J. Math. Sci. Univ. Tokyo
10 (2003), 391–419.

Brauer Groups and Tate-Shafarevich Groups

By Cristian D. Gonzalez-Aviles

Abstract. Let X_K be a proper, smooth and geometrically connected curve over a global field K. In this paper we generalize a formula of Milne relating the order of the Tate-Shafarevich group of the Jacobian of X_K to the order of the Brauer group of a proper regular model of X_K. We thereby partially answer a question of Grothendieck.

0. Introduction

Let K be a global field, i.e. K is a finite extension of $\mathbb{Q}$ (the "number field case") or is finitely generated and of transcendence degree 1 over a finite field (the "function field case"). In the number field case we let U denote a nonempty open subscheme of the spectrum of the ring of integers of K, and when K is a function field in one variable with finite field of constants k, we let U denote a nonempty open subscheme of the unique smooth complete curve over k whose function field is K. We will write S for the set of primes of K not corresponding to a point of U. Further, we will write $\overline{K}$ for the separable algebraic closure of K and Γ for the Galois group of $\overline{K}$ over K. The completion of K at a prime v will be denoted by K_v.

Assume now that a regular, connected, 2-dimensional scheme X is given together with a proper morphism $\pi\colon X \to U$ whose generic fiber $X_K = X \times_U \operatorname{Spec} K$ is a smooth geometrically connected curve over K. Let δ (resp. δ') denote the *index* (resp. *period*) of X_K. These integers may be defined as the least positive degree of a divisor on X_K and the least positive degree of a divisor class in $\operatorname{Pic}(X_{\overline{K}})^{\Gamma}$, respectively, where $X_{\overline{K}} = X_K \otimes_K \overline{K}$. For each prime v of K, we will write δ_v and δ'_v for the analogous quantities associated to the curve X_{K_v}. It is known that there are only finitely many

1991 *Mathematics Subject Classification.* 11G35, 14K15, 14G25.

Key words: Brauer groups, Tate-Shafarevich groups, Jacobian variety, index of a curve, period of a curve, Cassels-Tate pairing.

Supported by Fondecyt grants 195043 and 1000814.

primes v for which $\delta_v \neq 1$. Further, Lichtenbaum [16] has shown that δ_v equals either δ'_v or $2\delta'_v$ for each prime v (we note that the arguments of [16], which in principle apply only to finite extensions of $\mathbb{Q}_p$, are in fact valid for arbitrary local fields). We will write d for the number of primes v for which $\delta_v = 2\delta'_v$. These primes were called "deficient" in [27], and we sometimes refer to d as *the number of deficient primes of* X_K. Now let A be the Jacobian variety of X_K. It has long been known (see [34], §3) that there exist close connections between the Brauer group $\mathrm{Br}(X)$ of X and the Tate-Shafarevich group $Ш(A)$ of A. These connections were explored at length by Grothendieck in his paper [9], in a more general setting than the one considered here. The following theorem can be extracted from [9], p. 121.

Theorem (Grothendieck). *In the function field case, suppose that $\delta_v = 1$ for all primes v of K. Then there exist a finite group $T_2 \subset Ш(A)$ of order δ', a finite group T_3 of order dividing δ, and an exact sequence*

$$0 \to \mathrm{Br}(X) \to Ш(A)/T_2 \to T_3 \to 0.$$

Regarding this result, Grothendieck (op. cit., p. 122) asked for the exact order of T_3. This problem was solved by Milne [21] (see also [22], III.9.6), who used Cassels-Tate duality to compute the order of T_3. In order to state Milne's result, which covers both the function field and number field cases, we need the following definition. Let

$$\mathrm{Br}(X)' = \mathrm{Ker}\left[\mathrm{Br}(X) \to \bigoplus_{v\in S} \mathrm{Br}(X_{K_v})\right].$$

Thus $\mathrm{Br}(X)'$ is the group of elements in the Brauer group of X becoming trivial on X_{K_v} for all $v \in S$. Then one has the

Theorem (Milne). *Assume that $\delta_v = 1$ for all primes v of K and that $Ш(A)$ contains no nonzero infinitely divisible elements. Then the period of X_K equals its index, i.e. $\delta' = \delta$, and there is an exact sequence*

$$0 \to \mathrm{Br}(X)' \to Ш(A)/T_2 \to T_3 \to 0,$$

where T_2 and T_3 are finite groups of order δ. In particular, if one of $\text{Ш}(A)$ *or* $\mathrm{Br}(X)'$ *is finite, then so is the other, and their orders are related by*

$$\delta^2[\mathrm{Br}(X)'] = [\text{Ш}(A)].$$

Grothendieck (loc. cit.) went on to pose the problem of making explicit the relations between $\mathrm{Br}(X)$ and $\text{Ш}(A)$ when the integers δ_v are no longer assumed to be equal to one[1]. In this paper we generalize the methods developed by Milne in [21] to prove the above theorem and obtain the following stronger result, which may be viewed as a partial solution to Grothendieck's problem[2].

Main Theorem. *Assume that the integers δ'_v are relatively prime in pairs (i.e., $(\delta'_v, \delta'_{\bar{v}}) = 1$ for all $v \neq \bar{v}$) and that $\text{Ш}(A)$ contains no nonzero infinitely divisible elements. Then there is an exact sequence*

$$0 \to T_0 \to T_1 \to \mathrm{Br}(X)' \to \text{Ш}(A)/T_2 \to T_3 \to 0\,,$$

in which T_0, T_1, T_2 and T_3 are finite groups of orders

$$\begin{aligned}
[T_0] &= \delta/\delta' \\
[T_1] &= 2^e \\
[T_2] &= \delta'/\textstyle\prod \delta'_v \\
[T_3] &= \frac{\delta'/\prod \delta'_v}{2^f}\,,
\end{aligned}$$

where

$$e = \max(0, d-1)$$

and

$$f = \begin{cases} 1 & \textit{if } \delta'/\prod \delta'_v \textit{ is even and } d \geq 1 \\ 0 & \textit{otherwise.} \end{cases}$$

[1]To be precise, Grothendieck stated an equivalent form of this problem.

[2]In the last section of the paper we show that if a certain plausible conjecture holds then it is possible to give a complete answer to Grothendieck's question.

Here d is the number of deficient primes of X_K defined previously. In particular, if one of Ш(A) *or* $\mathrm{Br}(X)'$ *is finite, then so is the other, and their orders are related by*

$$\delta\delta'\,[\mathrm{Br}(X)'] = 2^{e+f}\prod(\delta_v')^2\,[\text{Ш}(A)].$$

Some immediate corollaries are

Corollary 1. *Suppose that one of* Ш(A) *or* $\mathrm{Br}(X)'$ *is finite. Assume also that $\delta_v = \delta_v'$ for all v (which holds for instance if X_K has genus 1*[3]*), and that these integers are relatively prime in pairs. Then $\delta = \delta'$ and*

$$\delta^2\,[\mathrm{Br}(X)'] = \prod\delta_v^2\,[\text{Ш}(A)].$$

In particular, if $\delta_v = 1$ for all v, then

$$\delta^2\,[\mathrm{Br}(X)'] = [\text{Ш}(A)].$$

Note that this last formula is precisely the formula of Milne stated before.

Corollary 2. *Assume that one of* Ш(A) *or* $\mathrm{Br}(X)'$ *is finite and that $\delta_v' = 1$ for all v (the latter holds for instance if X_K has genus 2). Then*

$$\delta\delta'\,[\mathrm{Br}(X)'] = 2^{e+f}\,[\text{Ш}(A)],$$

where e and f are as in the statement of the theorem.

In the function field case our main result, when combined with a formula of Gordon [5, p. 196], should imply the expected equivalence of the conjectures of Artin and Tate for X [34, Conj. C] and Birch and Swinnerton-Dyer for A [34, Conj. B] (this is Tate's "elementary" conjecture (d) of [34]), at least under the additional assumption that the structure morphism

[3]Regarding the parenthetical assertions in the statements of both corollaries, see [16]

$\pi\colon X \to U$ is cohomologically flat in dimension 0. We expect to address this issue in a separate publication.

Acknowledgements. I'm grateful to the people of Fondecyt for their financial support and their patience, and to the library staff of IMPA (Rio de Janeiro, Brasil) for their bibliographical assistance. I also thank the referee for carefully reading this paper and for pointing out the need to establish the compatibility of certain pairings (see the comments following (9) and the Appendix to the paper).

1. Preliminaries

We keep the notations introduced in the previous section. Thus in particular X is a regular connected scheme of dimension 2 equipped with a proper morphism $\pi\colon X \to U$ such that its generic fiber X_K is a smooth and geometrically connected curve over K.

Remark. The U-scheme X is a proper regular model of its generic fiber. Conversely, if we start with a (geometrically connected) proper and smooth curve X_K over K, then there is a closed immersion $X_K \to \mathbb{P}^n_K$ (for some n) and we can obtain a U-scheme X as above, with generic fiber X_K, by applying Lipman's desingularization process [2], [17] to the schematic image of X_K in $\mathbb{P}^n_U$.

The Picard scheme of X_K/K, $\mathrm{Pic}_{X_K/K}$, is a smooth group scheme over K whose identity component, $\mathrm{Pic}^0_{X_K/K}$, is an abelian variety, the Jacobian variety of X_K. Henceforth, we will write P for $\mathrm{Pic}_{X_K/K}$ and (in accordance with previous notations) A for $\mathrm{Pic}^0_{X_K/K}$. The following holds: if L is any field containing K such that $X_K(L)$ is nonempty, then $P(L) = \mathrm{Pic}(X_L)$ (for the basics on the relative Picard functor, see [3, §8.1] or [6]). Further, there is an exact sequence of Γ-modules

$$0 \to A(\overline{K}) \to P(\overline{K}) \xrightarrow{\deg} \mathbb{Z} \to 0,$$

where deg is the degree map on $P(\overline{K}) = \mathrm{Pic}(X_{\overline{K}})$. In particular $A(\overline{K})$ may be identified with $\mathrm{Pic}^0(X_{\overline{K}})$, the subgroup of $\mathrm{Pic}(X_{\overline{K}})$ consisting of divisor classes of degree zero (in this paper we shall regard the elements of

$\mathrm{Pic}(X_{\overline{K}})$ mainly as classes of divisors. See below). Now we observe that $P(K) = P(\overline{K})^{\Gamma} = \mathrm{Pic}(X_{\overline{K}})^{\Gamma}$. Further, there is an exact sequence (deduced from the preceding one by taking Γ-invariants)

$$0 \to A(K) \to P(K) \xrightarrow{\deg} \delta'\mathbb{Z} \to 0\,,$$

where δ' is the period of X_K as defined previously.

For any regular connected scheme Y, $R(Y)$ will denote the field of rational functions on Y. There is an exact sequence

$$0 \to R(X_{\overline{K}})^*/\overline{K}^* \to \mathrm{Div}^0(X_{\overline{K}}) \to \mathrm{Pic}^0(X_{\overline{K}}) \to 0$$

which induces an exact sequence

$$(1) \qquad 0 \to R(X_K)^*/K^* \to \mathrm{Div}^0(X_K) \to A(K) \to A(K)/\mathrm{Pic}^0(X_K) \to 0.$$

Observe that $A(K)/\mathrm{Pic}^0(X_K)$ is a finite abelian group, since it is finitely generated (by the Mordell-Weil theorem) and isomorphic to a subgroup of the torsion group $H^1(\Gamma, R(X_{\overline{K}})^*/\overline{K}^*)$. Similarly, $P(K)/\mathrm{Pic}(X_K)$ is finite and there is an exact sequence

$$(2) \qquad 0 \to R(X_K)^*/K^* \to \mathrm{Div}(X_K) \to P(K) \to P(K)/\mathrm{Pic}(X_K) \to 0.$$

We also have an exact sequence

$$0 \to \mathrm{Div}^0(X_K) \to \mathrm{Div}(X_K) \xrightarrow{\deg} \delta\,\mathbb{Z} \to 0\,,$$

where δ is the index of X_K as defined previously. We note that δ may also be defined as the greatest common divisor of the degrees of the fields L over K such that $X_K(L) \neq \emptyset$.

Lemma 1.1. *The period of X_K divides its index, i.e. $\delta' | \,\delta$, and*

$$[P(K) : \mathrm{Pic}(X_K)] = (\delta/\delta') \cdot [A(K) : \mathrm{Pic}^0(X_K)].$$

PROOF. The first assertion of the lemma follows from the definitions. Regarding the second, an application of the snake lemma to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Div}^0(X_K) & \longrightarrow & \mathrm{Div}(X_K) & \longrightarrow & \delta\,\mathbb{Z} & \longrightarrow & 0 \\ & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\ 0 & \longrightarrow & A(K) & \longrightarrow & P(K) & \longrightarrow & \delta'\,\mathbb{Z} & \longrightarrow & 0 \end{array}$$

yields, using (1) and (2) above, an exact sequence

$$0 \to A(K)/\mathrm{Pic}^0(X_K) \to P(K)/\mathrm{Pic}(X_K) \to \delta'\,\mathbb{Z}/\delta\,\mathbb{Z} \to 0.$$

The lemma is now immediate. $\square$

It is clear from the definitions that if L is any field containing K, then the index (resp. period) of X_L divides the index (resp. period) of X_K. Thus, for any prime v of K, $\delta_v | \, \delta$ and $\delta'_v | \, \delta'$, where δ_v and δ'_v are, respectively, the index and period of X_{K_v}. Further, since $X_K(K_v) \neq \emptyset$ for all but finitely many primes v (see for example [13, p. 249, Remark 1.6]), we conclude that there are only finitely many primes v with $\delta_v \neq 1$. The analogous statement with δ_v replaced by δ'_v holds true as well, since $\delta'_v | \, \delta_v$ for each v. We now write Δ (resp. Δ') for the least common multiple of the integers δ_v (resp. δ'_v). Clearly, $\Delta | \, \delta$ and $\Delta' | \, \delta'$. Further, since $\delta_v = \delta'_v$ or $2\delta'_v$ for each v as mentioned earlier, we have

$$\frac{\Delta}{\Delta'} = \begin{cases} 1 & \text{if } \delta_v = \delta'_v \text{ for all } v \\ 2 & \text{otherwise.} \end{cases}$$

We now consider the map

$$\Sigma : \bigoplus_v \delta_v^{-1}\mathbb{Z}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}\,, \quad (x_v) \mapsto \sum x_v\,.$$

LEMMA 1.2. *We have*

$$[\mathrm{Ker}\,\Sigma] = \prod \delta_v/\Delta$$

and

$$\operatorname{Im}\Sigma = \Delta^{-1}\mathbb{Z}\,/\,\mathbb{Z}.$$

Proof. That the image of Σ is exactly $\Delta^{-1}\mathbb{Z}/\mathbb{Z}$ follows easily from the fact that $\gcd\left(\frac{\Delta}{\delta_v}\right) = 1$. The rest of the lemma is clear. □

Next we consider the map

$$D : \mathbb{Z}/\delta'\mathbb{Z} \to \bigoplus_v \mathbb{Z}/\delta'_v\mathbb{Z}$$

given by $x \bmod \delta' \mapsto (x \bmod \delta'_v)$. Since the kernel of D is $\Delta'\mathbb{Z}/\delta'\mathbb{Z}$, the following lemma is clear.

Lemma 1.3. *We have*

$$[\operatorname{Ker} D] = \delta'/\Delta'$$

and

$$[\operatorname{Coker} D] = \prod \delta'_v/\Delta'.$$

For each prime v of K, we will write Γ_v for the Galois group of $\overline{K}_v$ over K_v. From the cohomology sequence associated to the exact sequence of Γ_v-modules $0 \to A(\overline{K}_v) \to P(\overline{K}_v) \xrightarrow{\deg} \mathbb{Z} \to 0$ we get an exact sequence

$$0 \to \mathbb{Z}/\delta'_v\mathbb{Z} \to H^1(\Gamma_v, A) \to H^1(\Gamma_v, P) \to 0. \tag{3}$$

Now by work of Lichtenbaum [16] and Milne [19] (see also [22, Remark I.3.7]), there exists a perfect pairing

$$H^0(\Gamma_v, A) \times H^1(\Gamma_v, A) \to \mathbb{Q}/\mathbb{Z}\,,$$

where $H^0(\Gamma_v, A)$ denotes $A(K_v)/N_{\overline{K}_v/K_v}A(\overline{K}_v)$ if v is archimedean and $A(K_v)$ otherwise. Relative to this pairing, the annihilator of the image of $\operatorname{Pic}^0(X_{K_v})$ in $H^0(\Gamma_v, A)$ under the canonical map $A(K_v) \to H^0(\Gamma_v, A)$

is exactly the image of $\mathbb{Z}/\delta'_v\mathbb{Z}$ in $H^1(\Gamma_v, A)$ under the map in (3). Consequently the following holds

LEMMA 1.4. *We have*

$$[A(K_v) : \mathrm{Pic}^0(X_{K_v})] = \delta'_v.$$

We now combine the previous lemma with an analogue of Lemma 1.1 to obtain

LEMMA 1.5. *We have*

$$[P(K_v) : \mathrm{Pic}(X_{K_v})] = \delta_v.$$

REMARK. The archimedean case of Lemma 1.5 was originally established by Witt in 1935 [38]. For an interesting review of this and other related classical results in terms of étale cohomology, see [30, §20.1].

We now derive a slight variant of the snake lemma (Proposition 1.6 below). It is one of the basic ingredients of the proof of our Main Theorem.

Consider the following exact commutative diagram in the category of abelian groups

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r, "f"] \arrow[d] & A_3 \arrow[r] \arrow[d, "\eta"] & A_4 \arrow[r] \arrow[d, "\lambda"] & A_5 \arrow[r] \arrow[d, "\mu"] & 0 \\
0 \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r, "g"] & B_3 \arrow[r] & B_4 \arrow[r] & B_5 \arrow[r] & 0
\end{tikzcd} \tag{4}$$

(we have labeled only those maps which are relevant to our purposes). There is an induced exact commutative diagram

$$\begin{tikzcd}
 & A_3 \arrow[r] \arrow[d, "\bar{\eta}"] & A_4 \arrow[r] \arrow[d, "\lambda"] & A_5 \arrow[r] \arrow[d, "\mu"] & 0 \\
0 \arrow[r] & B_3/\mathrm{Im}\, g \arrow[r] & B_4 \arrow[r] & B_5 \arrow[r] & 0
\end{tikzcd}$$

in which $\bar{\eta}$ is the composition of η with the canonical map $B_3 \to B_3/\mathrm{Im}\, g$. An application of the snake lemma to the above diagram (using the fact that $\mathrm{Im}\, f \subset \mathrm{Ker}\, \bar{\eta}$) yields

Proposition 1.6. *To any exact commutative diagram of the form (4) there is associated an exact sequence*

$$0 \to \mathrm{Im}\, f \to \mathrm{Ker}\, \bar{\eta} \to \mathrm{Ker}\, \lambda \to \mathrm{Ker}\, \mu \to \mathrm{Coker}\, \bar{\eta}\,,$$

where $\bar{\eta}$ is as defined above.

The following result supplements Proposition 1.6.

Lemma 1.7. *With the above notations, there is an exact sequence*

$$0 \to \mathrm{Ker}\, \eta \to \mathrm{Ker}\, \bar{\eta} \to \mathrm{Im}\, g \to \mathrm{Coker}\, \eta \to \mathrm{Coker}\, \bar{\eta} \to 0.$$

Proof. This is nothing more than the kernel-cokernel sequence [22, I.0.24] for the pair of maps $A_3 \xrightarrow{\eta} B_3 \to B_3/\mathrm{Im}\, g$. The maps in the exact sequence of the lemma are the natural ones, e.g. $\mathrm{Im}\, g \to \mathrm{Coker}\, \eta$ is the composite $\mathrm{Im}\, g \hookrightarrow B_3 \to B_3/\mathrm{Im}\, \eta = \mathrm{Coker}\, \eta$. $\square$

2. Proof of the Main Theorem

All cohomology groups below are either Galois cohomology groups or étale cohomology groups. We will view Γ_v as a subgroup of Γ in the standard way, i.e., by identifying it with the decomposition group of some fixed prime of $\overline{K}$ lying above v. For each v, $\mathrm{inv}_v : \mathrm{Br}(K_v) \to \mathbb{Q}/\mathbb{Z}$ will denote the usual invariant map of local class field theory. The n-torsion subgroup of an abelian group M will be denoted by $M_{n-\mathrm{tor}}$.

We begin by recalling a fundamental exact sequence. Since $H^q(X_{\overline{K}}, \mathbb{G}_m) = 0$ for all $q \geq 2$ [20, Ex. 2.23(b), p. 110], the Hochschild-Serre spectral sequence

$$H^p(\Gamma, H^q(X_{\overline{K}}, \mathbb{G}_m)) \Rightarrow H^{p+q}(X_K, \mathbb{G}_m)$$

yields (see [4, XV.5.11]) an exact sequence

$$0 \to \mathrm{Pic}(X_K) \to P(K) \to \mathrm{Br}(K) \to \mathrm{Br}(X_K) \to H^1(\Gamma, P) \to 0\,, \tag{5}$$

where the zero at the right-hand end comes from the fact that $H^3(\Gamma, \overline{K}^*) = 0$ [22, I.4.21]. (We have used here the well-known facts that $\mathrm{Pic}(X_K) = H^1(X_K, \mathbb{G}_m)$ and that the Brauer group of a regular scheme of dimension ≤ 2 agrees with the cohomological Brauer group of the scheme.) Similarly, for each prime v of K there is an exact sequence

$$0 \to \mathrm{Pic}(X_{K_v}) \to P(K_v) \xrightarrow{g_v} \mathrm{Br}(K_v) \to \mathrm{Br}(X_{K_v}) \to H^1(\Gamma_v, P) \to 0. \tag{6}$$

LEMMA 2.1. *For each prime v of K, we have*

$$\mathrm{Im}\, g_v = \mathrm{Br}(K_v)_{\delta_v-\mathrm{tor}}.$$

PROOF. By Lemma 1.5, $\mathrm{Im}\, g_v$ is a subgroup of $\mathrm{Br}(K_v)$ of order δ_v. On the other hand the invariant map inv_v induces an isomorphism $\mathrm{Br}(K_v)_{\delta_v-\mathrm{tor}} \xrightarrow{\sim} \delta_v^{-1}\mathbb{Z}/\mathbb{Z}$, whence the lemma follows. □

We now consider the commutative diagram

$$\begin{array}{ccccccccccc}
0 \to & \mathrm{Pic}(X_K) & \to & P(K) & \to & \mathrm{Br}(K) & \to & \mathrm{Br}(X_K) & \to & H^1(\Gamma, P) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow{\scriptstyle \eta} & & \downarrow & & \downarrow & \\
0 \to & \bigoplus_v \mathrm{Pic}(X_{K_v}) & \to & \bigoplus_v P(K_v) & \xrightarrow{g} & \bigoplus_v \mathrm{Br}(K_v) & \to & \bigoplus_v \mathrm{Br}(X_{K_v}) & \to & \bigoplus_v H^1(\Gamma_v, P) & \to 0
\end{array}$$

in which the direct sums extend over all primes of K, the vertical maps are the natural ones and $g = \oplus_v g_v$. This diagram is of the form considered at the end of Section 2, and we may therefore apply to it Proposition 1.6 above. Before doing so, however, we call upon

LEMMA 2.2. (a) *There is an exact sequence*

$$0 \to \mathrm{Br}(K) \xrightarrow{\eta} \bigoplus_v \mathrm{Br}(K_v) \xrightarrow{\sum \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0.$$

(b) *There is an exact sequence*

$$0 \to \mathrm{Br}(X) \to \mathrm{Br}(X_K) \to \bigoplus_{v \notin S} \mathrm{Br}(X_{K_v}),$$

where S is the set of primes of K not corresponding to a point of U.

Proof. Assertion (a) is one of the main theorems of class field theory. See [35, §11]. Assertion (b) is proved in Lemma 2.6 of [21]. □

We now apply Proposition 1.6 to the diagram above using the preceding lemma. We get an exact sequence

$$0 \to P(K)/\mathrm{Pic}(X_K) \to \mathrm{Ker}\,\bar{\eta} \to \mathrm{Br}(X)' \to Ш(P) \to \mathrm{Coker}\,\bar{\eta},$$

where $\bar{\eta}\colon \mathrm{Br}(K) \to \bigoplus_v \mathrm{Br}(K_v)\big/\,\mathrm{Im}\,g$ is induced by η and

$$\mathrm{Br}(X)' = \mathrm{Ker}\left[\mathrm{Br}(X) \to \bigoplus_{v \in S} \mathrm{Br}(X_{K_v})\right].$$

Now by Lemma 1.1, the order of $P(K)/\mathrm{Pic}(X_K)$ equals $(\delta/\delta') \cdot [A(K) : \mathrm{Pic}^0(X_K)]$. Regarding the kernel and cokernel of $\bar{\eta}$, the following holds.

Proposition 2.3. *We have*

$$[\mathrm{Ker}\,\bar{\eta}] = \prod \delta_v/\Delta\,,$$

and the map $\sum \mathrm{inv}_v : \bigoplus \mathrm{Br}(K_v) \to \mathbb{Q}/\mathbb{Z}$ induces an isomorphism

$$\mathrm{Coker}\,\bar{\eta} \simeq \mathbb{Q}/\Delta^{-1}\mathbb{Z}\,.$$

Proof. By combining Lemmas 1.7, 2.1 and 2.2 we obtain an exact sequence

$$0 \to \mathrm{Ker}\,\bar{\eta} \to \bigoplus \mathrm{Br}(K_v)_{\delta_v-\mathrm{tor}} \xrightarrow{\sum \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to \mathrm{Coker}\,\bar{\eta} \to 0.$$

Now, for each v, the invariant map inv_v induces an isomorphism $\mathrm{Br}(K_v)_{\delta_v-\mathrm{tor}} \simeq \delta_v^{-1}\mathbb{Z}/\mathbb{Z}$, whence it follows that the kernel and cokernel of $\sum \mathrm{inv}_v$ in the above exact sequence may be identified with the kernel and cokernel of the map Σ considered in Section 1. The proposition now follows from Lemma 1.2. □

We now summarize the results obtained so far.

COROLLARY 2.4. *There is an exact sequence*

$$0 \to T_0 \to T_1 \to \mathrm{Br}(X)' \to Ш(P) \to \mathbb{Q}/\Delta^{-1}\mathbb{Z}\,,$$

where T_0 and T_1 are finite groups of orders

$$\begin{aligned}[T_0] &= (\delta/\delta') \cdot [A(K) : \mathrm{Pic}^0(X_K)] \\ [T_1] &= \textstyle\prod \delta_v/\Delta\,.\end{aligned}$$

We now prove

THEOREM 2.5. *Assume that the integers δ_v' are relatively prime in pairs. Then*

$$A(K) = \mathrm{Pic}^0(X_K).$$

PROOF. There is an exact commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A(K)/\mathrm{Pic}^0(X_K) \arrow[r] \arrow[d] & \mathrm{Br}(K) \arrow[d, "\eta"] \\ 0 \arrow[r] & \bigoplus\limits_{\text{all } v} A(K_v)/\mathrm{Pic}^0(X_{K_v}) \arrow[r] & \bigoplus\limits_{\text{all } v} \mathrm{Br}(K_v)\,, \end{tikzcd}$$

in which the vertical maps are the natural ones and the nontrivial horizontal maps are induced by the maps $P(K)/\mathrm{Pic}(X_K) \to \mathrm{Br}(K)$ and $P(K_v)/\mathrm{Pic}(X_{K_v}) \to \mathrm{Br}(K_v)$ coming from (5) and (6). Arguing as in the proof of Lemma 2.1 (this time using Lemma 1.4 in place of Lemma 1.5), we conclude that for each v the image of $A(K_v)/\mathrm{Pic}^0(X_{K_v})$ in $\mathrm{Br}(K_v)$ equals $\mathrm{Br}(K_v)_{\delta'_v-\mathrm{tor}}$. It follows that $A(K)/\mathrm{Pic}^0(X_K)$ embeds into $\mathrm{Ker}\,\bar{\eta}'$, where

$$\bar{\eta}' : \mathrm{Br}(K) \to \bigoplus_v \mathrm{Br}(K_v)/\mathrm{Br}(K_v)_{\delta'_v-\mathrm{tor}}$$

is induced by η. We now argue as in the proof of Proposition 2.3 to conclude that the order of $\mathrm{Ker}\,\bar{\eta}'$ equals $\prod \delta'_v/\Delta'$. But this number is equal to 1 by hypothesis, and the theorem follows. □

We now recall from the introduction the integer d, which was defined to be the number of primes v of K for which $\delta_v = 2\delta'_v$. As noted just before the statement of Lemma 1.2, $\Delta/\Delta' = 1$ or 2 according as $d = 0$ or $d \geq 1$. We now observe that

$$\begin{aligned}\frac{\prod \delta_v}{\Delta} &= \left(\frac{\Delta}{\Delta'}\right)^{-1} \prod (\delta_v/\delta'_v) \cdot \frac{\prod \delta'_v}{\Delta'} \\ &= 2^e\, \frac{\prod \delta'_v}{\Delta'}\,,\end{aligned}$$

where

$$e = \max(0, d-1).$$

Therefore if the integers δ'_v are relatively prime in pairs, then $\prod \delta_v/\Delta = 2^e$. Thus Corollary 2.4 and Theorem 2.5 together imply

Corollary 2.6. *Assume that the integers δ'_v are relatively prime in pairs. Then there is an exact sequence*

$$0 \to T_0 \to T_1 \to \mathrm{Br}(X)' \to Ш(P) \xrightarrow{\phi} \mathbb{Q}/\Delta^{-1}\mathbb{Z}\,,$$

in which T_0 and T_1 are finite groups of orders

$$\begin{aligned}[][T_0] &= \delta/\delta' \\ [T_1] &= 2^e,\end{aligned}$$

where $e = \max(0, d-1)$.

We turn now to the problem of relating $Ш(P)$ to $Ш(A)$.

There is an exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/\delta'\mathbb{Z} & \longrightarrow & H^1(\Gamma, A) & \longrightarrow & H^1(\Gamma, P) & \longrightarrow & 0 \\
 & & \Big\downarrow D & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & \bigoplus\limits_{\text{all } v} \mathbb{Z}/\delta'_v\mathbb{Z} & \longrightarrow & \bigoplus\limits_{\text{all } v} H^1(\Gamma_v, A) & \longrightarrow & \bigoplus\limits_{\text{all } v} H^1(\Gamma_v, P) & \longrightarrow & 0\,,
\end{array}$$

in which D is the diagonal map of Section 1 and the rows come from the cohomology sequences of

$$0 \to A \to P \xrightarrow{\deg} \mathbb{Z} \to 0$$

over K and over K_v. Applying the snake lemma to the above diagram, we obtain the exact sequence

$$0 \to \operatorname{Ker} D \to Ш(A) \to Ш(P) \to \operatorname{Coker} D. \tag{7}$$

Now let $T_2 = \operatorname{Ker} D$. Then by Lemma 1.3,

$$[T_2] = \delta'/\Delta'\,. \tag{8}$$

Further, the order of $\operatorname{Coker} D$ equals $\prod \delta'_v/\Delta'$. Therefore the following holds

Proposition 2.7. *Suppose that the integers δ'_v are relatively primes in pairs. Then there is an exact sequence*

$$0 \to T_2 \to Ш(A) \xrightarrow{\rho} Ш(P) \to 0\,,$$

in which T_2 is a finite group of order

$$[T_2] = \delta'/\textstyle\prod \delta'_v.$$

In what follows we shall regard T_2 as a subgroup of $Ш(A)$ by identifying it with the kernel of the map ρ in Proposition 2.7.

Combining Corollary 2.6 and Proposition 2.7, we obtain

Corollary 2.8. *Assume that the integers δ'_v are relatively prime in pairs. Then there is an exact sequence*

$$0 \to T_0 \to T_1 \to \mathrm{Br}(X)' \to Ш(A)/T_2 \to \mathbb{Q}/\Delta^{-1}\mathbb{Z},$$

in which T_0, T_1 and T_2 are finite groups of orders

$$\begin{aligned}[][T_0] &= \delta/\delta' \\ [T_1] &= 2^e \\ [T_2] &= \delta'/\textstyle\prod \delta'_v\,,\end{aligned}$$

where $e = \max(0, d-1)$.

It remains only to compute T_3, the image of $Ш(A)/T_2$ in $\mathbb{Q}/\Delta^{-1}\mathbb{Z}$ under the map of Corollary 2.8, or equivalently, the image of the composite map $Ш(A) \xrightarrow{\rho} Ш(P) \xrightarrow{\phi} \mathbb{Q}/\Delta^{-1}\mathbb{Z}$, where ϕ and ρ are the maps in Corollary 2.6 and Proposition 2.7, respectively. We will show that T_3 has the order stated in the introduction by generalizing [21, Lemma 2.11].

We begin by recalling from [21, Remark 2.9] the explicit description of the map $\phi : Ш(P) \to \mathbb{Q}/\Delta^{-1}\mathbb{Z}$.

We write E for the canonical map $\mathrm{Div}(X_{\overline{K}}) \to \mathrm{Pic}(X_{\overline{K}}) = P(\overline{K})$. Represent $\alpha \in Ш(P)$ by a cocycle $a \in Z^1(\Gamma, P(\overline{K}))$, and let $\mathfrak{a} \in C^1(\Gamma, \mathrm{Div}(X_{\overline{K}}))$ be such that $E(\mathfrak{a}) = a$. Then $\partial\mathfrak{a} \in Z^2(\Gamma, \mathrm{Ker}\, E) = Z^2(\Gamma, R(X_{\overline{K}})^*/\overline{K}^*)$ and, because $H^3(\Gamma, \overline{K}^*) = 0$, it can be pulled back to an element $f \in Z^2(\Gamma, R(X_{\overline{K}})^*)$ (here ∂ = boundary map). On the other hand, $\mathrm{res}_v(a) = \partial a_v$ with $a_v \in C^0(\Gamma_v, P(\overline{K}_v))$. Let $\mathfrak{a}_v \in C^0(\Gamma_v, \mathrm{Div}(X_{\overline{K}_v}))$ be such that $E(\mathfrak{a}_v) = a_v$. Then $\mathrm{res}_v(\mathfrak{a}) = \partial\mathfrak{a}_v + \mathrm{div}(f_v)$ with $f_v \in C^1(\Gamma_v, R(X_{\overline{K}_v})^*)$, and $\mathrm{res}_v f/\partial f_v \in Z^2(\Gamma_v, \overline{K}_v^*)$. Let γ_v be the class of $\mathrm{res}_v f/\partial f_v$ in $\mathrm{Br}(K_v)$. Then

$$\phi(\alpha) = q\left(\sum_v \mathrm{inv}_v(\gamma_v)\right),$$

where q is the canonical map $\mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\Delta^{-1}\mathbb{Z}$ induced by the identity map on $\mathbb{Q}$.

We note that, for any divisor $\mathfrak{c}_v$ on X_{K_v} such that neither $\mathrm{res}_v f$ nor ∂f_v has a zero or a pole in the support of $\mathfrak{c}_v$,

$$(\mathrm{res}_v f)(\mathfrak{c}_v)/\partial f_v(\mathfrak{c}_v) = (\deg \mathfrak{c}_v)\, \mathrm{res}_v f/\partial f_v$$

(see for example [16, §4]). Here $(\mathrm{res}_v f)(\mathfrak{c}_v) = f(\mathfrak{c}_v) \in Z^2(\mathit{\Gamma}_v, \overline{K}_v^*)$ is the value at $(\mathrm{res}_v f, \mathfrak{c}_v)$ of the cup-product pairing

$$Z^2(\mathit{\Gamma}_v, R(X_{\overline{K}_v})^*) \times Z^0(\mathit{\Gamma}_v, \mathrm{Div}(X_{\overline{K}_v})) \to Z^2(\mathit{\Gamma}_v, \overline{K}_v^*)$$

which is induced by the evaluation pairing $R(X_{\overline{K}_v})^* \times \mathrm{Div}(X_{\overline{K}_v}) \to \overline{K}_v^*$, and similarly for $\partial f_v(\mathfrak{c}_v)$. Since $\partial f_v(\mathfrak{c}_v) = \partial(f_v(\mathfrak{c}_v))$ with $f_v(\mathfrak{c}_v) \in C^1(\mathit{\Gamma}_v, \overline{K}_v^*)$, we see that $(\deg \mathfrak{c}_v)\,\gamma_v$ is represented by $f(\mathfrak{c}_v)$. Now choose a divisor $\mathfrak{c}_v$ on X_{K_v} of degree δ_v such that neither f nor ∂f_v has a zero or a pole in the support of $\mathfrak{c}_v$ (this is always possible; see [14, VI, §4, Lemma 3]). Then $\delta_v \gamma_v$ is represented by $f(\mathfrak{c}_v)$.

We will now define a pairing

$$< \, , \, > : Ш(A) \times Ш(A) \to \mathbb{Q}/\mathbb{Z} \tag{9}$$

which agrees (up to sign) with the "Albanese-Picard" pairing of Poonen and Stoll [27] when $A = \mathrm{Pic}^0_{X_K/K}$ (from the first factor) is identified with $\mathrm{Alb}^0_{\mathrm{X_K/K}}$ via the canonical polarization. In an appendix to this paper we will show that the "Albanese-Picard" pairing alluded to above is compatible with the "Cassels-Tate" pairing of Milne [22, II.5.7(a)]. As the latter is known to annihilate only the divisible part of $Ш(A)$ (see [22, III.9.5]), it will follow that the same is true of our pairing (9). This fact will be used in the proof of Corollary 2.10.

Let $\alpha \in Ш(A)$ be represented by $a \in Z^1(\mathit{\Gamma}, A(\overline{K}))$, and let $\mathrm{res}_v(a) = \partial a_v$ with $a_v \in Z^0(\mathit{\Gamma}_v, A(\overline{K}_v))$. Write

$$\begin{aligned} a &= E(\mathfrak{a}), \qquad \mathfrak{a} \in C^1(\mathit{\Gamma}, \mathrm{Div}^0(X_{\overline{K}})) \\ a_v &= E(\mathfrak{a}_v), \qquad \mathfrak{a}_v \in C^0(\mathit{\Gamma}_v, \mathrm{Div}^0(X_{\overline{K}_v})). \end{aligned}$$

Then $\mathrm{res}_v(\mathfrak{a}) = \partial \mathfrak{a}_v + \mathrm{div}(f_v)$ in $C^1(\mathit{\Gamma}_v, \mathrm{Div}^0(X_{\overline{K}_v}))$ with $f_v \in C^1(\mathit{\Gamma}_v, R(X_{\overline{K}_v})^*)$. Moreover, $\partial \mathfrak{a} = \mathrm{div}(f)$ with $f \in Z^2(\mathit{\Gamma}, R(X_{\overline{K}})^*)$. Let α' be a second element of $Ш(A)$ and choose $\mathfrak{a}', \mathfrak{a}'_v, f'_v, f'$ as for α in such a way that $\mathrm{Supp}\,\mathfrak{a}^\sigma \cap \mathrm{Supp}\,(\mathfrak{a}')^\tau = \emptyset$ for all $\sigma, \tau \in \Gamma$. Define

$$\eta = f' \cup \mathfrak{a} - f \cup \mathfrak{a}' \quad \in C^3(\mathit{\Gamma}, \overline{K}^*).$$

Then $\partial\eta = 0$ so, since $H^3(\Gamma, \overline{K}^*) = 0$, $\eta = \partial\epsilon$ for some $\epsilon \in C^2(\Gamma, \overline{K}^*)$. Let

$$\gamma_v = f'_v \cup \mathrm{res}_v(\mathfrak{a}) - \mathfrak{a}'_v \cup \mathrm{res}_v f - \mathrm{res}_v(\epsilon) \quad \in C^2(\Gamma_v, \overline{K}_v^*),$$

where $\cup$ denotes the cup-product pairing induced by the evaluation pairing mentioned earlier. Then γ_v is a 2-cocycle representing an element $c_v \in \mathrm{Br}(K_v)$, and we define

$$< \alpha, \alpha' > = \sum_v \mathrm{inv}_v(c_v) \quad \in \mathbb{Q}/\mathbb{Z}.$$

It can be seen that this pairing is well-defined and independent of choices. Further, $< \alpha, \alpha > = 0$.

Now let $\varepsilon : \mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} \mathbb{Q}/\Delta^{-1}\mathbb{Z}$ be the natural isomorphism which is induced by multiplication by Δ^{-1} on $\mathbb{Q}$.

PROPOSITION 2.9. *Let β be a generator of $T_2 \subset Ш(A)$ and let $\alpha' = (\Delta/\Delta')\beta$. Then the composite*

$$Ш(A) \xrightarrow{\rho} Ш(P) \xrightarrow{\phi} \mathbb{Q}/\Delta^{-1}\mathbb{Z}$$

is $\alpha \mapsto \varepsilon\,(< \alpha, \alpha' >)$.

PROOF. Let $\alpha \in Ш(A)$ and define $\mathfrak{a}, \mathfrak{a}_v, f_v$ and f as above. Then $\phi(\rho(\alpha)) = \sum \mathrm{inv}_v(\gamma_v)$ where $\delta_v\gamma_v$ is represented by $f(\mathfrak{c}_v)$ for some divisor $\mathfrak{c}_v$ of degree δ_v on X_{K_v}.

On the other hand, we can choose $\alpha' \in Ш(A)$ to be represented by $a' = E(\mathfrak{a}')$ where $\mathfrak{a}' = \Delta\partial P$ and P is any closed point of $X_{\overline{K}}$. Define

$$\mathfrak{a}'_v = \Delta P - (\Delta/\delta_v)\mathfrak{c}_v \in C^0(\Gamma_v, \mathrm{Div}^0(X_{\overline{K}_v})).$$

Then $a'_v = E(\mathfrak{a}'_v)$ satisfies

$$\partial a'_v = E(\Delta\partial P) = E(\mathrm{res}_v\mathfrak{a}') = \mathrm{res}_v(a').$$

Further, $\mathrm{res}_v(\mathfrak{a}') = \partial\mathfrak{a}'_v$, which means that we may take $f'_v = 1$. Further, since $\partial\mathfrak{a}' = 0$, we can choose $f' = 1$. Then

$$\eta = -f \cup \mathfrak{a}' = -\Delta(f \cup \partial P) = -\Delta\partial(f(P)) = 0,$$

which means that we can take $\epsilon = 0$. Then $< \alpha, \alpha' > = -\sum \mathrm{inv}_v(\gamma_v')$, where γ_v' is represented by $f(\mathfrak{a}_v') = f(\Delta P - (\Delta/\delta_v)\mathfrak{c}_v) = f(P)^{\Delta}/\gamma_v^{\Delta}$. Consequently

$$\begin{aligned}\varepsilon\,(< \alpha, \alpha' >) &= \varepsilon\bigg(\Delta \sum_v \mathrm{inv}_v(\gamma_v)\bigg)\\ &= q\bigg(\sum_v \mathrm{inv}_v(\gamma_v)\bigg) = \phi(\rho(\alpha))\,,\end{aligned}$$

as claimed. $\square$

Remark. We should note that the pairing defined by Milne in [21, pp. 741-742] (or in [22, Remark I.6.12, p. 100]) seems to be incorrectly defined, as the elements γ_v defined there (resp. c_v) need not be cocycles. The correct definition of this pairing is given by Poonen and Stoll in [27], as a particular case of their definition of the "Albanese-Picard" pairing mentioned above. This explains why we have used the Poonen-Stoll pairing rather than that defined by Milne. We should also point out that, in spite of the problem with the definition of his pairing, Milne's proof of Lemma 2.11 of [21] (which we have generalized in order to prove our Proposition 2.9 above) remains valid nonetheless, because for the particular choices made in that proof the elements γ_v *are* indeed cocycles.

Corollary 2.10. *Assume that* Ш(A) *contains no nonzero infinitely divisible elements. Then*

$$[T_3] = \frac{\delta'/\Delta'}{\left(\frac{\Delta}{\Delta'}, \frac{\delta'}{\Delta'}\right)}.$$

Proof. By our hypothesis and previous comments, the pairing (9) is nondegenerate, and the proposition shows that T_3 is isomorphic to the dual of $< \alpha' > = (\Delta/\Delta')T_2$. The corollary now follows from (8). $\square$

Our Main Theorem, stated in the introduction, may now be obtained by combining Corollaries 2.8 and 2.10.

3. Concluding Remarks

As shown in the proof of Theorem 2.5, the order of $A(K)/\mathrm{Pic}^0(X_K)$ divides $\prod \delta_v'/\Delta'$. We believe that it is always equal to $\prod \delta_v'/\Delta'$. This is equivalent to the following plausible statement. Consider the nondegenerate pairing

$$\bigoplus_v \mathbb{Z}/\delta'_v\mathbb{Z} \times \bigoplus_v A(K_v)/\mathrm{Pic}^0(X_{K_v}) \to \mathbb{Q}/\mathbb{Z}\,,$$

which is the sum of the local pairings

$$\mathbb{Z}/\delta'_v\mathbb{Z} \times A(K_v)/\mathrm{Pic}^0(X_{K_v}) \to \mathbb{Q}/\mathbb{Z}$$

induced by Lichtenbaum duality (see the discussion preceding the statement of Lemma 1.4). Then, relative to this pairing, the image of the diagonal map

$$D : \mathbb{Z}/\delta'\mathbb{Z} \to \bigoplus_v \mathbb{Z}/\delta'_v\mathbb{Z}$$

is the exact annihilator of the image of the diagonal map

$$\mathcal{D} : A(K)/\mathrm{Pic}^0(X_K) \hookrightarrow \bigoplus A(K_v)/\mathrm{Pic}^0(X_{K_v}).$$

Assuming (for simplicity) that $\text{Ш}(A)$ is finite, the above conjectural statement implies in addition that the exact sequence

$$0 \to \mathrm{Ker}\, D \to \text{Ш}(A) \to \text{Ш}(P) \to \mathrm{Coker}\, D \to \mathrm{Coker}\,\lambda \to \mathrm{Coker}\,\mu \to 0$$

(which is the continuation of the exact sequence (7); here λ and μ are the natural maps $H^1(\Gamma, A) \to \bigoplus H^1(\Gamma_v, A)$ and $H^1(\Gamma, P) \to \bigoplus H^1(\Gamma_v, P)$) splits into two short exact sequences

$$0 \to \mathrm{Ker}\, D \to \text{Ш}(A) \to \text{Ш}(P) \to 0$$

and

$$0 \to \mathrm{Coker}\, D \to \mathrm{Coker}\,\lambda \to \mathrm{Coker}\,\mu \to 0,$$

the second of which is the dual (except possibly for p-components in characteristic $p \neq 0$) of

$$0 \to \mathrm{Pic}^0(X_K) \to A(K) \to \mathrm{Im}\,\mathcal{D} \to 0.$$

Thus if the above conjecture is true, then it is possible to give a complete answer to Grothendieck's question stated in the introduction.

Appendix

As announced in the main body of the paper (see the comments following (9)), we will check the compatibility of the "Cassels-Tate" pairing of [22, II.5.7(a)] with the "Albanese-Picard" pairing of Poonen and Stoll [27]. Some straightforward verifications will be ommited.

We begin by recalling from [22] the definition of the "Cassels-Tate" pairing mentioned above. Let A be an abelian variety defined over a global field K and let Y denote the spectrum of the ring of integers of K (in the number field case) or the unique smooth complete curve over the field of constants of K having function field K (in the function field case). Let $U \subset Y$ denote the complement of the finite set of points of Y where A has bad reduction. Then A extends to an abelian scheme $\mathcal{A}$ over U (see [3, Ch. 1, §1.4.3]). Similarly, the dual abelian variety A^t extends to an abelian scheme $\mathcal{A}^t$ over U. By [10, VIII.7.1(b)], the canonical Poincaré biextension of (A, A^t) by $\mathbb{G}_m$ extends to a biextension over U of $(\mathcal{A}, \mathcal{A}^t)$ by $\mathbb{G}_m$ (see [22, Appendix C] for basic facts regarding biextensions). Now, by [10, VII.3.6.5], (the isomorphism class of) this biextension corresponds to a map $\mathcal{A} \otimes^{\mathbf{L}} \mathcal{A}^t \to \mathbb{G}_m[1]$ in the derived category of the category of smooth sheaves on U (see below for an explicit description of this correspondence over a field. For derived categories, see [36, Ch. 10] and [12, Ch. I][4]). This map in turn induces (see [22, p.284]) a canonical "cup-product" pairing

$$H^1_{\mathrm{c}}(U, \mathcal{A}) \times H^1(U, \mathcal{A}^t) \to H^3_{\mathrm{c}}(U, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z},$$

where $H^r_{\mathrm{c}}(U, F)$ is Milne's "cohomology group with compact support" of the sheaf F (if K has no real primes, then these groups agree with the étale cohomology groups $H^r(Y, j_! F)$, where $j_!$ is the extension-by-zero functor). Now define

$$\begin{aligned} D^1(U, \mathcal{A}) &= \mathrm{Im}\,(H^1_{\mathrm{c}}(U, \mathcal{A}) \to H^1(U, \mathcal{A})) \\ &= \mathrm{Ker}\left(H^1(U, \mathcal{A}) \to \prod_{v \notin U} H^1(K_v, A) \right), \end{aligned}$$

and similarly for $\mathcal{A}^t$. Then the above pairing induces a pairing

$$D^1(U, \mathcal{A}) \times D^1(U, \mathcal{A}^t) \to \mathbb{Q}/\mathbb{Z}. \tag{10}$$

[4] In the order suggested.

Let $g\colon \operatorname{Spec} K \to U$ be the inclusion of the generic point into U. Then $g_*A = \mathcal{A}$ (as sheaves for the smooth topology on U) and the canonical map $H^1(U,\mathcal{A}) \to H^1(K,A)$ induces an isomorphism $D^1(U,\mathcal{A}) \simeq Ш(A)$ (see [22, II.5.5]). Similarly, g induces an isomorphism $D^1(U,\mathcal{A}^t) \simeq Ш(A^t)$. Therefore (10) induces a pairing

$$Ш(A) \times Ш(A^t) \to \mathbb{Q}/\mathbb{Z}, \tag{11}$$

and this is the "Cassels-Tate" pairing referred to above.

Now let V be a projective, smooth and geometrically connected scheme over K. Poonen and Stoll [27, p.1116] have defined an "Albanese-Picard" pairing

$$Ш(\mathrm{Alb}^0_{V/K}) \times Ш(\mathrm{Pic}^0_{V/K,\,\mathrm{red}}) \to \mathbb{Q}/\mathbb{Z}, \tag{12}$$

which will be shown to be compatible with the "Cassels-Tate" pairing (11) (for $A = \mathrm{Alb}^0_{V/K}$ and $A^t = \mathrm{Pic}^0_{V/K,\,\mathrm{red}}$). To this end, we will first review the alternative description of (12) in terms of hypercohomology, following [27, end of §3.2]. Write $A = \mathrm{Alb}^0_{V/K}$, $A^t = \mathrm{Pic}^0_{V/K,\,\mathrm{red}}$ and let $\mathcal{Z}^0(V_{\overline{K}})$ denote the group of zero-cycles of degree 0 on $V_{\overline{K}}$. Write $\mathcal{Y}^0(V_{\overline{K}})$ for the kernel of the canonical map $\mathcal{Z}^0(V_{\overline{K}}) \to A(\overline{K})$. Then there are exact sequences

$$0 \to \mathcal{Y}^0(V_{\overline{K}}) \to \mathcal{Z}^0(V_{\overline{K}}) \to A(\overline{K}) \to 0 \tag{13}$$

and

$$0 \to R(V_{\overline{K}})^*/\overline{K}^* \to \mathrm{Div}^0(V_{\overline{K}}) \to A^t(\overline{K}) \to 0, \tag{14}$$

where $\mathrm{Div}^0(V_{\overline{K}})$ is the group of divisors on $V_{\overline{K}}$ algebraically equivalent to zero. We will consider the complexes

$$\begin{aligned} M^\bullet_K &= \big(\mathcal{Y}^0(V_{\overline{K}}) \to \mathcal{Z}^0(V_{\overline{K}})\big) = (M^0_K \to M^1_K) \\ N^\bullet_K &= \Big(R(V_{\overline{K}})^*/\overline{K}^* \to \mathrm{Div}^0(V_{\overline{K}})\Big) = (N^0_K \to N^1_K) \\ P^\bullet_K &= \overline{K}^*[-1]. \end{aligned}$$

Now, for each finite extension L of K inside $\overline{K}$, let $\mathbb{A}_L$ denote the adele ring of L and define $\overline{\mathbb{A}}_K = \overline{K} \otimes_K \mathbb{A}_K = \bigcup_L \mathbb{A}_L$, where the union extends over all fields L as above. Then

$$A(\overline{\mathbb{A}}_K) = \bigcup_L \prod_w A(L_w)$$

where, for each L as above, the product extends over all primes w of L (see [37, Ch. I, esp. Remark on p. 2] for basic facts on varieties defined over adele rings). Now define

$$\begin{aligned}
\mathcal{Z}^0\left(V_{\overline{\mathbb{A}}_K}\right) &= \bigcup_L \prod_w \mathcal{Z}^0\left(V_{L_w}\right) \\
\mathcal{Y}^0\left(V_{\overline{\mathbb{A}}_K}\right) &= \bigcup_L \prod_w \mathcal{Y}^0\left(V_{L_w}\right) \\
\mathrm{Div}^0\left(V_{\overline{\mathbb{A}}_K}\right) &= \bigcup_L \prod_w \mathrm{Div}^0(V_{L_w}) \\
R\left(V_{\overline{\mathbb{A}}_K}\right)^* / \overline{\mathbb{A}}_K^* &= \bigcup_L \prod_w R(V_{L_w})^* / L_w^*.
\end{aligned}$$

Then, since for each L as above and each prime w of L there are sequences analogous to (13) and (14) with $\overline{K}$ replaced by L_w, and these sequences are exact for all but finitely many primes w (see [13, p. 249, Remark 1.6] and [24, Theorem 1.1, p. 168] for the case of curves), there are exact sequences

$$0 \to \mathcal{Y}^0(V_{\overline{\mathbb{A}}_K}) \to \mathcal{Z}^0(V_{\overline{\mathbb{A}}_K}) \to A(\overline{\mathbb{A}}_K) \to 0$$

and

$$0 \to R(V_{\overline{\mathbb{A}}_K})^* / \overline{\mathbb{A}}_K^* \to \mathrm{Div}^0(V_{\overline{\mathbb{A}}_K}) \to A^t(\overline{\mathbb{A}}_K) \to 0.$$

Consider now the complexes

$$\begin{aligned}
M_{\mathbb{A}}^{\bullet} &= \left(\mathcal{Y}^0\left(V_{\overline{\mathbb{A}}_K}\right) \to \mathcal{Z}^0\left(V_{\overline{\mathbb{A}}_K}\right)\right) = (M_{\mathbb{A}}^0 \to M_{\mathbb{A}}^1) \\
N_{\mathbb{A}}^{\bullet} &= \left(R\left(V_{\overline{\mathbb{A}}_K}\right)^* / \overline{\mathbb{A}}_K^* \to \mathrm{Div}^0\left(V_{\overline{\mathbb{A}}_K}\right)\right) = (N_{\mathbb{A}}^0 \to N_{\mathbb{A}}^1) \\
P_{\mathbb{A}}^{\bullet} &= \overline{\mathbb{A}}_K^*[-1].
\end{aligned}$$

In addition, define complexes $M^\bullet_{\mathbb{A}/K}$, $N^\bullet_{\mathbb{A}/K}$ and $P^\bullet_{\mathbb{A}/K}$ by the exactness of the sequences

$$
\begin{aligned}
0 \to M^\bullet_K \to M^\bullet_{\mathbb{A}} \to M^\bullet_{\mathbb{A}/K} \to 0,\\
0 \to N^\bullet_K \to N^\bullet_{\mathbb{A}} \to N^\bullet_{\mathbb{A}/K} \to 0,\\
0 \to P^\bullet_K \to P^\bullet_{\mathbb{A}} \to P^\bullet_{\mathbb{A}/K} \to 0.
\end{aligned}
$$

Now let F be any field containing K. Consider

$$
\begin{aligned}
E_{V/F} &= \{(x,D)\in \mathcal{Z}^0(V_F)\times \mathrm{Div}^0(V_F)\colon \mathrm{Supp}x\cap \mathrm{Supp}D=\emptyset\}\\
E^l_{V/F} &= \{(x,D)\in \mathcal{Y}^0(V_F)\times \mathrm{Div}^0(V_F)\colon \mathrm{Supp}x\cap \mathrm{Supp}D=\emptyset\}\\
E^r_{V/F} &= \{(x,D)\in \mathcal{Z}^0(V_F)\times R(V_F)^*\big/F^*\colon \mathrm{Supp}x\cap \mathrm{Supp}D=\emptyset\}\\
E^{lr}_{V/F} &= \{(x,D)\in \mathcal{Y}^0(V_F)\times R(V_F)^*\big/F^*\colon \mathrm{Supp}x\cap \mathrm{Supp}D=\emptyset\}.
\end{aligned}
$$

If J denotes either of these sets, we will write $G(J)$ for the abelian group with generators $[x,D]$, $(x,D)\in J$, and relations

$$[x_1+x_2,D]=[x_1,D]+[x_2,D],\qquad [x,D_1+D_2]=[x,D_1]+[x,D_2].$$

Poonen and Stoll [Op.Cit.] have defined pairings

$$\varphi_{V/F}\colon G(E^l_{V/F})\to F^* \tag{15}$$

and

$$\psi_{V/F}\colon G(E^r_{V/F})\to F^* \tag{16}$$

which agree on $G(E^{lr}_{V/F})$. We now define

$$
\begin{aligned}
G(E_{V/\mathbb{A}}) &= \bigcup_L \prod_w G(E_{V/L_w})\\
G(E^l_{V/\mathbb{A}}) &= \bigcup_L \prod_w G(E^l_{V/L_w})\\
G(E^r_{V/\mathbb{A}}) &= \bigcup_L \prod_w G(E^r_{V/L_w})\\
G(E^{lr}_{V/\mathbb{A}}) &= \bigcup_L \prod_w G(E^{lr}_{V/L_w}).
\end{aligned}
$$

Then we have pairings

$$\varphi_{V/\mathbb{A}}\colon G(E^{l}_{V/\mathbb{A}}) \to \overline{\mathbb{A}}^{*}_{K} \tag{17}$$

and

$$\psi_{V/\mathbb{A}}\colon G(E^{r}_{V/\mathbb{A}}) \to \overline{\mathbb{A}}^{*}_{K} \tag{18}$$

which agree on $G(E^{lr}_{V/\mathbb{A}})$. Consider next the complex

$$G^{\bullet}_{\mathbb{A}} = (G(E^{lr}_{V/\mathbb{A}}) \xrightarrow{d^0} G(E^{l}_{V/\mathbb{A}}) \oplus G(E^{r}_{V/\mathbb{A}}) \xrightarrow{d^1} G(E_{V/\mathbb{A}})),$$

where the maps d^0 and d^1 are induced by the corresponding maps of the complex $M^{\bullet}_{\mathbb{A}} \otimes N^{\bullet}_{\mathbb{A}}$ (see [22, p.10] for the definition of these latter maps). Then the pair of pairings (17)-(18) induces a map of complexes $G^{\bullet}_{\mathbb{A}} \to P^{\bullet}_{\mathbb{A}}$. On the other hand, [39, §2, p. 497] shows that there exists a quasi-isomorphism $G^{\bullet}_{\mathbb{A}} \xrightarrow{\sim} M^{\bullet}_{\mathbb{A}} \otimes N^{\bullet}_{\mathbb{A}}$ and therefore a map

$$M^{\bullet}_{\mathbb{A}} \otimes^{\mathbf{L}} N^{\bullet}_{\mathbb{A}} \to P^{\bullet}_{\mathbb{A}} \tag{19}$$

in the derived category of the category of abelian groups. Note that since the sequences (13) and (14) induce quasi-isomorphisms $M^{\bullet}_{\mathbb{A}}[1] \simeq A(\overline{\mathbb{A}}_K)$ and $N^{\bullet}_{\mathbb{A}}[1] \simeq A^t(\overline{\mathbb{A}}_K)$, (19) corresponds to a map

$$A(\overline{\mathbb{A}}_K) \otimes^{\mathbf{L}} A^t(\overline{\mathbb{A}}_K) \to P^{\bullet}_{\mathbb{A}}[2] = \mathbb{G}_m[1]. \tag{20}$$

Now it is not difficult to check that (19) maps $M^{\bullet}_{K} \otimes^{\mathbf{L}} N^{\bullet}_{K}$ into $P^{\bullet}_{K}$, whence there is a map $M^{\bullet}_{\mathbb{A}/K} \otimes^{\mathbf{L}} N^{\bullet}_{K} \to P^{\bullet}_{\mathbb{A}/K}$. Thus we have a "cup-product" pairing

$$\mathbb{H}^1\big(M^{\bullet}_{\mathbb{A}/K}\big) \times \mathbb{H}^2\big(N^{\bullet}_{K}\big) \to \mathbb{H}^3\big(M^{\bullet}_{\mathbb{A}/K} \otimes^{\mathbf{L}} N^{\bullet}_{K}\big) \to \mathbb{H}^3\big(P^{\bullet}_{\mathbb{A}/K}\big)$$

which induces a pairing

$$\operatorname{Ker}\big[\mathbb{H}^2(M^{\bullet}_{K}) \to \mathbb{H}^2(M^{\bullet}_{\mathbb{A}})\big] \times \operatorname{Ker}\big[\mathbb{H}^2(N^{\bullet}_{K}) \to \mathbb{H}^2(N^{\bullet}_{\mathbb{A}})\big] \to \mathbb{H}^3(P^{\bullet}_{\mathbb{A}/K}). \tag{21}$$

Now $\mathbb{H}^2(M^{\bullet}_{K}) = H^1(\varGamma, A(\overline{K}))$, $\mathbb{H}^2(M^{\bullet}_{\mathbb{A}}) = H^1(\varGamma, A(\overline{\mathbb{A}}_K))$ and

$$\operatorname{Ker}\big[\mathbb{H}^2(M^{\bullet}_{K}) \to \mathbb{H}^2(M^{\bullet}_{\mathbb{A}})\big] = Ш(A)$$

(and similarly for A^t). Further, $\mathbb{H}^3(P^{\bullet}_{\mathbb{A}/K}) = H^2(\Gamma, \overline{\mathbb{A}}_K^* / \overline{K}^*) = \mathbb{Q}/\mathbb{Z}$ by class field theory. Therefore (21) is a pairing

$$Ш(A) \times Ш(A^t) \to \mathbb{Q}/\mathbb{Z},$$

and this is the "Albanese-Picard" pairing (12).

It follows from the above description that in order to establish the compatibility of (11) and (12) we only need to check the compatibility of the maps $\mathcal{A} \otimes^{\mathbf{L}} \mathcal{A}^t \to \mathbb{G}_m[1]$ (coming from the canonical Poincaré biextension of (A, A^t) by $\mathbb{G}_m$) and (20) (which is induced by the pairings (15) and (16)). Using the formulas $g_v^*(\mathcal{A} \otimes^{\mathbf{L}} \mathcal{A}^t) = A_{K_v} \otimes^{\mathbf{L}} A^t_{K_v}$ for each prime v of K, where $g_v \colon \operatorname{Spec} K_v \to U$ is the natural map, we are easily reduced to checking the compatibility of (20) and the map

$$A(\overline{\mathbb{A}}_K) \otimes^{\mathbf{L}} A^t(\overline{\mathbb{A}}_K) \to \mathbb{G}_m[1] \tag{22}$$

induced by the canonical Poicaré biextension. To do so, it will be sufficient to describe (22) with $\overline{\mathbb{A}}_K$ replaced by a (local) field $L \supset K$. (This description, which we give below, seems to be well-known. See [39, §5] and [18, §2].)

Let L be a field containing K. The canonical Poincaré biextension of (A, A^t) by $\mathbb{G}_m$ is the $\mathbb{G}_m$-torsor over $A \times A^t$: $W = \mathcal{P} \setminus \{\text{zero section}\}$, where $\mathcal{P}$ is the Poincaré line bundle on $A \times A^t$. Now $W(L)$ is a biextension of $(A(L), A^t(L))$ by L^* whose points can be represented as triples (a, b, c), where $(a, b) \in A(L) \times A^t(L)$ and $c \in L^*$. The two partial addition laws on $W(L)$ are given as follows. Let

$$\tilde{\varphi}_{A/L} \colon G(\tilde{E}^l_{A/L}) \to L^* \tag{23}$$

and

$$\tilde{\psi}_{A/L} \colon G(\tilde{E}^r_{A/L}) \to L^* \tag{24}$$

be the pairings defined by Zarhin in [39, proof of Prop. 2, p. 496]. Then

$$(a_1, b, c_1) +_1 (a_2, b, c_2) = (a_1 + a_2,\, b,\, c_1 c_2 \tilde{\varphi}_{A/L}[(a_1 + a_2) - (a_1) - (a_2) + (0), D]),$$

where $D \in \mathrm{Div}^0(A_L)$ represents b, and

$$\begin{aligned}&(a, b_1, c_1) +_2 (a, b_2, c_2)\\ &\qquad = (a, b_1 + b_2, c_1 c_2 \tilde{\psi}_{A/L}[(a) - (0), {}^t\Delta((b_1 + b_2) - (b_1) - (b_2) + (\hat{0}))]),\end{aligned}$$

where Δ is a Poincaré divisor on $A \times A^t$ such that $(a, b_1 + b_2) \notin \mathrm{Supp}\Delta$ (see [39, p.501] for further details). Then the map $A(L) \otimes^{\mathbf{L}} A^t(L) \to L^*[1]$ corresponding to $W(L)$ under the isomorphism

$$\mathrm{Biext}^1(A(L), A^t(L); L^*)) \simeq \mathrm{Hom}_{\mathcal{D}}(A(L) \otimes^{\mathbf{L}} A^t(L), L^*[1])$$

of [10, VII.3.6.5] is induced by (23) and (24). Finally, the compatibility of the pairings (11) and (12) follows from [27, Prop. 31, p. 1144] and the easily verified fact that the pairings (15)-(16) (for $V = A$) and (23)-(24) *are the same.*

Remark. It is likely that Poonen and Stoll were unaware of the fact that their "partially defined" pairings of [27, p.1116] had (essentially) being already defined by Zarhin in [39]. We seem to be in the presence, therefore, of yet another example of rediscovery in Mathematics.

References

[1] Artin, M. and A. Grothendieck, *Théorie des Topos et Cohomologie Etale des Schémas*, Séminaire de Géométrie Algébrique du Bois Marie 1963–64 (SGA 4). Lecture Notes in Math., vol. 269, 270, 305, Springer-Verlag, Berlin Heidelberg New York, 1972–73.

[2] Artin, M., Lipman's Proof of Resolution of Singularities for Surfaces, in: Arithmetic Geometry. Cornell, G. and Silverman, J., Eds., Springer-Verlag, Berlin, 1986, pp. 267–287.

[3] Bosch, S., Lütkebohmert, W. and M. Raynaud, *Néron Models*, Springer Verlag, Berlin, 1989.

[4] Cartan, H. and S. Eilenberg, *Homological Algebra*, Princeton Univ. Press, 1956.

[5] Gordon, W., Linking the conjectures of Artin-Tate and Birch-Swinnerton-Dyer, Comp. Math. **38** (1979), no. 2, 163–199.

[6] Grothendieck, A., Technique de descente et théorèmes d'existence en Géométrie Algébrique, V, VI. Les Schémas de Picard, Sém. Bourbaki, exp. no. 232, 236 (1961–62).

[7] Grothendieck, A. and J. Dieudonné, *Eléments de Géométrie Algébrique I*, Springer-Verlag, Heidelberg, 1971.

[8] Grothendieck, A. and J. Dieudonné, Eléments de Géométrie Algébrique I-IV, Publ. Math. IHES **4, 8, 11, 17, 20, 24, 28, 32** (1960–67).

[9] Grothendieck, A., Le Groupe de Brauer I-III, in: Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, pp. 46–188.

[10] Grothendieck, A., *Groupes de Monodromie en Géométrie Algébrique I*, Séminaire de Géométrie Algébrique du Bois Marie 1967–69 (SGA 7 I). Lecture Notes in Math., vol. 288, Springer, Heidelberg, 1972.

[11] Hartshorne, R. *Algebraic Geometry*, Springer-Verlag, Berlin, 1977.

[12] Hartshorne, R., *Residues and Duality*, Lecture Notes in Math., vol. 20, Springer-Verlag, Berlin, 1966.

[13] Lang, S. (Ed.), *Number Theory III: Diophantine Geometry*, Encyclopaedia of Mathematical Sciences, vol. 60, Springer Verlag, Berlin, 1991.

[14] Lang, S., *Abelian Varieties*, Interscience, New York, 1959.

[15] Lichtenbaum, S., The period-index problem for elliptic curves, Am. J. Math. **80** (1968), 1209–1223.

[16] Lichtenbaum, S., Duality theorems for curves over P-adic fields, Invent. Math. **7** (1969), 120–126.

[17] Lipman, J., Desingularization of two-dimensional schemes, Ann. Math. **107** (1978), 151–207.

[18] Mazur, B. and J. Tate, *Canonical height pairings via biextensions*, Arithmetic Geometry. Progress in Math., vol. 35, Birkäuser, Boston, 1983, pp. 195–237.

[19] Milne, J. S., Weil-Chatelet groups over local fields, Ann. Sci. Éc. Norm. Sup. **3** (1970), 273–284; Addendum, Ibid. **5** (1972), 261–264.

[20] Milne, J. S., *Étale Cohomology*, Princeton University Press, Princeton, N.J., 1980.

[21] Milne, J. S., Comparison of the Brauer Group with the Tate-Safarevic group, J. Fac. Science Univ. Tokyo, Sec. IA **28** (1981), 735–743.

[22] Milne, J. S., *Arithmetic Duality Theorems*, Persp. in Math., vol. 1, Academic Press Inc., Orlando, 1986.

[23] Milne, J. S., Abelian varieties, in: Arithmetic Geometry. Cornell, G. and Silverman, J., Eds., Springer-Verlag, Berlin, 1986, pp. 103–150.

[24] Milne, J. S., Jacobian varieties, in: Arithmetic Geometry. Cornell, G. and Silverman, J., Eds., Springer-Verlag, Berlin, 1986, pp. 167–212.

[25] Milne, J. S., *Lectures on Étale Cohomology*, Notes for a course taught at the University of Michigan. Available from www.jmilne.org/math/.

[26] Mumford, D., *Abelian Varieties*, Oxford Univ. Press, Oxford, 1974.

[27] Poonen, B. and M. Stoll, The Cassels-Tate pairing on polarized abelian varieties, Ann. Math. **150** (1999), no. 3, 1109–1149.

[28] Raynaud, M., Caractéristique d'Euler-Poincaré d'un faisceau et cohomologie des variétés abéliennes, in: Dix Exposés sur la Cohomologie des Schémas,

North-Holland, Amsterdam, 1968, pp. 12–30.

[29] Raynaud, M., Spécialisation du foncteur de Picard, Publ. Math. IHES **38** (1970), 27–76.

[30] Scheiderer, C., *Real and Étale Cohomology*, Lecture Notes in Math., vol. 1588, Springer-Verlag, Berlin, 1994.

[31] Shatz, S., *Profinite groups, Arithmetic, and Geometry*, Annals of Math. Studies, vol. 67, Princeton Univ. Press, Princeton, 1972.

[32] Silverman, J., *Advanced Topics in the Arithmetic of elliptic Curves*, Springer-Verlag, New York, 1994.

[33] Tamme, G., *Introduction to Etale Cohomology*, Springer-Verlag, Berlin, 1994.

[34] Tate, J., On the conjecture of Birch and Swinnerton-Dyer and a geometric analogue, in: Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, pp. 189–214.

[35] Tate, J., Global Class Field Theory, in: Algebraic Number Theory, Cassels, J.W.S. and Fröhlich, A. (Eds.), Academic Press, London, 1967, pp. 162–203.

[36] Weibel, C., *An introduction to homological algebra. Cambridge Studies in Advanced Math.*, vol. 38, Cambridge University Press, Cambridge, 1994.

[37] Weil, A., *Adeles and Algebraic groups. Progress in Math.*, vol. 23, Birkhäuser, Boston, 1982.

[38] Witt, E., Über ein Gegenbeispiel zum Normensatz, Math. Z. **39** (1935), 462–467.

[39] Zarhin, Yu., Néron pairing and quasicharacters, Math. USSR Izv. **6** (1972), no. 3, 491–503.

(Received May 1, 2001)
(Revised October 31, 2002)

Av. Chile-España 210
Apt. 404, Ñuñoa
Santiago, Chile
E-mail: gonzomat2002@yahoo.es