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Continuous Dependence for Nonlinear Schrödinger Equation in H^s

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Abstract. This paper is concerned with the well-posedness, especially with the continuity of the solution map of the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = f(u), \quad u(x,0) = \phi(x)$$

on $\mathbf{R}^{n+1}$. Here, $f(u) = c_0|u|^\sigma u$, $c_0 \in \mathbf{C}$ and $\sigma > 0$. If $1 < s < \min(n/2, 2)$ and $0 < \sigma < 4/(n-2s)$, the solution map $\phi \mapsto u$ is continuous as a map from H^s to $C([0,T], H^s)$ for some $T > 0$. The proof is based on the estimates in the fractional order Besov spaces both for time and space variables.

1. Introduction

This paper is concerned with the well-posedness of the nonlinear Schrödinger equation:

$$i\partial_t u + \Delta u = f(u), \tag{1.1}$$

$$u(x,0) = \phi(x), \tag{1.2}$$

especially with the continuity of the solution map $\phi \mapsto u$. Here, $u : \mathbf{R}^{n+1} \to \mathbf{C}$, $f(u) = c_0|u|^\sigma u, c_0 \in \mathbf{C}$ and $\sigma > 0$. The corresponding integral equation is as follows:

$$u(t) = e^{it\Delta}\phi + (Gf(u))(t), \tag{1.3}$$

where

$$(Gg)(t) := -i\int_0^t e^{i(t-t')\Delta} g(t')dt'. \tag{1.4}$$

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The Cauchy problem (1.1)-(1.2) is said to be well-posed in H^s if the following are satisfied: (i) for any $\phi \in H^s$, there exists a solution $u \in C([0,T], H^s)$ to (1.3) for some $T > 0$; (ii) the solution is unique; (iii) the solution map $\phi \mapsto u$ is continuous as a map from H^s to $C([0,T], H^s)$.

There are a lot of works concerning the well-posedness of (1.1)-(1.2), for example [4, 6, 7, 8, 11, 13], and comprehensive lists of literature can be found in [2, 9]. Let $0 \leq s < n/2$ and $\sigma_0(s) < \sigma < 4/(n-2s)$, where

$$\sigma_0(s) = \begin{cases} 0, & 0 \leq s < 2, \\ s-2, & 2 \leq s < 4, \\ s-3, & s \geq 4. \end{cases}$$

Then, according to the previous results, (1.1)-(1.2) has a unique time local solution in H^s (precisely, in some cases, we need Strichartz type auxiliary spaces to show uniqueness). The condition for σ above is taken from [11]. Moreover, if we assume $\sigma \geq [s]+1$, the solution map is locally Lipschitz continuous [4]. It is also known that if $s = 0, 1, 2$ and $0 < \sigma < 4/(n-2s)$, the solution map is continuous at any $\phi \in H^s$ ([13] for $s = 0$, [6, 7] for $s = 1$ and [8] for $s = 2$). However in other cases, continuity of the solution map had been proved only in weaker sense, namely from H^s to $C([0,T], L^2) \cap L^\gamma([0,T], L^\rho)$, where (γ, ρ) is a suitable admissible pair. This is because we need $f \in C^{[s]+2}(\mathbf{C}, \mathbf{C})$ to show the Lipschitz continuity of $f(u)$ in Sobolev or Besov spaces of order s.

The aim of this paper is to prove the continuity of the solution map in full-strength in the case where $\sigma < [s]+1$. Recently, Cazenave-Fang-Han [3] affirmatively solved this problem when $0 < s < 1$. Therefore, we treat the case $1 < s < 2$ in the present paper. Our main theorem in this paper is the following:

THEOREM 1.1. *Let $n \geq 3$, $1 < s < \min\{n/2, 2\}$ and $0 < \sigma < 4/(n-2s)$. Then the solution map for (1.1)-(1.2) is continuous as a map from H^s to $C([0,T], H^s)$.*

The complete statement will be shown in Propositions 3.1-3.2. The difficulty of the proof for $s > 1$ comes from the fact that f might be differentiable only once, and we cannot estimate the difference of the nonlinear term in Sobolev/Besov spaces of order s. Instead, we use the estimate in

Besov spaces for time variable of order $s/2$, which are compared with s-th order spaces for spatial variables (Propositions 2.1-2.3).

In this paper We denote by $B^s_{p,q} = B^s_{p,q}(\mathbf{R}^n)$ the usual Besov spaces of order s on $\mathbf{R}^n$. Let $I \subset \mathbf{R}$ be an interval and X a Banach space. For $0 < \theta < 1$ and $1 < p, q < \infty$, X-valued Besov space on I is defined by

$$B^\theta_{p,q}(I, X) = (L^p(I, X), H^1_p(I, X))_{\theta,q},$$

where the right-hand side is the real interpolation. It is known that

$$\|u\|_{B^\theta_{p,q}(I,X)} \simeq \|u\|_{L^p(I,X)} + \left\{ \int_{-\infty}^{\infty} \|u(\cdot + \tau) - u\|^q_{L^p(I_\tau,X)} \tau^{-\theta q-1} d\tau \right\}^{1/q}.$$

Here $I_\tau = \{t \in I; t + \tau \in I\}$. For further detail, see [10] (see also [1, 12]). We denote by p' conjugate of the exponent p, that is, $1/p + 1/p' = 1$.

2. Linear Estimates

Definition 2.1. A pair (γ, ρ) is called *admissible* if

$$2 \le \rho < \frac{2n}{n-2} \quad \text{and} \quad \frac{2}{\gamma} = \frac{n}{2} - \frac{n}{\rho}. \tag{2.1}$$

Proposition 2.1. *Let (γ, ρ) be admissible. If $n \ge 3, 0 \le s < 2$, the following estimates hold:*

$$\|e^{it\Delta}\phi\|_{L^\gamma(\mathbf{R},B^s_{\rho,2})} \le c\|\phi\|_{H^s}, \tag{2.2}$$

$$\|e^{it\Delta}\phi\|_{B^{s/2}_{\gamma,2}(\mathbf{R},L^\rho)} \le c\|\phi\|_{H^s}. \tag{2.3}$$

Proof. [4] Theorems 2.1 and 2.2, [11] Proposition 2.5. □

Proposition 2.2. *Let $I \subset \mathbf{R}$ be an interval. If (γ, ρ) is an admissible pair and $(\gamma, \rho) \ne (\infty, 2), 0 < \theta < 1, \varepsilon > 0$ arbitrarily small, then the following estimate holds with c independent of I:*

$$\begin{aligned}\|Gg\|_{L^\gamma(I,B^{2\theta}_{\rho,2})} &+ \|Gg\|_{B^{\theta+\varepsilon}_{\gamma,2}(I,L^\rho)} \\ &\le c\Big(\|g\|_{B^{\theta+\varepsilon}_{\gamma',2}(I,L^{\rho'})} + \|g\|_{L^{\tilde\gamma(\theta+2\varepsilon)}(I,L^{\tilde\rho(\theta)}\cap L^{\tilde\rho(\theta+4\varepsilon)})} \\ &\qquad + \|g\|_{L^{\tilde\gamma(\theta)}(I,L^{\tilde\rho(\theta-\varepsilon)}\cap L^{\tilde\rho(\theta+\varepsilon)})}\Big),\end{aligned} \tag{2.4}$$

where $\dfrac{1}{\tilde\gamma(\theta)} = \dfrac{\theta}{\gamma} + \dfrac{1-\theta}{\gamma'}, \quad \dfrac{1}{\tilde\rho(\theta)} = \dfrac{\theta}{\rho} + \dfrac{1-\theta}{\rho'}.$

PROOF. [11] Proposition 2.6. □

PROPOSITION 2.3. *Let $I \subset \mathbf{R}$ be an interval. If $(\gamma,\rho) \neq (\infty,2)$ is an admissible pair, $0<\theta<1, \varepsilon>0$ arbitrarily small, then we have with c independent of I:*

$$\begin{aligned}(2.5)\quad & \|Gg\|_{L^\infty(I,H^{2\theta})} \\ & \le c\Big(\|g\|_{B^\theta_{\gamma',2}(I,L^{\rho'})} + \|g\|_{L^{\bar\gamma(\theta+\varepsilon)}(I,L^{\bar\rho(\theta+\varepsilon)})} + \|g\|_{L^{\bar\gamma(\theta-\varepsilon)}(I,L^{\bar\rho(\theta-\varepsilon)})}\Big),\end{aligned}$$

where $\dfrac{1}{\bar\gamma(\theta)} = \dfrac{1-\theta}{\gamma'}, \quad \dfrac{1}{\bar\rho(\theta)} = \dfrac{1-\theta}{\rho'} + \dfrac{\theta}{2}.$

Moreover, $u \in C(I,H^{2\theta})$ if the right-hand side of (2.5) *is finite.*

PROOF. For any $t_0 \in I$, we define $S_{t_0}g \equiv (Gg)(t_0)$ and interpolate between the following mappings:

$$S_{t_0} : L^{\gamma'}(I,L^{\rho'}) \to L^2,$$
$$S_{t_0} : H^1_{\gamma'}(I,L^{\rho'}) \cap L^\infty_0(I,L^2) \to H^2.$$

Here, $L^\infty_0(I,L^2)$ denotes the completion in the sup-norm of all functions $v(t)=\sum_j \chi_{E_j}(t)v_j, v_j \in L^2$, where the sum is finite and χ_{E_j} are the characteristic functions of disjoint measurable sets E_j with finite measure. One has $(L^2,H^2)_{\theta,2} = H^{2\theta}$. With the notation

$$A_0 = L^{\gamma'}(I,L^{\rho'}), \quad A_1^{(1)} = H^1_{\gamma'}(I,L^{\rho'}), \quad A_1^{(2)} = L^\infty_0(I,L^2)$$

and [11] Lemma 2.2 it holds $(A_0, A_1^{(1)} \cap A_1^{(2)})_{\theta,2} = (A_0,A_1^{(1)})_{\theta,2} \cap (A_0,A_1^{(2)})_{\theta,2}$. We have by [11] Lemma 2.1:

$$(A_0,A_1^{(1)})_{\theta,2} = B^\theta_{\gamma',2}(I,L^{\rho'}).$$

From [1] Theorem 3.4.1 (c), Theorem 4.7.1 and Theorem 5.1.2 it follows that

$$\begin{aligned}(A_0,A_1^{(2)})_{\theta,2} &\supset (A_0,A_1^{(2)})_{\theta-\varepsilon,\infty} \cap (A_0,A_1^{(2)})_{\theta+\varepsilon,\infty} \\ &\supset (A_0,A_1^{(2)})_{[\theta-\varepsilon]} \cap (A_0,A_1^{(2)})_{[\theta+\varepsilon]} \\ &= L^{\bar\gamma(\theta-\varepsilon)}(I,L^{\bar\rho(\theta-\varepsilon)}) \cap L^{\bar\gamma(\theta+\varepsilon)}(I,L^{\bar\rho(\theta+\varepsilon)}).\end{aligned}$$

Therefore we obtain that

$$\begin{aligned}\|(Gg)(t_0)\|_{H^{2\theta}} &= \|S_{t_0} g\|_{H^{2\theta}} \\ &\leq c\Big(\|g\|_{B^{\theta}_{\gamma',2}(I,L^{\rho'})} + \|g\|_{L^{\bar{\gamma}(\theta+\varepsilon)}(I,L^{\bar{\rho}(\theta+\varepsilon)})} \\ &\qquad + \|g\|_{L^{\bar{\gamma}(\theta-\varepsilon)}(I,L^{\bar{\rho}(\theta-\varepsilon)})}\Big),\end{aligned}$$

which yields the desired estimate. The continuity of u follows from density argument. □

3. The Nonlinear Problem

We consider the integral equation:

$$u(t) = e^{it\Delta}\phi + (Gf(u))(t), \tag{3.1}$$

where

$$f(u) = c_0|u|^\sigma u, c_0 \in \mathbf{C}, \tag{3.2}$$

$$\phi \in H^s. \tag{3.3}$$

Lemma 3.1. *If $s_1, s_2 \geq 0, 0 < k \leq 2, 0 < \eta < 1, \rho, \gamma \geq 1$, then the following estimate holds for sufficiently small $\varepsilon > 0$:*

$$\|u\|_{B^{k(\eta-\varepsilon)}_{\gamma,2}(I,B^{\eta s_1+(1-\eta)s_2}_{\rho,2})} \leq c\|u\|^{\eta}_{B^{k}_{\gamma,2}(I,B^{s_1}_{\rho,2})}\|u\|^{1-\eta}_{L^{\gamma}(I,B^{s_2}_{\rho,2})}. \tag{3.4}$$

Proof. [11] Lemma 4.1. □

Lemma 3.2. *The function f defined in* (3.2) *satisfies the following estimate:*

$$|f'(z_1) - f'(z_2)| \leq \begin{cases} c\left(|z_1|^{\sigma-1} \vee |z_2|^{\sigma-1}\right)|z_1 - z_2| & \text{if } \sigma \geq 1, \\ c|z_1 - z_2|^\sigma & \text{if } 0 < \sigma < 1. \end{cases} \tag{3.5}$$

Proof. [3] Remark 2.3. □

PROPOSITION 3.1. *Let* $n \geq 3, 1 < s < \min\{n/2, 2\}$ *and* $0 < \sigma < 4/(n-2s)$. *Let* (γ, ρ) *be an arbitrary admissible pair with*

$$\max\{\sigma + 2, 4(\sigma+2)/n\sigma\} < \gamma < 4(\sigma+2)/\sigma(n-2s). \tag{3.6}$$

Then there exists $T = T(\|\phi\|_{H^s}) > 0$ *such that* (1.3) *has a unique solution in* $X = C(I, H^s) \cap B^{s/2}_{\gamma,2}(I, L^\rho) \cap L^\gamma(I, B^{s-2\varepsilon}_{\rho,2})$, *where* $I = [0, T]$ *and* $\epsilon > 0$ *is sufficiently small. Moreover, let* $\phi_m \in H^s, m = 1, 2, \ldots$ *and* $\phi_m \to \phi$ *in* H^s *as* $m \to \infty$. *Then, for sufficiently large* m, (1.3) *with* ϕ *replaced by* ϕ_m *has a unique solution* u_m *in* X *and*

$$\|u_m - u\|_{L^\gamma(I,L^\rho)} \leq c\|\phi_m - \phi\|_{L^2} \to 0 \, (m \to \infty). \tag{3.7}$$

REMARK. (i) Proposition 3.1 shows the continuity of the solution map only in weaker sense. Actually, Pecher [11] did not state the continuity of the solution map explicitly as above, but it clearly follows from the proof of uniqueness. (ii) Precisely, Pecher [11] proved the case where $\gamma = 4(\sigma + 2)/\sigma(n-2s)$ and (γ, ρ) is the corresponding admissible pair. However, we can easily modify the proof and restate the theorem as above.

PROPOSITION 3.2. *Let* u *and* u_m *be solutions stated in Proposition 3.1. Then* $\|u_m - u\|_X \to 0$ *as* $m \to \infty$.

PROOF. Using Propositions 2.1-2.3, we get the following estimates:

$$\|e^{it\Delta}(\phi_m - \phi)\|_X \leq c\|\phi_m - \phi\|_{H^s}, \tag{3.8}$$

$$\begin{aligned} &\|G(f(u_m) - f(u))\|_{B^{s/2}_{\gamma,2}(I,L^\rho)} + \|G\left(f(u_m) - f(u)\right)\|_{L^\gamma(I,B^{s-2\varepsilon}_{\rho,2})} \\ &\quad \leq c\Big(\|f(u_m) - f(u)\|_{B^{s/2}_{\gamma',2}(I,L^{\rho'})} \\ &\qquad + \|f(u_m) - f(u)\|_{L^{\tilde\gamma(s/2+\varepsilon)}(I,L^{\tilde\rho(s/2-\varepsilon)} \cap L^{\tilde\rho(s/2+3\varepsilon)})} \\ &\qquad + \|f(u_m) - f(u)\|_{L^{\tilde\gamma(s/2-\varepsilon)}(I,L^{\tilde\rho(s/2-2\varepsilon)} \cap L^{\tilde\rho(s/2)})}\Big), \end{aligned} \tag{3.9}$$

$$\begin{aligned} \|G(f(u_m) - f(u))\|_{C(I,H^s)} \leq c\Big(&\|f(u_m) - f(u)\|_{B^{s/2}_{\gamma',2}(I,L^{\rho'})} \\ &+ \|f(u_m) - f(u)\|_{L^{\bar\gamma(s/2+\varepsilon)}(I,L^{\bar\rho(s/2+\varepsilon)})} \\ &+ \|f(u_m) - f(u)\|_{L^{\bar\gamma(s/2-\varepsilon)}(I,L^{\bar\rho(s/2-\varepsilon)})}\Big), \end{aligned} \tag{3.10}$$

where

$$\begin{aligned}\frac{1}{\tilde{\gamma}(s/2)} &= \frac{s}{2\gamma} + \frac{1}{\gamma'}\left(1-\frac{s}{2}\right) \equiv \frac{1}{\tilde{\gamma}}, \\ \frac{1}{\tilde{\rho}(s/2)} &= \frac{s}{2\rho} + \frac{1}{\rho'}\left(1-\frac{s}{2}\right) \equiv \frac{1}{\tilde{\rho}},\end{aligned} \tag{3.11}$$

$$\frac{1}{\bar{\gamma}(s/2)} = \frac{1}{\gamma'}\left(1-\frac{s}{2}\right) \equiv \frac{1}{\bar{\gamma}}, \quad \frac{1}{\bar{\rho}(s/2)} = \frac{1}{\rho'}\left(1-\frac{s}{2}\right) + \frac{s}{4} \equiv \frac{1}{\bar{\rho}}. \tag{3.12}$$

In the estimates of the last two terms in (3.9) and (3.10), we omit the small parameter ε and show

$$\|f(u_m) - f(u)\|_{L^{\tilde{\gamma}}(I,L^{\tilde{\rho}})} \le cT^\lambda \left(\|u_m\|_X^\sigma \vee \|u\|_X^\sigma\right) \|u_m - u\|_X, \tag{3.13}$$

$$\|f(u_m) - f(u)\|_{L^{\bar{\gamma}}(I,L^{\bar{\rho}})} \le cT^\lambda \left(\|u_m\|_X^\sigma \vee \|u\|_X^\sigma\right) \|u_m - u\|_X. \tag{3.14}$$

which make the proof much simpler. These are sufficient for our proof because all the conditions for the exponents to be fulfilled are given by strict inequalities.

If we have the inequalities

$$\frac{1}{\rho} > \frac{1}{(\sigma+1)\tilde{\rho}} > \frac{1}{\rho} - \frac{s(1-\eta)}{n}, \tag{3.15}$$

$$\frac{1}{(\sigma+1)\tilde{\gamma}} > \frac{1}{\gamma} - \frac{s}{2}\eta \tag{3.16}$$

for some $0 < \eta < 1$, we get the embedding

$$B^{s\eta/2}_{\gamma,2}(I, B^{(1-\eta)s}_{\rho,2}) \subset L^{(\sigma+1)\tilde{\gamma}}(I, L^{(\sigma+1)\tilde{\rho}}). \tag{3.17}$$

Since (γ,ρ) is admissible, calculations using (3.11) show that (3.15) is equivalent to

$$1-\eta > \frac{2}{s(\sigma+1)}\left(\frac{n\sigma}{4} - \frac{\sigma+2-s}{\gamma}\right) \equiv \alpha > 0. \tag{3.18}$$

On the other hand, (3.16) is equivalent to

$$1-\eta < 1 - \frac{2}{s(\sigma+1)}\left(\frac{\sigma+2-s}{\gamma} - \frac{2-s}{2}\right) \equiv \beta. \tag{3.19}$$

There exists $0 < \eta < 1$ satisfying (3.18) and (3.19) provided that $0 < \alpha < \beta$ and $\alpha < 1$. The first condition $0 < \alpha < \beta$ is fulfilled if $\gamma > 4(\sigma + 2 - s)/n\sigma$ and $\sigma < 4/(n-2s)$. The second condition $\alpha < 1$ is equivalent to

$$\frac{(n-2s)\sigma}{4} < \frac{\sigma+2}{\gamma} + s\left(\frac{1}{2} - \frac{1}{\gamma}\right),$$

which is verified by (3.6) and $\gamma > 2$. Therefore we can obtain the embedding (3.17). Using Hölder's inequality, (3.17) and (3.4) we deduce

$$\begin{aligned}
(3.20)\quad & \|f(u_m) - f(u)\|_{L^{\tilde\gamma}(I,L^{\tilde\rho})} \\
&\le c\left(\|u_m\|^{\sigma}_{L^{(\sigma+1)\tilde\gamma}(I,L^{(\sigma+1)\tilde\rho})} \vee \|u\|^{\sigma}_{L^{(\sigma+1)\tilde\gamma}(I,L^{(\sigma+1)\tilde\rho})}\right) \\
&\qquad \times \|u_m - u\|_{L^{(\sigma+1)\tilde\gamma}(I,L^{(\sigma+1)\tilde\rho})} \\
&\le cT^{\lambda}\left(\|u_m\|^{\sigma}_{B^{s\eta/2}_{\gamma,2}(I,B^{(1-\eta)(s-2\varepsilon)}_{\rho,2})} \vee \|u\|^{\sigma}_{B^{s\eta/2}_{\gamma,2}(I,B^{(1-\eta)(s-2\varepsilon)}_{\rho,2})}\right) \\
&\qquad \times \|u_m - u\|_{B^{s\eta/2}_{\gamma,2}(I,B^{(1-\eta)(s-2\varepsilon)}_{\rho,2})} \\
&\le cT^{\lambda}\left(\|u_m\|^{\sigma}_X \vee \|u\|^{\sigma}_X\right)\|u_m - u\|_X,
\end{aligned}$$

where $\lambda > 0$. In the second inequality we have used Hölder's inequality for time variable to gain some positive power of T, which is possible since (3.16) is still valid if we replace $(\sigma+1)\tilde\gamma$ by a slightly greater number.

Similarly we can get the following embedding:

$$(3.21)\qquad B^{s\eta/2}_{\gamma,2}(I, B^{s(1-\eta)}_{\rho,2}) \subset L^{(\sigma+1)\bar\gamma}(I, L^{(\sigma+1)\bar\rho})$$

so that we deduce (3.14) by using (3.4). Thus (3.13), (3.14) has been proved. Therefore combining (3.9) with (3.10) we have

$$\begin{aligned}
(3.22)\qquad \|G(f(u_m) - f(u))\|_X &\le c\|f(u_m) - f(u)\|_{B^{s/2}_{\gamma',2}(I,L^{\rho'})} \\
&\quad + cT^{\lambda}\left(\|u_m\|^{\sigma}_X \vee \|u\|^{\sigma}_X\right)\|u_m - u\|_X.
\end{aligned}$$

Next we estimate $f(u_m) - f(u)$ in $B^{s/2}_{\gamma',2}(I, L^{\rho'})$. From one of the equivalent norms in Besov spaces ([10], p. 327 or [12], Theorem 4.2.2), we have

$$\begin{aligned}
(3.23)\qquad & \|f(u_m) - f(u)\|_{B^{s/2}_{\gamma',2}(I,L^{\rho'})} \\
&= \|f(u_m) - f(u)\|_{L^{\gamma'}(I,L^{\rho'})} \\
&\quad + \left(\int_{-\infty}^{\infty} \frac{d\tau}{|\tau|^{1+s}} \|\delta\{f(u_m) - f(u)\}\|^2_{L^{\gamma'}(I_\tau,L^{\rho'})}\right)^{1/2}
\end{aligned}$$

with $I_\tau := \{t : t, t+\tau \in I\}$ and $\delta u(t) := u(t+\tau) - u(t)$. Hölder's inequality gives

$$\|f(u_m) - f(u)\|_{L^{\rho'}} \le c\left(\|u_m\|_{L^l}^\sigma \vee \|u\|_{L^l}^\sigma\right)\|u_m - u\|_{L^\rho}, \tag{3.24}$$

where $1/\rho' = \sigma/l + 1/\rho$. (3.6) implies $1/\rho > 1/l > 1/\rho - s/n$. Indeed, the first inequality is verified by $\gamma > 4(\sigma+2)/n\sigma$, the second inequality is satisfied if $\gamma < 4(\sigma+2)/(n-2s)\sigma$. Therefore we get the embedding

$$B^s_{\rho,2} \subset L^l. \tag{3.25}$$

From Hölder's inequality and (3.25) we obtain for sufficiently small $\varepsilon > 0$

$$\begin{aligned} &\|f(u_m) - f(u)\|_{L^{\gamma'}(I,L^{\rho'})} \\ &\quad \le cT^\kappa \left(\|u_m\|^\sigma_{L^\gamma(I,B^{s-2\varepsilon}_{\rho,2})} \vee \|u\|^\sigma_{L^\gamma(I,B^{s-2\varepsilon}_{\rho,2})}\right)\|u_m - u\|_{L^\gamma(I,L^\rho)}, \end{aligned} \tag{3.26}$$

where $\kappa = 1 - (\sigma+2)/\gamma > 0$. For $0 \le \theta \le 1$ we define

$$\begin{aligned} v_m(\theta) &:= \theta u_m(t+\tau) + (1-\theta)u_m(t), \\ v(\theta) &:= \theta u(t+\tau) + (1-\theta)u(t). \end{aligned} \tag{3.27}$$

By the mean value theorem, we have

$$\begin{aligned} \|\delta\{f(u_m) - f(u)\}\|_{L^{\rho'}} &\le \sup_{\theta\in[0,1]} \|f'(v_m(\theta))\,\delta u_m - f'(v(\theta))\,\delta u\|_{L^{\rho'}} \\ &\le c\|u_m\|^\sigma_{L^l}\|\delta(u_m - u)\|_{L^\rho} \\ &\quad + \sup_{\theta\in[0,1]} \|f'(v_m(\theta)) - f'(v(\theta))\|_{L^{l/\sigma}}\|\delta u\|_{L^\rho} \\ &\equiv p(t) + q(t). \end{aligned} \tag{3.28}$$

Hölder's inequality and (3.25) yield

$$\|p\|_{L^{\gamma'}(I_\tau)} \le cT^\kappa \|u_m\|^\sigma_{L^\gamma(I,B^{s-2\varepsilon}_{\rho,2})}\|\delta(u_m - u)\|_{L^\gamma(I_\tau,L^\rho)}. \tag{3.29}$$

In the estimate of $\|q\|_{L^{\gamma'}(I_\tau)}$, we distinguish the case $\sigma < 1$ from $\sigma \ge 1$. In the case $\sigma \ge 1$, from (3.5) we observe

$$q(t) \le c\left(\|u_m\|^{\sigma-1}_{L^l} \vee \|u\|^{\sigma-1}_{L^l}\right)\|u_m - u\|_{L^l}\|\delta u\|_{L^\rho}. \tag{3.30}$$

Taking $L^{\gamma'}$- norm in (3.30) we conclude

$$\|q\|_{L^{\gamma'}(I_\tau)} \leq cT^\kappa \left(\|u_m\|^{\sigma-1}_{L^\gamma(I,B^{s-2\varepsilon}_{\rho,2})} \vee \|u\|^{\sigma-1}_{L^\gamma(I,B^{s-2\varepsilon}_{\rho,2})} \right) \times \|u_m - u\|_{L^\gamma(I,B^{s-2\varepsilon}_{\rho,2})} \|\delta u\|_{L^\gamma(I_\tau,L^\rho)}. \tag{3.31}$$

Here we have used the embedding (3.25). Going back to the representation (3.23) and using the estimates (3.29), (3.31) together with (3.26), we see

$$\begin{aligned} \|f(u_m) - f(u)\|_{B^{s/2}_{\gamma',2}(I,L^\rho)} &\leq cT^\kappa (\|u_m\|^\sigma_X \vee \|u\|^\sigma_X) \|u_m - u\|_{B^{s/2}_{\gamma,2}(I,L^\rho)} \\ &\quad + cT^\kappa \left(\|u_m\|^{\sigma-1}_X \vee \|u\|^{\sigma-1}_X\right) \|u_m - u\|_X \|u\|_{B^{s/2}_{\gamma,2}(I,L^\rho)} \\ &\leq cT^\kappa (\|u_m\|^\sigma_X \vee \|u\|^\sigma_X) \|u_m - u\|_X. \end{aligned} \tag{3.32}$$

It follows from (3.1), (3.8), (3.22), (3.32) that

$$\|u_m - u\|_X \leq c\|\phi_m - \phi\|_{H^s} + c\left(T^\kappa + T^\lambda\right) (\|u_m\|^\sigma_X \vee \|u\|^\sigma_X) \|u_m - u\|_X. \tag{3.33}$$

If we choose $T > 0$ such that

$$\sup_m c\left(T^\kappa + T^\lambda\right) (\|u_m\|^\sigma_X \vee \|u\|^\sigma_X) < 1, \tag{3.34}$$

wee see

$$\|u_m - u\|_X \leq c\|\phi_m - \phi\|_{H^s}, \tag{3.35}$$

which yields the result for the case $\sigma \geq 1$.

In the case $0 < \sigma < 1$, using (3.5) and the Gagliardo-Nirenberg inequality

$$q(t) \leq c\|u_m - u\|^\sigma_{L^l} \|\delta u\|_{L^\rho} \leq c\|u_m - u\|^{\sigma\mu}_{B^s_{\rho,2}} \|u_m - u\|^{\sigma(1-\mu)}_{L^\rho} \|\delta u\|_{L^\rho} \tag{3.36}$$

for some $0 < \mu < 1$. (3.7) gives

$$\begin{aligned} \|q\|_{L^{\gamma'}(I_\tau)} &\leq cT^\kappa \|u_m - u\|^{\sigma\mu}_{L^\gamma(I,B^{s-2\varepsilon}_{\rho,2})} \|u_m - u\|^{\sigma(1-\mu)}_{L^\gamma(I,L^\rho)} \|\delta u\|_{L^\gamma(I_\tau,L^\rho)} \\ &\leq cT^\kappa \|u_m - u\|^{\sigma\mu}_X \|\phi_m - \phi\|^{\sigma(1-\mu)}_{L^2} \|\delta u\|_{L^\gamma(I_\tau,L^\rho)}. \end{aligned} \tag{3.37}$$

(3.23) and (3.26), (3.29), (3.37) imply

$$
\begin{aligned}
(3.38)\qquad &\|f(u_m)-f(u)\|_{B^{s/2}_{\gamma',2}(I,L^{\rho'})} \\
&\le cT^{\kappa}\left(\|u_m\|_X^{\sigma}\vee\|u\|_X^{\sigma}\right)\|u_m-u\|_{B^{s/2}_{\gamma,2}(I,L^{\rho})} \\
&\quad + cT^{\kappa}\|u_m-u\|_X^{\sigma\mu}\|\phi_m-\phi\|_{L^2}^{\sigma(1-\mu)}\|u\|_{B^{s/2}_{\gamma,2}(I,L^{\rho})} \\
&\le cT^{\kappa}\left(\|u_m\|_X^{\sigma}\vee\|u\|_X^{\sigma}\right)\|u_m-u\|_X \\
&\quad + cT^{\kappa}\|u_m-u\|_X^{\sigma\mu}\|u\|_X\|\phi_m-\phi\|_{H^s}^{\sigma(1-\mu)}.
\end{aligned}
$$

It follows from (3.1), (3.8), (3.22), (3.38) that

$$
\begin{aligned}
(3.39)\qquad \|u_m-u\|_X &\le c\|\phi_m-\phi\|_{H^s} \\
&\quad + c\left(T^{\kappa}+T^{\lambda}\right)\left(\|u_m\|_X^{\sigma}\vee\|u\|_X^{\sigma}\right)\|u_m-u\|_X \\
&\quad + cT^{\kappa}\|u_m-u\|_X^{\sigma\mu}\|u\|_X\|\phi_m-\phi\|_{H^s}^{\nu},
\end{aligned}
$$

where $\nu=\sigma(1-\mu), 0<\nu<1$. Taking $T>0$ such that (3.34) holds we have

$$
(3.40)\qquad \|u_m-u\|_X \le c\|\phi_m-\phi\|_{H^s}^{\nu}
$$

for m large enough, which completes the proof. □

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