[\frac{l}{2}\right]$. Let us take $\Sigma(m)=\{N_1, \cdots, N_{2m+1}\}$ and put $T_j=\frac{1}{\sqrt{2}}(N_j-N_{j+1})$ $j=1, \cdots, l-1$, $T_l=N_l$.

I-2 B_3 $(\alpha_1, \alpha_2)=(-1, 1), (-1, -1)$

T_i is the same as I-1 ii) and iii) respectively.

I-3 B_2 $\alpha_1=-1$

T_i is the same as I-1 iii).

II-1 D_l $l\geqq 5$

i) $(\alpha_1, \alpha_2)=(-1, 1)$

Let us take $\Lambda(l-1)=\{A_1, \cdots, A_{l-1}\}$ and put $T_i=A_i$ $i=1, \cdots, l-1$, $T_l=A_{l-1}$.

ii) $(\alpha_1, \alpha_2)=(-1, -1)$

Suppose $m=\left[\frac{l}{2}\right]$. Let us take $\Sigma(m)=\{N_1, \cdots, N_{2m+1}\}$ and put $T_j=\frac{1}{\sqrt{2}}(N_j-N_{j+1})$ $j=1, \cdots, l-1$, $T_l=\frac{1}{\sqrt{2}}(N_{l-1}+N_l)$.

In the following this system of matrices $\{T_1, \cdots, T_l\}$ will be denoted by $\Delta(l)=\{D_1, \cdots, D_l\}$, i.e. $T_i=D_i$.

II-2 D_4

i) $(\alpha_1, \alpha_2, \alpha_3)=(-1, -1, 1), (-1, -1, -1)$

We take T_i as the case II-1 i) and ii) respectively.

ii) $(\alpha_1, \alpha_2, \alpha_3)=(1, -1, -1)$

Let us take $\Lambda(3)=\{A_1, A_2, A_3\}$, and put $T_1=T_3=A_1$, $T_2=A_2$, $T_4=A_3$.

III-1 E_6, $\alpha_1=-1$

We take a $\Delta(5)$ associated to $\sum(3)$ by the same way as II-1 ii): $\sum(3)=\{N_1, \cdots, N_7\}$, $\Delta(5)=\{D_1, \cdots, D_5\}$. Put $T_i=D_i$ $i=1, 2, 3, 4, 5$, $T_6=\sum_{j=1}^{6} a_jN_j$ where $a_1=\cdots=a_5=-\dfrac{1}{2\sqrt{2}}$, $a_6=\dfrac{\sqrt{3}}{2\sqrt{2}}$.

III-2 E_7, $\alpha_1=-1$

We take a $\Delta(6)$ associated to a $\sum(3)$. Put $T_i=D_i$ $i=1, \cdots, 6$, $T_7=\sum_{j=1}^{7} a_jN_j$ where $a_i=-\dfrac{1}{2\sqrt{2}}$ $i=1, \cdots, 6$, $a_7=\dfrac{1}{2}$.

III-3 E_8, $\alpha_1=-1$

We take a $\Delta(7)$ associated to a $\sum(4)$. Put $T_i=D_i$, $i=1, \cdots, 7$, $T_8=\sum_{j=1}^{8} a_jN_j$, where $a_i=-\dfrac{1}{2\sqrt{2}}$ $i=1, \cdots, 8$.

IV F_4

i) $(\alpha_1, \alpha_2)=(-1, 1)$

Let us take $\sum(2)=\{N_1, N_2, N_3, N_4, N_5\}$ and put $T_1=N_1$, $T_2=-\dfrac{1}{2}N_1+\dfrac{\sqrt{3}}{2}N_2$, $T_3=N_3$, $T_4=-\dfrac{1}{2}N_3+\dfrac{\sqrt{3}}{2}N_4$

ii) $(\alpha_1, \alpha_2)=(-1, -1)$

Let us take $\sum(2)$ and put $T_1=\dfrac{1}{\sqrt{2}}(N_1-N_2)$, $T_2=\dfrac{1}{\sqrt{2}}(N_2-N_3)$, $T_3=N_3$, $T_4=-\dfrac{1}{2}\sum_{j=1}^{4} N_j$.

V $G_2^{(n)}$ $n=$even, $\alpha_1=-1$

By Lemma 8 there exists a complex number α_0 such that $(T_1T_2)^n=-I$ if $\{T_1, T_2\}=\alpha_0 I$, $T_1^2=T_2^2=I$. Let us take $\sum(1)$ and put $T_1=N_1$, $T_2=k_1N_1+k_2N_2$, where $k_1=\frac{1}{2}\alpha_0$ and k_2 is determined by the condition $k_1^2+k_2^2=1$.

VI-1 H_3, $\alpha_1=-1$

We remark that the equation in the proof of Lemma 8 is $f_3+f_2=0$, i.e. $\alpha^2+\alpha-1=0$ when $l=5$.

Let us take $\sum(1)$ and put $T_1=-N_1$, $T_2=\dfrac{1}{2}N_1+\dfrac{\sqrt{3}}{2}N_2$, $T_3=k_1N_2+k_2N_3$ where $k_1=\dfrac{\sqrt{5}-1}{2\sqrt{3}}$, $k_2=\dfrac{\sqrt{5}+1}{2\sqrt{3}}$.

VI-2 H_4, $\alpha_1=-1$

Let us take $\sum(2)$ and put $T_1=-N_1$, $T_2=\dfrac{1}{2}N_1+\dfrac{\sqrt{3}}{2}N_2$,

$T_3=-\frac{1}{\sqrt{3}}N_2-\frac{\sqrt{2}}{\sqrt{3}}N_3$, $T_4=k_1N_3+k_2N_4$ where $k_1=-\frac{\sqrt{3}(\sqrt{5}-1)}{4\sqrt{2}}$, $k_2=\frac{3+\sqrt{5}}{4\sqrt{2}}$.

Thus we have proved the theorem completely. The corollary is immediately deduced from the theorem and proposition 3.2 in Yamazaki's [7].

REMARK 1. The projective representations which are constructed by the systems of matrices given above are all irreducible. This is proved by the same reasoning as that of Schur ([5]) in the case of A_l. Especially for the Weyl group of any complex simple Lie algebra of rank l, there exists an irreducible projective representation over C of degree $2^{[l/2]}$.

REMARK 2. By above considerations, we have obtained a system of generators and defining relations for a representation-group of an irreducible finite reflection group. We denote by $\tilde{G}$ the group generated by $l+\kappa$ symbols $\{T_1, \cdots, T_l, \alpha_1, \cdots, \alpha_\kappa\}$ which has as defining relations $\alpha_j^2=e, \alpha_jT_i=T_i\alpha_j$ for $i=1, \cdots, l, j=1, \cdots, \kappa$ and the normalized relations in table II. e denotes the unit element. Then $\tilde{G}$ is a central extension of G with kernel N, the subgroup generated by all α_j, and for any projective representation of G there exists an ordinary representation of $\tilde{G}$ which has the property stated in § 0. We easily see the order of N equal to the order of $\mathrm{H}^2(G, C^*)$. $\tilde{G}$ is a representation-group of G.

REMARK 3. An upper bound for the number p of representation-groups (up to isomorphisms) of a finite group is given by Schur (See [4. p. 95]). If we take λ and κ as $G/[G, G]\cong(Z_2)^\lambda$ (cf. § 2) and $\mathrm{H}^2(G, C^*)=(Z_2)^\kappa$, we have the estimation $p\leqq 2^{\lambda\kappa}$.

University of Tokyo

References

[1] H. S. M. Coxeter, Discrete groups generated by reflections, Ann. of Math. **35** (1934), pp. 588-621.

[2] C. W. Curtis and I. Reiner, Representation theory of finite groups and associative algebras, New York (1962), Wiley.

[3] I. Schur, Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen, Crelles J. **127** (1904), pp. 20-50.

[4] ———, Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen, ibid., **132** (1907), pp. 85-137.

[5] ———, Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen, ibid., **139** (1911), pp. 155-250.

[6] E. Witt, Spiegerungsgruppen und Aufzählung halbeinfacher Liescher Ringe, Abh. Math. Sem. Univ. Hamburg, **14** (1941), pp. 289-322.

[7] K. Yamazaki, On projective representations and ring extensions of finite groups, J. Fac. Sci. Univ. Tokyo, **10** (1964), pp. 147-195.

(Received October 10 1964)