Rumor Transmission Models and Persistence Analysis
（流言伝播モデルとパーシステンス解析）

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Preface

As social conditions change, we communicate via language in order to grasp the change. However, such activity changes our mentality and behavior, and these micro-level changes can often trigger the alternation of social conditions. Hence we can consider it as a point of view which is useful for the stabilization of social conditions to investigate the dynamics of a remark spreading among people.

In this paper, "rumor" is defined as a remark which spreads among the general public in a short time through chain of word-of-mouth communication. We mathematically analyze the time change of the number of people who know a rumor and actively try to spread it, which gives a criterion on how bigly it gets around.

A basic mechanism of rumor transmission is that, when a person who does not know a rumor meets and talks with another person who knows and spreads it, the former person knows it through the story about it the latter person gives. On the other hand, a basic mechanism of the spread of infectious diseases is that, when a person who is not infected with a disease comes near another person who is infected with it, its pathogen moves from the latter person into the former one, who develops the disease. This analogy implies that we can establish the models for rumor transmission in the same way as those for infectious diseases.

By way of introduction, let us do a quick review of SIR model, which is most elementary in the models for infectious diseases. We classify the population into three subpopulations: susceptibles, infected (or infectious) people and recovered (or removed) people. We assume that people in each subpopulation behave in the same manner, a susceptible develops a disease by the contact with infected people, and infected people recover (or are quarantined) at a constant rate. Taking this assumption into consideration, we formulate SIR model as follows:

$$\begin{cases} \frac{d}{dt}S(t) = -\beta S(t)I(t) \\ \frac{d}{dt}I(t) = \beta S(t)I(t) - \gamma I(t) \\ \frac{d}{dt}R(t) = \gamma I(t) \end{cases},$$

where $S(t)$ is the population (or population density) of susceptibles at time t, $I(t)$ the population of infected people and $R(t)$ the population of recovered people. β represents the infection rate, γ the recovery rate and $\beta I(t)$ the force of infection. Although we classify the population according to their disease condition more minutely, we should notice that a R-state individual does not cause the state-change of other individuals he/she encounters because he/she does not spread the pathogen.

Then let us introduce the rumor-transmission models we are going to analyze in this paper. In Chapter 1, we classify the population according to their rumor state into three subpopulations: susceptibles (people who do not know a rumor), spreaders (people who know and spread it) and stiflers (people who know and prevent from spreading it). We assume three types of rumor-state change induced by contacts between individuals:

(i) A susceptible knows the rumor and becomes a spreader or a stifler with the contacts with spreaders.

(ii) A spreader gets tired of the rumor and becomes a stifler with the frequent contacts with spreaders.

(iii) When a spreader contacts a stifler, the spreader transmits the rumor at a constant frequency, and after hearing it, the stifler tries to remove it, because the stifler shows no interest in it or denies it. As a result, the spreader becomes a stifler.

(iii) can be translated in terms of SIR model as follows: a R-state individual is immune and gives the immunity to the spreaders he/she meets, which is unrealistic concerning infectious diseases. This is one of the differences between rumors and infectious diseases. Another important difference is that most rumors are modified in the communication process in a short time while, as far as infectious diseases are concerned, a pathogen does not often become mutated in a short time. With this in mind, we assume for some models in Chapter 1 that the modification of a rumor does not cause the state-change of spreaders because they always know the latest rumor through communication with each other, but cause the state-change of stiflers into susceptibles because they are not active enough to know the latest rumor, which is substantially far from the rumor they know.

In Chapter 1, we propose rumor-transmission models described with differential equation for the following three cases:

closed population Birth, death, emigration and immigration are ignored. This assumption is valid when we consider the temporary spread of a rumor.

constant emigration and immigration A certain number of newcomers are always immigrated as susceptibles, while some people are removed from the population at a constant rate on account of death or emigration.

age structure The frequency of contacts between individuals and the probability of state-change depend on age. Birth rate and death rate also depend on age and not on rumor-state.

For these differential equation systems we check their well-posedness and investigate their global dynamics, i.e., what the density (or the share) of each subpopulation is like after a long time.

As for the model of closed population, the system is substantially an autonomous nonlinear ordinary differential equation system on a 2–dim compact set. By using Dulac–Bendixson Criterion we can proof that the ω–set of any trajectory is an equilibrium. Hence we can virtually specify the global dynamics by examining the number of equilibria and their local stability. The model of constant emigration and immigration can be treated in the same way as that of closed population if we pay attention to the situation after long periods of time where the total population tends to a constant number.

There are two types of equilibria. One is rumor free equilibrium (RFE), where all individuals are susceptibles, and the other is rumor endemic equilibrium (REE), where susceptibles, spreaders and stiflers coexist. For ordinary equation systems we show that REE is globally asymptotically stable if and only if the number of newly added spreaders in a unit time when a very small number of spreaders invade RFE is larger than the extent of spreaders' decrease with emigration and immigration.

The age-structured rumor transmission model is described with partial differential equation system with boundary condition. This model has RFE, and the maximum number R_0 of newly added spreaders in a unit time when a very small number of spreaders invade RFE is obtained as a spectral radius of an operator on an infinite-dimensional function space. We show that if $R_0 < 1$ then the system has no REE and if $R_0 > 1$ then it has a REE. In addition, we show that REE bifurcates forward from RFE and is locally asymptotically stable if $R_0 > 1$ and $|R_0 - 1|$ is small enough.

The concept of persistence has attracted attention as an index of the global behavior of dynamical systems other than the stability of equilibria. It means whether a component of the population avoids extinction after long periods of time. In particular, the system is called uniformly strongly persistent about a component if there exists some $\varepsilon > 0$ such that, for any initial condition where the component exists, its amount is always over ε after long periods of time.

In Chapter 2, we quote a theorem which gives a sufficient condition for the system's uniform strong persistence and introduce two examples to which we can apply the theorem. Moreover, we apply the theorem to the age-structured rumor-transmission models in Chapter 1 and Chapter 4 and show that the system is uniformly strongly persistent about spreaders (and stiflers), which means that the number of spreaders (and stiflers) is always over a constant number after long periods of time if spreaders (and stiflers) exist at the beginning.

Well, we cannot ignore the effect of outside sources of information on rumor transmission although word-of-mouth has much impact on it. Mass media are examples of outside source of information, and here we identify with the mass media those who has much impact on other people through the mass media. And, while we implicitly assume in Chapter 1 that stiflers are not the first to talk about the rumor, they may be active to remove it, for example by spreading its rival rumor. We define such stiflers as "active stiflers" in distinction from the (passive) stiflers we consider in Chapter 1. In Chapter 3, we combine the following factors for a closed population and investigate the global behavior of the system:

- (passive) stiflers or active stiflers
- rumors are modified or not
- no mass media, rumor-spreading mass media or rumor-suppressing mass media

It is striking that in the model with active stiflers, variable rumor and mass media both forward bifurcation and backward bifurcation can occur according to parameters, which affects the set of goals for the suppress of the rumor.

In Chapter 4, we introduce an age-structured rumor-transmission model with active stiflers and investigate it in the same way as in Chapter 1 in order to find out more closely the possibility that the system shows different dynamics according to stiflers' behavior. In fact, unlike the age-structured model in Chapter 1, this model can have an equilibrium where only susceptibles and spreaders coexist, and an equilibrium where only susceptibles and active stiflers coexist. We can obtain the condition for their existence and their local stability.

Contents

Chapter 1

Deterministic models for rumor transmission

Abstract

In this chapter, we consider deterministic models for the transmission of a rumor. First, we investigate the age-independent case and introduce four models, which are classified according to whether the population is closed or not and whether the rumor is constant or variable. After formulating the models as finite-dimensional ODE systems, we show that the solutions converge to an equilibrium as $t \to \infty$. Next, we investigate a model for the transmission of a constant rumor in an age-structured population with age-dependent transmission coefficients. We formulate the model as an abstract Cauchy problem on an infinite-dimensional Banach space and show the existence and uniqueness of solutions. Then, under some appropriate assumptions, we examine the existence of its nontrivial equilibria and the stability of its trivial equilibrium. We show that the spectral radius $R_0 := r(\tilde{T})$ for some positive operator $\tilde{T}$ is the threshold. We also show sufficient conditions for the local stability of the nontrivial equilibria. Finally, we show that the model is uniformly strongly persistent if $R_0 > 1$.

Keywords: Rumor transmission; Threshold condition; Age-structured population; Rumor-free equilibrium; Rumor-endemic equilibrium; Global stability; Local stability; Uniform strong persistence

1.1 Introduction

In this chapter we apply models similar to those used in epidemiology to the "transmission of a rumor," which is the social phenomenon that a remark spreads on a large scale in a short time through chain of communication.

Rumor transmission is an example of social contagion processes. Pioneering contributions to their modeling, based on epidemiological models, date back to [2, 3, 4, 5]. In those days both deterministic models and stochastic models were used, and the former were so simple that they were solved analytically and regarded as the first approximation of the latter. Nearly a decade later, Daley and Kendall [6] explained the importance of dealing with stochastic rumor models rather than deterministic ones, henceforth stochastic models have been actively studied (see, for example, [7, 8, 9, 10, 11, 12, 13] and [14] for survey). The basic rumor transmission model which they used is called *Daley–Kendall model* after [6], and the simplified basic model is called *Maki–Thompson model* after [15]. We also refer the reader to [16, 17] for details.

Recently Pearce [18] and Gani [19] analyzed the probability generating functions in the stochastic rumor models by means of block-matrix methodology. In addition, Dickinson and Pearce [20] studied stochastic models for more general transient processes including epidemics.

Independently of this series of studies, deterministic models for rumor transmission have been studied sporadically. For example, Castillo-Chávez and Song [21] proposed the transmission models for a fanatic behavior based on the models for sexually transmitted diseases, and analyzed them qualitatively and numerically. Bettencourt et al [22] is another recent example, which deals with the spread of ideas.

Now, some rumors alternate propagation with cessation, momentarily modified in some cases. We can take "the rumor of Orléans" [23] for instance. We shall call such rumors *recursive rumors* for the meantime. We could attribute their occurrence to the distance in space and time. One mechanism might be as follows: The rumor spread locally in a region spills over out of the region where it has never been spread with movement of people and information. Another mechanism might be as follows: Since the power of fending off the rumor is weak, it survives in secret after its cessation. After a while, more and more people are unfamiliar with the rumor due to immigration from other areas, birth and the modification of the rumor. This might result in its repetition.

Noymer [24] proposed age-structured transmission models of "urban legends," which we could identify with rumor. In his models new people are constantly supplied through birth. Analyzing numerically, he found that the system seemed to converge to the steady state through damped oscillation but neither to behave periodically nor to show undamped oscillation. Note that he constructed his models based on the model for measles and hence ignored the law of mass action in the removal mechanism considered in the stochastic rumor models such as the Daley–Kendall model. Both age-structure and the law of mass action in the removal mechanism are considered to be important for rumor transmission.

In this chapter, we propose and mathematically analyze deterministic models for rumor transmission. In Section 1.2, we examine age-independent rumor transmission models, which are extensions of the deterministic Daley–Kendall model. In the last sections we introduce an age-structured rumor transmission model, We owe the argument there to [25]. We first establish the well-posedness of the time evolution problem. Next, introducing a positive operator $\tilde{T}$, we show that the system has at least one nontrivial equilibrium if and only if the spectral radius $r(\tilde{T})$ is larger than 1. We examine the asymptotical stability of the equilibria. We show that the model is uniformly strongly persistent if $R_0 > 1$. Finally we briefly discuss open problems and possible extensions of the basic model.

1.2 Age-independent models for the transmission of a constant rumor

Let $N(t)$ denote the total population at time t. We divide the population into three classes; the susceptible class, the spreader class and the stifler class, each of which we call *rumor-class*. Each population at time t is denoted by $X(t), Y(t), Z(t)$ respectively. Those who belong to the susceptible class, whom we call *susceptibles*, don't know about the rumor. Those who belong to the spreader class, whom we call *spreaders*, know about the rumor and spread it actively. Those who belong to the stifler class, whom we call *stiflers*, know about the rumor and don't spread it. By definition, we have

$$X(t) + Y(t) + Z(t) = N(t).$$

We assume that no transition of rumor-class happens unless a spreader contacts someone, since the two people who aren't spreaders don't talk about the rumor. That is, it is spreaders that are involved in the transition of rumor-class.

When a spreader contacts a susceptible, the spreader transmits the rumor at a constant frequency and the susceptible knows about it. Then the susceptible doesn't always become a spreader, but may doubt its credibility and consequently becomes a stifler. And so, we assume that $\alpha X(t)Y(t)\Delta t/N(t)$ susceptibles change their rumor-class and become spreaders at a constant rate $\theta \in (0,1]$ during the small interval $(t, t+\Delta t)$, where α is a positive constant number representing the product of the contact frequency and the probability of transmitting the rumor.

When two spreaders contact, both of them transmit the rumor at a constant frequency. Hearing it again and again, the spreader gets bored, gradually loses interest in it, and consequently becomes a stifler. And so, we assume that $\beta Y(t)^2\Delta t/N(t)$ spreaders become stiflers during the small interval $(t, t+\Delta t)$, where β is a positive constant number.

When a spreader contacts a stifler, the spreader transmits the rumor at a constant frequency, and after hearing it, the stifler tries to remove it, because the stifler shows no interest in it or denies it. As a result, the spreader becomes a stifler. And so, we assume that $\gamma Y(t)Z(t)\Delta t/N(t)$ spreaders become stiflers during the small interval $(t, t+\Delta t)$, where γ is a positive constant number.

For the meantime we assume that the rumor is "constant," that is, the same remark is transmitted at all times. Then stiflers don't change their rumor-class.

First, we consider the transmission of a constant rumor in a closed population, where people are neither born nor died. This assumption may be valid in the situation where the rumor spreads explosively in a short time and soon goes out. Then the dynamics of the population is governed by the following system:

$$\begin{cases} \dot{X}(t) = -\alpha X(t)\dfrac{Y(t)}{N(t)}, \\ \dot{Y}(t) = \alpha\theta X(t)\dfrac{Y(t)}{N(t)} - \beta Y(t)\dfrac{Y(t)}{N(t)} - \gamma Y(t)\dfrac{Z(t)}{N(t)}, \\ \dot{Z}(t) = \alpha(1-\theta)X(t)\dfrac{Y(t)}{N(t)} + \beta Y(t)\dfrac{Y(t)}{N(t)} + \gamma Y(t)\dfrac{Z(t)}{N(t)}, \end{cases} \tag{1.2.1}$$

where $\dot{}$ denotes the differentiation with respect to t. Now $N(t)$ is independent of time and can be denoted $N_0(>0)$.

We introduce new variables x, y, z by

$$x(t) := \frac{X(t)}{N(t)}, \quad y(t) := \frac{Y(t)}{N(t)}, \quad z(t) := \frac{Z(t)}{N(t)}.$$

Then we obtain the new system for x, y, z:

$$\begin{cases} \dot{x}(t) = -\alpha x(t)y(t), \\ \dot{y}(t) = \alpha\theta x(t)y(t) - \beta y(t)^2 - \gamma y(t)z(t), \\ \dot{z}(t) = \alpha(1-\theta)x(t)y(t) + \beta y(t)^2 + \gamma y(t)z(t). \end{cases}$$

Scaling time by setting $\tau := \alpha t$, we have

$$x' = -xy, \tag{1.2.2a}$$
$$y' = y(\theta x - by - cz), \tag{1.2.2b}$$
$$z' = y\{(1-\theta)x + by + cz\}, \tag{1.2.2c}$$

where $'$ denotes the differentiation with respect to τ and $b := \beta/\alpha,\ c := \gamma/\alpha$ are positive constants.

Let us consider the scaled system (1.2.2a) (1.2.2b) (1.2.2c). We rewrite τ as t. We define $\Omega \subset \mathbb{R}^3$ by

$$\Omega := \{(x, y, z) \in {\mathbb{R}_+}^3 \mid x + y + z = 1\}. \tag{1.2.3}$$

It is easy to show that the system (1.2.2a) (1.2.2b) (1.2.2c) has a unique solution on $(-\infty, \infty)$ in Ω. for any initial data in Ω. Note that $x(t) = 0$ for all $t \in \mathbb{R}$ if $x(0) = 0$ and that $y(t) = 0$ for all $t \in \mathbb{R}$ if $y(0) = 0$.

In the case $x(0) > 0$ and $y(0) > 0$, we have $x(t) > 0$ and $y(t) > 0$ for all $t \in \mathbb{R}$. Hence, for all $t \in \mathbb{R}$, $x'(t) < 0$ and so $x(t)$ is strictly decreasing. Since the set $\{x(t) \mid t \in \mathbb{R}\}$ is bounded, the limits

$$x(\infty) := \lim_{t\to\infty} x(t), \quad x(-\infty) := \lim_{t\to-\infty} x(t)$$

exist and satisfy $0 \leq x(\infty) < x(-\infty) \leq 1$. At the same time, we find $z'(t) > 0$ for all $t \in \mathbb{R}$ and a similar discussion yields that the limits $z(\infty), z(-\infty)$ exist and satisfy $0 \leq z(-\infty) < z(\infty) \leq 1$. Hence, as $t \to \pm\infty$, $y(t)$ converges. The limits $(x(\pm\infty), y(\pm\infty), z(\pm\infty))$ are the equilibria of the system in Ω, i.e., equal to $(t, 0, 1-t)$ $(0 \leq t \leq 1)$. Hence we have $y(\pm\infty) = 0$. In particular, the rumor goes out eventually.

Let

$$\begin{aligned} \Omega_1 &:= \{(x, y, z) \in \Omega \mid \theta x - by - cz < 0\}, \\ \Omega_2 &:= \{(x, y, z) \in \Omega \mid \theta x - by - cz > 0\}. \end{aligned}$$

In each domain $y(\theta x - by - cz)$ doesn't change its sign, so we find that the point $(x(t), y(t), z(t)) \in \Omega$ moves from Ω_2 into Ω_1 within a finite time, i.e., there exists some $T \in \mathbb{R}$ such that $(x(t), y(t), z(t)) \in \Omega_2$ whenever $t < T$ and $(x(t), y(t), z(t)) \in \Omega_1$ whenever $t > T$. Therefore, $y(t)$ takes its maximum at $t = T$.

Let

$$R_0 := \frac{c+\theta}{c}x(0) = \left(1 + \frac{\alpha\theta}{\gamma}\right)\frac{X(0)}{N_0}.$$

When $y(0) = Y(0)/N_0$ is sufficiently close to 0, $R_0 > 1$ means $x(0) > c/(c+\theta)$, so we see that $y(t)$ takes its maximum at $t = T > 0$, i.e., the rumor spreads and goes out just once. In contrast, $R_0 \leq 1$ means $x(0) \leq c/(c+\theta)$, so we see that $y(t)$ is strictly decreasing, i.e., the rumor doesn't spread. From this point of view, it can be safely said that R_0 is the threshold of this system.

Next, let us see the transmission of a constant rumor in a population with constant immigration and emigration. Let B be the sum of the population birth rate and the immigration rate, and μ the sum of the population death rate and the emigration rate. We assume that B, μ are positive constants, that the newcomers are all susceptibles, and that death and emigration are independent of rumor-class. These assumptions can be formulated as follows:

$$\begin{cases} \dot{X}(t) = B - \alpha X(t)\dfrac{Y(t)}{N(t)} - \mu X(t), \\ \dot{Y}(t) = \alpha\theta X(t)\dfrac{Y(t)}{N(t)} - \beta Y(t)\dfrac{Y(t)}{N(t)} - \gamma Y(t)\dfrac{Z(t)}{N(t)} - \mu Y(t), \\ \dot{Z}(t) = \alpha(1-\theta)X(t)\dfrac{Y(t)}{N(t)} + \beta Y(t)\dfrac{Y(t)}{N(t)} + \gamma Y(t)\dfrac{Z(t)}{N(t)} - \mu Z(t). \end{cases} \tag{1.2.4}$$

Adding these equations, we see that the total population $N(t)$ satisfies

$$\dot{N}(t) = B - \mu N(t).$$

$N(t)$ converges to $N_0 := B/\mu$ as $t \to \infty$.

First, let us consider the limit system

$$\begin{cases} \dot{X}(t) = \mu N_0 - \alpha X(t)\dfrac{Y(t)}{N_0} - \mu X(t), \\ \dot{Y}(t) = \alpha\theta X(t)\dfrac{Y(t)}{N_0} - \beta Y(t)\dfrac{Y(t)}{N_0} - \gamma Y(t)\dfrac{Z(t)}{N_0} - \mu Y(t). \end{cases}$$

Scaled in the same way as (1.2.1), this system takes the form

$$\begin{cases} x'(t) = d\{1 - x(t)\} - x(t)y(t), \\ y'(t) = y(t)\{\theta x(t) - by(t) - c(1 - x(t) - y(t)) - d\}, \end{cases} \tag{1.2.5}$$

where $d := \mu/\alpha$ is a positive constant and the scaled time τ is again called t for simplicity. It is easy to show that, for any initial data in Ω, there exists a unique solution x, y of the system (1.2.5) on $[0, \infty)$ in Ω.

Let

$$\begin{aligned} &f(x,y) := d(1-x) - xy, \quad g(x,y) := y\{\theta x - by - c(1 - x - y) - d\}, \\ &\mathbf{F}(x,y) := {}^t(f(x,y), g(x,y)) \end{aligned}$$

where ^{t} denotes the transpose of the vector. Let us explore the equilibria in Ω, i.e., the point $(x, y) \in \Omega$ satisfying $f(x,y) = g(x,y) = 0$.

From the latter equation $g(x,y) = 0$, we get $y = 0$ or $\theta x - by - c(1 - x - y) - d = 0$. If we substitute $y = 0$ into $f(x,y) = 0$, we obtain $x = 1$. The boundary equilibrium

$$\mathbf{x}^\circ = (x^\circ, y^\circ) := (1, 0)$$

is always in Ω regardless of the parameters b, c, d, θ. This equilibrium represents the situation that no one knows about the rumor. In this context, one could also call it *rumor-free equilibrium* (RFE). The Jacobian matrix at RFE is given by

$$D\mathbf{F}(\mathbf{x}^\circ) = \begin{pmatrix} -d & -1 \\ 0 & -d + \theta \end{pmatrix}$$

and its eigenvalues are $-d (< 0)$, $-d + \theta$.

${}^t(1, 0)$ is an eigenvector corresponding to $-d$. Now suppose $y(0) = 0$, then the solution is given by

$$x(t) = 1 - e^{-dt}(1 - x(0)), \quad y(t) = 0.$$

Hence the segment $I_0 := [0, 1] \times \{0\}$ included in x-axis is positively invariant, and for any initial data in I_0, $(x(t), y(t))$ converges to $\mathbf{x}^\circ$ as $t \to \infty$. That is, I_0 is the intersection of Ω and the stable manifold of $\mathbf{x}^\circ$.

Let $\Omega' := \Omega \setminus I_0$, then we find that that $0 < d < \theta$ implies that $\mathbf{x}^\circ$ is a saddle and any point sufficiently close to $\mathbf{x}^\circ$ in Ω' doesn't close to $\mathbf{x}^\circ$ eventually, so $\mathbf{x}^\circ$ is asymptotically unstable in Ω'. And we know that $d > \theta$ implies that $\mathbf{x}^\circ$ is a sink and so locally asymptotically stable in Ω.

Then, let us examine the existence of equilibria satisfying $y \neq 0$.

Suppose $(x, y) \in \Omega'$. Since $f(0, y) = d \neq 0$, any equilibrium $(x, y) \in \Omega'$, if it exists, satisfies $x \neq 0$. So, we can assume that $x > 0,\ y > 0,\ x + y \leq 1$.

In the case $d \geq 1$, taking into consideration the conditions $0 < x < 1 \leq d$ and $1 - x \geq y$, we have

$$\frac{(1-x)d}{x} > y \iff f(x,y) > 0,$$

from which it follows that no equilibrium exists in Ω'.

In the case $d < 1$, $f(x,y) = 0$ gives

$$y = \frac{(1-x)d}{x} \leq 1 - x,$$

which implies $d \le x < 1$, so we have $0 < y \le 1 - d$. On the other hand, $g(x, y) = 0$ and $y \ne 0$ lead to

$$x = \frac{(b - c)y + (c + d)}{\theta + c}$$

and by substituting it into $f(x, y) = 0$ we obtain an equation $h(y) = 0$ at most quadratic with respect to y, where

$$h(y) := (c - b)y^2 - (c + d - cd + \mathrm{d}b)y + d(\theta - d).$$

Note that

$$\begin{aligned} h(0) &= (\theta - d)d, \\ h(1 - d) &= -(1 - d)b - (1 - \theta)d < 0 \quad (\because\ b > 0). \end{aligned}$$

Suppose $\theta < d < 1$. If $c - b \ge 0$, the graph of $v = h(u)$ is concave, so $h(0) < 0$ and $h(1 - d) < 0$ yield that $h(y) < 0$ whenever $0 < y \le 1 - d$. Otherwise, i.e., if $c - b < 0$, since

$$c + d - cd + bd = c(1 - d) + b + bd > 0,$$

we find

$$h'(y) = 2(c - b)y - (c + d - cd + bd) < 0 \quad \text{whenever } 0 < y \le 1 - d.$$

Hence $h(y)$ is strictly decreasing and it follows that $h(y) < h(0) < 0$. In both cases, under the condition $\theta < d < 1$, $h(y) < 0$ holds whenever $0 < y \le 1 - d$ and we see that $h(y) = 0$ has no solution in the range.

Suppose $0 < d < \theta$, then we have

$$\begin{aligned} h(0) &= (\theta - d)d > 0, \\ h\left(\frac{\theta - d}{\theta + b}\right) &= -\frac{\{(1 - \theta)d + (1 - d)b\}(\theta + c)(\theta - d)}{(\theta + b)^2} < 0, \\ h(1 - d) &< 0. \end{aligned}$$

Hence, noting

$$(1 - d) - \frac{\theta - d}{\theta + b} = \frac{(1 - d)b + (1 - \theta)d}{\theta + b} > 0,$$

we find that the equation $h(y) = 0$ has the only solution y^* in $(0, 1 - d]$, which satisfies $0 < y^* < \frac{\theta - d}{\theta + b} (< 1 - d)$. Set

$$x^* := \frac{(b - c)y^* + c + d}{\theta + c},$$

then the inequalities

$$x^* = \frac{by^* + d + c(1 - y^*)}{\theta + c} > 0, \quad 1 - x^* - y^* = \frac{\theta - d - (\theta + b)y^*}{\theta + c} > 0$$

hold and it follows that $(x^*, y^*) \in \Omega'$, which is the only interior equilibrium. which we could call *rumor-endemic equilibrium* (REE).

Next, let us discuss the local stability of REE. Assuming $0 < d < \theta$, we examine the eigenvalues of the Jacobian matrix $M := D\mathbf{F}(\mathbf{x}^*)$ at $\mathbf{x}^*$, which is given by

$$M = \begin{pmatrix} -d - y^* & -x^* \\ (\theta + c)y^* & -d - c + (\theta + c)x^* + (-2b + 2c)y^* \end{pmatrix}.$$

Then a little calculation gives rise to

$$\det M = -y^* h'(y^*) > 0$$

and

$$(\theta - d)\operatorname{tr} M = y^* K, \quad \text{where } K := (c - b)y^* - \{c(1 - \theta) + (1 + b)\theta\}.$$

Now, let us examine the sign of K. If $c - b \le 0$ then it is clear that $K < 0$. Otherwise, we see that

$$y^* = \frac{d + c + bd - cd - \sqrt{D_0}}{2(c - b)},$$

where

$$D_0 := (d + c + bd - cd)^2 - 4(c - b)(\theta - d)d$$

$$= (-d + c + bd - cd)^2 + 4d\{c(1-\theta) + b\theta\}.$$

Since

$$2K = -d + c + bd - cd - \sqrt{D_0} - 2c(1-\theta) - 2(\theta - d) - 2b\theta,$$

we have $K < 0$ clearly if $-d + c + bd - cd \le 0$, and this holds even if $-d + c + bd - cd > 0$, because $\sqrt{D_0} > -d + c + bd - cd$, which follows from the form of D_0.

Hence we obtain $\operatorname{tr} M < 0$. From it and $\det M > 0$ we can conclude that the real part of any eigenvalue of M is negative and so REE is locally asymptotically stable.

Let us apply Dulac–Bendixson Criterion (cf. [29]) to exclude the possibility of a periodic orbit or a cyclic chain of equilibria, i.e., a piecewise smooth closed curve consisting of finitely many equilibria and of orbits connecting them. It is convenient to write the system in term of y and z as follows:

$$y' = y\{\theta(1 - y - z) - by - cz - d\},$$
$$z' = y\{(1-\theta)(1 - y - z) + by + cz\} - dz.$$

We define a Dulac function $\rho(y, z) := (yz)^{-1}$ on the domain $\{(y, z) \in \mathbb{R}_+{}^2 \mid y > 0, z > 0, y + z < 1\}$. Then we have

$$\frac{\partial}{\partial y}(\rho(y,z)y\{\theta(1-y-z) - by - cz - d\}) + \frac{\partial}{\partial z}(\rho(y,z)(y\{(1-\theta)(1-y-z) + by + cz\} - dz))$$
$$= -\left\{\frac{\theta + b}{z} + \frac{(1-\theta)(1-y) + by}{z^2}\right\},$$

and this is strictly negative on the domain. Hence we can use Dulac–Bendixson Criterion.

It follows from the above results and Poincaré–Bendixson trichotomy (cf. [29]) that, if $0 < d < \theta$, $\mathbf{x}^*$ is globally asymptotically stable in Ω'.

Therefore, we have the following results:

Theorem 1.2.1. *Concerning the system (1.2.5),*

(i) if $d > \theta$, then RFE is the only equilibrium and globally asymptotically stable in Ω, and

(ii) if $0 < d < \theta$, then the system has the only equilibrium $\mathbf{x}^* = (x^*, y^*)$ in Ω', which is globally asymptotically stable in Ω'.

By applying the theory of asymptotically autonomous differential equations (cf. [26, 27] and [28, Appendix F]), we find that the solutions of the system (1.2.4) show the same type of large-time behavior as the limit system, unless $\mu = \alpha\theta$. Hence $Y(t)$ converges to 0 or a positive number as $t \to \infty$ and this system has no undamped oscillation. And we also find that $R_0 := \theta/d = \alpha\theta/\mu$ is the threshold of the system (1.2.4). $\tau_Y := \mu^{-1}$ is the mean sojourn time spent in the spreader class with no rumor-class transition. Hence, $R_0 := (\alpha\theta)\tau_Y$ gives the average number of susceptibles that a spreader can let into spreader class during the time τ_Y, provided that the whole population is susceptible. In this context, one could also call it *basic reproduction number*.

1.3 Age-independent models for the transmission of a variable rumor

In this section, we consider the effect of some kind of modification of a rumor on its transmission.

So far we didn't take into consideration the possibility of the transition from the stifler class into the susceptible class, but in reality it is thought to exist. It is partly because the rumor gradually slips out of the memory of stiflers and so they become regarded as susceptibles when they hear the same rumor again.

However, what is thought to be more important for the transition is the effect of the modification of a rumor. Some rumors are modified in the communication process, which we call *variable*. Even if someone knows such a variable rumor at one time, substantially he doesn't know the modified one after a long period of time.

In general, the transition is considered to be dependent on how long he/she has been in the stifler class, his/her age and the speed at which the rumor is modified. For simplicity, these factors are ignored in the models below, and we assume that $\eta Z(t)\Delta t$ stiflers become susceptibles during the small interval $(t, t + \Delta t)$, where η is a positive constant number. This means that the transition rate from the stifler class into the susceptible class is independent of stifler's age and exponentially distributed with respect to the duration. For the above assumption we referred to those used in the influenza model [30, 31, 32].

First, we consider the transmission of a variable rumor in a closed population. Our model takes the following form:

$$\begin{cases} \dot{X}(t) = -\alpha X(t)\dfrac{Y(t)}{N(t)} + \eta Z(t), \\ \dot{Y}(t) = \alpha\theta X(t)\dfrac{Y(t)}{N(t)} - \beta Y(t)\dfrac{Y(t)}{N(t)} - \gamma Y(t)\dfrac{Z(t)}{N(t)}, \\ \dot{Z}(t) = \alpha(1-\theta)X(t)\dfrac{Y(t)}{N(t)} + \beta Y(t)\dfrac{Y(t)}{N(t)} + \gamma Y(t)\dfrac{Z(t)}{N(t)} - \eta Z(t). \end{cases} \tag{1.3.1}$$

Noting the total population $N(t)$ is a constant number N_0, after scaling time, we obtain the equations for the new scaled dependent variables $x(t) := X(t)/N(t)$, $y(t) := Y(t)/N(t)$, $z(t) := Z(t)/N(t)$:

$$\begin{cases} x' = -xy + k(1-x-y), \\ y' = y\{\theta x - by - c(1-x-y)\}, \end{cases} \tag{1.3.2}$$

where $k := \eta/\alpha$ is a positive constant number and $'$ denotes the derivative with respect to dimensionless time.

The equilibria and the dynamics of the system (1.3.2) are as follows:

Theorem 1.3.1. *The system (1.3.2) has RFE $\mathbf{x}^\circ = (1,0)$, which is asymptotically unstable in Ω'. And it has the unique REE $\mathbf{x}^* = (x^*, y^*)$ in Ω', which is asymptotically globally stable in Ω'.*

Next, we consider the transmission of a variable rumor in a population with constant immigration and emigration. Then we obtain the following system:

$$\begin{cases} \dot{X}(t) = B - \mu X(t) - \alpha X(t)\dfrac{Y(t)}{N(t)} + \eta Z(t), \\ \dot{Y}(t) = -\mu Y(t) + \alpha\theta X(t)\dfrac{Y(t)}{N(t)} - \beta Y(t)\dfrac{Y(t)}{N(t)} - \gamma Y(t)\dfrac{Z(t)}{N(t)}, \\ \dot{Z}(t) = -\mu Z(t) + \alpha(1-\theta)X(t)\dfrac{Y(t)}{N(t)} + \beta Y(t)\dfrac{Y(t)}{N(t)} + \gamma Y(t)\dfrac{Z(t)}{N(t)} - \eta Z(t). \end{cases} \tag{1.3.3}$$

As in the case of the system (1.2.4), first let us discuss the two-dimensional limiting scaled system

$$\begin{cases} x' = d(1-x) - xy + k(1-x-y), \\ y' = y\{\theta x - by - c(1-x-y) - d\}. \end{cases} \tag{1.3.4}$$

The equilibria and the dynamics of the system (1.3.4) are as follows:

Theorem 1.3.2. *Concerning the system (1.3.4),*

(i) if $d > \theta$, then RFE $\mathbf{x}^\circ = (1,0)$ is the only equilibrium and globally asymptotically stable in Ω, and

(ii) if $0 < d < \theta$, then the system has the unique REE $\mathbf{x}^* = (x^*, y^*)$ in Ω', which is globally asymptotically stable in Ω'. RFE is asymptotically unstable in Ω'

We omit the proof of Theorems 1.3.1 and 1.3.2, since it is the same as that of Theorem 1.2.1. See [1] for details.

1.4 Age-structured model for rumor transmission

Henceforth, we consider the transmission of a constant rumor in a closed age-structured population under the demographic growth. Let $a \in [0, \omega]$, where the number $\omega (< \infty)$ denotes the life span of the population, and $X(t,a), Y(t,a), Z(t,a)$ be the age-density functions at time t of the susceptible class, the spreader class, and the stifler class respectively. Let $P(t,a) := X(t,a) + Y(t,a) + Z(t,a)$ be the age-density of the total number of individuals, then the total size of the population is given by $N(t) := \int_0^\omega P(t,a)\,\mathrm{d}a$.

The basic system can be formulated as follows:

$$\begin{cases} (\partial_t + \partial_a)X(t,a) = -(\mu(a) + \lambda_1(t,a))X(t,a), \\ (\partial_t + \partial_a)Y(t,a) = \lambda_1(t,a)\theta(a)X(t,a) - (\mu(a) + \lambda_2(t,a))Y(t,a), \\ (\partial_t + \partial_a)Z(t,a) = \lambda_1(t,a)(1-\theta(a))X(t,a) + \lambda_2(t,a)Y(t,a) - \mu(a)Z(t,a), \\ \qquad X(t,0) = \displaystyle\int_0^\omega m(a)P(t,a)\,\mathrm{d}a, \\ \qquad Y(t,0) = 0, \quad Z(t,0) = 0, \\ \qquad X(0,a) = X_0(a), \quad Y(0,a) = Y_0(a), \quad Z(0,a) = Z_0(a). \end{cases} \tag{1.4.1}$$

$\mu(a), m(a)$ stand for the age-specific natural death rate and fertility rate respectively. $(X_0(a), Y_0(a), Z_0(a))$ is a given initial data. $\lambda_1(t,a)$ is the force of transition into the spreader class on a susceptible individual aged a at time t and defined by

$$\lambda_1(t,a) := \frac{1}{N(t)} \int_0^\omega \alpha(a,\sigma) Y(t,\sigma) \, \mathrm{d}\sigma,$$

where $\alpha(a,\sigma)$ is the transmission rate between a susceptible individual aged a and a spreader aged σ. $\theta(a)$ stands for the probability the susceptible individual aged a who knows about the rumor becomes a spreader. $\lambda_2(t,a)$ is the force of transition into the stifler class on a spreader aged a at time t and defined by

$$\lambda_2(t,a) := \frac{1}{N(t)} \int_0^\omega \{\beta(a,\sigma) Y(t,\sigma) + \gamma(a,\sigma) Z(t,\sigma)\} \, \mathrm{d}\sigma,$$

where $\beta(a,\sigma)$ is the transmission rate between a spreader aged a and another one aged σ, while $\gamma(a,\sigma)$ is the transmission rate between a spreader aged a and a stifler aged σ.

It follows from (1.4.1) that $P(t,a)$ satisfies the McKendrick equation

$$\begin{cases} (\partial_t + \partial_a) P(t,a) = -\mu(a) P(t,a), \\ P(t,0) = \displaystyle\int_0^\omega m(a) P(t,a) \, \mathrm{d}a, \\ P(0,a) = P_0(a) := X_0(a) + Y_0(a) + Z_0(a). \end{cases} \tag{1.4.2}$$

Note that we implicitly assume that there is no true interaction between demography and the spread of the rumor. Hence, it is convenient to introduce the fractional age distribution for each rumor-class as follows:

$$x(t,a) := \frac{X(t,a)}{P(t,a)}, \quad y(t,a) := \frac{Y(t,a)}{P(t,a)}, \quad z(t,a) := \frac{Z(t,a)}{P(t,a)}.$$

Then the new system for the fractional age distributions is given as follows:

$$\begin{cases} (\partial_t + \partial_a) x(t,a) = -\lambda_1(t,a) x(t,a), \\ (\partial_t + \partial_a) y(t,a) = \lambda_1(t,a) \theta(a) x(t,a) - \lambda_2(t,a) y(t,a), \\ (\partial_t + \partial_a) z(t,a) = \lambda_1(t,a)(1-\theta(a)) x(t,a) + \lambda_2(t,a) y(t,a), \\ \qquad x(t,0) = 1, \quad y(t,0) = 0, \quad z(t,0) = 0, \\ \qquad \lambda_1(t,a) = \displaystyle\int_0^\omega \alpha(a,\sigma) \psi(t,\sigma) y(t,\sigma) \, \mathrm{d}\sigma, \\ \qquad \lambda_2(t,a) = \displaystyle\int_0^\omega \psi(t,\sigma) \{\beta(a,\sigma) y(t,\sigma) + \gamma(a,\sigma) z(t,\sigma)\} \, \mathrm{d}\sigma, \end{cases} \tag{1.4.3}$$

where $\psi(t,a)$ is defined by

$$\psi(t,a) := \frac{P(t,a)}{\int_0^\omega P(t,a) \, \mathrm{d}a}.$$

According to the stable population theory (see, for example, [33, 34]), as $t \to \infty$, ψ converges to the persistent normalized age distribution uniformly with respect to a:

$$\lim_{t\to\infty} \psi(t,a) = c(a) := \frac{\mathrm{e}^{-\lambda_0 a} \mathcal{F}(a)}{\int_0^\omega \mathrm{e}^{-\lambda_0 a} \mathcal{F}(a) \, \mathrm{d}a},$$

where λ_0 denotes the intrinsic rate of natural increase, $\mathcal{F}(a)$ is the survival rate defined by

$$\mathcal{F}(a) := \exp\left(- \int_0^a \mu(\sigma) \, \mathrm{d}\sigma \right),$$

and $c(a)$ is called relatively stable age distribution. Note that $\int_0^\omega c(a) \, \mathrm{d}a = 1$.

In the following we assume that the stable age distribution is already attained. Then system (1.4.3) is rewritten as the autonomous system below:

$$\begin{cases} (\partial_t + \partial_a)x(t,a) = -\lambda_1(t,a)x(t,a), \\ (\partial_t + \partial_a)y(t,a) = \lambda_1(t,a)\theta(a)x(t,a) - \lambda_2(t,a)y(t,a), \\ (\partial_t + \partial_a)z(t,a) = \lambda_1(t,a)(1-\theta(a))x(t,a) + \lambda_2(t,a)y(t,a), \\ \qquad x(t,0) = 1, \quad y(t,0) = 0, \quad z(t,0) = 0, \\ \lambda_1(t,a) = \displaystyle\int_0^\omega \alpha(a,\sigma)c(\sigma)y(t,\sigma)\,\mathrm{d}\sigma, \\ \lambda_2(t,a) = \displaystyle\int_0^\omega c(\sigma)\{\beta(a,\sigma)y(t,\sigma) + \gamma(a,\sigma)z(t,\sigma)\}\,\mathrm{d}\sigma. \end{cases} \tag{1.4.4}$$

We mainly consider the system (1.4.4) under the condition

$$x(t,a) \geq 0, \; y(t,a) \geq 0, \; z(t,a) \geq 0, \; x(t,a) + y(t,a) + z(t,a) = 1. \tag{1.4.5}$$

Under this condition, we can formally exclude the susceptible class from the basic system. That is, instead of the basic system (1.4.4), we can consider the following system with linear homogeneous boundary conditions, which is more convenient to consider the well-definedness of the time evolution problem:

$$\begin{cases} (\partial_t + \partial_a)y(t,a) = \lambda_1(t,a)\theta(a)\{1 - y(t,a) - z(t,a)\} - \lambda_2(t,a)y(t,a), \\ (\partial_t + \partial_a)z(t,a) = \lambda_1(t,a)(1-\theta(a))\{1 - y(t,a) - z(t,a)\} + \lambda_2(t,a)y(t,a), \\ \qquad y(t,0) = 0, \quad z(t,0) = 0, \\ \lambda_1(t,a) = \displaystyle\int_0^\omega \alpha(a,\sigma)c(\sigma)y(t,\sigma)\,\mathrm{d}\sigma, \\ \lambda_2(t,a) = \displaystyle\int_0^\omega c(\sigma)\{\beta(a,\sigma)y(t,\sigma) + \gamma(a,\sigma)z(t,\sigma)\}\,\mathrm{d}\sigma. \end{cases} \tag{1.4.6}$$

The state space of this system is

$$\Omega = \{(y,z) \in (L^1_+(0,\omega))^2 \mid y + z \leq 1\},$$

where

$$L^1_+(0,\omega) := \{f \in L^1(0,\omega) \mid f(a) \geq 0 \text{ a.e.}\}$$

is a positive cone of $L^1(0,\omega)$, and $E_+ := (L^1_+(0,\omega))^2$ is a positive cone of $E := (L^1(0,\omega))^2$.

Let us define an unbounded linear operator A on E as follows:

$$(A\phi)(a) := \begin{pmatrix} -\mathrm{d}/\mathrm{d}a & 0 \\ 0 & -\mathrm{d}/\mathrm{d}a \end{pmatrix} \begin{pmatrix} \phi_1(a) \\ \phi_2(a) \end{pmatrix}, \tag{1.4.7}$$

where $\phi = {}^t(\phi_1, \phi_2)$ and the domain of A is defined by

$$\mathcal{D}(A) := \{\phi = {}^t(\phi_1,\phi_2) \in E \mid \phi_1, \phi_2 \text{ are absolutely continuous on } [0,\omega], \; \phi_1(0) = \phi_2(0) = 0\}. \tag{1.4.8}$$

Let F be a nonlinear operator on E defined by

$$F(\phi)(a) := \begin{pmatrix} \lambda_1[a \,|\, \phi_1]\theta(a)\{1 - \phi_1(a) - \phi_2(a)\} - \lambda_2[a \,|\, \phi_1,\phi_2]\phi_1(a) \\ \lambda_1[a \,|\, \phi_1]\{1-\theta(a)\}\{1 - \phi_1(a) - \phi_2(a)\} + \lambda_2[a \,|\, \phi_1,\phi_2]\phi_1(a) \end{pmatrix},$$

where $\lambda_1[a \,|\, \phi_1], \lambda_2[a \,|\, \phi_1,\phi_2]$ are defined by

$$\lambda_1[a \,|\, \phi] := \int_0^\omega \alpha(a,\sigma)c(\sigma)\phi(\sigma)\,\mathrm{d}\sigma,$$

$$\lambda_2[a \,|\, \phi_1,\phi_2] := \int_0^\omega c(\sigma)\{\beta(a,\sigma)\phi_1(\sigma) + \gamma(a,\sigma)\phi_2(\sigma)\}\,\mathrm{d}\sigma.$$

Let us define an E-valued function $u = {}^t(y,z)$. Then system (1.4.6) can be formulated as a semilinear Cauchy problem on the Banach space E:

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t) + F(u(t)), \quad u(0) = u_0. \tag{1.4.9}$$

Lemma 1.4.1. *The operator A generates a C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ and the state space Ω is positively invariant with respect to the semiflow defined by $\{e^{tA}\}_{t\geq 0}$.*

Proof. Let $\{T(t)\}_{t\geq 0}$ be a nilpotent translation C_0-semigroup on $L^1(0,\omega)$ induced by

$$(T(t)\phi)(a) := \begin{cases} 0, & t-a>0, \\ \phi(a-t), & a-t>0. \end{cases} \tag{1.4.10}$$

Furthermore, let $\{\tilde{T}(t)\}_{t\geq 0}$ be a C_0-semigroup on E defined by

$$\tilde{T}(t)\begin{pmatrix} \phi_1(a) \\ \phi_2(a) \end{pmatrix} := \begin{pmatrix} (T(t)\phi_1)(a) \\ (T(t)\phi_2)(a) \end{pmatrix}.$$

Then, it is clear that A is the generator of the C_0-semigroup $\{\tilde{T}(t)\}_{t\geq 0}$. Since $\{\tilde{T}(t)\}_{t\geq 0}$ is a translation semigroup and $(\tilde{T}(t)\phi)(a) = {}^t(0,0)$ whenever $t>a$, obviously Ω is positively invariant with respect to the semiflow induced by $\{e^{tA}\}_{t\geq 0}$. $\square$

Let $\|\cdot\|$ be the usual norm on $L^1(0,\omega)$ and $\|\cdot\|_E$ be a norm on E defined by

$$\|\phi\|_E := \max\{\|\phi_1\|, \|\phi_2\|\}, \quad \phi = {}^t(\phi_1,\phi_2) \in E.$$

Let $L^\infty_+(D)$ ($D \subset \mathbb{R}^N$) be a positive cone of $L^\infty(D)$ defined by

$$L^\infty_+(D) := \{f \in L^\infty(D) \mid f(a) \geq 0 \text{ a.e.}\}$$

Assumption 1.4.2. *$\alpha, \beta, \gamma \in L^\infty_+((0,\omega)\times(0,\omega))$, and $\theta, c \in L^\infty_+(0,\omega)$.*

Under this assumption, set

$$\alpha^\infty := \text{ess sup}\,\alpha,\ \beta^\infty := \text{ess sup}\,\beta,\ \gamma^\infty := \text{ess sup}\,\gamma,\ c^\infty := \text{ess sup}\,c.$$

Lemma 1.4.3. *Under Assumption 1.4.2, the map $F|_\Omega : \Omega \to E$ is Lipschitz continuous and there exists a number $k>0$ such that $(I+kF)(\Omega) \subset \Omega$.*

Proof. First, let us prove the first part. Let $\phi = {}^t(\phi_1,\phi_2) \in \Omega$, $\psi = {}^t(\psi_1,\psi_2) \in \Omega$ and $F(\phi) = {}^t(F_1(\phi), F_2(\phi))$. Applying the triangle inequality, we get the evaluation as follows:

$$\begin{aligned}
&\|F_1(\phi) - F_1(\psi)\| \\
&= \int_0^\omega \Big|\lambda_1[a|\phi_1]\theta(a)(1-\phi_1(a)-\phi_2(a)) - \lambda_2[a|\phi_1,\phi_2]\phi_1(a) \\
&\qquad - \lambda_1[a|\psi_1]\theta(a)(1-\psi_1(a)-\psi_2(a)) + \lambda_2[a|\psi_1,\psi_2]\psi_1(a)\Big|\,\mathrm{d}a \\
&\leq \int_0^\omega \Big\{\lambda_1[a|\phi_1]\theta(a)\Big|(1-\phi_1(a)-\phi_2(a)) - (1-\psi_1(a)-\psi_2(a))\Big| \\
&\qquad + \theta(a)(1-\psi_1(a)-\psi_2(a))\Big|\lambda_1[a|\phi_1] - \lambda_1[a|\psi_1]\Big| \\
&\qquad + \lambda_2[a|\phi_1,\phi_2]|\phi_1(a)-\psi_1(a)| + \psi_1(a)\Big|\lambda_2[a|\phi_1,\phi_2] - \lambda_2[a|\psi_1,\psi_2]\Big|\Big\}\,\mathrm{d}a.
\end{aligned}$$

Concerning the last line of the above evaluation, from the inequalities

$$\phi_1(a), \phi_2(a), \psi_1(a), \psi_2(a) \geq 0,\ \phi_1(a)+\phi_2(a) \leq 1,\ \psi_1(a)+\psi_2(a) \leq 1$$

we see that

$$\begin{aligned}
&\lambda_1[a|\phi_1] \leq \int_0^\omega \alpha^\infty c(\sigma)\,\mathrm{d}\sigma = \alpha^\infty, \qquad (1.4.11) \\
&\Big|\lambda_1[a|\phi_1] - \lambda_1[a|\psi_1]\Big| \leq \int_0^\omega \alpha(a,\sigma)c(\sigma)|\phi_1(\sigma)-\psi_1(\sigma)|\,\mathrm{d}\sigma \leq \alpha^\infty c^\infty \|\phi_1-\psi_1\|, \\
&\lambda_2[a|\phi_1,\phi_2] \leq \int_0^\omega c(\sigma)(\beta^\infty+\gamma^\infty)\,\mathrm{d}\sigma \leq \beta^\infty+\gamma^\infty, \\
&\Big|\lambda_2[a|\phi_1,\phi_2] - \lambda_2[a|\psi_1,\psi_2]\Big|
\end{aligned}$$

$$\leq \int_0^\omega c(\sigma)\beta(a,\sigma)|\phi_1(\sigma) - \psi_1(\sigma)|\,\mathrm{d}\sigma + \int_0^\omega c(\sigma)\gamma(a,\sigma)|\phi_2(\sigma) - \psi_2(\sigma)|\,\mathrm{d}\sigma$$
$$\leq c^\infty\beta^\infty\|\phi_1 - \psi_1\| + c^\infty\gamma^\infty\|\phi_2 - \psi_2\|.$$

Hence we have

$$\begin{aligned}
&\|F_1(\phi) - F_1(\psi)\| \\
&\leq \alpha^\infty(\|\phi_1 - \psi_1\| + \|\phi_2 - \psi_2\|) + \alpha^\infty c^\infty\omega\|\phi_1 - \psi_1\| \\
&\quad + (\beta^\infty + \gamma^\infty)\|\phi_1 - \psi_1\| + c^\infty\beta^\infty\omega\|\phi_1 - \psi_1\| + c^\infty\gamma^\infty\omega\|\phi_2 - \psi_2\| \\
&\leq (2\alpha^\infty + \alpha^\infty c^\infty\omega + \beta^\infty + \gamma^\infty + c^\infty\beta^\infty\omega + c^\infty\gamma^\infty\omega)\|\phi - \psi\|_E.
\end{aligned}$$

The same evaluation gives rise to

$$\|F_2(\phi) - F_2(\psi)\| \leq (2\alpha^\infty + \alpha^\infty c^\infty\omega + \beta^\infty + \gamma^\infty + c^\infty\beta^\infty\omega + c^\infty\gamma^\infty\omega)\|\phi - \psi\|_E.$$

Therefore, we obtain

$$\begin{aligned}
&\|F(\phi) - F(\psi)\|_E \leq K\|\phi - \psi\|_E, \\
&\quad \text{where } K := 2\alpha^\infty + \alpha^\infty c^\infty\omega + \beta^\infty + \gamma^\infty + c^\infty\beta^\infty\omega + c^\infty\gamma^\infty\omega,
\end{aligned}$$

which implies that $F|_\Omega$ is Lipschitz continuous.

To prove the second part, let $\phi = {}^t(\phi_1, \phi_2) \in \Omega$, then we have

$$\phi_1(a) + kF_1(\phi)(a) + \phi_2(a) + kF_2(\phi)(a) = k\lambda_1[a|\phi_1](1 - \phi_1(a) - \phi_2(a)) + \phi_1(a) + \phi_2(a),$$

which is less than or equal to 1 whenever $0 < k < (\alpha^\infty)^{-1}$ because of (1.4.11). It is obvious that $\phi_2(a) + kF_2(\phi)(a) \geq 0$. Since

$$\phi_1(a) + kF_1(\phi)(a) = (1 - k\lambda_2[a|\phi_1, \phi_2])\phi_1(a) + k\lambda_1[a|\phi_1]\theta(a)(1 - \phi_1(a) - \phi_2(a))$$

and $0 \leq \lambda_2[a|\phi_1, \phi_2] \leq \beta^\infty + \gamma^\infty$, $\phi_1(a) + kF_1(\phi)(a) \geq 0$ holds whenever $k < (\beta^\infty + \gamma^\infty)^{-1}$.

Therefore, if we choose $k > 0$ so small that

$$0 < k < \min\{(\alpha^\infty)^{-1},\ (\beta^\infty + \gamma^\infty)^{-1}\}, \tag{1.4.12}$$

then we find that $(I + kF)(\Omega) \subset \Omega$. □

According to [35], the Cauchy problem (1.4.9) can be rewritten as follows:

$$\frac{d}{dt}u(t) = \left(A - \frac{1}{k}\right)u(t) + \frac{1}{k}(\mathrm{Id} + kF)u(t), \quad u(0) = u_0 \in \Omega,$$

where k is chosen such as $0 < k < 1$ and (1.4.12) are satisfied. Its mild solution is given as the solution of the integral equation (the variation-of-constants formula, see [36, Chapter 6]):

$$u(t) = \mathrm{e}^{-k^{-1}t}\mathrm{e}^{tA}u_0 + k^{-1}\int_0^t \mathrm{e}^{-k^{-1}(t-s)}\mathrm{e}^{(t-s)A}\{u(s) + kF(u(s))\}\,\mathrm{d}s.$$

Let $\{S(t)u_0\}_{t\geq 0}$ be the semiflow induced by this mild solution. On the other hand, let $\{u^n\}_{n\in\mathbb{N}}$ be a sequence of E defined by $u^0(t) := u_0$ and

$$u^{n+1}(t) := \mathrm{e}^{-k^{-1}t}\mathrm{e}^{tA}u_0 + k^{-1}\int_0^t \mathrm{e}^{-k^{-1}(t-s)}\mathrm{e}^{(t-s)A}\{u^n(s) + kF(u^n(s))\}\,\mathrm{d}s.$$

If $u^n \in \Omega$, then $\mathrm{e}^{tA}u_0, \mathrm{e}^{(t-s)A}\{u^n(s) + kF(u^n(s))\} \in \Omega$, which implies $u^{n+1} \in \Omega$ since Ω is convex. It follows from the Lipschitz continuity of F that u^n uniformly converges to the mild solution $S(t)u_0$. Hence, u^n converges to $S(t)u_0$ in E and we have $S(t)u_0 \in \Omega$ since Ω is closed. Thus we obtain the following theorem:

Theorem 1.4.4. *The Cauchy problem (1.4.9) has a unique mild solution $S(t)u_0$, and Ω is positively invariant with respect to the semiflow $\{S(t)u_0\}_{t\geq 0}$. If $u_0 \in \mathcal{D}(A)$, then $S(t)u_0$ gives a classical solution.*

1.5 Existence of rumor-endemic equilibria

In this section, we consider the condition for the existence of rumor-endemic equilibria of the system (1.4.4). We denote the density vector at the rumor-endemic equilibrium by ${}^t(x^*, y^*, z^*)$, and the forces of rumor-class transition ${\lambda_1}^*, {\lambda_2}^*$. They must satisfy the following system:

$$\frac{\mathrm{d}}{\mathrm{d}a}x^*(a) = -{\lambda_1}^*(a)x^*(a), \tag{1.5.1a}$$
$$\frac{\mathrm{d}}{\mathrm{d}a}y^*(a) = {\lambda_1}^*(a)\theta(a)x^*(a) - {\lambda_2}^*(a)y^*(a), \tag{1.5.1b}$$
$$\frac{\mathrm{d}}{\mathrm{d}a}z^*(a) = {\lambda_1}^*(a)(1-\theta(a))x^*(a) + {\lambda_2}^*(a)y^*(a), \tag{1.5.1c}$$
$$x^*(0) = 1, \quad y^*(0) = 0, \quad z^*(0) = 0, \tag{1.5.1d}$$
$${\lambda_1}^*(a) = \int_0^\omega \alpha(a,\sigma)c(\sigma)y^*(\sigma)\,\mathrm{d}\sigma, \tag{1.5.1e}$$
$${\lambda_2}^*(a) = \int_0^\omega c(\sigma)\{\beta(a,\sigma)y^*(\sigma) + \gamma(a,\sigma)z^*(\sigma)\}\,\mathrm{d}\sigma. \tag{1.5.1f}$$

By formal integration, we obtain the following expressions:

$$x^*(a) = \mathrm{e}^{-\int_0^a {\lambda_1}^*(\sigma)\,\mathrm{d}\sigma},$$
$$y^*(a) = \int_0^a \mathrm{e}^{-\int_\sigma^a {\lambda_2}^*(\tau)\,\mathrm{d}\tau}{\lambda_1}^*(\sigma)\theta(\sigma)\mathrm{e}^{-\int_0^\sigma {\lambda_1}^*(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma,$$
$$z^*(a) = \int_0^a \left\{{\lambda_1}^*(b)(1-\theta(b))\mathrm{e}^{-\int_0^b {\lambda_1}^*(\sigma)\,\mathrm{d}\sigma} + {\lambda_2}^*(b)\int_0^b \mathrm{e}^{-\int_\sigma^b {\lambda_2}^*(\tau)\,\mathrm{d}\tau}{\lambda_1}^*(\sigma)\theta(\sigma)\mathrm{e}^{-\int_0^\sigma {\lambda_1}^*(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\right\}\mathrm{d}b.$$

Substituting them into (1.5.1e) and (1.5.1f) gives the following nonlinear integral equations:

$${\lambda_1}^*(a) = \int_0^\omega \alpha(a,\sigma)c(\sigma)\left\{\int_0^\sigma e^{-\int_b^\sigma {\lambda_2}^*(\tau)\,\mathrm{d}\tau}{\lambda_1}^*(b)\theta(b)e^{-\int_0^b {\lambda_1}^*(\tau)\,\mathrm{d}\tau}\,\mathrm{d}b\right\}\mathrm{d}\sigma, \tag{1.5.2a}$$
$$\begin{aligned}{\lambda_2}^*(a) = &\int_0^\omega \beta(a,\sigma)c(\sigma)\left\{\int_0^\sigma e^{-\int_b^\sigma {\lambda_2}^*(\tau)\,\mathrm{d}\tau}{\lambda_1}^*(b)\theta(b)e^{-\int_0^b {\lambda_1}^*(\tau)\,\mathrm{d}\tau}\,\mathrm{d}b\right\}\mathrm{d}\sigma\\ &+\int_0^\omega \gamma(a,\sigma')c(\sigma')\Bigg(\int_0^{\sigma'}\Big\{{\lambda_1}^*(b)(1-\theta(b))e^{-\int_0^b {\lambda_1}^*(\sigma)\,\mathrm{d}\sigma}\\ &+{\lambda_2}^*(b)\int_0^b e^{-\int_\sigma^b {\lambda_2}^*(\tau)\,\mathrm{d}\tau}{\lambda_1}^*(\sigma)\theta(\sigma)e^{-\int_0^\sigma {\lambda_1}^*(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\Big\}\,\mathrm{d}b\Bigg)\,\mathrm{d}\sigma'.\end{aligned} \tag{1.5.2b}$$

Let Φ be the nonlinear operator on E defined by

$$\Phi_1(u)(a) := \int_0^\omega \alpha(a,\sigma)c(\sigma)\left\{\int_0^\sigma \mathrm{e}^{-\int_b^\sigma u_2(\tau)\,\mathrm{d}\tau}u_1(b)\theta(b)\mathrm{e}^{-\int_0^b u_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}b\right\}\mathrm{d}\sigma, \tag{1.5.3a}$$
$$\begin{aligned}\Phi_2(u)(a) := &\int_0^\omega \beta(a,\sigma)c(\sigma)\left\{\int_0^\sigma \mathrm{e}^{-\int_b^\sigma u_2(\tau)\,\mathrm{d}\tau}u_1(b)\theta(b)\mathrm{e}^{-\int_0^b u_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}b\right\}\mathrm{d}\sigma\\ &+\int_0^\omega \gamma(a,\sigma')c(\sigma')\Bigg(\int_0^{\sigma'}\Big\{u_1(b)(1-\theta(b))\mathrm{e}^{-\int_0^b u_1(\sigma)\,\mathrm{d}\sigma}\\ &+u_2(b)\int_0^b \mathrm{e}^{-\int_\sigma^b u_2(\tau)\,\mathrm{d}\tau}u_1(\sigma)\theta(\sigma)\mathrm{e}^{-\int_0^\sigma u_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\Big\}\,\mathrm{d}b\Bigg)\,\mathrm{d}\sigma',\end{aligned} \tag{1.5.3b}$$
$$\Phi(u) = {}^t(\Phi_1(u),\, \Phi_2(u)), \quad u = {}^t(u_1,\, u_2) \in E. \tag{1.5.3c}$$

We find that Φ is a positive operator on E and $\Phi(0) = 0$. Let $T : E \to E$ be the Fréchet derivative of Φ at 0, then T is given as follows:

$$Tu = {}^t(T_1u,\, T_2u), \quad u \in E,$$
$$(T_1u)(a) := \int_0^\omega \phi_1(a,b)u_1(b)\,\mathrm{d}b, \tag{1.5.4a}$$
$$(T_2u)(a) := \int_0^\omega \phi_2(a,b)u_1(b)\,\mathrm{d}b, \tag{1.5.4b}$$

$$\phi_1(a,b) := \theta(b)\int_b^\omega \alpha(a,\sigma)c(\sigma)\,\mathrm{d}\sigma, \tag{1.5.4c}$$

$$\phi_2(a,b) := \theta(b)\int_b^\omega \beta(a,\sigma)c(\sigma)\,\mathrm{d}\sigma + (1-\theta(b))\int_b^\omega \gamma(a,\sigma)c(\sigma)\,\mathrm{d}\sigma \tag{1.5.4d}$$

Now, let us define a linear operator $\tilde{T}$ on $L^1(0,\omega)$ by

$$(\tilde{T}u)(a) := T_1\,{}^t(u,0)(a) = \int_0^\omega \phi_1(a,b)u(b)\,\mathrm{d}b, \quad u \in L^1(0,\omega). \tag{1.5.5}$$

If $v = {}^t(v_1, v_2) \in E$ is an eigenvector of T corresponding to $\lambda \neq 0$, then v_1 is an eigenvector of $\tilde{T}$ corresponding to λ since

$$(\tilde{T}v_1)(a) = (T_1 v)(a) = \lambda v_1(a).$$

Meanwhile, if $v_1 \in L^1(0,\omega)$ is an eigenvector of $\tilde{T}$ corresponding to $\lambda \neq 0$, then $v = {}^t(v_1,v_2) \in E$ is an eigenvector of T corresponding to λ, where v_2 is expressed in terms of v_1 as follows:

$$v_2(a) := \lambda^{-1}\int_0^\omega \phi_2(a,b)v_1(b)\,\mathrm{d}b$$

In particular, $v = {}^t(v_1,v_2) \in E$ is a positive eigenvector of T corresponding to $\lambda \neq 0$ if and only if $v_1 \in L^1(0,\omega)$ is a positive eigenvector of $\tilde{T}$ corresponding to $\lambda \neq 0$. In addition, T doesn't have any eigenvectors corresponding to 1 in E_+ if and only if $\tilde{T}$ doesn't have any eigenvectors corresponding to 1 in $L^1_+(0,\omega)$.

In the following we make some assumptions deriving some important properties of $\tilde{T}$ and Φ:

Assumption 1.5.1. *(i) $\theta(a) > 0$ and $c(a) > 0$ for almost all $a \in (0,\omega)$.*

(ii) There exist a number $b_0 \in (0,\omega)$ and a number $\varepsilon_0 > 0$ such that $\alpha(a,\sigma) \geq \varepsilon_0$ for almost all $(a,\sigma) \in (0,\omega) \times (\omega - b_0, \omega)$.

(iii) $\alpha(a,\sigma), \beta(a,\sigma), \gamma(a,\sigma), \theta(a)$ are extended as 0 when a or σ is in $\mathbb{R} \setminus [0,\omega]$, then the following holds uniformly with respect to σ:

$$\lim_{h\to 0}\int_0^\omega |\alpha(a+h,\sigma) - \alpha(a,\sigma)|\,\mathrm{d}a = 0,$$
$$\lim_{h\to 0}\int_0^\omega |\beta(a+h,\sigma) - \beta(a,\sigma)|\,\mathrm{d}a = 0,$$
$$\lim_{h\to 0}\int_0^\omega |\gamma(a+h,\sigma) - \gamma(a,\sigma)|\,\mathrm{d}a = 0,$$
$$\lim_{h\to 0}\int_0^\omega |\theta(a+h) - \theta(a)|\,\mathrm{d}a = 0.$$

Now, let us summarize some ideas from positive operator theory. For more detail, the reader may refer to [37]. As for basic results on Banach lattices and positive operators, see, for example, [38].

Let X be an ordered vector space and X_+ its positive cone. For $u, v \in X$, $u \leq v$ if and only if $v - u \in X_+$. X_+ is called *total* if $X_+ - X_+$ is dense in X. Let Y be an ordered vector space and Y_+ its positive cone. The dual of X is denoted by X^*. An operator $T : X \to Y$ is called *positive* if $TX_+ \subset Y_+$. A subset of X^* consisting of all positive linear functionals on X is called the *dual cone*, which we denote by X^*_+.

We write $F(u)$ as $\langle F, u\rangle$ for $u \in X$, $F \in X^*$. A positive linear functional $f \in X^*_+$ is called *strictly positive* if $\langle f, u\rangle > 0$ for all $u \in X_+ \setminus \{0\}$. $u \in X_+$ is called *nonsupporting point* if $\langle f, u\rangle > 0$ for all $f \in X^*_+ \setminus \{0\}$. Let T be a positive linear bounded operator on X. T is called *nonsupporting* if for all $u \in X_+ \setminus \{0\}$ and for all $f \in X^*_+ \setminus \{0\}$ there exists a positive integer p depending on u, f such that $\langle f,\ T^n u\rangle > 0$ whenever $n \geq p$. T is called *semi-nonsupporting* if for all $u \in X_+ \setminus \{0\}$ and for all $f \in X^*_+ \setminus \{0\}$ there exists a positive integer p depending on u, f such that $\langle f,\ T^p u\rangle > 0$. It is obvious that, if T is nonsupporting, it is semi-nonsupporting.

Given a linear operator T on E, $r(T)$ stands for its spectral radius, $s(T)$ the spectral bound of T, $\rho(T)$ the resolvent set of T, $\sigma(T)$ the spectrum of T and $P_\sigma(T)$ the point spectrum of T. T^* stands for the dual operator of T.

Theorem 1.5.2 (Klein-Rutman, [39]). *Let X be a Banach space and X_+ its positive cone, which is total. Let T be a compact positive linear operator on X satisfying $r(T) > 0$.*

Then $r(T)$ is an eigenvalue of T and the corresponding eigenvector $\psi \in X_+ \setminus \{0\}$ exists.

Theorem 1.5.3 ([37]). *Let X be a Banach space and X_+ its positive cone, which is total. Let T be a semi-nonsupporting positive linear operator on X. We assume that $r(T)$ is a pole of the resolvent $R(\lambda, T) = (\lambda - T)^{-1}$. Then the following holds:*

(i) $r(T) \in P_\sigma(T) \setminus \{0\}$ and $r(T)$ is a simple pole of the resolvent.

(ii) The eigenspace corresponding to $r(T)$ is one-dimensional subspace of X spanned by a quasi-interior point $\psi \in X_+$. If $\phi \in X_+$, $c \in \mathbb{R}$ satisfy $T\phi = c\phi$, then $c = r(T)$ and there exists a real number $k > 0$ such that $\phi = k\psi$.

(iii) $r(T) \in P_\sigma(T^*)$ and the eigenspace of T^* corresponding to $r(T)$ is a one-dimensional subspace of X^* spanned by a strictly positive functional f.

By applying the method of the Krasnoselskii's fixed point theorem [40, Theorem 4.11], we have the following (see [25, Proposition 4.6]):

Theorem 1.5.4. *Let Ψ be a positive nonlinear operator on a real Banach space X with a positive cone X_+. We assume that $\Psi(0) = 0$ and Ψ has the strong Fréchet derivative $T := \Psi'(0)$, which has a positive eigenvector $v_0 \in X_+$ corresponding to the eigenvalue $\lambda_0 > 1$ and no eigenvector corresponding to the eigenvalue 1 in X_+. In addition, we assume that Ψ is completely continuous and $\Psi(X_+)$ is bounded.*

Then, Ψ has a non-zero positive fixed point.

Let X be an ordered vector space. X is called a *vector lattice* if $x \vee y := \sup\{x, y\}, x \wedge y := \inf\{x, y\}$ exist for all $x, y \in X$. The norm $\| \ \|$ on a vector lattice X is called a *lattice norm* if $|x| \le |y|$ implies $\|x\| \le \|y\|$, where $|x| := x \vee (-x)$. A *Banach lattice* is a Banach space X endowed with an ordering $\le$ such that $(X, \le)$ is a vector lattice and the norm on X is a lattice norm.

The following comparison theorem is due to [41].

Theorem 1.5.5. *Let X be a Banach lattice. For positive linear bounded operators S, T on X, the following holds:*

(i) $S \le T$ implies $r(S) \le r(T)$.

(ii) Moreover, if S, T are semi-nonsupporting and compact, then $S \le T$, $S \ne T$, $r(T) \ne 0$ imply $r(S) < r(T)$.

After the above preparations, we firstly consider the properties of $\tilde{T}$ defined by (1.5.5).

Lemma 1.5.6. *$\tilde{T}$ is nonsupporting.*

Proof. Since $\phi_1(a, b) \ge 0$ holds for all $a, b \in (0, \omega)$, if $u \ge 0$ then $\tilde{T}u \ge 0$, that is, $\tilde{T}$ is positive. Let

$$s(\xi) := \begin{cases} \varepsilon_0, & \text{if } \xi \in (\omega - b_0, \omega), \\ 0, & \text{otherwise}, \end{cases} \tag{1.5.6}$$

then $\alpha(a, \sigma) \ge s(\sigma)$ for all $a, \sigma \in \mathbb{R}$. Let f_0 be a linear functional on $L^1(0, \omega)$ defined by

$$\langle f_0, u \rangle := \int_0^\omega \theta(b) \left(\int_b^\omega s(\sigma) c(\sigma) \, d\sigma \right) u(b) \, db. \tag{1.5.7}$$

The assumption $s, c, \theta \in L^\infty_+(0, \omega)$ implies that $f_0 \in (L^1(0, \omega))^*_+$. Moreover, since

$$\int_b^\omega s(\sigma) c(\sigma) \, d\sigma > 0, \text{ for all } b \in (0, \omega)$$

holds, we find that f_0 is strictly positive.

Let $u \in L^1_+(0, \omega)$. $\alpha(a, \sigma) \ge s(\sigma)$ yields that $\tilde{T}u \ge \langle f_0, u \rangle e$, where $e(a) \equiv 1, \ a \in \mathbb{R}$. Hence

$$\tilde{T}^{n+1} u \ge \langle f_0, u \rangle \langle f_0, e \rangle^n e, \text{ for all } n \in \mathbb{N},$$

and for all $F \in (L^1(0, \omega))^*_+ \setminus \{0\}$ and for all $n \in \mathbb{N}$ the following holds:

$$\langle F, \tilde{T}^n u \rangle \ge \langle f_0, u \rangle \langle f_0, e \rangle^{n-1} \langle F, e \rangle > 0.$$

Therefore, $\tilde{T}$ is nonsupporting. □

Lemma 1.5.7. *$\tilde{T}$ is compact.*

Proof. First observe that

$$\begin{aligned}\int_0^\omega |\phi_1(a+h,b)-\phi_1(a,b)|\,\mathrm{d}a &= \int_0^\omega \left|\theta(b)\int_b^\omega (\alpha(a+h,\sigma)-\alpha(a,\sigma))c(\sigma)\,\mathrm{d}\sigma\right| \mathrm{d}a \\ &\le \int_0^\omega \left(\int_b^\omega |\alpha(a+h,\sigma)-\alpha(a,\sigma)|c(\sigma)\,\mathrm{d}\sigma\right) \mathrm{d}a \\ &\le \int_0^\omega \left\{\int_0^\omega |\alpha(a+h,\sigma)-\alpha(a,\sigma)|\,\mathrm{d}a\right\} c(\sigma)\,\mathrm{d}\sigma.\end{aligned}$$

Assumption 1.5.1 implies that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_0^\omega |\alpha(a+h,\sigma)-\alpha(a,\sigma)|\,\mathrm{d}a < \varepsilon, \quad \text{for all } \sigma \in \mathbb{R}$$

holds whenever $|h| < \delta$. This gives

$$\int_0^\omega |\phi_1(a+h,b)-\phi_1(a,b)|\,\mathrm{d}a < \varepsilon \int_0^\omega c(\sigma)\,\mathrm{d}\sigma = \varepsilon.$$

Now, let $\{u_\lambda \mid \lambda \in \Lambda\}$ be a bounded subset of $L^1(0,\omega)$. Note that $\|u_\lambda\| < C_0$ for some positive constant C_0. Then we have

$$\begin{aligned}\int_0^\omega |(\tilde{T}u_\lambda)(a+h)-(\tilde{T}u_\lambda)(a)|\,\mathrm{d}a &\le \int_0^\omega \left(\int_0^\omega |\phi_1(a+h,b)-\phi_1(a,b)|\,\mathrm{d}a\right)|u_\lambda(b)|\,\mathrm{d}b \\ &\le \varepsilon\|u_\lambda\| < C_0\varepsilon.\end{aligned}$$

From the above evaluation and the well-known compactness criteria in L^1 (see, for example, [42, p.275]), we see that $\{\tilde{T}u_\lambda \mid \lambda \in \Lambda\}$ is relatively compact in $L^1(0,\omega)$, hence $\tilde{T}$ is compact. □ □

According to Lemmas 1.5.6, 1.5.7 and Theorems 1.5.2, 1.5.3, the spectral radius $r(\tilde{T})$ of the operator $\tilde{T}$ is the only positive eigenvalue with a positive eigenvector $u_0 \in L^1_+(0,\omega)$ which is a nonsupporting point. Moreover, $r(\tilde{T})$ is an eigenvalue of $\tilde{T}^*$ with a strictly positive eigenfunctional F_0.

Secondly, let us consider the properties of Φ defined by (1.5.3c).

Lemma 1.5.8. *Φ is completely continuous, and there exists a constant $M_0 > 0$ such that $\|\Phi(u)\| \le M_0$ whenever $u \in E_+$.*

Proof. Let $\{u_\lambda = {}^t(u^1_\lambda, u^2_\lambda) \in E \,|\, \lambda \in \Lambda\}$ be a bounded subset of E. By definion $\|u_\lambda\| < C_0$ holds for some constant $C_0 > 0$. Then we have

$$\begin{aligned}&\int_0^\omega |\Phi_1(u_\lambda)(a+h)-\Phi_1(u_\lambda)(a)|\,\mathrm{d}a \\ &\le \int_0^\omega \mathrm{d}a \int_0^\omega |\alpha(a+h,\sigma)-\alpha(a,\sigma)|c(\sigma)\left\{\int_0^\sigma \mathrm{e}^{-\int_b^\sigma u^2_\lambda(\tau)\,\mathrm{d}\tau}|u^1_\lambda(b)|\theta(b)\mathrm{e}^{-\int_0^b u^1_\lambda(\tau)\,\mathrm{d}\tau}\,\mathrm{d}b\right\}\mathrm{d}\sigma.\end{aligned}$$

Now, if $0 \le b \le \sigma \le \omega$ then it follows from

$$\left|-\int_0^b u^1_\lambda(\tau)\,\mathrm{d}\tau\right| \le \int_0^b |u^1_\lambda(\tau)|\,\mathrm{d}\tau \le \|u^1_\lambda\| \le C_0$$

that $\mathrm{e}^{-\int_0^b u^1_\lambda(\tau)\,\mathrm{d}\tau} \le \mathrm{e}^{C_0}$. Similarly we have $\mathrm{e}^{-\int_b^\sigma u^2_\lambda(\tau)\,\mathrm{d}\tau} \le \mathrm{e}^{C_0}$. Hence it follows that

$$\begin{aligned}&\int_0^\omega |\Phi_1(u_\lambda)(a+h)-\Phi_1(u_\lambda)(a)|\,\mathrm{d}a \\ &\le \mathrm{e}^{2C_0}\int_0^\omega \mathrm{d}a \int_0^\omega |\alpha(a+h,\sigma)-\alpha(a,\sigma)|c(\sigma)\left\{\int_0^\sigma |u^1_\lambda(b)|\theta(b)\,\mathrm{d}b\right\}\mathrm{d}\sigma \\ &= \mathrm{e}^{2C_0}\int_0^\omega \mathrm{d}a \int_0^\omega \mathrm{d}b\left\{\int_b^\omega |\alpha(a+h,\sigma)-\alpha(a,\sigma)|c(\sigma)\,\mathrm{d}\sigma\right\}\theta(b)|u^1_\lambda(b)|\,\mathrm{d}b.\end{aligned}$$

Assumption 1.5.1 implies that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_0^\omega |\alpha(a+h,\sigma)-\alpha(a,\sigma)|\,da < \varepsilon \text{ for all } \sigma \in \mathbb{R} \text{ whenever } |h| < \delta.$$

With this, a similar estimation as in the proof of Lemma 1.5.7 gives:

$$\int_0^\omega |\Phi_1(u_\lambda)(a+h) - \Phi_1(u_\lambda)(a)|\,\mathrm{d}a < \mathrm{e}^{2C_0}C_0\varepsilon.$$

Hence, $A_1 := \{\Phi_1(u_\lambda) \mid \lambda \in \Lambda\}$ is relatively compact in $L^1(0,\omega)$. Similarly we find that $A_2 := \{\Phi_2(u_\lambda) \mid \lambda \in \Lambda\}$ is also relatively compact in $L^1(0,\omega)$.

Let $A := \{\Phi(u_\lambda) \mid \lambda \in \Lambda\}$, then $A \subset A_1 \times A_2$ implies $\bar{A} \subset \overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2}$, where $\bar{A}$ stands for the closure of A. Since $\overline{A_1}, \overline{A_2}$ are compact, $\overline{A_1} \times \overline{A_2}$ is compact, so $\bar{A}$, which is a closed subset of $\overline{A_1} \times \overline{A_2}$, is also compact. The above proves that Φ is completely continuous.

Next, for any $u = {}^t(u_1, u_2) \in E_+$, it follows that

$$\begin{aligned}
\|\Phi_1(u)\| &= \int_0^\omega \mathrm{d}a \left\{ \int_0^\omega \alpha(a,\sigma)c(\sigma) \left(\int_0^\sigma \mathrm{e}^{-\int_b^\sigma u_2(\tau)\,\mathrm{d}\tau} u_1(b)\theta(b)\mathrm{e}^{-\int_0^b u_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}b \right) \mathrm{d}\sigma \right\} \\
&\le \int_0^\omega \mathrm{d}a \left\{ \int_0^\omega \alpha(a,\sigma)c(\sigma) \left(\int_0^\sigma u_1(b)\mathrm{e}^{-\int_0^b u_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}b \right) \mathrm{d}\sigma \right\} \\
&= \int_0^\omega \mathrm{d}a \left\{ \int_0^\omega \alpha(a,\sigma)c(\sigma) \left[-\mathrm{e}^{-\int_0^b u_1(\tau)\,\mathrm{d}\tau} \right]_{b=0}^{b=\sigma} \mathrm{d}\sigma \right\} \\
&= \int_0^\omega \mathrm{d}a \left\{ \int_0^\omega \alpha(a,\sigma)c(\sigma) \left(1 - \mathrm{e}^{-\int_0^\sigma u_1(\tau)\,\mathrm{d}\tau} \right) \mathrm{d}\sigma \right\} \\
&\le \alpha^\infty \int_0^\omega \mathrm{d}a \int_0^\omega c(\sigma)\,\mathrm{d}\sigma = \alpha^\infty \omega.
\end{aligned}$$

Similarly we find that

$$\|\Phi_2(u)\| \le (\beta^\infty + 2\gamma^\infty)\omega.$$

Let $M_0 := (\alpha^\infty + \beta^\infty + 2\gamma^\infty)\omega$, then we see that $\|\Phi(u)\| \le M_0$. □

After the above preparations, we can prove the following threshold results:

Theorem 1.5.9. *(i) If $r(\tilde{T}) \le 1$, then $u = 0$ is the only solution of $u = \Phi(u)$ in E_+, i.e., RFE is the only equilibrium of the system.*

(ii) If $r(\tilde{T}) > 1$, $u = \Phi(u)$ has at least one solution in $E_+ \setminus \{0\}$, i.e., the system has at least one REE.

Proof. Suppose $r(\tilde{T}) \le 1$. We assume that $u = \Phi(u)$ for some $u = {}^t(u_1, u_2) \in E_+ \setminus \{0\}$. Then we have $u_1 = \Phi_1(u)$. Note that $u_1 \in L^1_+(0,\omega) \setminus \{0\}$, because $u_1 = 0$ implies $u = \Phi(u) = 0$, which contradicts the assumption $u \in E_+ \setminus \{0\}$.

Since $0 < \sigma < b$ implies $\mathrm{e}^{-\int_b^\sigma u_2(\tau)\,\mathrm{d}\tau}\mathrm{e}^{-\int_0^b u_1(\tau)\,\mathrm{d}\tau} < 1$, it follows that

$$\tilde{T}u_1 - u_1 = \tilde{T}u_1 - \Phi_1(u) \in L^1_+(0,\omega) \setminus \{0\}.$$

hence $\langle F_0, \tilde{T}u_1 - u_1 \rangle > 0$ because F_0 is strictly positive. On the other hand, observe

$$\begin{aligned}
\langle F_0, \tilde{T}u_1 - u_1 \rangle &= \langle \tilde{T}^* F_0, u_1 \rangle - \langle F_0, u_1 \rangle \\
&= (r(T) - 1)\langle F_0, u_1 \rangle.
\end{aligned}$$

Since $r(T) - 1 \le 0$ and $\langle F_0, u_1 \rangle > 0$, we have $\langle F_0, \tilde{T}u_1 - u_1 \rangle \le 0$, which is a contradiction. Therefore $u = 0$ is the only solution of $u = \Phi(u)$ in E_+.

Next suppose $r(\tilde{T}) > 1$. Then we see that T has a positive eigenvector corresponding to $r(\tilde{T})$ and no eigenvector in E_+ corresponding to the eigenvalue 1. In addition, Φ has the properties stated in Lemma 1.5.8. Therefore, Theorem 1.5.4 implies that Φ has a non-zero positive fixed point. □

From the above result, we can regard $r(\tilde{T})$ as the basic reproduction number of this system, which is denoted by R_0 in the following.

1.6 Stability of rumor-free equilibrium

In this section, we consider the stability of RFE.

The first element $y(t)$ of $u(t)$ in (1.4.9) satisfies the abstract equation on $L^1(0,\omega)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = By(t) + Py(t) \cdot (1 - y(t) - z(t)) - \lambda_2[a \,|\, y, z]y(t),$$

$$y(0) = y_0 \in L^1(0,\omega),$$

where $z(t)$ is regarded as given and we define as follows:

$$\mathcal{D}(B) := \{u \in L^1(0,\omega) \mid u \text{ is absolutely continuous on } [0,\omega],\ u(0) = 0\}, \tag{1.6.1}$$

$$Bu(a) := -\frac{\mathrm{d}}{\mathrm{d}a}u(a),\ u \in \mathcal{D}(B), \tag{1.6.2}$$

$$Pu(a) := \theta(a)\int_0^\omega \alpha(a,\sigma)c(\sigma)u(\sigma)\,\mathrm{d}\sigma.$$

Let $\{T(t)\}_{t\geq 0}$ be the nilpotent translation semigroup on $L^1(0,\omega)$ defined by (1.4.10), which is generated by B. Let $C(t)$ be a bounded operator on $L^1(0,\omega)$ defined by

$$C(t)u := (Pu)\cdot(1 - y(t) - z(t)) - \lambda_2[a \mid y, z]y(t).$$

For any $u \in L^1_+(0,\omega)$ we have $C(t)u \leq Pu$, because

$$x(t) = 1 - y(t) - z(t) \leq 1, \quad Pu(a) \geq 0, \quad \lambda_2[a \mid y, z]y(t) \geq 0.$$

Since $T(t),\ t \geq 0$ is a positive operator on $L^1(0,\omega)$, we have

$$T(s)C(t)u \leq T(s)Pu, \quad \text{for all } s,t \geq 0 \text{ and for all } u \in L^1_+(0,\omega).$$

Hence, rewriting the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = By(t) + C(t)y(t)$$

with the variation-of-constants formula gives

$$\begin{aligned} y(t) &= T(t)y_0 + \int_0^t T(t-s)C(s)y(s)\,\mathrm{d}s \\ &\leq T(t)y_0 + \int_0^t T(t-s)Py(s)\,\mathrm{d}s \end{aligned}$$

for all $t \geq 0$. Hence, if we denote the C_0-semigroup generated by $B + P$ by $\{W(t)\}_{t\geq 0}$, we obtain

$$0 \leq y(t) \leq W(t)y_0, \quad \text{for all } t \in \mathbb{R}_+. \tag{1.6.3}$$

Now, let us consider the spectrum of $B + P$.

Lemma 1.6.1. *For any $\lambda \in \rho(B+P)$, the resolvent $R(\lambda, B+P)$ is compact and expressed with a compact operator $\hat{T}_\lambda$ as follows:*

$$(R(\lambda, B+P)u)(a) = \int_0^a \mathrm{e}^{-\lambda(a-\tau)}((I - \hat{T}_\lambda)^{-1}u)(\tau)\,\mathrm{d}\tau, \tag{1.6.4}$$

$$\sigma(B+P) = P_\sigma(B+P) = \{\lambda \in \mathbb{C} \mid 1 \in \sigma(\hat{T}_\lambda)\}, \tag{1.6.5}$$

where $u \in L^1(0,\omega),\ a \in \mathbb{R}_+$.

Proof. For $u \in L^1(0,\omega)$, let $v := R(\lambda, B+P)u$ and let us express v in terms of u.
$(\lambda - (B+P))v = u$ is rewritten as follows:

$$v'(a) = -\lambda v(a) + X(a) + u(a),$$
$$\text{where } X(a) := \theta(a)\int_0^\omega \alpha(a,\sigma)c(\sigma)v(\sigma)\,\mathrm{d}\sigma,\ a \in \mathbb{R}_+.$$

$v \in \mathcal{D}(B)$ implies $v(0) = 0$, so the above integral-differential equation can be transformed as follows:

$$v(a) = \int_0^a \mathrm{e}^{-\lambda(a-\tau)}(X(\tau) + u(\tau))\,\mathrm{d}\tau.$$

If we multiply $\alpha(\tau', a)c(a)$ on both sides of the above equation, integrate with respect to a on $(0,\omega)$ and multiply $\theta(\tau')$, we get

$$X(\tau') = \theta(\tau')\int_0^\omega \alpha(\tau', a)c(a)\left(\int_0^a \mathrm{e}^{-\lambda(a-\tau)}(X(\tau) + u(\tau))\,\mathrm{d}\tau\right)\mathrm{d}a$$

$$= \int_0^\omega \tilde{\phi}_\lambda(\tau',\tau)(X(\tau)+u(\tau))\,\mathrm{d}\tau,$$

$$\text{where} \quad \tilde{\phi}_\lambda(a,b) := \theta(a)\int_b^\omega \mathrm{e}^{-\lambda(\sigma-b)}\alpha(a,\sigma)c(\sigma)\,\mathrm{d}\sigma.$$

For $f \in L^1(0,\omega)$, let

$$(\hat{T}_\lambda f)(a) := \int_0^\omega \tilde{\phi}_\lambda(a,b)f(b)\,\mathrm{d}b,$$

then we get the equation with respect to X as follows:

$$X = \hat{T}_\lambda X + \hat{T}_\lambda u \Leftrightarrow (I-\hat{T}_\lambda)X = \hat{T}_\lambda u$$

Hence, X is uniquely determined if and only if $1 \in \rho(\hat{T}_\lambda)$, when

$$X = (I-\hat{T}_\lambda)^{-1}\hat{T}_\lambda u.$$

Noting that

$$X+u = (I-\hat{T}_\lambda)^{-1}(\hat{T}_\lambda + I - \hat{T}_\lambda)u = (I-\hat{T}_\lambda)^{-1}u,$$

we derive the equation (1.6.4).

With Assumption 1.5.1, the same evaluation as in the proof of Lemma 1.5.7 yields that $\hat{T}_\lambda$ is compact. From (1.6.4) we see that $R(\lambda, B+P)$ is the integral operator whose integral kernel is a continuous function, hence it is a compact operator on $L^1(0,\omega)$. So, $B+P$ has a compact resolvent, from which (1.6.5) follows . □

Let $\Sigma := \{\lambda \in \mathbb{C} \mid 1 \in \sigma(\hat{T}_\lambda)\}$.

Assumption 1.6.2. *There exist an interval $(b_1, b_2) \subset (\omega - b_0, \omega)$ and a number $\varepsilon' > 0$ for which*

$$\theta(b) \geq \varepsilon', \quad \text{for all } b \in (b_1, b_2).$$

Theorem 1.6.3. *$r(\hat{T}_{s(B+P)}) = 1$ holds, where $s(B+P)$ means the spectral bound of $B+P$.*

To show the above theorem, we prepare the following lemmas in advance:

Lemma 1.6.4. $\lim_{\lambda\to-\infty} r(\hat{T}_\lambda) = +\infty$ *and* $\lim_{\lambda\to\infty} r(\hat{T}_\lambda) = 0$.

Proof. We define $s(\xi)$ by (1.5.6) and f_λ by

$$\langle f_\lambda, u\rangle := \int_0^\omega \theta(b)\left(\int_b^\omega s(\sigma)c(\sigma)\mathrm{e}^{-\lambda(\sigma-b)}\,\mathrm{d}\sigma\right)u(b)\,\mathrm{d}b.$$

Note that, when $\lambda = 0$, f_λ equals to f_0 defined by (1.5.7). The same argument as in the proof of Lemma 1.5.6 shows that f_λ is a strictly positive linear functional on $L^1(0,\omega)$ and

$$\hat{T}_\lambda u \geq \langle f_\lambda, u\rangle e,$$

from which it follows that $\hat{T}_\lambda$ is nonsupporting. In addition, in Lemma 1.6.1 we saw that $\hat{T}_\lambda$ is compact. Hence, $r(\hat{T}_\lambda)$ is an eigenvalue of $\hat{T}_\lambda^*$ with a strictly positive eigenfunctional F_λ. Then we have

$$\langle F_\lambda, \hat{T}_\lambda u\rangle \geq \langle f_\lambda, u\rangle\langle F_\lambda, e\rangle \;\Leftrightarrow\; r(\hat{T}_\lambda)\langle F_\lambda, u\rangle \geq \langle f_\lambda, u\rangle\langle F_\lambda, e\rangle.$$

Suppose $u = e$, then F_λ is strictly positive, from which $\langle F_\lambda, e\rangle > 0$ follows. Hence we get

$$r(\hat{T}_\lambda) \geq \langle f_\lambda, e\rangle.$$

With Assumptions 1.5.1 and 1.6.2, we see that

$$\begin{aligned}
\langle f_\lambda, e\rangle &= \int_0^\omega s(\sigma)c(\sigma)\left(\int_0^\sigma \theta(b)\mathrm{e}^{-\lambda(\sigma-b)}\,\mathrm{d}b\right)\mathrm{d}\sigma \\
&\geq \int_{b_1}^\omega \varepsilon c(\sigma)\left(\int_{b_1}^{b_2} \varepsilon'\mathrm{e}^{-\lambda(\sigma-b)}\,\mathrm{d}b\right)\mathrm{d}\sigma \\
&= \int_{b_1}^\omega \varepsilon c(\sigma)\varepsilon'\cdot\frac{\mathrm{e}^{-\lambda(\sigma-b_1)}(1-\mathrm{e}^{-\lambda(b_1-b_2)})}{-\lambda}\,\mathrm{d}\sigma.
\end{aligned}$$

As λ tends to $-\infty$, $\mathrm{e}^{-\lambda(\sigma-b_1)}(1-e^{-\lambda(b_1-b_2)})/(-\lambda)$ tends to infinity, hence $\langle f_\lambda, e\rangle$ tends to infinity from Fatou's lemma, and $r(\hat{T}_\lambda)$ also tends to infinity.

Next, observe that

$$(\hat{T}_\lambda u)(a) \le \alpha^\infty \int_0^\omega \left(\int_b^\omega e^{-\lambda(\sigma-b)} c(\sigma)\,\mathrm{d}\sigma\right) u(b)\,\mathrm{d}b.$$

The same argument as above gives

$$r(\hat{T}_\lambda) \le \alpha^\infty \int_0^\omega \left(\int_b^\omega \mathrm{e}^{-\lambda(\sigma-b)} c(\sigma)\,\mathrm{d}\sigma\right) \mathrm{d}b = \alpha^\infty \int_0^\omega c(\sigma)\frac{1-\mathrm{e}^{-\lambda\sigma}}{\lambda}\,\mathrm{d}\sigma.$$

Then, Lebesgue's dominated convergence theorem yields that $\int_0^\omega c(\sigma)(1-\mathrm{e}^{-\lambda\sigma})/\lambda\,\mathrm{d}\sigma \to 0$ as $\lambda\to\infty$, from which we have $r(\hat{T}_\lambda)\to 0$. □

Lemma 1.6.5. *$\lambda \mapsto r(\hat{T}_\lambda)$ is continuous and strictly decreasing. In addition, $r(\hat{T}_\lambda) = 1$ has the unique root λ_0 in $\mathbb{R}$, which satisfies $\lambda_0 \in \Sigma$.*

Proof. $r(\hat{T}_\lambda)$ is a point spectrum of $\hat{T}_\lambda$ but not an accumulating point of $\sigma(\hat{T}_\lambda)$. The mapping $\mathbb{R}\ni\lambda\mapsto \hat{T}_\lambda \in \mathcal{L}(L^1(0,\omega))$, where $\mathcal{L}(L^1(0,\omega))$ is endowed with the topology induced by the operator norm, is continuous. Hence $\lambda\mapsto r(\hat{T}_\lambda)$ is continuous (see, for example, [43, IV-§3.5]).

In addition, $\hat{T}_\lambda$ is nonsupporting and compact for all $\lambda\in\mathbb{R}$, and if $\lambda<\lambda'$ then $\hat{T}_\lambda \ge \hat{T}_{\lambda'}$, $\hat{T}_\lambda \ne \hat{T}_{\lambda'}$ and $r(\hat{T}_\lambda)>0$, so Theorem 1.5.5 implies that $r(\hat{T}_\lambda) > r(\hat{T}_{\lambda'})$.

Hence we find that $\lambda\mapsto r(\hat{T}_\lambda)$ is continuous and strictly decreasing. From it and Lemma 1.6.4, we see that $r(\hat{T}_\lambda)=1$ has the unique root λ_0 in $\mathbb{R}$. Since $\hat{T}_{\lambda_0}$ is nonsupporting and compact, we have $1=r(\hat{T}_{\lambda_0})\in\sigma(\hat{T}_{\lambda_0})$, which yields that $\lambda_0\in\Sigma$. □

We referred the idea of the proof of the following lemma to [44, Theorem 6.13].

Lemma 1.6.6. $\operatorname{Re}\lambda < \lambda_0$ *for all* $\lambda\in\Sigma\setminus\{\lambda_0\}$.

Proof. For all $\lambda\in\Sigma$, $1=\sigma(\hat{T}_\lambda)$ and so $\hat{T}_\lambda v = v$ for some $v\in L^1(0,\omega)$.

For $u\in L^1(0,\omega)$ we define $|u|\in L^1_+(0,\omega)$ by $|u|(a):=|u(a)|$. Then we see that

$$\begin{aligned}
|v|(a) &= |\hat{T}_\lambda v(a)| \\
&\le \int_0^\omega |v(b)|\theta(a)\left(\int_b^\omega |\mathrm{e}^{-\lambda(\sigma-b)}|\alpha(a,\sigma)c(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}b \\
&= \hat{T}_{\operatorname{Re}\lambda}|v|(a), \quad \text{for all } a\in(0,\omega),
\end{aligned}$$

i.e., $|v| = |\hat{T}_\lambda v| \le \hat{T}_{\operatorname{Re}\lambda}|v|$. Let $F_{\operatorname{Re}\lambda}$ be a strictly positive eigenfunctional of $\hat{T}^*_{\operatorname{Re}\lambda}$ corresponding to $r(\hat{T}_{\operatorname{Re}\lambda})$, then we have

$$r(\hat{T}_{\operatorname{Re}\lambda})\langle F_{\operatorname{Re}\lambda}, |v|\rangle \ge \langle F_{\operatorname{Re}\lambda}, |v|\rangle.$$

From $\langle F_{\operatorname{Re}\lambda}, |v|\rangle > 0$ it follows that $r(\hat{T}_{\operatorname{Re}\lambda})\ge 1$. Since $\lambda\mapsto r(\hat{T}_\lambda)$ is strictly decreasing by Lemma 1.6.5, we obtain that $\operatorname{Re}\lambda\le\lambda_0$.

Now suppose $\operatorname{Re}\lambda=\lambda_0$. Then we have $|v|\le \hat{T}_{\lambda_0}|v|$. If $\hat{T}_{\lambda_0}|v| > |v|$, letting F_{λ_0} be a strictly positive eigenfunctional of $\hat{T}^*_{\lambda_0}$ corresponding to $r(\hat{T}_{\lambda_0})=1$, we see that

$$\langle F_{\lambda_0}, |v|\rangle < \langle F_{\lambda_0}, \hat{T}_{\lambda_0}|v|\rangle = \langle \hat{T}^*_{\lambda_0}F_{\lambda_0}, |v|\rangle = \langle F_{\lambda_0}, |v|\rangle,$$

which is a contradiction. Hence we have $\hat{T}_{\lambda_0}|v| = |v|$. Let $v_0\in L^1_+(0,\omega)$ be a nonsupporting eigenvector of $\hat{T}_{\lambda_0}$ corresponding to the eigenvalue 1, then $|v| = c_0v_0$ for some $c_0>0$. Hence

$$v(a) = v_0(a)\cdot c_0\mathrm{e}^{ik(a)}$$

for some function $k:(0,\omega)\to\mathbb{R}$. Observe

$$|\hat{T}_\lambda v|(a) = |v|(a) = c_0v_0(a) = c_0\hat{T}_{\lambda_0}v_0(a),$$

which is rewritten in terms of integration as follows:

$$\left|\int_0^\omega \mathrm{d}b\int_b^\omega g(\sigma,b)\,\mathrm{d}\sigma\right| = \int_0^\omega \mathrm{d}b\int_b^\omega |g(\sigma,b)|\,\mathrm{d}\sigma, \tag{1.6.6}$$
$$\text{where}\quad g(\sigma,b) := \theta(a)v_0(b)c_0\mathrm{e}^{ik(b)}\mathrm{e}^{-\lambda(\sigma-b)}\alpha(a,\sigma)c(\sigma).$$

From (1.6.6) it follows that $g(\sigma, b) = g_0(\sigma, b)\mathrm{e}^{ik_1}$ for some positive function $g_0(\sigma, b)$ and some constant real number k_1 independent of σ, b (see, for example, [45, Theorem 1.33]). Hence we may assume

$$k(b) - (\operatorname{Im}\lambda)(\sigma - b) = k_1,$$

then the following holds:

$$\begin{aligned}
\hat{T}_\lambda v(a) &= \int_0^\omega \theta(a)v_0(b)c_0\mathrm{e}^{ik(b)} \left(\int_b^\omega \mathrm{e}^{-\lambda(\sigma-b)}\alpha(a,\sigma)c(\sigma)\,\mathrm{d}\sigma \right) \mathrm{d}b \\
&= \int_0^\omega \theta(a)v_0(b)c_0 \left(\int_b^\omega \mathrm{e}^{-\operatorname{Re}\lambda(\sigma-b)+ik_1}\alpha(a,\sigma)c(\sigma)\,\mathrm{d}\sigma \right) \mathrm{d}b \\
&= \mathrm{e}^{ik_1}c_0\hat{T}_{\lambda_0}v_0(a) \\
&= c_0\mathrm{e}^{ik_1}v_0(a),
\end{aligned}$$

from which it follows that

$$c_0\mathrm{e}^{ik(a)}v_0(a) = v(a) = \hat{T}_\lambda v(a) = c_0\mathrm{e}^{ik_1}v_0(a),$$

i.e., $\mathrm{e}^{(\operatorname{Im}\lambda)(\sigma-b)} = 1$ holds whenever $0 \le b \le \sigma \le \omega$. Therefore we have $\operatorname{Im}\lambda = 0$ and this completes the proof. □

Proof of Theorem 1.6.3. Lemma 1.6.1 implies that $s(B+P) = \sup\{\operatorname{Re}\lambda \mid \lambda \in \Sigma\}$, hence from Lemma 1.6.6 we have $\lambda_0 = s(B+P)$. Lemma 1.6.5 implies that $1 = r(\hat{T}_{\lambda_0})$, therefore we obtain the conclusion. □

After the above preparations we can prove the global stability for RFE in the case $R_0 < 1$.

Theorem 1.6.7. *If $R_0 = r(\tilde{T}) < 1$, the trivial equilibrium $(x, y, z) = (1, 0, 0)$ of the system (1.4.4) is globally asymptotically stable in the state space $\{(x, y, z) \in (L^1_+(0, \omega)^3) \mid x + y + z = 1\}$.*

Proof. The translation C_0-semigroup $\{T(t)\}_{t\ge0}$ is nilpotent, so it is eventually norm continuous. In addition, with Assumption 1.5.1, the same evaluation as in the proof of Lemma 1.5.7 yields that P is compact. Hence $\{W(t)\}_{t\ge0}$, generated by $B + P$, is also eventually norm continuous ([38, A-II, Theorem 1.30]). Therefore we can apply the spectral mapping theorem [38, A-III, Theorem 6.6] and we obtain $\omega_0(B + P) = s(B + P)$, where $\omega_0(B + P)$ denotes the growth bound of the semigroup $\{W(t)\}_{t\ge0}$.

On the other hand, observe that

$$r(\hat{T}_0) = r(\tilde{T}) < 1 = r(\hat{T}_{s(B+P)}).$$

Since $\lambda \mapsto r(\hat{T}_\lambda)$ is strictly decreasing, we have $s(B + P) < 0$.

Hence, $\omega_0(B + P) < 0$ and $\|W(t)\| \to 0$ as $t \to \infty$. From (1.6.3) it follows that $y(t) \to 0$ as $t \to \infty$. It is easily seen from (1.4.4) that $\lambda_1(t, \cdot) \to 0$ and $x(t) \to 1$. This completes the proof. □

Finally, in the case $R_0 > 1$, the following holds:

Theorem 1.6.8. *If $R_0 > 1$, then the trivial equilibrium $(x, y, z) = (1, 0, 0)$ of system (1.4.4) is unstable.*

Proof. Let us consider the abstract equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = By(t) + Py(t)\cdot(1 - y(t) - z(t)) - \lambda_2[a \,|\, y, z]y(t),$$

where $y(0) = y_0 \in L^1(0, \omega)$. The linearization of its right-hand side at 0 gives $(B + P)y(t)$. Since

$$r(\hat{T}_0) = R_0 > 1 = r(\hat{T}_{s(B+P)}),$$

we have $s(B + P) > 0$, from which it follows that $B + P$ has an eigenvalue whose real part is positive. This completes the proof. □

1.7 Stability of rumor-endemic equilibria

Let us investigate the local stability of REE (x^*, y^*, z^*) under the condition $R_0 > 1$ and the proportionate mixing assumption (PMA):

Assumption 1.7.1. *α, β, γ are expressed as follows:*

$$\alpha(a, \sigma) = \alpha_1(a)\alpha_2(\sigma), \quad \beta(a, \sigma) = \beta_1(a)\beta_2(\sigma), \quad \gamma(a, \sigma) = \gamma_1(a)\gamma_2(\sigma).$$

$\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in L^\infty_+(0, \omega)$, and β_1 and γ_1 are linearly independent in $L^\infty(0, \omega)$.

The assumption $\alpha(a,\sigma)=\alpha_1(a)\alpha_2(\sigma)$ means that there is no correlation between the age of a susceptible individual and the age of a spreader in their contact. The interpretation of the other assumptions is the same. Although PMA is simplistic because people usually mix differently, our analysis would be far more difficult without PMA.

Let $\alpha_1^\infty := \operatorname{ess\,sup}\alpha_1$ etc.. Notice that we can calculate R_0 under PMA. In fact, we see that

$$(\tilde{T}u)(a)=\alpha_1(a)\int_0^\omega u(b)\theta(b)\left(\int_b^\omega \alpha_2(\sigma)c(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}b$$

and substituting α_1 for u gives

$$(\tilde{T}\alpha_1)(a)=\alpha_1(a)\int_0^\omega \alpha_1(b)\theta(b)\left(\int_b^\omega \alpha_2(\sigma)c(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}b,$$

which yields through the change of integral order that

$$R_0=\int_0^\omega \alpha_2(\sigma)c(\sigma)\left(\int_0^\sigma \alpha_1(b)\theta(b)\,\mathrm{d}b\right)\mathrm{d}\sigma. \tag{1.7.1}$$

In this section, we fix the coefficients $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2$ and θ so that $R_0=1$ holds, and we rewrite this α_1 as $\alpha_1^\star$. By definition we have

$$\int_0^\omega \alpha_2(\sigma)c(\sigma)\left(\int_0^\sigma \alpha_1^\star(b)\theta(b)\,\mathrm{d}b\right)\mathrm{d}\sigma=1. \tag{1.7.2}$$

Let $\alpha_1(a):=\varepsilon\alpha_1^\star(a)$, where ε is the bifurcation parameter here and $R_0=\varepsilon$. PMA implies that ${\lambda_1}^*$ and ${\lambda_2}^*$ have finite-dimensional ranges, i.e.,

$${\lambda_1}^*(a)=c_1\alpha_1(a)=c_1\varepsilon\alpha_1^\star(a),\quad {\lambda_2}^*(a)=c_2\beta_1(a)+c_3\gamma_1(a),$$

where

$$c_1:=\int_0^\omega \alpha_2(\sigma)c(\sigma)y^*(\sigma)\,\mathrm{d}\sigma,\quad c_2:=\int_0^\omega \beta_2(\sigma)c(\sigma)y^*(\sigma)\,\mathrm{d}\sigma,\quad c_3:=\int_0^\omega \gamma_2(\sigma)c(\sigma)z^*(\sigma)\,\mathrm{d}\sigma.$$

Note that the condition $c_1=0$ implies $c_2=c_3=0$ and corresponds to RFE and that the condition $c_1>0$ corresponds to REE. Then, (1.5.2a) and (1.5.2b) is expressed as a nonlinear system for c_1,c_2,c_3 corresponding to REE :

$$\Theta_j(c_1,c_2,c_3;\varepsilon)=0,\ j=1,2,3,$$

where

$$\Theta_1(c_1,c_2,c_3;\varepsilon):=\varepsilon\int_0^\omega \alpha_2(\sigma)c(\sigma)\left(\int_0^\sigma \mathrm{e}^{-c_2\int_b^\sigma\beta_1(\tau)\mathrm{d}\tau-c_3\int_b^\sigma\gamma_1(\tau)\mathrm{d}\tau}\alpha_1^\star(b)\theta(b)\mathrm{e}^{-c_1\varepsilon\int_0^b\alpha_1^\star(\tau)\mathrm{d}\tau}\mathrm{d}b\right)\mathrm{d}\sigma-1,$$

$$\Theta_2(c_1,c_2,c_3;\varepsilon):=c_1\varepsilon\int_0^\omega \beta_2(\sigma)c(\sigma)\left(\int_0^\sigma \mathrm{e}^{-c_2\int_b^\sigma\beta_1(\tau)\mathrm{d}\tau-c_3\int_b^\sigma\gamma_1(\tau)\mathrm{d}\tau}\alpha_1^\star(b)\theta(b)\mathrm{e}^{-c_1\varepsilon\int_0^b\alpha_1^\star(\tau)\mathrm{d}\tau}\mathrm{d}b\right)\mathrm{d}\sigma-c_2,$$

$$\begin{aligned}\Theta_3(c_1,c_2,c_3;\varepsilon):=c_1\varepsilon\int_0^\omega \gamma_2(\sigma')c(\sigma')\Big\{&\int_0^\sigma \alpha_1^\star(b)(1-\theta(b))\mathrm{e}^{-c_1\varepsilon\int_0^b\alpha_1^\star(\tau)\mathrm{d}\tau}\\&+(c_2\beta_1(b)+c_3\gamma_1(b))\left(\int_0^b \mathrm{e}^{-c_2\int_b^\sigma\beta_1(\tau)\mathrm{d}\tau-c_3\int_b^\sigma\gamma_1(\tau)\mathrm{d}\tau}\alpha_1^\star(\sigma)\theta(\sigma)\mathrm{e}^{-c_1\varepsilon\int_0^b\alpha_1^\star(\tau)\mathrm{d}\tau}\mathrm{d}\sigma\right)\mathrm{d}b\Big\}\mathrm{d}\sigma'-c_3.\end{aligned}$$

By using (1.7.2), we see that $\Theta_j(0,0,0;1)=0$ for $j=1,2,3$. Let us use the implicit function theorem to find a solution $(c_1(\varepsilon),c_2(\varepsilon),c_3(\varepsilon))$ bifurcated at the point $\varepsilon=1$ from the trivial solution $(c_1,c_2,c_3)=(0,0,0)$. Let $M(c_1,c_2,c_3;\varepsilon)$ be the Jacobian matrix of the mapping

$$\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}\mapsto\begin{pmatrix}\Theta_1(c_1,c_2,c_3;\varepsilon)\\\Theta_2(c_1,c_2,c_3;\varepsilon)\\\Theta_3(c_1,c_2,c_3;\varepsilon)\end{pmatrix},$$

i.e.,

$$M(c_1,c_2,c_3;\varepsilon):=\left(\frac{\partial\Theta_i}{\partial c_j}(c_1,c_2,c_3;\varepsilon)\right)_{i,j=1,2,3}.$$

At the bifurcation point, this matrix takes the form

$$M(0,0,0;1) = \begin{pmatrix} \frac{\partial\Theta_1}{\partial c_1}(0,0,0;1) & \frac{\partial\Theta_1}{\partial c_2}(0,0,0;1) & \frac{\partial\Theta_1}{\partial c_3}(0,0,0;1) \\ \frac{\partial\Theta_2}{\partial c_1}(0,0,0;1) & -1 & 0 \\ \frac{\partial\Theta_3}{\partial c_1}(0,0,0;1) & 0 & -1 \end{pmatrix},$$

where

$$\begin{aligned}
\frac{\partial\Theta_1}{\partial c_1}(0,0,0;1) &= -\int_0^\omega \alpha_2(\sigma)c(\sigma)\int_0^\sigma \alpha_1^\star(b)\theta(b)\int_0^b \alpha_1^\star(\tau)\,\mathrm{d}\tau\,\mathrm{d}b\,\mathrm{d}\sigma < 0,\\
\frac{\partial\Theta_1}{\partial c_2}(0,0,0;1) &= -\int_0^\omega \alpha_2(\sigma)c(\sigma)\int_0^\sigma \alpha_1^\star(b)\theta(b)\int_0^b \beta_1(\tau)\,\mathrm{d}\tau\,\mathrm{d}b\,\mathrm{d}\sigma \le 0,\\
\frac{\partial\Theta_1}{\partial c_3}(0,0,0;1) &= -\int_0^\omega \alpha_2(\sigma)c(\sigma)\int_0^\sigma \alpha_1^\star(b)\theta(b)\int_0^b \gamma_1(\tau)\,\mathrm{d}\tau\,\mathrm{d}b\,\mathrm{d}\sigma \le 0,\\
\frac{\partial\Theta_2}{\partial c_1}(0,0,0;1) &= \int_0^\omega \beta_2(\sigma)c(\sigma)\int_0^\sigma \alpha_1^\star(b)\theta(b)\,\mathrm{d}b\,\mathrm{d}\sigma \ge 0,\\
\frac{\partial\Theta_3}{\partial c_1}(0,0,0;1) &= \int_0^\omega \gamma_2(\sigma)c(\sigma)\int_0^\sigma \alpha_1^\star(b)(1-\theta(b))\,\mathrm{d}b\,\mathrm{d}\sigma \ge 0.
\end{aligned}$$

It follows that

$$D_0 := \det M(0,0,0;1) = \frac{\partial\Theta_1}{\partial c_1}(0,0,0;1) + \frac{\partial\Theta_1}{\partial c_2}(0,0,0;1)\frac{\partial\Theta_2}{\partial c_1}(0,0,0;1) + \frac{\partial\Theta_1}{\partial c_3}(0,0,0;1)\frac{\partial\Theta_3}{\partial c_1}(0,0,0;1)$$

is strictly negative. Hence, we can apply the implicit function theorem to show the existence of a branching solution $(c_1(\varepsilon), c_2(\varepsilon), c_3(\varepsilon))$ with $c_1(1) = c_2(1) = c_3(1) = 0$ when $\varepsilon > 1$ is small enough. In addition, we see that

$$\begin{pmatrix} c_1'(1) \\ c_2'(1) \\ c_3'(1) \end{pmatrix} = -M(0,0,0;1)^{-1}\begin{pmatrix} \frac{\partial\Theta_1}{\partial\varepsilon}(0,0,0;1) \\ \frac{\partial\Theta_2}{\partial\varepsilon}(0,0,0;1) \\ \frac{\partial\Theta_3}{\partial\varepsilon}(0,0,0;1) \end{pmatrix} = -\frac{1}{D_0}\begin{pmatrix} 1 \\ \frac{\partial\Theta_2}{\partial c_1}(0,0,0;1) \\ \frac{\partial\Theta_3}{\partial c_1}(0,0,0;1) \end{pmatrix}.$$

REE (x^*, y^*, z^*) depends on ${\lambda_1}^*$ and ${\lambda_2}^*$, i.e., $c_1(\varepsilon), c_2(\varepsilon), c_3(\varepsilon)$ when $\varepsilon > 1$ is small enough, which admits such expression as

$$x^*(a) = x^*(a;\varepsilon), \quad y^*(a) = y^*(a;\varepsilon), \quad z^*(a) = z^*(a;\varepsilon).$$

Then, we have

$$\begin{aligned}
\frac{\partial x^*}{\partial\varepsilon}(a;1) &= -c_1'(1)\int_0^a \alpha_1^\star(\sigma)\,\mathrm{d}\sigma,\\
\frac{\partial y^*}{\partial\varepsilon}(a;1) &= c_1'(1)\int_0^a \alpha_1^\star(\sigma)\theta(\sigma)\,\mathrm{d}\sigma,\\
\frac{\partial z^*}{\partial\varepsilon}(a;1) &= c_1'(1)\int_0^a \alpha_1^\star(\sigma)(1-\theta(\sigma))\,\mathrm{d}\sigma.
\end{aligned}$$

Then, let us return into the discussion of the local stability of REE. Let

$$x(t,a) = x^*(a) + \bar{x}(t,a),\ y(t,a) = y^*(a) + \bar{y}(t,a),\ z(t,a) = z^*(a) + \bar{z}(t,a)$$

be a solution of system (1.4.4). $(\bar{x}(t,a),\ \bar{y}(t,a),\ \bar{z}(t,a))$ denote the small perturbations from REE. Note that

$$\bar{x}(t,0) = \bar{y}(t,0) = \bar{z}(t,0) = 0, \tag{1.7.3}$$
$$\bar{x}(t,a) + \bar{y}(t,a) + \bar{z}(t,a) = 0. \tag{1.7.4}$$

The small perturbations satisfy the following equations:

$$\begin{cases}
(\partial_t + \partial_a)\bar{x}(t,a) = -\bar{x}(t,a)({\lambda_1}^*(a) + \bar{\lambda}_1(t,a)) - x^*(a)\bar{\lambda}_1(t,a),\\
(\partial_t + \partial_a)\bar{y}(t,a) = \bar{x}(t,a)\theta(a)({\lambda_1}^*(a) + \bar{\lambda}_1(t,a)) + x^*(a)\theta(a)\bar{\lambda}_1(t,a)\\
\qquad\qquad - \bar{y}(t,a)({\lambda_2}^*(a) + \bar{\lambda}_2(t,a)) - y^*(a)\bar{\lambda}_2(t,a),\\
(\partial_t + \partial_a)\bar{z}(t,a) = \bar{x}(t,a)(1-\theta(a))({\lambda_1}^*(a) + \bar{\lambda}_1(t,a)) + x^*(a)(1-\theta(a))\bar{\lambda}_1(t,a)\\
\qquad\qquad + \bar{y}(t,a)({\lambda_2}^*(a) + \bar{\lambda}_2(t,a)) + y^*(a)\bar{\lambda}_2(t,a),
\end{cases} \tag{1.7.5}$$

where

$$\bar{\lambda}_1(t,a) = \int_0^\omega \alpha(a,\sigma)c(\sigma)\bar{y}(t,\sigma)\,\mathrm{d}\sigma,$$
$$\bar{\lambda}_2(t,a) = \int_0^\omega c(\sigma)\{\beta(a,\sigma)\bar{y}(t,\sigma) + \gamma(a,\sigma)\bar{z}(t,\sigma)\}\,\mathrm{d}\sigma.$$

We can formulate (1.7.5) as an abstract semilinear problem on the Banach space $(L^1(0,\omega))^3$:

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t) + G(u(t)), \quad u(t) = {}^t(\bar{x}(t,\cdot), \bar{y}(t,\cdot), \bar{z}(t,\cdot)). \tag{1.7.6}$$

The generator A is defined by

$$(A\phi)(a) := \begin{pmatrix} -\mathrm{d}/\mathrm{d}a & 0 & 0 \\ 0 & -\mathrm{d}/\mathrm{d}a & 0 \\ 0 & 0 & -\mathrm{d}/\mathrm{d}a \end{pmatrix} \begin{pmatrix} \phi_1(a) \\ \phi_2(a) \\ \phi_3(a) \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \tag{1.7.7}$$

where the domain of A is defined by

$$\mathcal{D}(A) := \{\phi = {}^t(\phi_1, \phi_2, \phi_3) \in (L^1(0,\omega))^3 \mid \phi_1, \phi_2, \phi_3 \text{ are absolutely continuous on } [0,\omega],$$
$$\phi_1(0) = \phi_2(0) = \phi_3(0) = 0,\ \phi_1(a) + \phi_2(a) + \phi_3(a) = 0\}. \tag{1.7.8}$$

The nonlinear term G is defined by

$$\begin{aligned}
G(u) &:= {}^t(G_1(u), G_2(u), G_3(u)),\\
G_1(u) &:= -u_1(\lambda_1{}^* + P_\alpha u_2) - x^* P_\alpha u_2,\\
G_2(u) &:= u_1\theta(\lambda_1{}^* + P_\alpha u_2) + x^*\theta P_\alpha u_2 - u_2(\lambda_2{}^* + P_\beta u_2 + P_\gamma u_3) - y^*(P_\beta u_2 + P_\gamma u_3),\\
G_3(u) &:= u_1(1-\theta)(\lambda_1{}^* + P_\alpha u_2) + x^*(1-\theta)P_\alpha u_2 + u_2(\lambda_2{}^* + P_\beta u_2 + P_\gamma u_3) + y^*(P_\beta u_2 + P_\gamma u_3),
\end{aligned}$$

where, for $f \in L^1(0,\omega)$,

$$\begin{cases} (P_\alpha f)(a) := \displaystyle\int_0^\omega \alpha(a,\sigma)c(\sigma)f(\sigma)\,\mathrm{d}\sigma, \\ (P_\beta f)(a) := \displaystyle\int_0^\omega \beta(a,\sigma)c(\sigma)f(\sigma)\,\mathrm{d}\sigma, \\ (P_\gamma f)(a) := \displaystyle\int_0^\omega \gamma(a,\sigma)c(\sigma)f(\sigma)\,d\sigma. \end{cases} \tag{1.7.9}$$

The linearized equation of (1.7.6) around $u = 0$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = (A + C)u(t),$$

where the bounded linear operator C is the Fréchet derivative of $G(u)$ at $u = 0$ given by

$$Cu := \begin{pmatrix} -u_1\lambda_1{}^* - x^* P_\alpha u_2 \\ u_1\theta\lambda_1{}^* + x^*\theta P_\alpha u_2 - u_2\lambda_2{}^* - y^*(P_\beta u_2 + P_\gamma u_3) \\ u_1(1-\theta)\lambda_1{}^* + x^*(1-\theta)P_\alpha u_2 + u_2\lambda_2{}^* + y^*(P_\beta u_2 + P_\gamma u_3) \end{pmatrix}.$$

Now let us consider the resolvent equation for $A + C$:

$$(\zeta - (A + C))v = u, \quad v \in \mathcal{D}(A),\ u \in (L^1(0,\omega))^3,\ \zeta \in \mathbb{C}. \tag{1.7.10}$$

Then we have

$$v_1'(a) = -\zeta v_1(a) - \lambda_1{}^*(a)v_1(a) - x^*(a)(P_\alpha v_2)(a) + u_1(a), \tag{1.7.11a}$$
$$\begin{aligned} v_2'(a) = &-\zeta v_2(a) + \lambda_1{}^*(a)\theta(a)v_1(a) + \theta(a)x^*(a)(P_\alpha v_2)(a) - \lambda_2^*(a)v_2(a) \\ &- y^*(a)(P_\beta v_2)(a) - y^*(a)(P_\gamma v_3)(a) + u_2(a), \end{aligned} \tag{1.7.11b}$$
$$\begin{aligned} v_3'(a) = &-\zeta v_3(a) + \lambda_1{}^*(a)(1-\theta(a))v_1(a) + (1-\theta(a))x^*(a)(P_\alpha v_2)(a) \\ &+ \lambda_2^*(a)v_2(a) + y^*(a)(P_\beta v_2)(a) + y^*(a)(P_\gamma v_3)(a) + u_3(a). \end{aligned} \tag{1.7.11c}$$

From (1.7.11a) and $v_1(0) = 0$, we obtain

$$v_1(a) = \int_0^a \{-x^*(\tau)(P_\alpha v_2)(\tau) + u_1(\tau)\}\mathrm{e}^{-\int_\tau^a (\zeta + \lambda_1{}^*(r))\,\mathrm{d}r}\,\mathrm{d}\tau. \tag{1.7.12}$$

From (1.7.11b) and $v_2(0) = 0$, we have

$$v_2(a) = \int_0^a \{\lambda_1{}^*(\sigma)\theta(\sigma)v_1(\sigma) + \theta(\sigma)x^*(\sigma)(P_\alpha v_2)(\sigma) - y^*(\sigma)(P_\beta v_2)(\sigma) \\ - y^*(\sigma)(P_\gamma v_3)(\sigma) + u_2(\sigma)\}\mathrm{e}^{-\zeta(a-\sigma)}\mathrm{e}^{-\int_\sigma^a \lambda_2{}^*(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma. \quad (1.7.13)$$

Let

$$\xi_1 := \int_0^\omega \alpha_2(\sigma)c(\sigma)v_2(\sigma)\,\mathrm{d}\sigma, \quad \xi_2 := \int_0^\omega \beta_2(\sigma)c(\sigma)v_2(\sigma)\,\mathrm{d}\sigma, \quad \xi_3 := \int_0^\omega \gamma_2(\sigma)c(\sigma)v_3(\sigma)\,\mathrm{d}\sigma, \quad (1.7.14)$$

Assumption 1.7.1 implies

$$(P_\alpha v_2)(a) = \xi_1\alpha_1(a), \quad (P_\beta v_2)(a) = \xi_2\beta_1(a), \quad (P_\gamma v_3)(a) = \xi_3\gamma_1(a).$$

Inserting (1.7.12) and (1.7.13) into (1.7.14) yields a three-dimensional system as

$$(I - \Phi(\zeta,\varepsilon))\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix} = \begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix}, \quad (1.7.15)$$

where I is the 3×3 unit matrix and $\Phi(\zeta,\varepsilon) = (\phi_{ij}(\zeta,\varepsilon))_{1\le i,j\le 3}$ a 3×3 matrix. $\phi_{ij}(\zeta,\varepsilon)$ and η_i $(1 \le i,j \le 3)$ are defined as follows:

$$\phi_{11}(\zeta,\varepsilon) = \int_0^\omega \alpha_2(r)c(r)\Big(\int_0^r \theta(\sigma)x^*(\sigma;\varepsilon)\varepsilon\alpha_1^\star(\sigma)\mathrm{e}^{-\zeta(r-\sigma)}\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r\beta_1(\tau)\,\mathrm{d}\tau - c_3(\varepsilon)\int_\sigma^r\gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\Big)\,\mathrm{d}r \\ - \int_0^\omega \alpha_2(r)c(r)\Big\{\int_0^r c_1(\varepsilon)\varepsilon\alpha_1^\star(\sigma)\theta(\sigma)\mathrm{e}^{-\zeta(r-\sigma)}\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r\beta_1(\tau)\,\mathrm{d}\tau - c_3(\varepsilon)\int_\sigma^r\gamma_1(\tau)\,\mathrm{d}\tau} \\ \times\Big(\int_0^\sigma \varepsilon\alpha_1^\star(\tau')\mathrm{e}^{-\zeta(\sigma-\tau')}\mathrm{e}^{-c_1(\varepsilon)\int_0^\sigma\varepsilon\alpha_1^\star(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\tau'\Big)\,\mathrm{d}\sigma\Big\}\,\mathrm{d}r,$$

$$\phi_{12}(\zeta,\varepsilon) = -\int_0^\omega \alpha_2(r)c(r)\Big(\int_0^r y^*(\sigma;\varepsilon)\beta_1(\sigma)\mathrm{e}^{-\zeta(r-\sigma)}\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r\beta_1(\tau)\,\mathrm{d}\tau - c_3(\varepsilon)\int_\sigma^r\gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\Big)\,\mathrm{d}r,$$

$$\phi_{13}(\zeta,\varepsilon) = -\int_0^\omega \alpha_2(r)c(r)\Big(\int_0^r y^*(\sigma;\varepsilon)\gamma_1(\sigma)\mathrm{e}^{-\zeta(r-\sigma)}\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r\beta_1(\tau)\,\mathrm{d}\tau - c_3(\varepsilon)\int_\sigma^r\gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\Big)\,\mathrm{d}r,$$

$$\phi_{21}(\zeta,\varepsilon) = \int_0^\omega \beta_2(r)c(r)\Big(\int_0^r \theta(\sigma)x^*(\sigma;\varepsilon)\varepsilon\alpha_1^\star(\sigma)\mathrm{e}^{-\zeta(r-\sigma)}\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r\beta_1(\tau)\,\mathrm{d}\tau - c_3(\varepsilon)\int_\sigma^r\gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\Big)\,\mathrm{d}r \\ - \int_0^\omega \beta_2(r)c(r)\Big\{\int_0^r c_1(\varepsilon)\varepsilon\alpha_1^\star(\sigma)\theta(\sigma)\mathrm{e}^{-\zeta(r-\sigma)}\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r\beta_1(\tau)\,\mathrm{d}\tau - c_3(\varepsilon)\int_\sigma^r\gamma_1(\tau)\,\mathrm{d}\tau} \\ \times\Big(\int_0^\sigma \varepsilon\alpha_1^\star(\tau')\mathrm{e}^{-\zeta(\sigma-\tau')}\mathrm{e}^{-c_1(\varepsilon)\int_0^\sigma\varepsilon\alpha_1^\star(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\tau'\Big)\,\mathrm{d}\sigma\Big\}\,\mathrm{d}r,$$

$$\phi_{22}(\zeta,\varepsilon) = -\int_0^\omega \beta_2(r)c(r)\Big(\int_0^r y^*(\sigma;\varepsilon)\beta_1(\sigma)\mathrm{e}^{-\zeta(r-\sigma)}\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r\beta_1(\tau)\,\mathrm{d}\tau - c_3(\varepsilon)\int_\sigma^r\gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\Big)\,\mathrm{d}r,$$

$$\phi_{23}(\zeta,\varepsilon) = -\int_0^\omega \beta_2(r)c(r)\Big(\int_0^r y^*(\sigma;\varepsilon)\gamma_1(\sigma)\mathrm{e}^{-\zeta(r-\sigma)}\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r\beta_1(\tau)\,\mathrm{d}\tau - c_3(\varepsilon)\int_\sigma^r\gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\Big)\,\mathrm{d}r,$$

$$\phi_{31}(\zeta,\varepsilon) = \int_0^\omega \gamma_2(r)c(r)\Big(\int_0^r x^*(\sigma;\varepsilon)\varepsilon\alpha_1^\star(\sigma)\mathrm{e}^{-\zeta(r-\sigma)}\mathrm{e}^{-c_1(\varepsilon)\int_\sigma^r\varepsilon\alpha_1^\star(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\Big)\,\mathrm{d}r \\ - \int_0^\omega \gamma_2(r)c(r)\Big(\int_0^r \theta(\sigma)x^*(\sigma;\varepsilon)\varepsilon\alpha_1^\star(\sigma)\mathrm{e}^{-\zeta(r-\sigma)}\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r\beta_1(\tau)\,\mathrm{d}\tau - c_3(\varepsilon)\int_\sigma^r\gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\Big)\,\mathrm{d}r \\ + \int_0^\omega \gamma_2(r)c(r)\Big\{\int_0^r c_1(\varepsilon)\varepsilon\alpha_1^\star(\sigma)\theta(\sigma)\mathrm{e}^{-\zeta(r-\sigma)}\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r\beta_1(\tau)\,\mathrm{d}\tau - c_3(\varepsilon)\int_\sigma^r\gamma_1(\tau)\,\mathrm{d}\tau} \\ \times\Big(\int_0^\sigma \varepsilon\alpha_1^\star(\tau')\mathrm{e}^{-\zeta(\sigma-\tau')}\mathrm{e}^{-c_1(\varepsilon)\int_0^\sigma\varepsilon\alpha_1^\star(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\tau'\Big)\,\mathrm{d}\sigma\Big\}\,\mathrm{d}r,$$

$$\phi_{32}(\zeta,\varepsilon) = \int_0^\omega \gamma_2(r)c(r)\Big(\int_0^r y^*(\sigma;\varepsilon)\beta_1(\sigma)\mathrm{e}^{-\zeta(r-\sigma)}\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r\beta_1(\tau)\,\mathrm{d}\tau - c_3(\varepsilon)\int_\sigma^r\gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\Big)\,\mathrm{d}r,$$

$$\phi_{33}(\zeta,\varepsilon) = \int_0^\omega \gamma_2(r)c(r)\Big(\int_0^r y^*(\sigma;\varepsilon)\gamma_1(\sigma)\mathrm{e}^{-\zeta(r-\sigma)}\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r\beta_1(\tau)\,\mathrm{d}\tau - c_3(\varepsilon)\int_\sigma^r\gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\Big)\,\mathrm{d}r,$$

$$\eta_1 = \int_0^\omega \alpha_2(r)c(r)\Big(\int_0^r u_2(\sigma)\mathrm{e}^{-\zeta(r-\sigma)}\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r\beta_1(\tau)\,\mathrm{d}\tau - c_3(\varepsilon)\int_\sigma^r\gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\Big)\,\mathrm{d}r$$

$$
\begin{aligned}
&+\int_0^\omega \alpha_2(r)c(r)\left\{\int_0^r c_1(\varepsilon)\varepsilon\alpha_1^\star(\sigma)\theta(\sigma)\mathrm{e}^{-\zeta(r-\sigma)}\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r\beta_1(\tau)\,\mathrm{d}\tau-c_3(\varepsilon)\int_\sigma^r\gamma_1(\tau)\,\mathrm{d}\tau}\right.\\
&\quad\left.\times\left(\int_0^\sigma u_1(\tau')\mathrm{e}^{-\zeta(\sigma-\tau')}\mathrm{e}^{-c_1(\varepsilon)\int_{\tau'}^\sigma\varepsilon\alpha_1^\star(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\tau'\right)\mathrm{d}\sigma\right\}\mathrm{d}r,\\
\eta_2 &= \int_0^\omega \beta_2(r)c(r)\left(\int_0^r u_2(\sigma)\mathrm{e}^{-\zeta(r-\sigma)}\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r\beta_1(\tau)\,\mathrm{d}\tau-c_3(\varepsilon)\int_\sigma^r\gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\right)\mathrm{d}r\\
&+\int_0^\omega \beta_2(r)c(r)\left\{\int_0^r c_1(\varepsilon)\varepsilon\alpha_1^\star(\sigma)\theta(\sigma)\mathrm{e}^{-\zeta(r-\sigma)}\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r\beta_1(\tau)\,\mathrm{d}\tau-c_3(\varepsilon)\int_\sigma^r\gamma_1(\tau)\,\mathrm{d}\tau}\right.\\
&\quad\left.\times\left(\int_0^\sigma u_1(\tau')\mathrm{e}^{-\zeta(\sigma-\tau')}\mathrm{e}^{-c_1(\varepsilon)\int_{\tau'}^\sigma\varepsilon\alpha_1^\star(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\tau'\right)\mathrm{d}\sigma\right\}\mathrm{d}r,\\
\eta_3 &= -\int_0^\omega \gamma_2(r)c(r)\left(\int_0^r u_1(\sigma)\mathrm{e}^{-\zeta(r-\sigma)}\mathrm{e}^{-c_1(\varepsilon)\int_\sigma^r\varepsilon\alpha_1^\star(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\right)\mathrm{d}r\\
&-\int_0^\omega \gamma_2(r)c(r)\left(\int_0^r u_2(\sigma)\mathrm{e}^{-\zeta(r-\sigma)}\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r\beta_1(\tau)\,\mathrm{d}\tau-c_3(\varepsilon)\int_\sigma^r\gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma\right)\mathrm{d}r\\
&-\int_0^\omega \gamma_2(r)c(r)\left\{\int_0^r c_1(\varepsilon)\varepsilon\alpha_1^\star(\sigma)\theta(\sigma)\mathrm{e}^{-\zeta(r-\sigma)}\mathrm{e}^{-c_2(\varepsilon)\int_\sigma^r\beta_1(\tau)\,\mathrm{d}\tau-c_3(\varepsilon)\int_\sigma^r\gamma_1(\tau)\,\mathrm{d}\tau}\right.\\
&\quad\left.\times\left(\int_0^\sigma u_1(\tau')\mathrm{e}^{-\zeta(\sigma-\tau')}\mathrm{e}^{-c_1(\varepsilon)\int_{\tau'}^\sigma\varepsilon\alpha_1^\star(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\tau'\right)\mathrm{d}\sigma\right\}\mathrm{d}r.
\end{aligned}
$$

$\phi_{ij}(0,1)$ are calculated as follows:

$$
\begin{aligned}
\phi_{11}(0,1) &= \int_0^\omega \alpha_2(r)c(r)\left(\int_0^r \theta(\sigma)\alpha_1^\star(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}r = 1,\\
\phi_{21}(0,1) &= \int_0^\omega \beta_2(r)c(r)\left(\int_0^r \theta(\sigma)\alpha_1^\star(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}r = \frac{\partial\Theta_2}{\partial c_1}(0,0,0;1),\\
\phi_{31}(0,1) &= \int_0^\omega \gamma_2(r)c(r)\left(\int_0^r (1-\theta(\sigma))\alpha_1^\star(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}r = \frac{\partial\Theta_3}{\partial c_1}(0,0,0;1),\\
\phi_{ij}(0,1) &= 0 \quad (i=1,2,3,\ j=2,3).
\end{aligned}
$$

In addition, we observe that

$$
\begin{aligned}
\frac{\partial\phi_{11}}{\partial\zeta}(0,1) &= \int_0^\omega \alpha_2(r)c(r)\left(\int_0^r \theta(\sigma)\alpha_1(\sigma)(-r+\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}r < 0,\\
\frac{\partial\phi_{ij}}{\partial\zeta}(0,1) &= 0 \quad (i=1,2,3,\ j=2,3),\\
\frac{\partial\phi_{11}}{\partial\varepsilon}(0,1) &= \frac{\partial\Theta_1}{\partial c_1}(0,0,0;1)\{1+c_1'(1)\}+1+\frac{\partial\Theta_1}{\partial c_2}(0,0,0;1)+\frac{\partial\Theta_1}{\partial c_3}(0,0,0;1),\\
\frac{\partial\phi_{12}}{\partial\varepsilon}(0,1) &= c_1'(1)\frac{\partial\Theta_1}{\partial c_2}(0,0,0;1),\\
\frac{\partial\phi_{13}}{\partial\varepsilon}(0,1) &= c_1'(1)\frac{\partial\Theta_1}{\partial c_3}(0,0,0;1).
\end{aligned}
$$

Now we denote the determinant of $I-\Phi(\zeta,\varepsilon)$ by $f(\zeta,\varepsilon)$. We see that the roots of $f(\zeta,1)$ give the eigenvalues of the linearized system at RFE. If $\varepsilon=1$, then $f(0,1)=0$ and we can see that

$$
\frac{\partial f}{\partial\zeta}(0,1) = -\frac{\partial\phi_{11}}{\partial\zeta}(0,1) > 0.
$$

Then the implicit function theorem implies that the equation $f(\zeta,\varepsilon)=0$ can be solved locally as $\zeta=\zeta(\varepsilon)$ with $\zeta(1)=0$. At the same time, it is rather easy to see that

$$
\begin{aligned}
\frac{\partial f}{\partial\varepsilon}(0,1) &= -\frac{\partial\phi_{11}}{\partial\varepsilon}(0,1) - \frac{\partial\phi_{12}}{\partial\varepsilon}(0,1)\phi_{21}(0,1) - \frac{\partial\phi_{13}}{\partial\varepsilon}(0,1)\phi_{31}(0,1)\\
&= -D_0c_1'(1) - 1 - \frac{\partial\Theta_1}{\partial c_1}(0,0,0;1) - \frac{\partial\Theta_1}{\partial c_2}(0,0,0;1) - \frac{\partial\Theta_1}{\partial c_3}(0,0,0;1)\\
&= -\frac{\partial\Theta_1}{\partial c_1}(0,0,0;1) - \frac{\partial\Theta_1}{\partial c_2}(0,0,0;1) - \frac{\partial\Theta_1}{\partial c_3}(0,0,0;1) > 0,
\end{aligned}
$$

which means that the dominant eigenvalue goes to the left half complex plane as ε increases small enough from 1. The well-known technique based on the Rouché's theorem (see [33, Proposition 4.1]) yields that $\zeta(\varepsilon)$ is the dominant root of f as long as $|\varepsilon - 1|$ is small enough.

Now, let us examine the relationship between the dominant eigenvalue and the local stability of REE. Let $\Sigma^* = \sigma(A + C)$ be the spectrum of $A + C$, then the following holds:

Lemma 1.7.2. *(i) Σ^* can be rewritten as follows:*

$$\Sigma^* = P_\sigma(A + C) = \{\zeta \in \mathbb{C} \mid f(\zeta, \varepsilon) = 0\}.$$

(ii) The linearized semigroup $\{e^{t(A+C)}\}_{t\geq 0}$ is eventually compact and

$$\omega_0(A + C) = s(A + C) = \sup\{\mathrm{Re}\,\zeta \mid \zeta \in P_\sigma(A + C)\}.$$

Proof. The resolvent equation (1.7.10) is solvable if and only if $I - \Phi(\zeta, \varepsilon)$ is invertible, i.e., $f(\zeta, \varepsilon) = 0$. Hence we have

$$\rho(A + C) = \{\zeta \in \mathbb{C} \mid f(\zeta, \varepsilon) \neq 0\}, \quad \Sigma^* = \{\zeta \in \mathbb{C} \mid f(\zeta, \varepsilon) = 0\}.$$

It is easy to show that the resolvent $R(\zeta, A + C)$ is compact for any $\zeta \in \rho(A + C)$, from which it follows that $\Sigma^* = P_\sigma(A + C)$ and we have shown the first assertion.

Let us prove the second assertion. The linearized generator $A + C$ is decomposed as $A + C_1 + C_2$, where

$$C_1 u := {}^t(-\lambda_1{}^* u_1, \theta\lambda_1{}^* u_1 - \lambda_2{}^* u_2, (1 - \theta)\lambda_1{}^* u_1 + \lambda_2{}^* u_2)$$

and

$$C_2 u := \begin{pmatrix} -x^*(P_\alpha u_2) \\ \theta x^*(P_\alpha u_2) - y^*(P_\beta u_2 + P_\gamma u_3) \\ (1 - \theta)x^*(P_\alpha u_2) + y^*(P_\beta u_2 + P_\gamma u_3) \end{pmatrix}.$$

We easily find that the operator $A + C_1$ is the generator of a multistate stable population with a finite age interval (see [34]). Hence it follows that the population semigroup generated by $A + C_1$ is eventually compact and its essential growth $\omega_1(A + C_1)$ is $-\infty$. Since C_2 is compact due to Assumption 1.5.1, we have $\omega_1(A + C_1 + C_2) = \omega_1(A + C_1) = -\infty$ (see [46, Proposition 4.14]). Then the second assertion is obtained from [46, Proposition 4.13]. □

From the above results and the principle of linearized stability (see, for example, [46, Proposion 4.17]), we conclude:

Theorem 1.7.3. *Under Assumption 1.7.1, the rumor-endemic equilibrium bifurcates forward from RFE and is locally asymptotically stable if $R_0 > 1$ and $|R_0 - 1|$ is small enough.*

1.8 Uniform strong persistence

In this section we show that $R_0 > 1$ implies uniform strong rumor persistence under PMA.

PMA with Assumption 1.5.1 (ii) implies that $\alpha_1(a) > 0$ for almost every $a \in (0, \omega)$ and $\alpha_2(\sigma) > 0$ for almost every $\sigma \in (\omega - b_0, \omega)$. Moreover, it implies that

$$\lambda_1(t, a) = \alpha_1(a)\psi_1(t), \quad \text{where } \psi_1(t) := \int_0^\omega \alpha_2(\sigma)c(\sigma)y(t, \sigma)\,d\sigma \tag{1.8.1}$$

and

$$\lambda_2(t, a) = \beta_1(a)\psi_2(t) + \gamma_1(a)\psi_3(t)$$

where

$$\psi_2(t) := \int_0^\omega \beta_2(\sigma)c(\sigma)y(t, \sigma)\,d\sigma, \quad \psi_3(t) := \int_0^\omega \gamma_2(\sigma)c(\sigma)z(t, \sigma)\,d\sigma. \tag{1.8.2}$$

By integrating along characteristics we obtain from the partial derivative equation for $y(t, a)$ in (1.4.4) that

$$y(t, a) = \int_0^a \alpha_1(\sigma)\psi_1(t - a + \sigma)\theta(\sigma)x(t - a + \sigma, \sigma)e^{-\int_\sigma^a \lambda_2(t-a+\tau,\tau)\,d\tau}\,d\sigma \tag{1.8.3}$$

if $t > a$, and

$$y(t, a) = y_0(a - t)e^{-\int_0^t \lambda_2(\tau, a-t+\tau)\,d\tau} + x_0(a - t)\int_0^t \{\alpha_1(a - t + s)\psi_1(s)$$

$$\times\,\theta(a-t+s)\mathrm{e}^{-\int_0^s \lambda_1(\tau,a-t+\tau)\,\mathrm{d}\tau-\int_s^t \lambda_2(\tau,a-t+\tau)\,\mathrm{d}\tau}\Big\}\,\mathrm{d}s$$

if $t < a$. If $t > \omega$, by substituting this into (1.8.1) and letting $r = s - \sigma$, we have

$$\psi_1(t) = \int_0^\omega \alpha_2(s)c(s)\Big\{\int_0^s \alpha_1(s-r)\psi_1(t-r)\theta(s-r)x(t-r,s-r)\mathrm{e}^{-\int_{s-r}^s \lambda_2(t-s+\tau,\tau)\,\mathrm{d}\tau}\,\mathrm{d}r\Big\}\,\mathrm{d}s$$
$$= \int_0^\omega \psi_1(t-r)\Big\{\int_r^\omega \alpha_2(s)c(s)\alpha_1(s-r)\theta(s-r)x(t-r,s-r)\mathrm{e}^{-\int_{s-r}^s \lambda_2(t-s+\tau,\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\Big\}\,\mathrm{d}r.$$

if $t < \omega$. Let $c(a) = 0$ for $a > \omega$, then we have

$$\psi_1(t) = \int_0^t \psi_1(t-r)\Big\{\int_r^t \alpha_2(s)c(s)\alpha_1(s-r)\theta(s-r)x(t-r,s-r)\mathrm{e}^{-\int_{s-r}^s \lambda_2(t-s+\tau,\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\Big\}\,\mathrm{d}r \qquad (1.8.4)$$

if $t > \omega$. Similarly, if $t < \omega$, then we have

$$\psi_1(t) = \int_0^t \psi_1(t-r)\Big\{\int_r^t \alpha_2(s)c(s)\alpha_1(s-r)\theta(s-r)x(t-r,s-r)\mathrm{e}^{-\int_{s-r}^s \lambda_2(t-s+\tau,\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\Big\}\,\mathrm{d}r$$
$$+\int_t^\omega \alpha_2(a)c(a)y_0(a-t)\mathrm{e}^{-\int_0^t \lambda_2(\tau,a-t+\tau)\,\mathrm{d}\tau}\,\mathrm{d}a$$
$$+\int_0^t \psi_1(t-r)\Big\{\int_t^\omega \alpha_2(s)c(s)\alpha_1(s-r)\theta(s-r)x_0(s-t)\mathrm{e}^{-\int_0^{t-r} \lambda_1(\tau,s-t+\tau)\,\mathrm{d}\tau-\int_{t-r}^t \lambda_2(\tau,s-t+\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\Big\}\,\mathrm{d}r.$$

Let $\psi_{1,b}(t) := \psi_1(b+t)$ for $b \geq 0$, then from the way $c(a)$ is extended for $a > \omega$ we see that

$$\psi_{1,b}(t) = \int_0^t \psi_{1,b}(t-r)\Big\{\int_r^t \alpha_2(s)c(s)\alpha_1(s-r)\theta(s-r)x(t+b-r,s-r)$$
$$\times\,\mathrm{e}^{-\int_{s-r}^s \lambda_2(t-s+\tau+b,\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\Big\}\,\mathrm{d}r \qquad (1.8.5)$$

for sufficiently large t.

Lemma 1.8.1. *(1) If $t > a$, then $x(t,a) > 0$. And if $t < a$ and $x_0(a-t) > 0$, then $x(t,a) > 0$.*

(2) If $y(t,a) > 0$, then $y(t+\sigma, a+\sigma) > 0$ for all $\sigma \in (0, \omega - a)$.

Proof. From the partial derivative equation for $x(t,a)$ in (1.4.4) we have

$$x(t,a) = x(t-a,0)\mathrm{e}^{-\int_0^a \lambda_1(t-a+\tau,\tau)\,\mathrm{d}\tau}$$

for $t > a$. This and $x(t-a,0) = 1$ imply that $x(t,a) > 0$ for $t > a$. The case $t < a$ and $x_0(a-t) > 0$ is as well.

From the partial derivative equation for $y(t,a)$ in (1.4.4) and $\lambda_1(t,a)\theta(a)x(t,a) \geq 0$, we have

$$(\partial_t + \partial_a)y(t,a) \geq -\lambda_2(t,a)y(t,a).$$

Integrating along characteristics gives

$$y(t+\sigma,a+\sigma) \geq y(t,a)\mathrm{e}^{-\int_0^\sigma \lambda_2(t+\tau,a+\tau)\,\mathrm{d}\tau} > 0,$$

thus we have the conclusion. □

The following lemma states that spreaders never clear out unless there are no spreaders from start.

Lemma 1.8.2. *If $\|y_0\| > 0$, then $\|y(t)\| > 0$ for all $t > 0$.*

Proof. Suppose $\|y(t_0)\| = 0$ for some $t = t_0 > 0$. Then we have $y(t_0,a) = 0$ for almost every $a \in (0,\omega)$.

If t_0 is sufficiently large, then we find from Lemma 1.8.1 that $y(t_0 - t_1, a) = 0$ for all $t_1 \in (0, b_0)$ and almost every $a \in (0, \omega - t_1)$, where b_0 is defined in Assumption 1.5.1 (ii). In addition, the integrand of (1.8.3) is strictly positive except ψ_1 for almost every $\sigma \in (0,a)$, hence we obtain $\psi_1(t) = 0$ for almost every $t \in (t_0 - a, t_0)$. Since this holds for almost every $a \in (0,\omega)$, we find $\psi_1(t) = 0$ for almost every $t \in (t_0 - \omega, t_0)$. The fact that $\alpha_2(\sigma)c(\sigma)$ in (1.8.1) is strictly positive for almost every $\sigma \in (\omega - b_0, \omega)$ implies that $y(t,a) = 0$ for almost every $t \in (t_0 - \omega, t_0)$ and almost every $a \in (\omega - b_0, \omega)$. Then we find again from Lemma 1.8.1 that $y(t_0 - t_1, a) = 0$ for all $t_1 \in (0, b_0)$ and almost every $a \in (\omega - t_1, \omega)$. Therefore it follows that $y(t,a) = 0$ for all $t \in (t_0 - b_0, t_0]$ and almost every $a \in (0,\omega)$.

By iterating this discussion finite times, we obtain $y(0,a) = 0$ for almost every $a \in (0,\omega)$, where the above discussion should be modified a little when $t < \omega$. This contradicts the assumption $\|y_0\| > 0$, thus we obtain the conclusion. □

Here, we introduce the concept of rumor persistence. Persistence in the general meaning is detailed in Sections 2.1 and 2.2, and consistent with the following definition of rumor persistence. The rumor is called *uniformly weakly persistent* if there exists some $\varepsilon > 0$, which does not depend on the initial data $(x(0,\cdot), y(0,\cdot), z(0,\cdot))$ and satisfies

$$\int_0^\omega c(a)y(0,a)\,\mathrm{d}a > 0 \ \Rightarrow\ \limsup_{t\to\infty} \int_0^\omega c(a)y(t,a)\,\mathrm{d}a > \varepsilon.$$

The rumor is called *uniformly strongly persistent* if there exists some $\varepsilon > 0$, which does not depend on the initial data $(x(0,\cdot), y(0,\cdot), z(0,\cdot))$ and satisfies

$$\int_0^\omega c(a)y(0,a)\,\mathrm{d}a > 0 \ \Rightarrow\ \liminf_{t\to\infty} \int_0^\omega c(a)y(t,a)\,\mathrm{d}a > \varepsilon.$$

It is clear that uniform strong persistence implies uniform weak persistence.

First, let us show the weak result. We referred the idea of its proof to [47].

Theorem 1.8.3. *If $R_0 > 1$, then the rumor is uniformly weakly persistent.*

Proof. Assume that for any $\varepsilon > 0$ there exist some $T_0 > 0$ and some appropriate initial condition such that $\int_0^\omega c(a)y(t,a)\,\mathrm{d}a \le \varepsilon$ for all $t \ge T_0$. We can choose T_0 to be so large that (1.8.5) holds for all $t \ge T_0$. By the definition of $\psi_1(t)$ and $\psi_2(t)$, it is easily seen that

$$\psi_1(t) \le \alpha_2^\infty \varepsilon, \quad \psi_2(t) \le \beta_2^\infty \varepsilon$$

for $t \ge T_0$. Next, let us obtain the upper bound for $\psi_3(t)$. The conditions that $z(t,a) \le 1$ and $\int_0^\omega c(\sigma)\,d\sigma = 1$ imply that

$$\lambda_1(t,a) \le \alpha^\infty \varepsilon, \quad \lambda_2(t,a) \le \beta^\infty \varepsilon + \gamma^\infty$$

for $t \ge T_0$. It follows that

$$(\partial_t + \partial_a) z(t,a) \le \alpha^\infty \varepsilon + (\beta^\infty \varepsilon + \gamma^\infty) y(t,a)$$

for $t \ge T_0$. Hence, if $t - a \ge T_0$, integrating the above inequality along characteristics gives

$$z(t,a) \le \alpha^\infty a \varepsilon + (\beta^\infty \varepsilon + \gamma^\infty) \int_0^a y(t-a+\tau, \tau)\,\mathrm{d}\tau.$$

Therefore, if $t \ge T_0 + \omega$, we have

$$\begin{aligned}
\psi_3(t) &\le \alpha^\infty \varepsilon \cdot \frac{1}{2}\omega^2 + (\beta^\infty \varepsilon + \gamma^\infty) \int_0^\omega \left(\int_0^\sigma y(t-\sigma+\tau, \tau)\,\mathrm{d}\tau \right) \mathrm{d}\sigma \\
&= \alpha^\infty \varepsilon \cdot \frac{1}{2}\omega^2 + (\beta^\infty \varepsilon + \gamma^\infty) \int_0^\omega \left(\int_0^{\omega-u} y(t-u, \tau)\,\mathrm{d}\tau \right) \mathrm{d}u \\
&\le \alpha^\infty \varepsilon \cdot \frac{1}{2}\omega^2 + (\beta^\infty \varepsilon + \gamma^\infty) \int_0^\omega \varepsilon\,\mathrm{d}u \\
&= \alpha^\infty \varepsilon \cdot \frac{1}{2}\omega^2 + (\beta^\infty \varepsilon + \gamma^\infty) \omega \varepsilon.
\end{aligned}$$

In what follows, we can assume for simplification that, for any $\varepsilon > 0$, there exist some $T_0 > 0$ and some appropriate initial condition such that (1.8.5) and

$$\int_0^\omega y(t,a)\,\mathrm{d}a \le \varepsilon, \quad \psi_i(t) \le \varepsilon,\ i = 1,2,3$$

hold for $t \ge T_0$. Integrating the partial derivative equation for $x(t,a)$ in (1.4.4) along characteristics gives

$$\begin{aligned}
x(t+b-r, s-r) &= \mathrm{e}^{-\int_0^{s-r} \psi_1(t+b-s+\tau)\alpha_1(\tau)\,\mathrm{d}\tau} \\
&\ge \mathrm{e}^{-\varepsilon \int_0^{s-r} \alpha_1(\tau)\,\mathrm{d}\tau}
\end{aligned} \tag{1.8.6}$$

for $t > T_0$, $0 \le r \le s \le t$ and $b \ge 0$. Hence it follows from (1.8.5) that

$$\psi_{1,b}(t) \ge \int_0^t \psi_{1,b}(t-r) \left\{ \int_r^t \alpha_2(s)c(s)\alpha_1(s-r)\theta(s-r) \mathrm{e}^{-\varepsilon \int_0^{s-r} \alpha_1(\tau)\,\mathrm{d}\tau} e^{-\varepsilon \int_{s-r}^s (\beta_1(\tau)+\gamma_1(\tau))\,d\tau}\,\mathrm{d}s \right\} \mathrm{d}r.$$

This implies that

$$
\begin{aligned}
&\psi_{1,b+T}(t) \\
&= \psi_{1,b}(t+T) \\
&\geq \int_0^{t+T} \psi_{1,b+T}(t-r) \left\{ \int_r^{t+T} \alpha_2(s)c(s)\alpha_1(s-r)\theta(s-r)\mathrm{e}^{-\varepsilon\int_0^{s-r}\alpha_1(\tau)\,\mathrm{d}\tau}\mathrm{e}^{-\varepsilon\int_{s-r}^{s}(\beta_1(\tau)+\gamma_1(\tau))\,\mathrm{d}\tau}\,\mathrm{d}s \right\} \mathrm{d}r \\
&\geq \int_0^{t} \psi_{1,b+T}(t-r) \left\{ \int_r^{r+T} \alpha_2(s)c(s)\alpha_1(s-r)\theta(s-r)\mathrm{e}^{-\varepsilon\int_0^{s-r}\alpha_1(\tau)\,\mathrm{d}\tau}\mathrm{e}^{-\varepsilon\int_{s-r}^{s}(\beta_1(\tau)+\gamma_1(\tau))\,\mathrm{d}\tau}\,\mathrm{d}s \right\} \mathrm{d}r
\end{aligned}
$$

for all $t \geq 0$ if we take sufficiently large $T > 0$. Taking Laplace transforms leads to

$$
\widehat{\psi_{1,b+T}}(\lambda) \geq \widehat{\psi_{1,b+T}}(\lambda) F(\varepsilon, \lambda, T), \tag{1.8.7}
$$

where

$$
\begin{aligned}
F(\varepsilon,\lambda,T) := \int_0^\infty \mathrm{e}^{-\lambda r} \Bigg\{ \int_r^{r+T} & \alpha_2(s)c(s)\alpha_1(s-r)\theta(s-r) \\
& \times \mathrm{e}^{-\varepsilon\int_0^{s-r}\alpha_1(\tau)\,\mathrm{d}\tau - \varepsilon\int_{s-r}^{s}(\beta_1(\tau)+\gamma_1(\tau))\,\mathrm{d}\tau}\,\mathrm{d}s \Bigg\} \mathrm{d}r.
\end{aligned} \tag{1.8.8}
$$

Since ψ_1 is bounded, the Laplace transform of $\psi_{1,b+T}(\lambda)$ is defined for all $\lambda \geq 0$. Moreover,

$$
\begin{aligned}
\lim_{T\to\infty} F(0,0,T) &= \int_0^\infty \left\{ \int_r^\infty \alpha_2(s)c(s)\alpha_1(s-r)\theta(s-r)\,\mathrm{d}s \right\} \mathrm{d}r \\
&= \int_0^\omega \left\{ \int_r^\omega \alpha_2(s)c(s)\alpha_1(s-r)\theta(s-r)\,\mathrm{d}s \right\} \mathrm{d}r,
\end{aligned}
$$

which is equal to $R_0 > 1$ through the change of variables: $\sigma = s,\ b = s - r$. Since $F(\varepsilon, \lambda, T)$ is continuous, we see that $F(\varepsilon, \lambda, T) > 1$ if $\varepsilon, \lambda > 0$ are chosen small enough and T large enough. Then (1.8.7) implies that $\widehat{\psi_{1,b+T}}(\lambda) = 0$, which means $\psi_{1,b+T} = 0$ a.e. on $[0, \infty)$, i.e., $\psi_1(t) = 0$ for almost every $t > b + T$.

Take some $t = T_1 > b + T$ for which $\psi_1(T_1) = 0$ holds, then we have $y(T_1, a) = 0$ for almost every $a \in (\omega - b_0, \omega)$. On the other hand, if we take any p, q such that $0 < p < q < \omega - b_0$ and $q - p < b_0$, then $(p + \tau, q + \tau) \subset (\omega - b_0, \omega)$ for any $\tau \in (\omega - b_0 - p, \omega - q)$. There exists some $\tau \in (\omega - b_0 - p, \omega - q)$ such that $\psi_1(T_1 + \tau) = 0$, so that $y(T_1 + \tau, a) = 0$ for almost every $a \in (p + \tau, q + \tau)$, which implies by means of Lemma 1.8.1 that $y(T_1, a) = 0$ for almost every $a \in (p, q)$. Hence $y(T_1, a) = 0$ for almost every $a \in (0, \omega)$, from which it follows that $\|y(T_1)\| = 0$.

However, we have $\|y(T_1)\| > 0$, because of the assumption $\|y_0\| > 0$ and Lemma 1.8.2. This contradicts the above result. □

Next, let us show that uniform weak persistence implies uniform strong persistence for the system (1.4.4).

Let $\Phi : \mathbb{R}_+ \times \tilde{\Omega} \to \tilde{\Omega}$ be the semiflow induced by the system (1.4.4) via

$$
\Phi(t, {}^t(x_0, y_0, z_0)) = {}^t(x(t,\cdot), y(t,\cdot), z(t,\cdot)),
$$

where

$$
\tilde{\Omega} := \{ {}^t(x, y, z) \in (L^1_+(0,\omega))^3 \mid x + y + z = 1 \}
$$

is the state space of the system.

For sufficiently large $T_0 (> \omega)$, set

$$
B := \{ \Phi(T_0, {}^t(x_0, y_0, z_0)) \mid {}^t(x_0, y_0, z_0) \in \tilde{\Omega} \}.
$$

Then we have the following:

Lemma 1.8.4. *B is relatively compact in $\tilde{\Omega}$.*

Proof. It is obvious that B is bounded. Observe that Assumption 1.5.1 with PMA implies that for any ε there exists $\delta > 0$ such that

$$
\int_0^\omega |\alpha_1(a+h) - \alpha_1(a)|\,\mathrm{d}a < \varepsilon,
$$

$$\int_0^\omega |\theta(a+h)-\theta(a)|\,\mathrm{d}a < \varepsilon$$

whenever $|h| < \delta$. We can assume that $\delta < \varepsilon$.

Let $\Phi(t, {}^t(x_0, y_0, z_0)) = {}^t(x(t,\cdot), y(t,\cdot), z(t,\cdot)) \in B$. Let $P_j\,{}^t(u_1, u_2, u_3) := u_j$ $(j = 1, 2, 3)$ be the projection operator on $(L^1(0,\omega))^3$.

First, we find that, if $a, a+h \in (0,\omega)$, then

$$\begin{aligned}
|x(t,a+h)-x(t,a)| &= |\mathrm{e}^{-\int_0^{a+h}\lambda_1(t-a-h+\tau,\tau)\,\mathrm{d}\tau} - \mathrm{e}^{-\int_0^a \lambda_1(t-a+\tau,\tau)\,\mathrm{d}\tau}| \\
&\le \left|\int_0^{a+h}\lambda_1(t-a-h+\tau,\tau)\,\mathrm{d}\tau - \int_0^a \lambda_1(t-a+\tau,\tau)\,\mathrm{d}\tau\right| \\
&= \left|\int_0^h \lambda_1(t-a-h+\tau,\tau)\,\mathrm{d}\tau + \int_0^a (\alpha_1(\tau+h)-\alpha_1(\tau))\psi_1(t-a+\tau)\,\mathrm{d}\tau\right| \\
&\le \left|\int_0^h \lambda_1(t-a-h+\tau,\tau)\,\mathrm{d}\tau\right| + \int_0^a |\alpha_1(\tau+h)-\alpha_1(\tau)|\psi_1(t-a+\tau)\,\mathrm{d}\tau \\
&\le \alpha^\infty|h| + \alpha_2^\infty\varepsilon \\
&\le (\alpha^\infty + \alpha_2^\infty)\varepsilon.
\end{aligned}$$

This implies that, if $h > 0$, then

$$\begin{aligned}
\int_0^\omega |x(t,a+h)-x(t,a)|\,\mathrm{d}a &= \int_0^{\omega-h}|x(t,a+h)-x(t,a)|\,\mathrm{d}a + \int_{\omega-h}^\omega |0-x(t,a)|\,\mathrm{d}a \\
&\le (\alpha^\infty+\alpha_2^\infty)\omega\varepsilon + h \\
&\le \{(\alpha^\infty+\alpha_2^\infty)\omega + 1\}\varepsilon.
\end{aligned} \tag{1.8.9}$$

The case $h < 0$ is as well. Hence, Theorem 2.2.4 yields that P_1B is relatively compact in $P_1\tilde{\Omega}$.

Next, let us see that the same argument leads to the fact that P_2B is relatively compact in $P_2\tilde{\Omega}$. If $a, a+h \in (0,\omega)$, then

$$\begin{aligned}
&y(t,a+h)-y(t,a) \\
&= \int_0^h \lambda_1(t-a-h+\sigma,\sigma)\theta(\sigma)x(t-a-h+\sigma,\sigma)\mathrm{e}^{-\int_\sigma^{a+h}\lambda_2(t-a-h+\tau,\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma \\
&+ \int_0^a \psi_1(t-a+\sigma)\{\alpha_1(\sigma+h)\theta(\sigma+h)x(t-a+\sigma,\sigma+h) \\
&\quad \times \mathrm{e}^{-\int_\sigma^a \lambda_2(t-a+\tau,\tau+h)\,\mathrm{d}\tau} - \alpha_1(\sigma)\theta(\sigma)x(t-a+\sigma,\sigma)\mathrm{e}^{-\int_\sigma^a\lambda_2(t-a+\tau,\tau)\,\mathrm{d}\tau}\}\,\mathrm{d}\sigma.
\end{aligned}$$

It follows that

$$\begin{aligned}
&|y(t,a+h)-y(t,a)| \\
&\le \alpha^\infty|h| + \alpha_2^\infty\int_0^a|\alpha_1(\sigma+h)-\alpha_1(\sigma)|\,\mathrm{d}\sigma + \alpha^\infty\int_0^a|\theta(\sigma+h)-\theta(\sigma)|\,\mathrm{d}\sigma \\
&\quad + \alpha^\infty\int_0^a |x(t-a+\sigma,\sigma+h)-x(t-a+\sigma,\sigma)|\,\mathrm{d}\sigma \\
&\quad + \alpha^\infty\int_0^a |\mathrm{e}^{-\int_\sigma^a\lambda_2(t-a+\tau,\tau+h)\,\mathrm{d}\tau} - \mathrm{e}^{-\int_\sigma^a\lambda_2(t-a+\tau,\tau)\,\mathrm{d}\tau}|\,\mathrm{d}\sigma \\
&\le \alpha^\infty\varepsilon + \alpha_2^\infty\varepsilon + \alpha^\infty\varepsilon + \{(\alpha^\infty+\alpha_2^\infty)a+1\}\varepsilon \\
&\quad + \alpha^\infty\int_0^a\left\{\int_\sigma^a |\lambda_2(t-a+\tau,\tau+h)-\lambda_2(t-a+\tau,\tau)|\,\mathrm{d}\tau\right\}\mathrm{d}\sigma,
\end{aligned}$$

where the evaluation similar to (1.8.9) is made. Let us evaluate the last term of the above inequality. We have

$$\begin{aligned}
&\lambda_2(t-a+\tau,\tau+h)-\lambda_2(t-a+\tau,\tau) \\
&= \psi_2(t-a+\tau)(\beta_1(\tau+h)-\beta_1(\tau)) + \psi_3(t-a+\tau)(\gamma_1(\tau+h)-\gamma_1(\tau)),
\end{aligned}$$

and the evaluation

$$\int_\sigma^a |\psi_2(t-a+\tau)(\beta_1(\tau+h)-\beta_1(\tau))|\,\mathrm{d}\tau \le \beta_2^\infty\varepsilon \tag{1.8.10}$$

holds whenever $|h| < \delta$ if $\delta > 0$ is chosen small enough. This is true if $\beta_2^\infty = 0$, because then $\beta_2(\sigma) = 0$ for almost every $\sigma \in (0, \omega)$ and it follows that $\psi_2(t) = 0$ for all t. Otherwise, $\beta_2(\sigma) > 0$ for some $\sigma \in (0, \omega)$, so that Assumption 1.5.1 implies that

$$\int_0^\omega |\beta_1(a+h) - \beta_1(a)| \,\mathrm{d}a < \varepsilon$$

holds whenever $|h| < \delta$ if $\delta > 0$ is chosen small enough. This gives the evaluation (1.8.10). The same argument leads to the evaluation

$$\int_\sigma^a |\psi_3(t-a+\tau)(\gamma_1(\tau+h) - \gamma_1(\tau))| \,\mathrm{d}\tau \le \gamma_2^\infty \varepsilon.$$

whenever $|h| < \delta$ if $\delta > 0$ is chosen small enough. Hence we have

$$\begin{aligned} \int_0^a \left\{ \int_\sigma^a |\lambda_2(t-a+\tau, \tau+h) - \lambda_2(t-a+\tau, \tau)| \,\mathrm{d}\tau \right\} \mathrm{d}\sigma &\le \int_0^a (\beta_2^\infty + \gamma_2^\infty)\varepsilon \,\mathrm{d}\sigma \\ &\le (\beta_2^\infty + \gamma_2^\infty)\varepsilon a, \end{aligned}$$

which yields that $|y(t, a+h) - y(t, a)| \le C_0 \varepsilon$ for some constant $C_0 > 0$.

The same evaluation as (1.8.9) implies that

$$\int_0^\omega |y(t, a+h) - y(t, a)| \,\mathrm{d}a \le (C_0 \omega + 1)\varepsilon.$$

Therefore we obtain that $P_2 B$ is relatively compact in $P_2 \tilde{\Omega}$.

Then it is obvious that $P_3 B$ is relatively compact in $P_3 \tilde{\Omega}$ and we have the conclusion. □

Theorem 1.8.5. *If $R_0 > 1$, then the rumor is uniformly strongly persistent.*

Proof. Since we have already shown that the rumor is uniformly weakly persistent, the assertion is a direct consequence of Theorem 2.2.3. All we have to do is to make sure the "compactness condition."

It is clear that the autonomous semiflow Φ is continuous. We define $\rho : \tilde{\Omega} \to \mathbb{R}_+$ by

$$\rho({}^t(x, y, z)) := \int_0^\omega c(a) y(a) \,\mathrm{d}a, \quad {}^t(x, y, z) \in \tilde{\Omega}.$$

Then we find that ρ is uniformly continuous on $\tilde{\Omega}$. Lemma 1.8.2 implies that

$$\rho(\Phi(t, {}^t(x, y, z)) > 0 \text{ for all } t > 0 \text{ whenever } \rho(\Phi({}^t(x, y, z)) > 0.$$

By the definition of B, we have $\Phi(t, {}^t(x, y, z)) \to \bar{B}$ as $t \to \infty$. In addition, Lemma 1.8.4 implies that $\bar{B}$ is compact. Therefore we obtain the conclusion. □

1.9 Discussion

In this chapter we have examined rumor transmission models motivated by S-I-R type epidemic models.

We have derived the global behavior of the age-independent rumor transmission models which are extensions of the deterministic Daley–Kendall model. The result is that there is no undamped oscillation and the solution converges to some equilibrium as $t \to \infty$.

In addition, we have shown that, in the age-structured transmission model of a constant rumor, there exists a threshold value $R_0 := r(\tilde{T})$ given as the spectral radius of the positive linear operator $\tilde{T}$ and that RFE is the only equilibrium and globally asymptotically stable if $R_0 < 1$ and at least one REE exists if and only if $R_0 > 1$. Moreover, assuming PMA, we have shown that $R_0 > 1$ implies that the rumor is uniformly strongly persistent and REE is locally asymptotically stable if $|R_0 - 1|$ is small enough.

Assuming PMA is a huge simplification. Without PMA, even uniform strong persistence or the local stability of REE is a difficult problem.

As for the age-structured model considered in this chapter, how many REEs exist in the case $R_0 > 1$ is left as open problems, which should be investigated in the near future. It would be also an interesting open problem whether the stable REE could lose its stability and lead a bifurcation of periodic or chaotic solutions. Moreover, the age-structured model for a variable rumor remains to be analyzed.

Our rumor transmission models could be extended to several directions: one way is to introduce more fine structures such as how the transition rate from the stifler into the susceptible class depends on duration in the case of a variable rumor. Another important way of extension is to introduce the effect on mass communication, which could be considered as both rumor-source and rumor-"vaccination." We will discuss it in Chapter 3. Moreover, it would be interesting to consider the case that several conflicting rumors are transmitted ([20, 21, 50, 51]), which would be useful in the control of a rumor which is troublesome for an individual or an organization.

Chapter 2

A note on persistence about structured population models

Abstract

In this chapter we report some results on persistence in two structured population models: a chronic-age-structured epidemic model and an age-duration-structured epidemic model. Regarding these models we observe that the system is uniformly strongly persistent, which means, roughly speaking, that the proportion of infected subpopulation is bounded away from 0 and the bound does not depend on the initial data after a sufficient long time, if the basic reproduction ratio is larger than 1. We derive this by adopting Thieme's technique, which requires some conditions about positivity and compactness. Although the compactness condition is rather difficult to show in general infinite-dimensional function spaces, we can apply Fréchet–Kolmogorov L^1-compactness criteria to our models. The two examples that we study illuminate a useful method to show persistence in structured population models.

Keywords: structured population model; persistence; compactness condition

2.1 Introduction

Persistence and permanence are considered as important concepts of dynamical systems and of systems in ecology, epidemics etc. These concepts concern the long-term survival of each component of a system of interacting components, for example of species in an ecological community, genotypes, strategies in an evolutionary game, or polynucleotides competing for energy sources. In epidemics, they are related to the question whether the disease is never eradicated after sufficiently long time. Persistence means the long-term survival of some of all components of a system, while permanence addresses the limits of growth for some (or all) components of the system. We refer to Hutson and Schmitt [52] and Thieme [53] for background information and references.

An important theorem concerning them was obtained by Thieme [49](or see [48]), which shows that uniform weak persistence implies uniform strong persistence. This means in the words of population dynamics that, if the size of a population, while it may come arbitrarily close to 0 every now and then, always climbs back to a level that is eventually independent of the initial data, then the size is bounded away from 0 and the bound is eventually independent of the initial data. The theorem has been gradually used in structured population models (for example, see [54, 55, 56, 57]). However, the number of systems proved to have the property of persistence (permanence) has been at present far fewer than those proved to have the property of the local asymptotic stability (or instability) of an equilibrium. Its major reason is that the former property was mathematically established later than the latter. Another reason is seemingly that it is difficult to find an attracting (or absorbing) set and to show compactness conditions linked with persistence and permanence. However, we consider the problems here on a finite domain for the structural variable which require an approach somewhat different from those used so far — that is, the method of using Thieme's technique [47] and Fréchet–Kolmogorov L^1-compactness criteria. The method is fully presented in [1, §8].

This chapter is structured as follows. Section 2.2 presents some ideas from persistence theory. We define the concept of persistence rigorously in the words of dynamical systems, and see what theorems are useful for showing uniform strong persistence in structured population models. We illuminate the method by proving uniform strong persistence in two models in the following sections. In Section 2.3 we deal with a chronic-age-structured epidemic model which was first proposed by Martcheva and Castillo-Chávez [58].

In Section 2.4 we discuss an age-duration-structured epidemic model which was first proposed by Inaba [59]. Section 2.5 summarizes our discussions and presents an open problem.

2.2 Mathematical tools used in this chapter

Let X be a metric space with metric d. Let $\Phi : [0, \infty) \times X \to X$ be a semiflow on X, i.e., $\Phi_t \circ \Phi_s = \Phi_{t+s}$ for all $t, s \geq 0$, where Φ_t denotes the mapping $\Phi(t, \cdot) : X \to X$. Let $\rho : X \to [0, \infty)$ be a nonnegative uniformly continuous functional on X. We assume that the composition $\sigma := \rho \circ \Phi$ is a continuous mapping from $[0, \infty) \times X \to \mathbb{R}$.

Definition . *(i) Φ is called* weakly ρ-persistent *if* $\limsup_{t\to\infty} \sigma(t, x) > 0$ *whenever* $\rho(x) > 0$.

(ii) Φ is called strongly ρ-persistent *if* $\liminf_{t\to\infty} \sigma(t, x) > 0$ *whenever* $\rho(x) > 0$.

(iii) Φ is called uniformly weakly ρ-persistent *if there exists some $\varepsilon > 0$ such that* $\limsup_{t\to\infty} \sigma(t, x) > \varepsilon$ *whenever* $\rho(x) > 0$.

(iv) Φ is called uniformly strongly ρ-persistent *if there exists some $\varepsilon > 0$ such that* $\liminf_{t\to\infty} \sigma(t, x) > \varepsilon$ *whenever* $\rho(x) > 0$.

If no misunderstanding about the functional ρ is possible, we use persistent *rather than ρ-persistent.*

Example 2.2.1. Let us consider the prey-predator model proposed by Lotka and Volterra:

$$\begin{cases} x'(t) = x(t)\{\alpha - \beta y(t)\}, \\ y'(t) = -y(t)\{\gamma - \delta x(t)\}, \end{cases}$$
$$x(0) = x_0 \geq 0, \quad y(0) = y_0 \geq 0,$$

where $x(t)$ denotes the population density of the prey at time t, $y(t)$ the population density of the predator at time t and α, β, γ, δ are strictly positive constants. This system induces a continuous semiflow Φ on the state space $\Omega := \mathbb{R}_+ \times \mathbb{R}_+$, where $\mathbb{R}_+$ denotes the set of nonnegative real numbers.

For example, we define ρ_1 as $\rho_1(x, y) := x$, then $\sigma_1(t, (x_0, y_0)) := \rho_1(\Phi(t, (x_0, y_0)))$ represents the population density of the preys at time t under the condition that the population density of the prey and that of the predator at time 0 are x_0 and y_0 respectively. We can easily find that Φ is strongly ρ_1-persistent, that it is uniformly weakly ρ_1-persistent with $\limsup_{t\to\infty} \sigma_1(t, (x, y)) > \gamma/(2\delta)$ whenever $\rho_1(x, y) > 0$ and that it is not uniformly strongly ρ_1-persistent because for any $\varepsilon \in (0, \gamma/\delta)$ $\liminf_{t\to\infty} \sigma_1(t, (\varepsilon, \beta/\alpha)) = \varepsilon$.

On the other hand, let us define ρ_2 as $\rho_2(x, y) := y$, then $\sigma_2(t, (x_0, y_0)) := \rho_2(\Phi(t, (x_0, y_0)))$ represents the population density of the predator at time t under the condition that the population density of the prey and that of the predator at time 0 are x_0 and y_0 respectively. Note that Φ is not weakly ρ_2-persistent, because the predators eventually die out if there are no preys at time $t = 0$.

We consider the following compactness condition (C):

There exist some $\varepsilon_0 > 0$ and a closed subset B of X with the following properties:

(i) If $\rho(x) \leq \varepsilon_0$, then $\lim_{t\to\infty} d(\Phi_t(x), B) = 0$, that is, for every $\varepsilon > 0$, there exists some $t > 0$ such that for every $s > t$ some $b \in B$ can be found with $d(\Phi(s, x), b) < \varepsilon$.

(ii) For every $\varepsilon_1 \in (0, \varepsilon_0)$, the intersection $B \cap \rho^{-1}[\varepsilon_1, \varepsilon_0]$ is compact.

Remark 2.2.2. If we take B as $\Phi_{t_0}(\Omega)$, where t_0 is a nonnegative constant, then condition (C)-a) is satisfied by itself for any ε. The reason is as follows: take any $x \in \Omega$, then for $t > t_0$ we have $\Phi_t(x) = \Phi_{t_0}(\Phi_{t-t_0}(x)) \in \Phi_{t_0}(\Omega) = B$, which gives rise to the fact that $d(\Phi_t(x), B) = 0$ for any $t > t_0$.

The next theorem is due to Thieme [49].

Theorem 2.2.3. *Let the compactness condition (C) hold and assume that $\sigma(t, x) > 0$ for all $t \geq 0$ whenever $\rho(x) > 0$. Then uniform weak persistence implies uniform strong persistence.*

The reader may easily find the above theorem's proof in [49, 48].

When we apply this theorem to some structured population models, the metric space (X, d) is often taken as the finite product space of $L^1(S)$ for some $S \subset \mathbb{R}^n$, where $\mathbb{R}^n$ denotes Euclidian n-space. Hence, it is necessary to have some compactness criteria in function spaces in order to verify the compactness condition (C)-b). Fréchet–Kolmogorov's Theorem generalised to $\mathbb{R}^n$ ([60, Theorem IV.8.21]) is a useful criterion:

Theorem 2.2.4. *Let S be a subset of $\mathbb{R}^n$, $\mathcal{B}$ the σ-algebra of Borel subsets of S, μ the Lebesgue measure on $\mathcal{B}$, and $1 \le p < \infty$. Then a subset K of $L^p(S, \mathcal{B}, \mu)$ is relatively compact if and only if it is bounded in $L^p(S, \mathcal{B}, \mu)$ and the following limits are uniform for $f \in K$:*

$$\lim_{h_1,\dots,h_n \to 0} \int_S |f(x_1 + h_1, \dots, x_n + h_n) - f(x_1, \dots, x_n)|^p \, dx_1 \cdots dx_n = 0, \tag{2.2.1}$$

$$\lim_{M \to \infty} \int_{S \setminus C_M} |f(x)|^p \, dx = 0, \tag{2.2.2}$$

where C_M is the cube $-M \le x_1, \dots, x_n \le M$.

In the following, the notation of $\mathcal{B}$ and μ is omitted. If S is an interval of $\mathbb{R}$ and K is a subset of $L^1(S)$ consisting of absolutely continuous functions, condition (2.2.1) can be substituted into that of uniform boundedness of the derivative of the functions in K, which is easier to prove. We will use the following corollary in the next section.

Corollary 2.2.5. *Let S be an interval of $\mathbb{R}$ and K a subset of $L^1(S)$ consisting of absolutely continuous functions. Then K is relatively compact if condition (2.2.2) holds and K and $\{f' \mid f \in K\}$ are bounded in $L^1(S)$.*

Proof. We have only to consider condition (2.2.1), since the other conditions hold. From our assumption, there exists a positive constant C such that $\|f'\|_{L^1(S)} \le C$ holds for any $f \in K$. Then we have for $x \in S$ and $h > 0$

$$|f(x+h) - f(x)| = \left| \int_x^{x+h} f'(y) \, dy \right| \le \int_x^{x+h} |f'(y)| \, dy.$$

By changing the order of integrals we obtain

$$\int_S |f(x+h) - f(x)| \le \int_S |f'(y)| h \, dy = \|f'\|_{L^1(S)} h \le Ch.$$

The case $h < 0$ is the same. Hence we find that condition (2.2.1) holds and the limit is uniform for $f \in K$. $\square$

2.3 Persistence in a chronic-age-structured population model

Let us consider the following system proposed by Martcheva and Castillo-Chávez [58, (3.2)(3.4)(3.5)] as an epidemiological model of hepatitis C with a chronic infectious class structured by age-since-infection:

$$\begin{cases} s' = b(1-s) - \gamma i s - s D_u(t) + A_u(t), \\ i' = \gamma i s + s D_u(t) - (b+k) i, \\ (\partial_t + \partial_\theta) u(\theta, t) = -(b + \alpha(\theta)) u(\theta, t), \\ u(0, t) = k i, \\ s(0) = s_0; \; i(0) = i_0; \; u(\theta, 0) = u_0(\theta) \end{cases} \tag{2.3.1}$$

with the relation

$$s(t) + i(t) + \int_0^\infty u(\theta, t) = 1,$$

where

$$D_u(t) := \int_0^\infty \delta(\theta) u(\theta, t) \, d\theta, \; A_u(t) := \int_0^\infty \alpha(\theta) u(\theta, t) \, d\theta.$$

$s(t)$ denotes the portion of the susceptible population in the total population at time t, $i(t)$ the portion of the population infected with acute hepatitis C in the total population at time t and $u(\theta, t)$ the age-since-infection density at time t — normalised by divided by the total population — of individuals infected with chronic hepatitis C (with or without cirrhosis), where θ means the time spent in the chronic stage. b means birth/recruitment rate, γ the effective contact rate of individuals with acute hepatitis C, k the rate of progression to chronic stage, $\alpha(\theta)$ the age-since-infection structured treatment/recovery rate for the chronic stage, $\delta(\theta)$ the age-since-infection structured effective contact rate of individuals with chronic hepatitis C.

In the following, we change the notation of s, i, u into x, y, z respectively for convenience. Moreover, we introduce the finite maximum age-since-infection ω, which the original paper [58] assumed infinite. The main reason for this modification is to simplify the discussion. In reality, it is reasonable to assume that the maximum age-since-infection is finite, since the human life span is finite. So, the system is rewritten as follows:

$$\begin{cases} x'(t) = b\{1 - x(t)\} - \gamma x(t)y(t) - x(t)D_z(t) + A_z(t), \\ y'(t) = \gamma x(t)y(t) + x(t)D_z(t) - (b+k)y(t), \\ (\partial_t + \partial_\theta)z(\theta,\, t) = -\{b + \alpha(\theta)\}z(\theta, t), \\ z(0,t) = ky(t), \\ x(0) = x_0;\ y(0) = y_0;\ z(\theta, 0) = z_0(\theta) \end{cases} \tag{2.3.2}$$

with

$$x(t) + y(t) + \int_0^\omega z(\theta, t)\,\mathrm{d}\theta = 1,$$

$$D_z(t) := \int_0^\omega \delta(\theta)z(\theta,t)\,\mathrm{d}\theta,\ A_z(t) := \int_0^\omega \alpha(\theta)z(\theta,t)\,\mathrm{d}\theta.$$

Assumption 2.3.1. *(i) b, k, γ are strictly positive constants.*

(ii) $\delta(\theta)$ is bounded above and strictly positive a.e. on $[0,\omega)$. We denote by δ^∞ the supremum of δ.

(iii) $\alpha(\cdot)$ is nonnegative, locally integrable on $[0,\omega)$ and $\int_0^\omega \alpha(\theta)\,\mathrm{d}\theta = \infty$, which means that every individual at z-stage returns into x-stage by the time the age-since-infection θ reaches ω.

For this system, the basic reproduction ratio is defined as follows:

$$R_0 := \frac{\gamma + \int_0^\omega \delta(\theta)k\mathcal{L}(\theta)\,\mathrm{d}\theta}{b+k}, \tag{2.3.3}$$

where $\mathcal{L}(\theta) := \mathrm{e}^{-b\theta}\exp(-\int_0^\theta \alpha(\tau)\,\mathrm{d}\tau)$.

Martcheva and Castillo-Chávez [58] proved the following theorem, in which it makes no difference whether ω is finite:

Theorem 2.3.2. *Let Assumption 2.3.1 hold, then we have:*

(i) If $R_0 < 1$ then the trivial equilibrium $(x,y,z) = (1,0,0)$ is globally asymptotically stable.

(ii) If $R_0 > 1$ the trivial equilibrium is unstable and there is a unique nontrivial equilibrium, which is locally asymptotically stable if $\int_0^\omega k\mathcal{L}(\theta)\,\mathrm{d}\theta < 1$.

Next, let us consider the semiflow induced by system (2.3.2). We define its state space Ω as follows:

$$\Omega := \left\{(x,y,z) \in \mathbb{R}_+ \times \mathbb{R}_+ \times L^1_+(0,\omega)\ \middle|\ x + y + \|z\| = 1,\ \int_0^\omega \alpha(\theta)z(\theta)\,\mathrm{d}\theta < \infty\right\}.$$

Then we can prove the following proposition. It means the system's well-posedness, i.e., that system (2.3.2) has a unique continuous solution with values in Ω and the solution depends on its initial value continuously.

Proposition 2.3.3. *System (2.3.2) induces a continuous semiflow Φ on the state space Ω.*

Proof. From the third equation of system (2.3.2) we obtain

$$z(\theta,t) = \begin{cases} z_0(\theta - t)\mathrm{e}^{-bt}\Gamma(\theta)/\Gamma(\theta - t) & \text{if } \omega > \theta > t, \\ ky(t-\theta)\mathcal{L}(\theta) & \text{if } \theta < \min\{t,\omega\}, \end{cases} \tag{2.3.4}$$

where $\Gamma(\theta) := \exp(-\int_0^\theta \alpha(\tau)\,\mathrm{d}\tau)$. We define $\tilde{\alpha}(\theta) := \alpha(\theta)\mathcal{L}(\theta)$ and $\tilde{\delta}(\theta) := \delta(\theta)\mathcal{L}(\theta)$, and the asterisk denotes the convolution operation defined by

$$(f * g)(t) := \int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau.$$

Substituting (2.3.4) into the equation for x and y in system (2.3.2), we get

$$\begin{cases} x'(t) = b\{1 - x(t)\} - \gamma x(t)y(t) - x(t)\{k(\tilde{\delta} * y)(t) + F_1(t)\} + k(\tilde{\alpha} * y)(t) + F_2(t), \\ y'(t) = \gamma x(t)y(t) + x(t)\{k(\tilde{\delta} * y)(t) + F_1(t)\} - (b+k)y(t), \end{cases} \tag{2.3.5}$$

where we define $\tilde{\alpha}(\theta) = \tilde{\delta}(\theta) = 0$ for $\mathbb{R} \setminus [0, \omega]$ and $F_1(t)$, $F_2(t)$ are given functions depending on the initial data as

$$F_1(t) := \int_t^{\omega} \delta(\theta) z_0(\theta - t) \mathrm{e}^{-bt} \Gamma(\theta)/\Gamma(\theta - t)\, \mathrm{d}\theta,$$
$$F_2(t) := \int_t^{\omega} \alpha(\theta) z_0(\theta - t) \mathrm{e}^{-bt} \Gamma(\theta)/\Gamma(\theta - t)\, \mathrm{d}\theta$$

for $t < \omega$ and $F_1(t) := 0,\ F_2(t) := 0$ for $t \geq \omega$. Equations (2.3.5) are regarded as a two-dimensional finitely-retarded functional differential equation, since the minimum of retard is $-\omega$. Then we can obtain the conclusion by applying standard local existence and uniqueness results for retarded functional differential equation (see Hale [61, Section 2.2]) to the integro-differential equation (2.3.5). Global existence and continuous dependency on the initial value will follow immediately. □

Now, we come around to prepare for the proof of uniform strong persistence. Let

$$\rho(x, y, z) := y + \|z\|, \quad \sigma(t, (x, y, z)) := \rho(\Phi(t, (x, y, z)))$$

for $(x, y, z) \in \Omega$ and $t \geq 0$. We can easily find that $\rho : \Omega \to \mathbb{R}$ is nonnegative and uniformly continuous.

Lemma 2.3.4. *We have $z(\theta, t) > 0$ if either of the following two conditions hold:*

(i) $\omega > \theta > t$ and $z_0(\theta - t) > 0$, or

(ii) $\theta < \min\{t, \omega\}$ and $y(t - \theta) > 0$.

Proof. Let us see (2.3.4). Noticing that $\Gamma(s) > 0$ for all $0 \leq s < \omega$, we have the conclusion. □

Lemma 2.3.5. *If $y_0 > 0$, then $y(t) > 0$ for all $t \geq 0$.*

Proof. From the second equation of system (2.3.2) and the non-negativity of x, y, z, we have the inequality

$$y'(t) \geq -(b + k) y(t),$$

which leads to the inequality

$$y(t) \geq y_0 \mathrm{e}^{-(b+k)t} > 0.$$

□

Lemma 2.3.6. *$\sigma(t, (x_0, y_0, z_0)) > 0$ for all $t \geq 0$ whenever $\rho(x_0, y_0, z_0) > 0$.*

Proof. If $y_0 > 0$, then the assertion is obvious from Lemma 2.3.5. Let us consider the case $y_0 = 0$ and $\|z_0\| > 0$.

Assume that, for some $t_0 > 0$, $\sigma(t_0, (x_0, y_0, z_0)) = 0$ holds, i.e., $y(t_0) = 0$ and $z(\theta, t_0) = 0$ for a.e. $\theta \in [0, \omega]$. Then we can find from Lemma 2.3.4 (ii) that $y(t) = 0$ for all $t \in [0, t_0]$. On the other hand, the assumption of $\|z_0\| > 0$, Lemma 2.3.4 and the continuity of $D_z(t)$ imply that there exists some $t_1 > 0$ such that $D_z(t) > 0$ for all $t \in [0, t_1)$. In addition, it is easily seen from the first equation of the system (2.3.2) that $x(t) > 0$ for all $t > 0$. Hence, $x(t) D_z(t) > 0$ for all $t \in (0, t_1)$, which leads to $y(t) > 0$ for sufficiently small $t > 0$. This is contradictory. □

Lemma 2.3.7. *System (2.3.2) satisfies the compactness condition (C) with $\varepsilon_0 = 1$ and B the closure of $\Phi_{T_0}(\Omega)$, where $T_0 > \omega$ is a sufficiently large positive number.*

Proof. Taking Remark 2.2.2 into consideration, we find that we have only to prove that B is relatively compact in Ω. Let P_1, P_2, P_3 be projections defined by

$$P_1(x, y, z) = x, \ P_2(x, y, z) = y, \ P_3(x, y, z) = z$$

for $(x, y, z) \in \mathbb{R} \times \mathbb{R} \times L^1(0, \omega)$. We have $P_1 B,\ P_2 B \subset [0, 1]$, so $P_1 B$ and $P_2 B$ are relatively compact. Then, all we have to do is to prove that $P_3 B$ is relatively compact, since the direct product space of compact spaces is compact (Tychonoff's Theorem) and the direct product space of the closure of topological spaces X_λ $(\lambda \in \Lambda)$ is the closure of $\prod_{\lambda \in \Lambda} X_\lambda$. To this aim, we use Corollary 2.2.5.

Let $(x(t), y(t), z(\cdot, t))$ be a solution of system (2.3.2), where we take an initial condition $(x_0, y_0, z_0(\cdot))$ arbitrarily. Noticing that $x(t),\ y(t),\ \|z(\cdot, t)\| \in [0,\ 1]$, it follows from the equation for y in (2.3.2) that

$$|y'(t)| \leq \gamma + \left| \int_0^{\omega} \delta(\theta) z(\theta,\, t)\, \mathrm{d}\theta \right| + b + k$$

$$\begin{aligned}&\leq \gamma+\delta^{\infty}\int_0^{\omega} z(\theta,\, t)\, \mathrm{d}\theta + b + k\\ &\leq \gamma+\delta^{\infty} + b + k\end{aligned} \tag{2.3.6}$$

for all $t \geq 0$. And by the definition of $\mathcal{L}(\theta)$ we have that

$$\begin{aligned}\int_0^{\omega} |\mathcal{L}'(\theta)|\, \mathrm{d}\theta &= -\int_0^{\omega} \mathcal{L}'(\theta)\, \mathrm{d}\theta\\ &= \mathcal{L}(0) - \mathcal{L}(\omega) = 1.\end{aligned} \tag{2.3.7}$$

By using (2.3.4), (2.3.6) and (2.3.7), we obtain for any $t \geq T_0$

$$\begin{aligned}\|\partial_\theta z(\cdot,\, t)\| &= \int_0^{\omega} |-ky'(t-\theta)\mathcal{L}(\theta) + ky(t-\theta)\mathcal{L}'(\theta)|\, \mathrm{d}\theta\\ &\leq k(\gamma+\delta^{\infty}+b+k)\|\mathcal{L}\| + k,\end{aligned}$$

which implies the boundedness of the set $\{z' \mid (x,\, y,\, z(\cdot)) \in B\}$. From this fact with the boundedness of $P_3 B$, we can apply Corollary 2.2.5 to derive that $P_3 B$ is relatively compact. This completes our proof. □

And now let us prove the result on persistence with the above preparations, which imply that system (2.3.2) satisfies all assumptions in Theorem 2.2.3. In the words of epidemiology, the rate of individuals infected with acute or chronic hepatitis C is bounded away from 0 and the bound is eventually independent of the initial data.

Theorem 2.3.8. *System (2.3.2) is uniformly strongly ρ-persistent if $R_0 > 1$.*

Proof. Thanks to Theorem 2.2.3, we have only to show the uniform weak ρ-persistence for system (2.3.2).

Let $\phi_d(t) := \gamma y(t+d) + D_z(t+d)$ for $d \geq 0$. By applying the method of variation of constant into equations (2.3.2) we have

$$\begin{aligned}y(t+d) &= y_0 \mathrm{e}^{-(b+k)(t+d)} + \int_0^{t+d} x(\tau)\{\gamma y(\tau) + D_z(\tau)\}\mathrm{e}^{-(b+k)(t+d-\tau)}\, \mathrm{d}\tau\\ &\geq \int_d^{t+d} x(\tau)\{\gamma y(\tau) + D_z(\tau)\}\mathrm{e}^{-(b+k)(t+d-\tau)}\, \mathrm{d}\tau\\ &= \int_0^{t} \phi_d(t-r)x(t+d-r)\mathrm{e}^{-(b+k)r}\, \mathrm{d}r,\end{aligned} \tag{2.3.8}$$

where $r := t+d-\tau$. Using this and (2.3.4), for sufficiently large t we have

$$\begin{aligned}D_z(t+d) &= \int_0^{\omega} \delta(\theta)ky(t+d-\theta)\mathcal{L}(\theta)\, \mathrm{d}\theta\\ &\geq \int_0^{\omega} \delta(\theta)k\mathcal{L}(\theta)\int_0^{t+d-\theta} x(\tau)\{\gamma y(\tau) + D_z(\tau)\}\mathrm{e}^{-(b+k)(t+d-\theta-\tau)}\, \mathrm{d}\tau\, \mathrm{d}\theta\\ &= \int_0^{\omega} \delta(\theta)k\mathcal{L}(\theta)\int_\theta^{t+d} x(t+d-r)\{\gamma y(t+d-r) + D_z(t+d-r)\}\mathrm{e}^{-(b+k)(r-\theta)}\, \mathrm{d}r\, \mathrm{d}\theta\\ &= \int_0^{t} \phi_d(t-r)x(t+d-r)\mathrm{e}^{-(b+k)r}\int_0^{r} \delta(\theta)k\mathcal{L}(\theta)\mathrm{e}^{(b+k)\theta}\mathrm{d}\theta\, \mathrm{d}r,\end{aligned} \tag{2.3.9}$$

where in the last line we changed the order of integrals and used the setting of $\delta(\theta)\mathcal{L}(\theta) = \tilde{\delta}(\theta)$: $\tilde{\delta}(\theta) = 0$ for $\mathbb{R} \setminus [0, \omega]$. We substitute (2.3.8) and (2.3.9) into the expression of $\phi_d(t)$ to obtain

$$\phi_d(t) \geq \int_0^{t} \phi_d(t-r)x(t+d-r)\mathrm{e}^{-(b+k)r}\left\{\gamma + \int_0^{r} \delta(\theta)k\mathcal{L}(\theta)\mathrm{e}^{(b+k)\theta}\, \mathrm{d}\theta\right\}\mathrm{d}r. \tag{2.3.10}$$

Suppose that system (2.3.2) is not uniformly weakly ρ-persistent, then for any $\varepsilon \in (0, 1)$ we can choose some initial condition (x_0, y_0, z_0) and some $T > 0$ which satisfy $\rho(x_0, y_0, z_0) > 0$ and

$$y(t) + \int_0^{\omega} z(\theta, t)\, \mathrm{d}\theta \leq \varepsilon, \quad x(t) \geq 1-\varepsilon, \quad \forall t \geq T.$$

If $d > T$ is sufficiently large, then we have

$$\phi_d(t) \geq \int_0^{t} \phi_d(t-r)\cdot(1-\varepsilon)\mathrm{e}^{-(b+k)r}\left\{\gamma + \int_0^{r} \delta(\theta)k\mathcal{L}(\theta)\mathrm{e}^{(b+k)\theta}\, \mathrm{d}\theta\right\}\mathrm{d}r$$

for all $t \geq 0$. Taking Laplace transforms leads to

$$\hat{\phi}_d(\lambda) \geq \hat{\phi}_d(\lambda) F(\varepsilon, \lambda), \tag{2.3.11}$$

where

$$F(\varepsilon, \lambda) := \int_0^\infty \mathrm{e}^{-\lambda r}(1-\varepsilon)\mathrm{e}^{-(b+k)r}\left\{\gamma + \int_0^r \delta(\theta) k \mathcal{L}(\theta) \mathrm{e}^{(b+k)\theta}\, \mathrm{d}\theta\right\} \mathrm{d}r.$$

Since ϕ_d is bounded, $\hat{\phi}_d(\lambda)$ is defined for all $\lambda \geq 0$. We can easily find through the change of integral variables that $F(0,0) = R_0 > 1$. Since $F(\varepsilon, \lambda)$ is continuous, we see that $F(\varepsilon, \lambda) > 1$ if $\varepsilon, \lambda > 0$ are chosen small enough. Then (2.3.11) implies that $\hat{\phi}_d(\lambda) = 0$, which means $\phi_d(t) = 0$ a.e. on $[0, \infty)$, i.e., $y(t) = D_z(t) = 0$ for almost every $t \geq T$. This contradicts Lemma 2.3.6. □

2.4 Persistence in an age-duration-structured population model

Let us consider the following system proposed by Inaba [59] as an age-duration-structured population model for HIV infection in a homosexual community:

$$\begin{cases} (\partial_t + \partial_a)x(t,a) = -\lambda(t,a)x(t,a), \\ (\partial_t + \partial_\tau)y(t,\tau;a) = 0, \\ x(t,0) = 1, \quad y(t,0;a) = \lambda(t,a)x(t,a), \\ \lambda(t,a) = \tilde{C}(P(t)) \int_0^\omega \int_0^b K(a,b,\tau)y(t,\tau;b-\tau)\,\mathrm{d}\tau\,\mathrm{d}b, \\ P(t) = \int_0^\omega Bl(a)\left\{x(t,a) + \int_0^a \Gamma(\tau;a-\tau)y(t,\tau;a-\tau)\,\mathrm{d}\tau\right\}\mathrm{d}a, \\ x(0,a) = x_0(a), \quad y(0,\tau;a) = y_0(\tau;a). \end{cases} \tag{2.4.1}$$

First, we shall follow [59] in order to derive (2.4.1). We divide the homosexual population into three groups: S (uninfected but susceptible), I (HIV infected) and A (fully developed AIDS symptoms). Let $S(t,\, a)$ be the age-density of the S-population at time t and age a, $I(t, \tau;\, a)$ the density of the I-population at time t and age-since-infection (duration) τ which were infected at age a, and $A(t,\, \tau;\, a)$ the density of the A-population at time t and age-since-infection τ for individuals who have developed AIDS at age a. Let B be the birth rate of the S-population, $\mu(a)$ the age-specific natural death rate, $\gamma(\tau;\, a)$ the rate of developing AIDS at disease-age τ for the S-individuals at age a, $\delta(\tau;\, a)$ the death rate at duration τ due to AIDS for the individuals who have developed AIDS at age a and $\lambda(t,\, a)$ the infection rate (the force of infection) at age a and time t. Then, the dynamics of the population is governed by the following system:

$$\begin{cases} (\partial_t + \partial_a)S(t,a) = -\{\mu(a) + \lambda(t,a)\}S(t,a), \\ (\partial_t + \partial_\tau)I(t,\tau;a) = -\{\mu(a+\tau) + \gamma(\tau;\, a)\}I(t,\, \tau;\, a), \\ (\partial_t + \partial_\tau)A(t,\tau;a) = -\{\mu(a+\tau) + \delta(\tau;\, a)\}A(t,\, \tau;\, a), \\ S(t,0) = B, \quad I(t,0;a) = \lambda(t,a)S(t,a), \\ A(t,\, 0;\, a) = \int_0^a \gamma(\tau;\, a-\tau)I(t,\, \tau;\, a-\tau)\,\mathrm{d}\tau, \\ S(0,a) = S_0(a), \quad I(0,\tau;a) = I_0(\tau;a), \quad A(0,\tau;a) = A_0(\tau;a). \end{cases} \tag{2.4.2}$$

$\lambda(t,\, a)$ is assumed to have the following expression:

$$\lambda(t,\, a) = \int_0^\omega \int_0^b \beta(a,\, b,\, \tau)\frac{C(P(t))}{P(t)} I(t,\, \tau;\, b-\tau)\,\mathrm{d}\tau\,\mathrm{d}b,$$

where $\beta(a,\, b,\, \tau)$ is the transmission probability that a S-person of age a becomes infected by sexual contact with an infected partner of age b and disease-age τ, ω denotes the upper bound of age of the sexually active population, $P(t)$ is the total size of sexually active population given by

$$P(t) = \int_0^\omega \left\{S(t,\, \sigma) + \int_0^\sigma I(t,\, \tau;\, \sigma-\tau)\,\mathrm{d}\tau\right\}\mathrm{d}\sigma$$

and $C(P)$ denotes the mean number of sexual partners an average individual has per unit time when the population size is P.

Let us simplify system (2.4.2) by introducing new functions x, y by

$$S(t,\, a) = x(t,\, a)Bl(a), \quad I(t,\, \tau;\, a) = i(t,\, \tau;\, a)Bl(a+\tau)\Gamma(\tau;\, a),$$

where $l(a)$, $\Gamma(a)$ are the survival functions defined by

$$l(a) := \mathrm{e}^{-\int_0^a \mu(\sigma)\,\mathrm{d}\sigma}, \quad \Gamma(\tau;\, a) := \mathrm{e}^{-\int_0^\tau \gamma(\sigma;\, a)\,\mathrm{d}\sigma}.$$

Then, we obtain a simplified system (2.4.1) for $(x,\, y)$, where

$$\tilde{C}(P) := C(P)/P, \quad K(a,\, b,\, \tau) := \beta(a,\, b,\, \tau)Bl(b)\Gamma(\tau;\, b-\tau).$$

In the following, we make some assumptions for the coefficients and the functions appearing in (2.4.1), some of which can be satisfied when we impose appropriate restrictions on μ, γ, β etc.

Assumption 2.4.1. *(i) B, ω are strictly positive constants.*

(ii) $l : [0,\omega] \to \mathbb{R}$ is a monotone non-increasing Lebesgue-measurable function and $l(0) = 1$ and $l(\omega) = 0$.

(iii) $\Gamma : {\mathbb{R}_+}^2 \to [0,1]$ is Lebesgue-measurable and $\Gamma(\cdot\,; a)$ is monotone non-increasing for any fixed $a \in \mathbb{R}_+$.

(iv) $K(a,b,\tau)$ is expressed as $k_1(a)k_2(b,\tau)$, where

- *(a) $k_1(a) > 0$ if $a \in [0,\omega]$, and otherwise $k_1(a) = 0$. In addition, $k_1 \in L^\infty_+(\mathbb{R})$ and we denote its essential upper bound by k_1^∞.*
- *(b) If $k_2(b,\tau) \neq 0$, then $(b,\tau) \in \Delta_0 := \{(b,\tau) \in \mathbb{R}^2 \,|\, 0 \le \tau \le b \le \omega\}$. In addition, there exists some constant $a_\dagger \in (0,\omega]$ such that $k_2(b,\tau) > 0$ whenever $(b,\tau) \in D_0 := \{(b,\tau) \in \Delta_0 \,|\, b-\tau \le a_\dagger\}$.*
- *(c) $k_2 \in L^\infty_+(\mathbb{R}^2)$ and we denote its essential upper bound by k_2^∞.*

(v) $\tilde{C}(P)$ is strictly positive, monotone non-increasing for $P > 0$ and there exists some constant $L > 0$ such that for all $P, Q > 0$ we have

$$|\tilde{C}(P) - \tilde{C}(Q)| \le L|P - Q|.$$

Let us generate additional ideas about Assumption 2.4.1 to supplement. (i) states that ω — the upper bound of age of the sexually active population — is finite as in system (2.3.2) in Section 3, which makes the following discussion easier. (iv) is the proportionate mixing assumption, that is, there is no correlation between the age of susceptibles and the age of infectives, hence it is not necessarily realistic but very helpful for theoretical analysis. (iv)-b) is a technical assumption, stating that individuals who have infected by age $a_\dagger$ have non-zero force of infection. (v) implies that $C_0 := \lim_{P\downarrow 0} \tilde{C}(P)$ exists and is finite.

Mathematical well-posedness of the time evolution problem (2.4.1) is proved in [59, Appendix A], where the semigroup solution can be constructed by using the perturbation method of non-densely defined operators [62].

For system (2.4.1), the basic reproduction ratio is defined as follows:

$$R_0 := \tilde{C}(P_0)\int_0^\omega k_1(z)\int_z^\omega k_2(b, b-z)\,\mathrm{d}b\,\mathrm{d}z, \tag{2.4.3}$$

where $P_0 := \int_0^\omega Bl(a)\,\mathrm{d}a$.

Lemma 2.4.2. *The following relations hold for $t > \omega$:*

$$x(t,a) + \int_0^a y(t,\tau;a-\tau)\,\mathrm{d}\tau = 1, \tag{2.4.4}$$

$$x(t,a) + \int_0^a \Gamma(\tau;a-\tau)y(t,\tau;a-\tau)\,\mathrm{d}\tau \le 1, \tag{2.4.5}$$

$$P(t) \le P_0. \tag{2.4.6}$$

$$\tilde{C}(P_0) \le \tilde{C}(P(t)) \le C_0. \tag{2.4.7}$$

Proof. Let

$$f(\alpha) := x(t-a+\alpha,\alpha) + \int_0^\alpha y(t-a+\alpha, \alpha-\tau;\tau)\,\mathrm{d}\tau,$$

then it is easy to show that $f'(\alpha) = 0$, so $f(\alpha)$ does not depend on α and $f(a) = f(0) = x(t-a,0) = 1$. The other relations are obvious from (2.4.1), (2.4.4) and Assumption 2.4.1. □

Let us consider the semiflow induced by this system. We define its state space $\Omega := L^1_+(0,\omega) \times L^1_+(\Delta)$, where $\Delta := \{(a,b) \in \mathbb{R}^2 \mid a \ge 0,\ b \ge 0,\ a+b \le \omega\}$. Mathematical well-posedness of the time evolution problem (2.4.1) implies the following proposition:

Proposition 2.4.3. *System (2.4.1) induces a continuous semiflow Φ on the state space Ω.*

Let

$$\phi(t) := \tilde{C}(P(t)) \int_0^\omega \int_0^b k_2(b,\tau)\, y(t,\tau; b-\tau)\, \mathrm{d}\tau\, \mathrm{d}b.$$

Then we have $\lambda(t,a) = k_1(a)\phi(t)$ and for any $t > \tau$

$$\begin{aligned} y(t,\tau;a) &= y(t-\tau,0;a) = \lambda(t-\tau,a)x(t-\tau,a) \\ &= k_1(a)\phi(t-\tau)x(t-\tau,a). \end{aligned} \tag{2.4.8}$$

Let

$$\begin{aligned} \rho(x,y) &:= \int_0^\omega \int_0^b k_2(b,\tau)\, y(\tau; b-\tau)\, \mathrm{d}\tau\, \mathrm{d}b, \\ \sigma(t,(x,y)) &:= \rho(\Phi(t,(x,y))) \end{aligned}$$

for $(x,y) \in \Omega$ and $t \geq 0$. We can easily find that $\rho : \Omega \to \mathbb{R}$ is nonnegative and uniformly continuous. In addition, we find that

$$\phi(t) = \tilde{C}(P(t))\sigma(t,(x,y)). \tag{2.4.9}$$

Then, let us prove the positivity results. First, the upper-boundedness of λ and the equation for x in system (2.4.1) lead to

$$x(t,a) > 0 \quad \text{whenever} \quad t > a. \tag{2.4.10}$$

Lemma 2.4.4. *If $\rho(x_0,y_0) > 0$, then $\sigma(t,(x_0,y_0)) > 0$ for all $t \geq 0$.*

Proof. Let $(x(t,\cdot), y(t,\cdot;\cdot)) = \Phi(t,(x_0,y_0))$. Suppose there exists some $t_0 > 0$ such that $\sigma(t_0,(x_0,y_0)) = 0$. Then, $y(t_0,\tau;b-\tau) = 0$ for a.e. $(b,\tau) \in D_0$.

If $0 \leq \tau \leq b < t_0$, from (2.4.8) we have the relation

$$y(t_0,\tau;b-\tau) = k_1(b-\tau)\phi(t_0-\tau)x(t_0-\tau,b-\tau) \tag{2.4.11}$$

and $x(t_0-\tau,b-\tau) > 0$. Hence, we have $\phi(t_0-\tau) = 0$ for almost every (b,τ) satisfying $0 \leq \tau \leq b < t_0$ and $(b,\tau) \in D_0$. From this and the continuity of ϕ, we have $\phi(0) = 0$, and this and the relation (2.4.9) give rise to $\rho(x_0,y_0) = \sigma(0,(x_0,y_0)) = 0$, which contradicts the assumption $\rho(x_0,y_0) > 0$. □

Now, let us prove that system (2.4.1) satisfies the compactness condition (C). While we can apply a multi-dimensional version of Corollary 2.2.5, it might be good to illuminate the technique of using Theorem 2.2.4, which is slightly different from that in Section 2.3.

Lemma 2.4.5. *System (2.4.1) satisfies the compactness condition (C) with $\varepsilon_0 = 1$ and B the closure of $\Phi_{T_0}(\Delta)$, where $T_0 > 2\omega$ is a sufficiently large positive number.*

Proof. Taking Remark 2.2.2 into consideration, we find that we have only to prove that B is relatively compact in Ω. Let P_1, P_2 be projections defined by

$$P_1(x,y) = x, \;\; P_2(x,y) = y \quad \text{for } (x,y) \in L^1(0,\omega) \times L^1(\Delta).$$

Our aim is to prove that $P_1 B$ and $P_2 B$ are relatively compact.

Let $(x(t,\cdot), y(t,\cdot;\cdot))$ be the solution of system (2.4.1), where we take an initial condition $(x_0(\cdot), y_0(\cdot;\cdot))$ arbitrarily. For convenience, we define $x(t,a) = 0$ for $a \in \mathbb{R} \setminus [0,\omega]$ and $y(t,\tau;a) = 0$ for $(\tau,a) \in \mathbb{R}^2 \setminus \Delta$.

Since $k_1 \in L^1(\mathbb{R})$ and ϕ is uniformly continuous on $[0,T_0]$, which is based on its continuity on $[0,T_0]$, for any $\varepsilon > 0$, there exists some $\delta = \delta(\varepsilon) \in (0,\varepsilon)$ such that

$$\int_{\mathbb{R}} |k_1(a+h) - k_1(a)|\, \mathrm{d}a < \varepsilon$$

for any h with $|h| < \delta$ and

$$|\phi(t+h) - \phi(t)| < \varepsilon$$

whenever $h < \delta$ and $t, t+h \in [0,T_0]$. In the following, we fix such ε and δ.

Let $0 < a < \omega - h$ and $0 < h < \delta$. Using (2.4.1), Lemma 2.4.2 and

$$|\phi(t)| \leq C_0 \int_0^\omega k_2^\infty \int_0^b y(t,\tau,b-\tau)\, \mathrm{d}\tau\, \mathrm{d}b \leq C_0 k_2^\infty \omega,$$

we can evaluate as follows:

$$\begin{aligned}
&|x(T_0, a+h) - x(T_0, a)| \\
&= \left| \mathrm{e}^{-\int_0^{a+h} \lambda(T_0-a-h+r,r)\,\mathrm{d}r} - \mathrm{e}^{-\int_0^a \lambda(T_0-a+r,r)\,\mathrm{d}r} \right| \\
&\leq \left| \int_0^{a+h} \lambda(T_0 - a - h + r, r)\,\mathrm{d}r - \int_0^a \lambda(T_0 - a + r, r)\,\mathrm{d}r \right| \\
&\leq \int_0^a |\lambda(T_0 - a + r, r + h) - \lambda(T_0 - a + r, r)|\,\mathrm{d}r + \int_0^h |\lambda(T_0 - a - h + r, r)|\,\mathrm{d}r \\
&= \int_0^a |k_1(r+h) - k_1(r)|\phi(T_0 - a + r)\,\mathrm{d}r + \int_0^h k_1(r)\phi(T_0 - a - h + r)\,\mathrm{d}r \\
&\leq C_0 k_2^\infty \omega(1 + k_1^\infty)\varepsilon.
\end{aligned} \tag{2.4.12}$$

Then we have

$$\begin{aligned}
\int_0^\omega |x(T_0, a+h) - x(T_0, a)|\,\mathrm{d}a &= \int_0^{\omega-h} |x(T_0, a+h) - x(T_0, a)|\,\mathrm{d}a + \int_{\omega-h}^\omega |x(T_0, a)|\,\mathrm{d}a \\
&\leq C_0 k_2^\infty (1 + k_1^\infty)\omega^2 \varepsilon + h \\
&\leq \{C_0 k_2^\infty (1 + k_1^\infty)\omega^2 + 1\}\varepsilon.
\end{aligned}$$

The case $h < 0$ is the same. Hence, it follows from Theorem 2.2.4 that $P_1 B$ is relatively compact.

Next, let $h_1, h_2 \in (0, \delta)$ and $(\tau, a), (\tau + h_1, a + h_2) \in \Delta$, i.e.,

$$(\tau, a) \in \Delta_{h_1,h_2} := \{(a, b) \in \mathbb{R}^2 \mid a \geq 0,\ b \geq 0,\ a + b \leq \omega - h_1 - h_2\}.$$

Then we find that

$$\begin{aligned}
&|y(T_0, \tau + h_1; a + h_2) - y(t, \tau; a)| \\
&= |y(T_0 - \tau - h_1, 0; a + h_2) - y(T_0 - \tau, 0; a)| \\
&\leq |k_1(a + h_2) - k_1(a)|\phi(T_0 - \tau - h_1)x(T_0 - \tau - h_1, a + h_2) \\
&\quad + k_1(a)|\phi(T_0 - \tau - h_1) - \phi(T_0 - \tau)|x(T_0 - \tau - h_1, a + h_2) \\
&\quad + k_1(a)\phi(T_0 - \tau)|x(T_0 - \tau - h_1, a + h_2) - x(T_0 - \tau, a)| \\
&\leq |k_1(a + h_2) - k_1(a)|C_0 k_2^\infty \omega \cdot 1 + k_1^\infty \varepsilon \cdot 1 + k_1^\infty \cdot C_0 k_2^\infty \omega \cdot C_0 k_2^\infty \omega (1 + k_1^\infty)\varepsilon,
\end{aligned}$$

where a similar evaluation as (2.4.12) and the relation (2.4.8) are used. Hence we have

$$\int_{\Delta_{h_1,h_2}} |y(T_0, \tau + h_1; a + h_2) - y(t, \tau; a)|\,\mathrm{d}\tau\,\mathrm{d}a \leq C_1 \varepsilon,$$

where

$$C_1 := C_0 k_2^\infty \omega^2 + \{k_1^\infty + C_0^2 (k_2^\infty)^2 k_1^\infty (1 + k_1^\infty)\omega^2\}\omega^2/2.$$

It follows that

$$\begin{aligned}
&\int_\Delta |y(T_0, \tau + h_1; a + h_2) - y(t, \tau; a)|\,\mathrm{d}\tau da \\
&= \int_{\Delta_{h_1,h_2}} |y(T_0, \tau + h_1; a + h_2) - y(t, \tau; a)|\,\mathrm{d}\tau\,\mathrm{d}a + \int_{\Delta \setminus \Delta_{h_1,h_2}} y(t, \tau; a)\,\mathrm{d}\tau\,\mathrm{d}a \\
&\leq (C_1 + 2)\varepsilon.
\end{aligned}$$

Other cases, for example $h_1 < 0$ and $h_2 > 0$, are about the same, whose discussion is omitted here. Hence, it follows from Theorem 2.2.4 that $P_2 B$ is relatively compact. This completes our proof. □

And now let us prove the result on persistence with the above preparations, which imply that system (2.4.1) satisfies all assumptions in Theorem 2.2.3. Roughly speaking, the following theorem states that the rate of infecteds is bounded away from 0 and the bound is eventually independent of the initial data.

Theorem 2.4.6. *System (2.4.1) is uniformly strongly ρ-persistent if $R_0 > 1$.*

Proof. Thanks to Theorem 2.2.3, we have only to show the uniform weak ρ-persistence for system (2.4.1).

Suppose that system (2.4.1) is not uniformly weakly ρ-persistent, then for any $\varepsilon \in (0,1)$ we can choose some initial condition (x_0, y_0) and some sufficiently large $T_0 > 0$ which satisfy $\rho(x_0, y_0) > 0$ and

$$\int_0^\omega \int_0^a k_2(a,\tau) y(t,\tau; a-\tau)\,\mathrm{d}\tau\,\mathrm{d}a \le \varepsilon \quad \text{for all } t \ge T_0.$$

Then we obtain

$$\lambda(t,a) \le k_1(a)\tilde{C}(P(t))\varepsilon \quad \text{for all } t \ge T_0,$$

which implies that

$$\begin{aligned} x(t,a) &= x(t-a,0)\exp\left(-\int_0^a \lambda(t-a+r,r)\,\mathrm{d}r\right) \\ &\ge \exp\left(-\varepsilon \int_0^a k_1(r)\tilde{C}(P(t-a+r))\,\mathrm{d}r\right) \end{aligned} \tag{2.4.13}$$

for any $t \ge T_0$.

At the same time, if $t > \omega$, then we have

$$\begin{aligned} \phi(t) &= \tilde{C}(P(t)) \int_0^\omega \int_0^b k_2(b,\tau)k_1(b-\tau)\phi(t-\tau)x(t-\tau,b-\tau)\,\mathrm{d}\tau\,\mathrm{d}b \\ &= \int_0^\omega \phi(t-\tau)\left\{\tilde{C}(P(t))\int_\tau^\omega k_2(b,\tau)k_1(b-\tau)x(t-\tau,b-\tau)\,\mathrm{d}b\right\}\mathrm{d}\tau \\ &= \int_0^t \phi(t-\tau)\left\{\tilde{C}(P(t))\int_\tau^t k_2(b,\tau)k_1(b-\tau)x(t-\tau,b-\tau)\,\mathrm{d}b\right\}\mathrm{d}\tau, \end{aligned}$$

where we have used the fact that $k_2(b,\tau) = 0$ if $b > \omega$ or $\tau > \omega$, which is obvious from Assumption 2.4.1 iv)b). In the same reason, $\phi_d(t) := \phi(t+d)$ $(d \ge 0)$ satisfies that

$$\begin{aligned} \phi_d(t) &= \int_0^{t+d} \phi(t+d-\tau)\left\{\tilde{C}(P(t+d))\int_\tau^{t+d} k_2(b,\tau)k_1(b-\tau)x(t+d-\tau,b-\tau)\,\mathrm{d}b\right\}\mathrm{d}\tau \\ &= \int_0^t \phi_d(t-\tau)\left\{\tilde{C}(P(t+d))\int_\tau^t k_2(b,\tau)k_1(b-\tau)x(t+d-\tau,b-\tau)\,\mathrm{d}b\right\}\mathrm{d}\tau \end{aligned} \tag{2.4.14}$$

for sufficiently large $t (> \omega)$

(2.4.7), (2.4.13) and (2.4.14) imply that

$$\phi_d(t) \ge \int_0^t \phi_d(t-\tau)\left\{\tilde{C}(P_0)\int_\tau^t k_2(b,\tau)k_1(b-\tau)\exp\left(-\varepsilon\int_0^{b-\tau} C_0 k_1(r)\,\mathrm{d}r\right)\mathrm{d}b\right\}\mathrm{d}\tau.$$

If d is larger than a sufficiently large positive number T, we find that

$$\begin{aligned} \phi_d(t) &\ge \int_0^{t+T} \phi_d(t-\tau)\left\{\tilde{C}(P_0)\int_\tau^{t+T} k_2(b,\tau)k_1(b-\tau)\exp\left(-\varepsilon\int_0^{b-\tau} C_0 k_1(r)\,\mathrm{d}r\right)\mathrm{d}b\right\}\mathrm{d}\tau \\ &\ge \int_0^t \phi_d(t-\tau)\left\{\tilde{C}(P_0)\int_\tau^{\tau+T} k_2(b,\tau)k_1(b-\tau)\exp\left(-\varepsilon\int_0^{b-\tau} C_0 k_1(r)\,\mathrm{d}r\right)\mathrm{d}b\right\}\mathrm{d}\tau \end{aligned}$$

for all $t \ge 0$. Taking Laplace transforms leads to

$$\hat{\phi}_d(\lambda) \ge \hat{\phi}_d(\lambda)F(\varepsilon,\lambda,T), \tag{2.4.15}$$

where

$$F(\varepsilon,\lambda,T) := \int_0^\infty e^{-\lambda\tau}\left\{\tilde{C}(P_0)\int_\tau^{\tau+T} k_2(b,\tau)k_1(b-\tau)\exp\left(-\varepsilon\int_0^{b-\tau} C_0 k_1(r)\,\mathrm{d}r\right)\mathrm{d}b\right\}\mathrm{d}\tau.$$

Since ϕ_d is bounded, $\hat{\phi}_d(\lambda)$ is defined for all $\lambda \ge 0$. Moreover,

$$\lim_{T\to\infty} F(0,0,T) = \int_0^\infty \tilde{C}(P_0)\left\{\int_\tau^\infty k_2(b,\tau)k_1(b-\tau)\,\mathrm{d}b\right\}\mathrm{d}\tau$$

$$
\begin{aligned}
&= \int_0^\omega \tilde{C}(P_0) \left\{ \int_\tau^\omega k_2(b,\tau) k_1(b-\tau)\, \mathrm{d}b \right\} \mathrm{d}\tau \\
&= R_0 > 1.
\end{aligned}
$$

Since $F(\varepsilon, \lambda, T)$ is continuous, we see that $F(\varepsilon, \lambda, T) > 1$ if $\varepsilon, \lambda > 0$ are chosen small enough and T large enough. Then (2.4.15) implies that $\hat{\phi}_d(\lambda) = 0$, which means $\phi_d(t) = 0$ a.e. on $[0, \infty)$, i.e., $\phi(t) = 0$ for almost every $t > d + T$. This contradicts Lemma 2.4.4. □

2.5 Discussion

In this paper we have shown uniform strong persistence in two structured population models. Uniform weak persistence has been proved by using positivity result and the Laplace transform of the force of infection $\phi_d(t)$, where we have seen the model's basic reproduction ratio R_0 plays an important role. Then we have applied the theorem stating uniform weak persistence implies uniform strong persistence under the compactness condition (C) to derive the two model's uniform strong persistence. In order to verify the compactness condition (C)-b) we have made use of Fréchet–Kolmogorov's Theorem that states the compactness criteria for the subset of L^p-space.

In Section 2.3, we do not use $\phi(t)$, whose alternative is $\sigma(t,\ (x, y, z)) = y(t) + \|z(t,\ \cdot)\|$. Assumption 2.3.1 and $\gamma > 0$ assure that $\sigma(t,\ (x,\ y,\ z)) > 0$ if and only if $\phi_0(t) > 0$. From this point of view, the discussion in Section 2.3 and that in Section 2.4 are virtually identical.

Assuming that the upper bound of age or age-since-infection is finite facilitates the proof of persistence, but we see no problem in that the assumption is realistic and reasonable. The case that ω is infinite could be handled by a splitting introduced by Schappcher (see [54, 56, 57, 46]).

Chapter 3

The impact of mass media on rumor transmission

3.1 Introduction

Rumor transmission is considered as a social phenomenon that a remark spreads on a large scale in a short time through chain of communication. Its mathematical models, most of which are similar to the models describing the spread of infectious diseases because word-of-mouth and infectious diseases have much in common, have been constructed and investigated since the 1950s. They contain deterministic models expressed in terms of ODE or PDE system such as those discussed in Chapter 1. On the other hand, the effect of outside source, mass media for example, can have a considerable effect on rumor transmission, although it is not taken into consideration in most of the existing rumor transmission models. For example, while many people, getting more conscious of global warming, take measures to it, some researchers voice on skepticism for it through mass media so extensively that the number of people who agree with the researchers' skepticism is gradually increasing.

In this chapter, we regard mass media or the people in the population who have much influence by using them as outside sources, and discuss how they affect the dynamics of rumor transmission in the population.

3.2 Rumor-spreading mass media : simple extensions

As in Section 1.2, we denote by $N(t)$ the total population at time t, which consists of susceptibles, spreaders and stiflers. Each population at time t is denoted by $X(t)$, $Y(t)$, $Z(t)$ respectively.

First, we assume that mass media constantly spread a rumor and as a result $pX(t)\Delta t$ susceptibles and $qZ(t)\Delta t$ stiflers become spreaders during the small interval $(t,\, t+\Delta t)$, where p, q are strictly positive constants and p does not necessarily equal q, which means that it is valid to regard the impact of mass media on susceptibles as different from that on stiflers. We also assume that the spreaders do not change their behavior after they rediscover the rumor in mass media.

3.2.1 Constant rumor with rumor-spreading mass media

In the case of the transmission of a constant rumor in a closed population, if the dynamics of the population without mass media is governed by system (1.2.1), its dynamics taking the effect of mass media into consideration is governed by the following system:

$$\begin{cases} \dot{X}(t) = -\alpha X(t)\dfrac{Y(t)}{N(t)} - pX(t), \\ \dot{Y}(t) = \alpha\theta X(t)\dfrac{Y(t)}{N(t)} - \beta Y(t)\dfrac{Y(t)}{N(t)} - \gamma Y(t)\dfrac{Z(t)}{N(t)} + pX(t) + qZ(t), \\ \dot{Z}(t) = \alpha(1-\theta)X(t)\dfrac{Y(t)}{N(t)} + \beta Y(t)\dfrac{Y(t)}{N(t)} + \gamma Y(t)\dfrac{Z(t)}{N(t)} - qZ(t), \end{cases} \tag{3.2.1}$$

Taking into consideration that $N(t)$ is constant, we can rewrite it in the terms of

$$x(t) := \frac{X(t)}{N(t)},\ y(t) := \frac{Y(t)}{N(t)},\ z(t) := \frac{Z(t)}{N(t)}$$

as follow:

$$\begin{cases} \dot{x}(t) = -\alpha x(t)y(t) - px(t), \\ \dot{y}(t) = \alpha\theta x(t)y(t) - \beta\{y(t)\}^2 - \gamma y(t)z(t) + px(t) + qz(t), \\ \dot{z}(t) = \alpha(1-\theta)x(t)y(t) + \beta\{y(t)\}^2 + \gamma y(t)z(t) - qz(t), \end{cases} \tag{3.2.2}$$

It is easy to show that system (3.2.2) has a unique solution on $(-\infty, \infty)$ in

$$\Omega := \{(x, y, z) \in \mathbb{R_+}^3 \mid x + y + z = 1\}$$

for any initial data in Ω.

Substituting $x(t) = 1 - y(t) - z(t)$ into (3.2.2), we can obtain the equations for y and z only:

$$\begin{cases} \dot{y} = \alpha\theta(1-y-z)y - \beta y^2 - \gamma yz + p(1-y-z) + qz, \\ \dot{z} = \alpha(1-\theta)(1-y-z)y + \beta y^2 + \gamma yz - qz. \end{cases}$$

We define a Dulac function $\rho(y, z) := (yz)^{-1}$ on the domain $\{(y, z) \in \mathbb{R_+}^2 \mid y > 0,\ z > 0,\ y + z < 1\}$. Then we have

$$\begin{aligned} &\frac{\partial}{\partial y}(\rho(y, z)\{\alpha\theta(1-y-z)y - \beta y^2 - \gamma yz + p(1-y-z) + qz\}) \\ &\quad + \frac{\partial}{\partial z}(\rho(y, z)\{\alpha(1-\theta)(1-y-z)y + \beta y^2 + \gamma yz - qz\}) \\ &= -\left\{\frac{\alpha\theta + \beta}{z} + \frac{\alpha(1-\theta)(1-y) + \beta y}{z^2} + \frac{p(1-z)}{y^2 z} + \frac{q}{y^2}\right\}, \end{aligned}$$

which is strictly negative on the domain. Hence we can apply Dulac–Bendixson Criterion to exclude the possibility of a periodic orbit or a cyclic chain of equilibria.

Now, let us explore the equilibria in Ω.

$$-\alpha xy - px = 0 \ \Leftrightarrow\ x(\alpha y + p) = 0$$

gives $x = 0$. Then, substituting $x = 0$ and $z = 1 - y$ into

$$\alpha\theta xy - \beta y^2 - \gamma yz + px + qz = 0,$$

we find that $f(y) = 0$, where

$$f(y) := -\beta y^2 - (\gamma y - q)(1 - y).$$

$f(y)$ is at most quadratic with respect to y and $f(0) = q > 0,\ f(1) = -\beta < 0$. Hence, $f(y)$ has a unique root y^* in the open interval $(0, 1)$.

Therefore, we have the following results:

Theorem 3.2.1. *System (3.2.2) has only one equilibrium* $(0, y^*, z^*)$ *with* $y^* > 0,\ z^* > 0,\ y^* + z^* = 1,$ *which is globally asymptotically stable in* Ω.

3.2.2 Variable rumor with rumor-spreading mass media

Next, in the case of the transmission of a variable rumor in a closed population, if the dynamics of the population without mass media is governed by system (1.3.1), its dynamics taking the effect of mass media into consideration is governed by the following system:

$$\begin{cases} \dot{X}(t) = -\alpha X(t)\dfrac{Y(t)}{N(t)} + \eta Z(t) - pX(t), \\ \dot{Y}(t) = \alpha\theta X(t)\dfrac{Y(t)}{N(t)} - \beta Y(t)\dfrac{Y(t)}{N(t)} - \gamma Y(t)\dfrac{Z(t)}{N(t)} + pX(t) + qZ(t), \\ \dot{Z}(t) = \alpha(1-\theta)X(t)\dfrac{Y(t)}{N(t)} + \beta Y(t)\dfrac{Y(t)}{N(t)} + \gamma Y(t)\dfrac{Z(t)}{N(t)} - \eta Z(t) - qZ(t). \end{cases} \tag{3.2.3}$$

Taking into consideration that $N(t)$ is constant, we can rewrite it in the terms of

$$x(t) := \frac{X(t)}{N(t)},\ y(t) := \frac{Y(t)}{N(t)},\ z(t) := \frac{Z(t)}{N(t)}$$

as follow:

$$\begin{cases} \dot{x}(t) = -\alpha x(t)y(t) + \eta z(t) - px(t), \\ \dot{y}(t) = \alpha\theta x(t)y(t) - \beta\{y(t)\}^2 - \gamma y(t)z(t) + px(t) + qz(t), \\ \dot{z}(t) = \alpha(1-\theta)x(t)y(t) + \beta\{y(t)\}^2 + \gamma y(t)z(t) - \eta z(t) - qz(t), \end{cases} \tag{3.2.4}$$

It is easy to show that system (3.2.2) has a unique solution on $(-\infty, \infty)$ in

$$\Omega := \{(x, y, z) \in \mathbb{R}_+{}^3 \mid x + y + z = 1\}$$

for any initial data in Ω.

Substituting $x(t) = 1 - y(t) - z(t)$ into (3.2.4), we can obtain the equations for y and z only:

$$\begin{cases} \dot{y} = \alpha\theta(1-y-z)y - \beta y^2 - \gamma yz + p(1-y-z) + qz, \\ \dot{z} = \alpha(1-\theta)(1-y-z)y + \beta y^2 + \gamma yz - \eta z - qz. \end{cases}$$

We define a Dulac function $\rho(y, z) := (yz)^{-1}$ on the domain $\{(y, z) \in \mathbb{R}_+{}^2 \mid y > 0,\ z > 0,\ y + z < 1\}$. Then we have

$$\begin{aligned} &\frac{\partial}{\partial y}(\rho(y, z)\{\alpha\theta(1-y-z)y - \beta y^2 - \gamma yz + p(1-y-z) + qz\}) \\ &\quad + \frac{\partial}{\partial z}(\rho(y, z)\{\alpha(1-\theta)(1-y-z)y + \beta y^2 + \gamma yz - \eta z - qz\}) \\ &= -\left\{\frac{\alpha\theta+\beta}{z} + \frac{\alpha(1-\theta)(1-y)+\beta y}{z^2} + \frac{p(1-z)}{y^2 z} + \frac{q}{y^2} + \frac{\eta}{y}\right\}, \end{aligned}$$

which is strictly negative on the domain. Hence we can apply Dulac–Bendixson Criterion to exclude the possibility of a periodic orbit or a cyclic chain of equilibria.

Now, let us explore the equilibria in Ω. By substituting $z = 1 - x - y$ into $-\alpha xy + \eta z - px = 0$, we have

$$-\alpha xy + \eta(1-x-y) - px = 0 \Leftrightarrow x = \frac{\eta(1-y)}{\alpha y + p + \eta}.$$

Substituting this and $z = 1-x-y$ into $\alpha\theta xy - \beta y^2 - \gamma yz + px + qz = 0$, we obtain the equation $f(y) = 0$, where

$$f(y) := \{(\alpha\theta + \gamma)y + p - q\}\eta(1-y) + \{-\beta y^2 - \gamma y(1-y) + q(1-y)\}(\alpha y + p + \eta).$$

We find that

$$f(0) = p(q+\eta) > 0, \quad f(1) = -\beta(\alpha + p + \eta) < 0.$$

Then, noting that $f(y)$ is of at most third degree, we obtain that $f(y)$ has at most three roots in the open interval $(0, 1)$.

Since it would be complicated to find out in a precise sense how many roots of $f(y)$ exist in $(0, 1)$ in general, let us assume $\theta = 1$ for simplicity. Then

$$f\left(-\frac{p}{\alpha}\right) = -\frac{\beta p^2}{\alpha^2} < 0$$

and we can conclude that $f(y)$ has a unique root in $(0, 1)$ and in $(-\frac{p}{\alpha}, 0)$ respectively. Let y^* be the root in $(0, 1)$, then

$$\begin{aligned} x^* &:= \frac{\eta(1-y^*)}{\alpha y^* + p + \eta}, \\ z^* &:= 1 - x^* - y^* = (1-y^*)\cdot\frac{\alpha y^* + p}{\alpha y^* + p + \eta} \end{aligned}$$

satisfies the conditions $x^* > 0,\ z^* > 0$.

The above argument is valid when θ is sufficiently close to 1 because of the continuity of the function $\theta \mapsto f\left(-\frac{p}{\alpha}\right)$. Therefore we have the following:

Theorem 3.2.2. *If θ is sufficiently close to 1, then System (3.2.4) has only one equilibrium (x^*, y^*, z^*) with $x^* > 0,\ y^* > 0,\ z^* > 0,\ x^* + y^* + z^* = 1$, which is globally asymptotically stable in Ω.*

3.3 Active stiflers

So far, we have examined the dynamics of rumor transmission on the assumption that stiflers do not show their judgment towards the rumor unless they hear of it. For example, when a stifler contacts a susceptible, the stifler is not the first to transmit the rumor and the susceptible remains not to know it. In this viewpoint, we may call such a passive stifler *skeptic.* On the other hand, it could be that some people actively deny the rumor, whom we can call "active stiflers." They voluntarily tell susceptibles and spreaders that the rumor is false, while their stories may be based on hearsay information obtained through their acquaintances or mass media. In other words, active stiflers prevent the spread of the rumor by spreading its rival rumor. In this section, we investigate the transmission of a rumor in a closed population consisting of susceptibles, spreaders and active stiflers, and the impact of mass media on the transmission.

3.3.1 Constant rumor without mass media

First, as the simplest case, we treat the transmission of a constant rumor, and take into consideration the possibility of three kinds of transition; from susceptible class into spreader class, from susceptible class into active stifler class, and between spreader class and active stifler class. We can express our model in the framework of ODE as follows:

$$\begin{cases} \dot{X}(t) = -\alpha X(t)\dfrac{Y(t)}{N(t)} - \delta X(t)\dfrac{Z(t)}{N(t)}, \\ \dot{Y}(t) = \alpha X(t)\dfrac{Y(t)}{N(t)} - \gamma Y(t)\dfrac{Z(t)}{N(t)}, \\ \dot{Z}(t) = \gamma Y(t)\dfrac{Z(t)}{N(t)} + \delta X(t)\dfrac{Z(t)}{N(t)}, \end{cases} \tag{3.3.1}$$

where α is the susceptible–spreader interaction parameter, δ the susceptible–stifler interaction parameter, γ the spreader–stifler interaction parameter. We assume that α and δ are strictly positive, and that γ can take any value in $\mathbb{R} \setminus \{0\}$.

Since the total population $N(t) := X(t) + Y(t) + Z(t)$ is constant, we can rewrite the system (3.3.1) in the terms of

$$x(t) := \frac{X(t)}{N(t)},\ y(t) := \frac{Y(t)}{N(t)},\ z(t) := \frac{Z(t)}{N(t)}$$

as follows:

$$\begin{cases} \dot{x}(t) = -\alpha x(t)y(t) - \delta x(t)z(t), \\ \dot{y}(t) = \alpha x(t)y(t) - \gamma y(t)z(t), \\ \dot{z}(t) = \gamma y(t)z(t) + \delta x(t)z(t), \end{cases} \tag{3.3.2}$$

It is easy to show that system (3.3.2) has a unique solution on $(-\infty, \infty)$ in

$$\Omega := \{(x,\, y,\, z) \in {\mathbb{R}_+}^3 \mid x + y + z = 1\}$$

for any initial data in Ω. Substituting $x(t) = 1 - y(t) - z(t)$ into (3.3.2), we can obtain the equations for y and z only:

$$\begin{cases} \dot{y} = \alpha(1 - y - z)y - \gamma yz, \\ \dot{z} = \gamma yz + \delta(1 - y - z)z. \end{cases} \tag{3.3.3}$$

We define a Dulac function $\rho(y,\, z) := (yz)^{-1}$ on the domain $\{(y,\, z) \in {\mathbb{R}_+}^2 \mid y > 0,\ z > 0,\ y + z < 1\}$. Then we have

$$\frac{\partial}{\partial y}(\rho(y,\, z)\{\alpha(1 - y - z)y - \gamma yz\}) + \frac{\partial}{\partial z}(\rho(y,\, z)\{\gamma yz + \delta(1 - y - z)z\}) = -\left(\frac{\alpha}{z} + \frac{\delta}{y}\right),$$

which is strictly negative on the domain. Hence we can apply Dulac–Bendixson Criterion to exclude the possibility of a periodic orbit or a cyclic chain of equilibria.

We can easily find out that the equilibria of (3.3.2) in Ω are

$$E_0(1,\, 0,\, 0),\ E_y(0,\, 1,\, 0),\ E_z(0,\, 0,\, 1).$$

The Jacobian matrices of system (3.3.3) at these equilibria are

$$M|_{E_0} = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix},\quad M|_{E_y} = \begin{pmatrix} -\alpha & -\alpha - \gamma \\ 0 & \gamma \end{pmatrix},\quad M|_{E_z} = \begin{pmatrix} -\gamma & 0 \\ \gamma - \delta & -\delta \end{pmatrix},$$

Therefore we have the following:

Theorem 3.3.1. *System (3.3.2) has three boundary equilibria E_0, E_y, E_z in Ω.*

If $\gamma > 0$, then $\{(x, y, z) \in \Omega \mid z = 0\} \setminus E_0$ is the stable manifold of the saddle E_y, and the sink E_z is globally asymptotically stable in $\Omega \setminus I_{xy}$.

If $\gamma < 0$, then $\{(x, y, z) \in \Omega \mid y = 0\} \setminus E_0$ is the stable manifold of the saddle E_z, and the sink E_y is globally asymptotically stable in $\Omega \setminus I_{xz}$.

3.3.2 Constant rumor with rumor-spreading mass media

Next, we add the effect of mass media that constantly spread the rumor to system (3.3.1) as system (3.2.1) to obtain

$$\begin{cases} \dot{X}(t) = -\alpha X(t)\dfrac{Y(t)}{N(t)} - \delta X(t)\dfrac{Z(t)}{N(t)} - pX(t), \\ \dot{Y}(t) = \alpha X(t)\dfrac{Y(t)}{N(t)} - \gamma Y(t)\dfrac{Z(t)}{N(t)} + pX(t) + qZ(t), \\ \dot{Z}(t) = \gamma Y(t)\dfrac{Z(t)}{N(t)} + \delta X(t)\dfrac{Z(t)}{N(t)} - qZ(t), \end{cases} \tag{3.3.4}$$

where p and q are strictly positive constants. Since the total population $N(t) := X(t) + Y(t) + Z(t)$ is constant, we can rewrite the system (3.3.4) in the terms of

$$x(t) := \frac{X(t)}{N(t)},\ y(t) := \frac{Y(t)}{N(t)},\ z(t) := \frac{Z(t)}{N(t)}$$

as follows:

$$\begin{cases} \dot{x}(t) = -\alpha x(t)y(t) - \delta x(t)z(t) - px(t), \\ \dot{y}(t) = \alpha x(t)y(t) - \gamma y(t)z(t) + px(t) + qz(t), \\ \dot{z}(t) = \gamma y(t)z(t) + \delta x(t)z(t) - qz(t), \end{cases} \tag{3.3.5}$$

It is easy to show that system (3.3.5) has a unique solution on $(-\infty, \infty)$ in

$$\Omega := \{(x, y, z) \in {\mathbb{R}_+}^3 \mid x + y + z = 1\}$$

for any initial data in Ω. Substituting $x(t) = 1 - y(t) - z(t)$ into (3.3.5), we can obtain the equations for y and z only:

$$\begin{cases} \dot{y} = \alpha(1 - y - z)y - \gamma yz + p(1 - y - z) + qz, \\ \dot{z} = \gamma yz + \delta(1 - y - z)z - qz. \end{cases} \tag{3.3.6}$$

We define a Dulac function $\rho(y, z) := (yz)^{-1}$ on the domain $\{(y, z) \in {\mathbb{R}_+}^2 \mid y > 0,\ z > 0,\ y + z < 1\}$. Then we have

$$\begin{aligned} &\frac{\partial}{\partial y}(\rho(y, z)\{\alpha(1 - y - z)y - \gamma yz + p(1 - x - y) + qz\}) + \frac{\partial}{\partial z}(\rho(y, z)\{\gamma yz + \delta(1 - y - z)z - qz\}) \\ &= -\left(\frac{\alpha}{z} + \frac{\delta}{y} + \frac{p(1 - z)}{y^2 z} + \frac{q}{y^2}\right), \end{aligned}$$

which is strictly negative on the domain. Hence we can apply Dulac–Bendixson Criterion to exclude the possibility of a periodic orbit or a cyclic chain of equilibria.

We can easily find out that the equilibria of (3.3.5) in Ω are

$$E_y(0, 1, 0),\ E_{yz}\left(0, \frac{q}{\gamma}, 1 - \frac{q}{\gamma}\right),$$

where E_{yz} exists in Ω if and only if $\gamma \geq q$. The Jacobian matrices of system (3.3.6) at these equilibria are

$$M|_{E_y} := \begin{pmatrix} -\alpha - p & -\alpha - \gamma - p + q \\ 0 & \gamma - q \end{pmatrix},\quad M|_{E_{yz}} := \begin{pmatrix} -\frac{\alpha q}{\gamma} + q - \gamma - p & -\frac{\alpha q}{\gamma} - p \\ \frac{(\gamma-\delta)(\gamma-q)}{\gamma} & \frac{-\delta(\gamma-q)}{\gamma} \end{pmatrix},$$

If $\gamma > q (> 0)$, then we have

$$\begin{aligned} \operatorname{tr} M|_{E_{yz}} &= -\frac{\alpha q}{\gamma} + q - \gamma - p - \frac{\delta(\gamma - q)}{\gamma} < 0, \\ \det M|_{E_{yz}} &= \frac{(\gamma - q)\{\delta(\gamma - q) + p\gamma + \alpha q\}}{\gamma} > 0, \end{aligned}$$

which implies that the two eigenvalues of $M|_{E_{yz}}$ have strictly negative real parts.

Therefore we have the following:

Theorem 3.3.2. *System (3.3.5) has two equilibria E_y, E_{yz} in Ω, while E_{yz} exists in Ω if and only if $\gamma \geq q$.*

If $\gamma < q$, then E_y is globally asymptotically stable in Ω.

If $\gamma > q$, then the segment $I_{xy} = \{(x, y, z) \in \Omega \mid z = 0\}$ is the stable manifold of the saddle E_y, and E_{yz} is globally asymptotically stable in $\Omega \setminus I_{xy}$.

3.3.3 Variable rumor without mass media

Let us consider the transmission of a variable rumor. Unlike Section 1.3 and system (3.2.3), we assume that both spreaders and active stiflers have the possibility of transiting into the susceptible class because the rumor is modified in the communication process momentarily.

We can express our model in the framework of ODE as follows:

$$\begin{cases} \dot{X}(t) = -\alpha X(t)\dfrac{Y(t)}{N(t)} - \delta X(t)\dfrac{Z(t)}{N(t)} + \eta_1 Y(t) + \eta_2 Z(t), \\ \dot{Y}(t) = \alpha X(t)\dfrac{Y(t)}{N(t)} - \gamma Y(t)\dfrac{Z(t)}{N(t)} - \eta_1 Y(t), \\ \dot{Z}(t) = \gamma Y(t)\dfrac{Z(t)}{N(t)} + \delta X(t)\dfrac{Z(t)}{N(t)} - \eta_2 Z(t), \end{cases} \tag{3.3.7}$$

where η_1 and η_2 are strictly positive constants and denote the transition rates from each rumor-class into the susceptible class.

Since the total population $N(t) := X(t) + Y(t) + Z(t)$ is constant, we can rewrite the system (3.3.7) in the terms of

$$x(t) := \frac{X(t)}{N(t)},\ y(t) := \frac{Y(t)}{N(t)},\ z(t) := \frac{Z(t)}{N(t)}$$

as follows:

$$\begin{cases} \dot{x}(t) = -\alpha x(t)y(t) - \delta x(t)z(t) + \eta_1 y(t) + \eta_2 z(t), \\ \dot{y}(t) = \alpha x(t)y(t) - \gamma y(t)z(t) - \eta_1 y(t), \\ \dot{z}(t) = \gamma y(t)z(t) + \delta x(t)z(t) - \eta_2 z(t), \end{cases} \tag{3.3.8}$$

It is easy to show that system (3.3.8) has a unique solution on $(-\infty, \infty)$ in

$$\Omega := \{(x, y, z) \in \mathbb{R}_+{}^3 \mid x + y + z = 1\}$$

for any initial data in Ω. Substituting $x(t) = 1 - y(t) - z(t)$ into (3.3.8), we can obtain the equations for y and z only:

$$\begin{cases} \dot{y} = \alpha(1 - y - z)y - \gamma yz - \eta_1 y, \\ \dot{z} = \gamma yz + \delta(1 - y - z)z - \eta_2 z. \end{cases} \tag{3.3.9}$$

We define a Dulac function $\rho(y, z) := (yz)^{-1}$ on the domain $\{(y, z) \in \mathbb{R}_+{}^2 \mid y > 0,\ z > 0,\ y + z < 1\}$. Then we have

$$\frac{\partial}{\partial y}(\rho(y, z)\{\alpha(1 - y - z)y - \gamma yz - \eta_1 y\}) + \frac{\partial}{\partial z}(\rho(y, z)\{\gamma yz + \delta(1 - y - z)z - \eta_2 z\}) = -\left(\frac{\alpha}{z} + \frac{\delta}{y}\right),$$

which is strictly negative on the domain. Hence we can apply Dulac–Bendixson Criterion to exclude the possibility of a periodic orbit or a cyclic chain of equilibria.

In the following, we assume

$$\gamma \neq 0,\ \gamma \neq \delta - \alpha$$

for simplicity. Let us explore the equilibria of system (3.3.8). Possible equilibria are three boundary equilibria

$$E_0(1, 0, 0),\ E_1(1 - y_1, y_1, 0),\ E_2(1 - z_2, 0, z_2)$$

and an interior equilibrium $E^*(x^*, y^*, z^*)$, where

$$y_1 := 1 - \frac{\eta_1}{\alpha}(< 1),\ z_2 := 1 - \frac{\eta_2}{\delta}(< 1),$$
$$x^* := \frac{\gamma + \eta_1 - \eta_2}{\gamma + \alpha - \delta},\ y^* := \frac{\alpha\eta_2 - \delta\eta_1 - \gamma(\delta - \eta_2)}{\gamma(\gamma + \alpha - \delta)},\ z^* := \frac{(\alpha - \eta_1)\gamma - (\alpha\eta_2 - \delta\eta_1)}{\gamma(\gamma + \alpha - \delta)}.$$

For simplicity, we assume that $x^* \neq 0,\ y^* \neq 0,\ z^* \neq 0$.

The Jacobian matrices of system (3.3.9) at these equilibria are

$$
\begin{aligned}
M|_{E_0} &:= \begin{pmatrix} \alpha-\eta_1 & 0 \\ 0 & \delta-\eta_2 \end{pmatrix} = \begin{pmatrix} \alpha y_1 & 0 \\ 0 & \delta z_2 \end{pmatrix}, \\
M|_{E_1} &:= \begin{pmatrix} \eta_1-\alpha & -(\alpha+\gamma)\left(1-\frac{\eta_1}{\alpha}\right) \\ 0 & \frac{(\alpha-\eta_1)\gamma-(\alpha\eta_2-\delta\eta_1)}{\alpha} \end{pmatrix} = \begin{pmatrix} -\alpha y_1 & -(\alpha+\gamma)y_1 \\ 0 & \frac{\gamma(\gamma+\alpha-\delta)}{\alpha}z^* \end{pmatrix}, \\
M|_{E_2} &:= \begin{pmatrix} \frac{\alpha\eta_2-\delta\eta_1-\gamma(\delta-\eta_2)}{\delta} & 0 \\ (\gamma-\delta)\left(1-\frac{\eta_2}{\delta}\right) & \eta_2-\delta \end{pmatrix} = \begin{pmatrix} \frac{\gamma(\gamma+\alpha-\delta)}{\delta}y^* & 0 \\ (\gamma-\delta)z_2 & -\delta z_2 \end{pmatrix}, \\
M|_{E^*} &:= \begin{pmatrix} -\alpha y^* & -(\alpha+\gamma)y^* \\ (\gamma-\delta)z^* & -\delta z^* \end{pmatrix}.
\end{aligned}
$$

We find that

$$\operatorname{tr} M|_{E^*} = -\alpha y^* - \delta z^*, \quad \det M|_{E^*} = y^* z^* \gamma(\gamma+\alpha-\delta).$$

In the following, we classify the phase-plane for system (3.3.9) in

$$\Delta := \{(y,\, z) \in \mathbb{R}^2 \mid y \geq 0,\ z \geq 0,\ y+z \leq 1\}$$

and discuss the existence and the local stability of its equilibria. We identify the projection of E_0 onto yz-plane with E_0 and so on. Let

$$
\begin{aligned}
&\operatorname{Int}(\Delta) := \{(y,\, z) \in \mathbb{R}^2 \mid y > 0,\ z > 0,\ y+z < 1\} \\
&I_y := \{(y,\, z) \in \Delta \mid z = 0\}, \quad I_z := \{(y,\, z) \in \Delta \mid y = 0\}.
\end{aligned}
$$

On the phase-plane, the y–nullcline is

$$y = 0,\ f_1(y,\, z) := -(\alpha+\gamma)y - \alpha z + \alpha - \eta_1 = 0,$$

and the z–nullcline is

$$z = 0,\ f_2(y,\, z) := (\gamma-\delta)y - \delta z + \delta - \eta_2 = 0.$$

The line $f_1(y,\, z) = 0$ passes through E_1 and E^*, and the line $f_2(y,\, z) = 0$ E_2 and E^*. The slopes of the lines $f_1(y,\, z) = 0$ and $f_2(y,\, z) = 0$ are, if they exist,

$$T_1 := -\frac{\alpha}{\alpha+\gamma}, \quad T_2 := \frac{\gamma-\delta}{\delta}.$$

If $T_1 > 0$ and $T_2 \geq 0$, then

$$\alpha+\gamma < 0 \quad \text{and} \quad \gamma-\delta \geq 0,$$

which are inconsistent because α, δ are strictly positive.

Case 1. $y_1 < 0$ and $z_2 < 0$.

E_0 is a sink in Δ and locally asymptotically stable (LAS). E_y and E_z do not exist in Δ.

If $E^* \in \operatorname{Int}(\Delta)$, then the slopes T_1 and T_2 must be both strictly positive, which is a contradiction. Hence $E^* \notin \operatorname{Int}(\Delta)$.

Therefore, E_0 is globally asymptotically stable (GAS) in Δ (Fig. 3.1 A).

Case 2. $y_1 > 0$ and $z_2 < 0$ (i.e., $\alpha > \eta_1$ and $\delta < \eta_2$).

E_0 is a saddle in Δ, and its stable manifold contains I_z. $E_1 \in I_y$ and $E_2 \notin I_z$.

If $E^* \in \operatorname{Int}(\Delta)$, then the slope T_2 must be strictly positive, i.e.,

$$T_2 > 0 \Leftrightarrow \gamma > \delta, \tag{3.3.10}$$

and under this condition, z^* must be strictly positive, i.e.,

$$(\alpha-\eta_1)\gamma - (\alpha\eta_2 - \delta\eta_1) > 0. \tag{3.3.11}$$

Conversely, if (3.3.11) holds, then

$$\gamma - \delta > \frac{\alpha\eta_2 - \delta\eta_1}{\alpha-\eta_1} - \delta = \frac{\alpha(\eta_2-\delta)}{\alpha-\eta_1} = \frac{-\delta z_2}{y_1} > 0,$$

from which it follows that

$$-1 < -\frac{\alpha}{\alpha+\delta} < T_1 < 0,$$

and that $E^* \in \text{Int}(\Delta)$. Hence, $E^* \in \text{Int}(\Delta)$ iff (3.3.11) holds, when E_1 is a saddle, whose stable manifold contains $I_y \setminus E_0$. E^* is GAS in $\Delta \setminus (I_y \cup I_z)$ (Fig. 3.1 B).

If $E^* \notin \text{Int}(\Delta)$, (3.3.11) does not hold, and from the assumption $z^* \neq 0$, we have

$$(\alpha - \eta_1)\gamma - (\alpha\eta_2 - \delta\eta_1) < 0.$$

Hence, E_1 is GAS in $\Delta \setminus I_z$ (Fig. 3.1 C).

Case 3. $y_1 < 0$ and $z_2 > 0$ (i.e., $\alpha < \eta_1$ and $\delta > \eta_2$).

E_0 is a saddle in Δ, and its stable manifold contains I_y. $E_1 \notin I_y$ and $E_2 \in I_z$.

A similar discussion as in Case 2, we can easy find that $E^* \in \text{Int}(\Delta)$ iff

$$\alpha\eta_2 - \delta\eta_1 - \gamma(\delta - \eta_2) > 0, \tag{3.3.12}$$

when E_2 is a saddle, whose stable manifold contains $I_z \setminus E_0$. E^* is GAS in $\Delta \setminus (I_y \cup I_z)$. (Fig. 3.1 D)

If $E^* \notin \text{Int}(\Delta)$, (3.3.12) does not hold, and from the assumption $y^* \neq 0$, we have

$$\alpha\eta_2 - \delta\eta_1 - \gamma(\delta - \eta_2) < 0.$$

Hence, E_2 is GAS in $\Delta \setminus I_y$ (Fig. 3.1 E).

Case 4. $y_1 > 0$ and $z_2 > 0$ (i.e., $\alpha > \eta_1$ and $\delta > \eta_2$).

E_0 is a source in Δ. $E_1 \in I_y$ and $E_2 \in I_z$.

If $y^* < 0$ and $z^* < 0$, then E^* exists in the third quadrant of yz-plane, and the slopes T_1 and T_2 must be both strictly positive, which is a contradiction.

Assume that $y^* z^* < 0$, then $E^* \notin \text{Int}(\Delta)$. The sign of $\frac{(\alpha-\eta_1)\gamma-(\alpha\eta_2-\delta\eta_1)}{\alpha}$ is different from that of $\frac{\alpha\eta_2-\delta\eta_1-\gamma(\delta-\eta_2)}{\delta}$. Hence, if $\frac{(\alpha-\eta_1)\gamma-(\alpha\eta_2-\delta\eta_1)}{\alpha} < 0$, then E_1 is a saddle, whose stable manifold contains $I_y \setminus E_0$, and E_2 is GAS in $\Delta \setminus I_y$ (Fig. 3.1 F). Otherwise, E_2 is a saddle, whose stable manifold contains $I_z \setminus E_0$, and E_1 is GAS in $\Delta \setminus I_z$ (Fig. 3.1 G).

Assume that $y^* > 0$ and $z^* > 0$.

4-i) $\gamma(\gamma + \alpha - \delta) > 0$.
Then we have

$$(\alpha - \eta_1)\gamma > \alpha\eta_2 - \delta\eta_1 > \gamma(\delta - \eta_2). \tag{3.3.13}$$

If $\gamma > 0$, then $\gamma + \alpha - \delta > 0$ and it follows from (3.3.13) that

$$\alpha - \eta_1 + \eta_2 - \delta > 0$$

and

$$\gamma + \eta_1 - \eta_2 > \frac{\alpha\eta_2 - \delta\eta_1}{\alpha - \eta_1} + \eta_1 - \eta_2 = \frac{\eta_1(\alpha - \eta_1 + \eta_2 - \delta)}{\alpha - \eta_1} > 0.$$

Otherwise, $\gamma + \alpha - \delta < 0$ holds and it follows from (3.3.13) that

$$\alpha - \eta_1 + \eta_2 - \delta > 0$$

and

$$\gamma + \eta_1 - \eta_2 < \frac{\alpha\eta_2 - \delta\eta_1}{\delta - \eta_2} + \eta_1 - \eta_2 = \frac{\eta_2(\alpha - \eta_1 + \eta_2 - \delta)}{\delta - \eta_2} < 0.$$

In both cases, we see that $x^* > 0$ and $E^* \in \text{Int}(\Delta)$. (3.3.13) implies that E_1 is a saddle, whose stable manifold contains $I_y \setminus E_0$, and E_2 is also a saddle, whose stable manifold contains $I_z \setminus E_0$. We find

$$\text{tr}\, M|_{E^*} < 0, \quad \det M|_{E^*} > 0,$$

which implies that $M|_{E^*}$ has two eigenvalues whose real parts are both strictly negative. Hence E^* is GAS in $\Delta \setminus (I_y \cup I_z)$ (Fig. 3.1 H).

4-ii) $\gamma(\gamma+\alpha-\delta)<0$.
Then we have

$$(\alpha-\eta_1)\gamma > \alpha\eta_2-\delta\eta_1 > \gamma(\delta-\eta_2),$$

which implies that E_1, E_2 are both LAS (bistable).
A similar discussion as in the case $\gamma(\gamma+\alpha-\delta)>0$, it is easily found that $x^*>0$ and $E^*\in \mathrm{Int}(\Delta)$. We have $\det M|_{E^*}<0$, which implies that E^* is a saddle. The stable manifold of E^* is the separatrix which splits the basin of attraction of E_1 and that of E_2 (Fig. 3.1 I).

3.3.4 Variable rumor with rumor-spreading mass media

Now, we add to system (3.3.7) the effect of mass media that constantly spread the rumor to obtain

$$\begin{cases} \dot X(t) = -\alpha X(t)\dfrac{Y(t)}{N(t)} - \delta X(t)\dfrac{Z(t)}{N(t)} + \eta_1 Y(t) + \eta_2 Z(t) - pX(t), \\ \dot Y(t) = \alpha X(t)\dfrac{Y(t)}{N(t)} - \gamma Y(t)\dfrac{Z(t)}{N(t)} - \eta_1 Y(t) + pX(t) + qZ(t), \\ \dot Z(t) = \gamma Y(t)\dfrac{Z(t)}{N(t)} + \delta X(t)\dfrac{Z(t)}{N(t)} - \eta_2 Z(t) - qZ(t), \end{cases} \tag{3.3.14}$$

where γ can take any value in $\mathbb{R}\setminus\{0\}$, $\eta_1\geq 0$, $\eta_2\geq 0$ and the other parameters are strictly positive.
Since the total population $N(t) := X(t)+Y(t)+Z(t)$ is constant, we can rewrite the system (3.3.14) in the terms of

$$x(t) := \frac{X(t)}{N(t)},\ y(t) := \frac{Y(t)}{N(t)},\ z(t) := \frac{Z(t)}{N(t)}$$

as follows:

$$\begin{cases} \dot x(t) = -\alpha x(t)y(t) - \delta x(t)z(t) + \eta_1 y(t) + \eta_2 z(t) - px(t), \\ \dot y(t) = \alpha x(t)y(t) - \gamma y(t)z(t) - \eta_1 y(t) + px(t) + qz(t), \\ \dot z(t) = \gamma y(t)z(t) + \delta x(t)z(t) - \eta_2 z(t) - qz(t), \end{cases} \tag{3.3.15}$$

It is easy to show that system (3.3.15) has a unique solution on $(-\infty,\infty)$ in

$$\Omega := \{(x,\,y,\,z)\in {\mathbb{R}_+}^3 \mid x+y+z=1\}$$

for any initial data in Ω. Substituting $x(t)=1-y(t)-z(t)$ into (3.3.15), we can obtain the equations for y and z only:

$$\begin{cases} \dot y = \alpha(1-y-z)y - \gamma yz - \eta_1 y + p(1-x-y) + qz, \\ \dot z = \gamma yz + \delta(1-y-z)z - \eta_2 z - qz. \end{cases} \tag{3.3.16}$$

We define a Dulac function $\rho(y,\,z) := (yz)^{-1}$ on the domain $\{(y,\,z)\in {\mathbb{R}_+}^2 \mid y>0,\ z>0,\ y+z<1\}$. Then we have

$$\begin{aligned} &\frac{\partial}{\partial y}(\rho(y,\,z)\{\alpha(1-y-z)y - \gamma yz - \eta_1 y + p(1-x-y) + qz\}) \\ &\quad + \frac{\partial}{\partial z}(\rho(y,\,z)\{\gamma yz + \delta(1-y-z)z - \eta_2 z - qz\}) \\ &= -\left(\frac{\alpha}{z} + \frac{\delta}{y} + \frac{p(1-z)}{y^2 z} + \frac{q}{y^2}\right), \end{aligned}$$

which is strictly negative on the domain. Hence we can apply Dulac–Bendixson Criterion to exclude the possibility of a periodic orbit or a cyclic chain of equilibria.

Next, let us investigate the equilibria of system (3.3.15). $(x^*,\,y^*,\,z^*)$ is one of them if and only if it satisfies $(x^*,\,y^*,\,z^*)\in\Omega$ and

$$f^1(x^*,\,y^*,\,z^*) = f^2(x^*,\,y^*,\,z^*) = f^3(x^*,\,y^*,\,z^*) = 0,$$

where

$$f^1(x,\,y,\,z) := -\alpha xy - \delta xz + \eta_1 y + \eta_2 z - px,$$

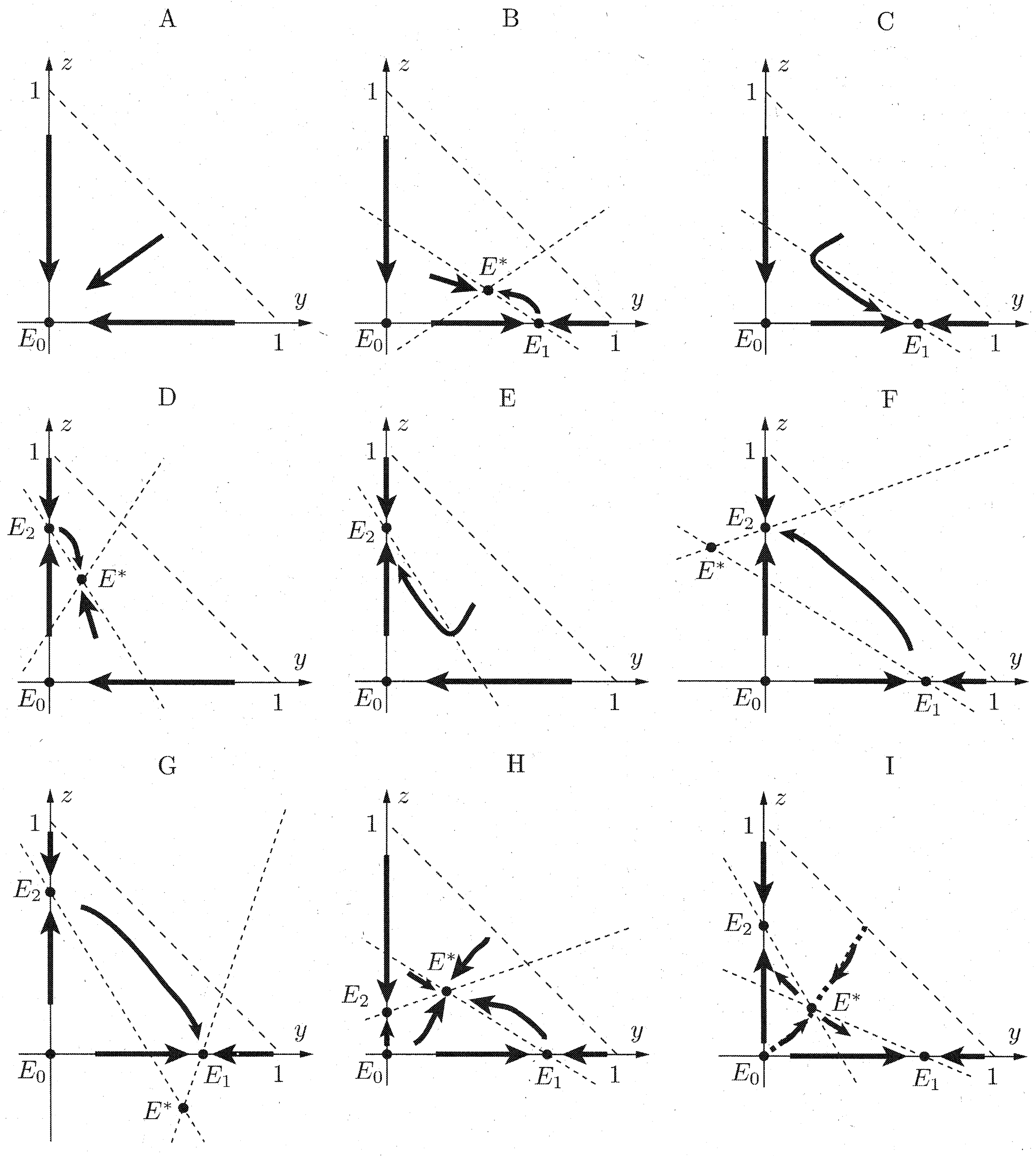

Fig. 3.1: yz-phase plane of system (3.3.9). (A) corresponds to Case 1. (B) corresponds to $E^* \in \text{Int}(\Delta)$ in Case 2. (C) corresponds to $E^* \notin \text{Int}(\Delta)$ in Case 2. (D) corresponds to $E^* \in \text{Int}(\Delta)$ in Case 3. (E) corresponds to $E^* \notin \text{Int}(\Delta)$ in Case 3. (F) corresponds to $y^*z^* < 0$ and $\frac{(\alpha-\eta_1)\gamma-(\alpha\eta_2-\delta\eta_1)}{\alpha} < 0$ in Case 4. (G) corresponds to $y^*z^* < 0$ and $\frac{(\alpha-\eta_1)\gamma-(\alpha\eta_2-\delta\eta_1)}{\alpha} > 0$ in Case 4. (H) corresponds to Case 4-i). (I) corresponds to Case 4-ii).

$$f^2(x, y, z) := \alpha xy - \gamma yz - \eta_1 y + px + qz,$$
$$f^3(x, y, z) := \gamma yz + \delta xz - \eta_2 z - qz.$$

Note that $f^1(x, y, z) + f^2(x, y, z) + f^3(x, y, z) = 0$.

$f^3(x^*, y^*, z^*) = 0$ implies

$$z^* = 0 \quad \text{or} \quad \gamma y^* + \delta x^* - \eta_2 - q = 0.$$

In the case $z^* = 0$, it follows from $f^2(x^*, y^*, 0) = 0$ and $(x^*, y^*, z^*) \in \Omega$ that

$$\alpha x^* y^* + px^* - \eta_1 y^* = 0 \quad \text{and} \quad x^* + y^* = 1.$$

Eliminating y^* from these equations, we have $g^1(x^*) = 0$, where

$$g^1(x) := \alpha x(1 - x) + px - \eta_1(1 - x).$$

Since $g^1(x)$ is quadratic with the coefficient of its quadratic term negative, and $g^1(0) = -\eta_1 < 0$, $g^1(1) = p > 0$, we find that $g^1(x)$ has a unique root x_0 in the interval $(0, 1)$, where

$$x_0 = \frac{\alpha + p + \eta_1 - \sqrt{(\alpha + p + \eta_1)^2 - 4\alpha\eta_1}}{2\alpha},$$
$$y_0 := 1 - x_0 = \frac{\alpha - p - \eta_1 + \sqrt{(\alpha + p + \eta_1)^2 - 4\alpha\eta_1}}{2\alpha}.$$

Hence, $E_0(x_0, y_0, 0)$ is an equilibrium of system (3.3.15).

Here, the Jacobian matrix of (3.3.16) at the equilibrium (x^*, y^*, z^*) is

$$M = \begin{pmatrix} \alpha(1 - 2y^* - z^*) - \gamma z^* - p - \eta_1 & -(\alpha + \gamma)y^* - p + q \\ (\gamma - \eta)z^* & \gamma y^* + \delta(1 - y^* - 2z^*) - \eta_2 - q \end{pmatrix}.$$

Hence we have

$$M|_{E_0} = \begin{pmatrix} -\sqrt{(\alpha + p + \eta_1)^2 - 4\alpha\eta_1} & -(\alpha + \gamma)y_0 - p + q \\ 0 & \eta x_0 + \gamma y_0 - \eta_2 - q \end{pmatrix}.$$

Observing that x_0 and y_0 do not depend on γ, we see that the function $\gamma \mapsto \delta x_0 + \gamma y_0 - \eta_2 - q$ is strictly monotone increasing. So, we can find a constant γ_0 which satisfies $\delta x_0 + \gamma_0 y_0 - \eta_2 - q = 0$, and it follows that, if $\gamma < \gamma_0$ then E_0 is LAS, and if $\gamma > \gamma_0$ then E_0 is a saddle, whose stable manifold is $\{(x, y, z) \in \Omega \mid z = 0\}$. γ_0 is expressed as follows:

$$\gamma_0 = -\frac{(\eta_2 + q - \delta)\left(\alpha - p - \eta_1 - \sqrt{(\alpha + p + \eta_1)^2 - 4\alpha\eta_1}\right)}{2p} + \delta$$

Next, we shall examine the case $\gamma y^* + \delta x^* - \eta_2 - q = 0$. Let

$$\Delta := \{(x, y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0,\ x + y \leq 1\}.$$

Case 1. $\delta \leq \eta_2 + q$.

Since it follows that

$$(\alpha + p - \eta_1)^2 - \{(\alpha + p + \eta_1)^2 - 4\alpha\eta_1\} = -4p\eta_1 < 0,$$

we find that

$$\alpha - p - \eta_1 - \sqrt{(\alpha + p + \eta_1)^2 - 4\alpha\eta_1} < -2p,$$

hence

$$\gamma_0 > -\frac{(\eta_2 + q - \delta)(-2p)}{2p} + \delta = \eta_2 + q.$$

1-i) $\gamma < \eta_2 + q$.

Then we have (Fig. 3.2 A)

$$\{(x, y) \in \mathbb{R}^2 \mid \gamma y + \delta x - \eta_2 - q = 0\} \cap \Delta \subset \{(1, 0)\}.$$

The point $(1, 0, 0)$ cannot be an equilibrium of system (3.3.15) because $f^1(1, 0, 0) = -p \neq 0$. Hence, system (3.3.15) has no equilibria except E_0, which is GAS in Ω because $\gamma < \gamma_0$. The xy-phase plane is as Fig. 3.2 B.

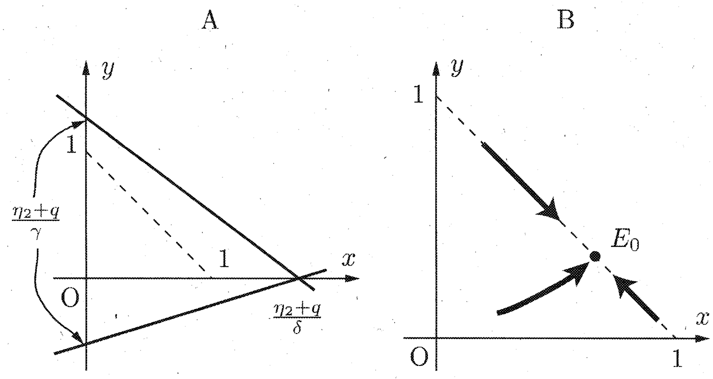

Fig. 3.2: xy-phase plane in Case 1-i). (A) represents the position of the line $\gamma y + \delta x - \eta_2 - q = 0$ and Δ. (B) represents the global dynamics of system (3.3.15).

1-ii) $\delta = \gamma = \eta_2 + q$.
$\gamma y^* + \delta x^* - \eta_2 - q = 0$ implies $x^* + y^* = 1$, i.e., $z^* = 0$. Then system (3.3.15) has no equilibria except E_0, which is GAS in Ω.

1-iii) $\delta < \eta_2 + q \leq \gamma$.
Since $(x^*, y^*) \in \{(x, y) \in \mathbb{R}^2 \mid \gamma y + \delta x - \eta_2 - q = 0\} \cap \Delta$, it must satisfy (Fig. 3.3 A)

$$0 \leq x^* \leq x_\dagger, \quad y_\dagger \leq y^* \leq \frac{\eta_2 + q}{\gamma},$$

where

$$x_\dagger := \frac{\gamma - \eta_2 - q}{\gamma - \delta}, \quad y_\dagger := 1 - x_\dagger = \frac{\delta - \eta_2 - q}{\delta - \gamma}.$$

Let

$$g^2(x) := f^1\left(x, \frac{\eta_2 + q - \delta x}{\gamma}, 1 - x - \frac{\eta_2 + q - \delta x}{\gamma}\right),$$

then the equation x^* must satisfy is $g^2(x^*) = 0$. Here $g^2(x)$ is at most quadratic, whose quadratic term is

$$\left\{-\alpha\left(-\frac{\delta}{\gamma}\right) - \delta\left(-1 + \frac{\delta}{\gamma}\right)\right\} x^2 = \frac{\delta(\gamma - \delta + \alpha)}{\gamma} x^2,$$

and it follows that

$$\begin{aligned} g^2(0) &= \eta_2 \cdot \frac{\gamma - \eta_2 - q}{\gamma} + \eta_1 \cdot \frac{\eta_2 + q}{\gamma} > 0, \\ g^2(x_\dagger) &= -\alpha x_\dagger(1 - x_\dagger) - p x_\dagger + \eta(1 - x_\dagger) = -g^1(x_\dagger). \end{aligned} \tag{3.3.17}$$

Since the function $[\eta_2 + q, \infty) \ni \gamma \mapsto x_\dagger = x_\dagger(\gamma)$ is strictly monotone increasing and its range is $[0, 1)$, there exists a unique $\gamma = \gamma_1$ such that $x_\dagger(\gamma_1) = x_0$. Then we have

$$\delta x_0 + \gamma_0 y_0 - \eta_2 - q = \delta x_\dagger(\gamma_1) + \gamma_1\{1 - x_\dagger(\gamma_1)\} - \eta_2 - q = 0,$$

which implies $\gamma_1 = \gamma_0$ because $y_0 > 0$.
Hence, if $\gamma > \gamma_0$, then $x_\dagger > x_0$ and $g^2(x_\dagger) < 0$, which implies that $g^2(x)$ has only one root x_1 in the interval $[0, x_\dagger]$. System (3.3.15) has just two equilibria E_0 and $E_1(x_1, y_1, z_1)$. The latter satisfies

$$g^2(x_1) = 0, \quad 0 < x_1 < x_\dagger, \quad y_1 = \frac{\eta_2 + q - \delta x_1}{\gamma}, \quad z_1 = 1 - x_1 - y_1.$$

We have

$$M|_{E_1} = \begin{pmatrix} \alpha(x_1 - y_1) - \gamma z_1 - p - \eta_1 & -(\alpha + \gamma) y_1 - p + q \\ (\gamma - \delta) z_1 & -\delta z_1 \end{pmatrix}$$

and a little calculation gives rise to the following:

$$\det M|_{E_1} = -\gamma z_1 \cdot (g^2)'(x_1),$$

$$\begin{aligned}\operatorname{tr} M|_{E_1} &= (\alpha x_1 - \gamma z_1 - \eta_1) - \alpha y_1 - p - \delta z_1 \\ &= -\frac{p x_1 + q z_1}{y_1} - \alpha y_1 - p - \delta z_1\end{aligned}$$

We find that $\det M|_{E_1} > 0$ and $\operatorname{tr} M|_{E_1} < 0$, which implies that $M|_{E_1}$ has two eigenvalues whose real parts are both strictly negative. Hence E_1 is GAS in $\{(x,\, y,\, z) \in \Omega \mid z \neq 0\}$ (Fig. 3.3 B).

On the other hand, if $\eta_2 + q \leq \gamma < \gamma_0$, then $g^2(x_\dagger) > 0$, hence there are two possibilities: $g^2(x)$ has no root in the interval $[0,\, x_\dagger]$, or $g^2(x)$ has two roots x_1, x_2 there. In the former case, the bifurcation diagram of $z = z^*(\gamma)$ with γ the bifurcation parameter is as Fig. 3.4 A, which shows that a supercritical bifurcation occurs at $\gamma = \gamma_0$. In the latter case, the bifurcation diagram is as Fig. 3.4 B, which shows that a subcritical bifurcation occurs as $\gamma = \gamma_0$. The middle branch in the range $\gamma_2 < \gamma < \gamma_0$ represents the equilibrium $E_2(x_2,\, y_2,\, z_2)$, which satisfies

$$0 < z_2 < z_1, \quad 0 < x_1 < x_2 < x_\dagger.$$

Hence

$$\det M|_{E_1} > 0, \quad \det M|_{E_2} = -\gamma z_2 \cdot (g^2)'(x_2) < 0$$

which implies that E_1 is LAS and E_2 is a saddle (Fig. 3.3 C).

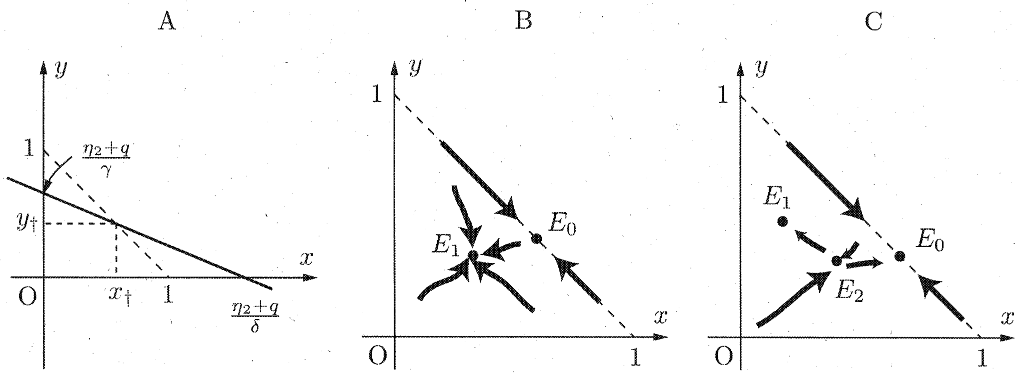

Fig. 3.3: xy-phase plane in Case 1-iii). (A) represents the position of the line $\gamma y + \delta x - \eta_2 - q = 0$ and Δ. (B) and (C) represent the global dynamics of system (3.3.15).

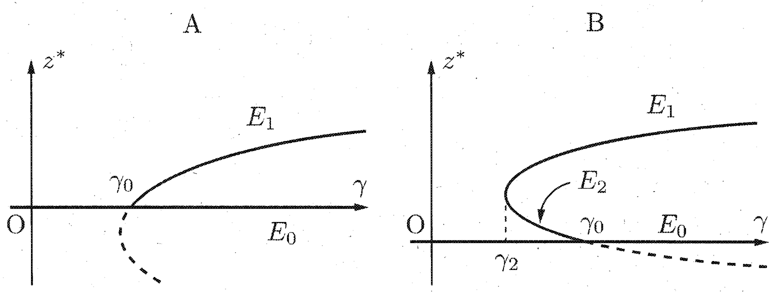

Fig. 3.4: Bifurcation diagram in Case 1.

Case 2. $\delta > \eta_2 + q$.

It is easily seen as in Case 1 that $\gamma_0 < \eta_2 + q$.

2-i) $\gamma > \eta_2 + q$.

Since $(x^*,\, y^*) \in \{(x,\, y) \in \mathbb{R}^2 \mid \gamma y + \delta x - \eta_2 - q = 0\} \cap \Delta$, it must satisfy (Fig. 3.5 A)

$$0 \leq x^* \leq \frac{\eta_2 + q}{\delta}, \quad 0 \leq y^* \leq \frac{\eta_2 + q}{\gamma}.$$

$g^2(x)$ is at most quadratic and

$$g^2(0) > 0,$$
$$g^2\left(\frac{\eta_2+q}{\delta}\right) = -q\cdot\frac{\delta-\eta_2-q}{\delta} - p\cdot\frac{\eta_2+q}{\delta} < 0. \tag{3.3.18}$$

Hence $g^2(x)$ has only one root x_1 in the interval $[0, \frac{\eta_2+q}{\delta}]$ and the equilibrium E_1 satisfies $\det M|_{E_1} > 0$ and $\mathrm{tr}M|_{E_1} < 0$. Therefore, E_0 is a saddle and E_1 is GAS in $\{(x, y, z) \mid z \neq 0\}$.

2-ii) $\gamma < \eta_2 + q$.
Since $(x^*, y^*) \in \{(x, y) \in \mathbb{R}^2 \mid \gamma y + \delta x - \eta_2 - q = 0\} \cap \Delta$, x^* must exist between $x_\dagger$ and $\frac{\eta_2+q}{\delta}$ (Fig. 3.5 B or Fig. 3.5 C).

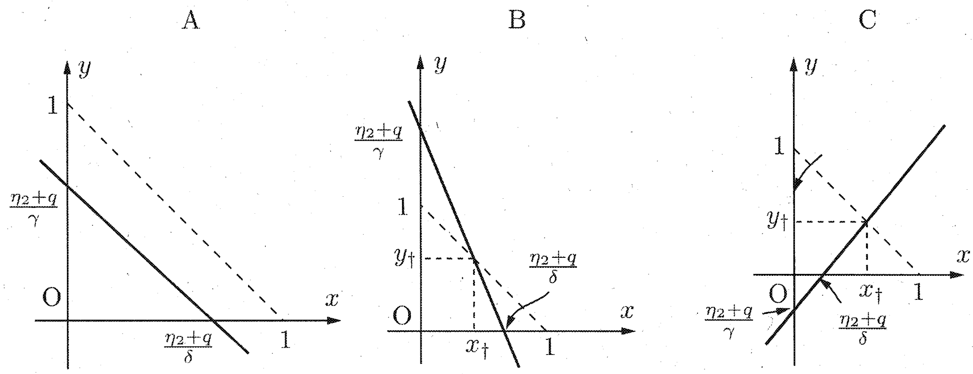

Fig. 3.5: xy-phase plane in Case 2. Each figure represents the position of the line $\gamma y + \delta x - \eta_2 - q = 0$ and Δ. (A) corresponds to Case 2-i). (B) corresponds to $0 < \gamma < \eta_2 + q$ in Case 2-ii). (C) corresponds to $\gamma < 0$ in Case 2-ii).

The function $(-\infty, \eta_2 + q) \ni \gamma \mapsto x_\dagger$ is strictly decreasing and its range is $(0, 1)$. Then it follows that there exists a unique $\gamma = \gamma_1$ such that $x_\dagger(\gamma_1) = x_0$, and we can easily show as in Case 1 that $\gamma_1 = \gamma_0$.
If $\gamma_0 < \gamma < \eta_2 + q$, then $0 < x_\dagger < x_0$, which implies $g^1(x_\dagger) < 0$ and $g^2(x_\dagger) > 0$. From this and (3.3.18) we observe that $g^2(x)$ has only one equilibrium x_1 between $x_\dagger$ and $\frac{\eta_2+q}{\delta}$. If $\gamma > 0$, then

$$x_\dagger < x_1 < \frac{\eta_2+q}{\delta} \quad \text{and} \quad (g^2)'(x_1) < 0.$$

Otherwise, if $\gamma < 0$, then

$$\frac{\eta_2+q}{\delta} < x_1 < x_\dagger \quad \text{and} \quad (g^2)'(x_1) > 0.$$

Both cases imply that $\det M|_{E_1} > 0$ and $\mathrm{tr}\, M|_{E_1} < 0$, therefore the global behavior of system (3.3.15) is the same as Case 1-iii).
If $\gamma < \gamma_0$, then $x_0 < x_\dagger < 1$, which implies $g^1(x_\dagger) > 0$ and $g^2(x_\dagger) < 0$. From this and (3.3.18) we observe that $g^2(x)$ has no root between $x_\dagger$ and $\frac{\eta_2+q}{\delta}$ under the condition $\gamma(\gamma - \delta + \alpha) \geq 0$. In this case, E_0 is GAS in Ω. Otherwise, i.e., under the condition $\gamma(\gamma - \delta + \alpha) < 0$, there are two possibilities: $g^2(x)$ has no root in the interval $[0, x_\dagger]$, or $g^2(x)$ has two roots x_1, x_2 there, which satisfies

$$\begin{cases} x_\dagger < x_2 < x_1 < \frac{\eta_2+q}{\delta} & (\gamma > 0) \\ \frac{\eta_2+q}{\delta} < x_1 < x_2 < x_\dagger & (\gamma < 0) \end{cases},$$

which implies

$$\begin{cases} (g^2)'(x_1) < 0 < (g^2)'(x_2) & (\gamma > 0) \\ (g^2)'(x_2) < 0 < (g^2)'(x_1) & (\gamma < 0) \end{cases}.$$

In the case $g^2(x)$ has no root, the bifurcation diagram of $z = z^*(\gamma)$ with γ the bifurcation parameter is as Fig. 3.4 A, which shows that a supercritical bifurcation occurs at $\gamma = \gamma_0$. In the other case, the bifurcation diagram is as Fig. 3.4 B, which shows that a subcritical bifurcation occurs as $\gamma = \gamma_0$. The middle branch in the range $\gamma_2 < \gamma < \gamma_0$ represents the equilibrium $E_2(x_2, y_2, z_2)$. The dynamics is the same as in the case $\delta \leq \eta_2 + q \leq \gamma < \gamma_0$.

3.4 Rumor-suppressing mass media

So far we have treated the situation where mass media spread rumor. In contrast, we shall discuss the situation where mass media constantly suppress rumor in the following. We assume that rumor-suppressing mass media change $pX(t)\Delta t$ susceptibles and $qY(t)\Delta t$ spreaders into stiflers during the small interval $(t,\, t+\Delta t)$, where p, q are strictly positive constants and p does not necessarily equal to q. We also assume that the stiflers do not change their behavior after they rediscover the rumor in mass media.

3.4.1 Constant rumor with rumor-suppressing mass media

In the case of the transmission of a constant rumor in a closed population, if the dynamics of the population without mass media is governed by system (1.2.1), its dynamics taking the effect of mass media into consideration is governed by the following system:

$$\begin{cases} \dot{X}(t) = -\alpha X(t)\dfrac{Y(t)}{N(t)} - pX(t), \\ \dot{Y}(t) = \alpha\theta X(t)\dfrac{Y(t)}{N(t)} - \beta Y(t)\dfrac{Y(t)}{N(t)} - \gamma Y(t)\dfrac{Z(t)}{N(t)} - qY(t), \\ \dot{Z}(t) = \alpha(1-\theta)X(t)\dfrac{Y(t)}{N(t)} + \beta Y(t)\dfrac{Y(t)}{N(t)} + \gamma Y(t)\dfrac{Z(t)}{N(t)} + pX(t) + qY(t), \end{cases} \tag{3.4.1}$$

Taking into consideration that $N(t)$ is constant, we can rewrite it in the terms of

$$x(t) := \frac{X(t)}{N(t)},\ y(t) := \frac{Y(t)}{N(t)},\ z(t) := \frac{Z(t)}{N(t)}.$$

as follow:

$$\begin{cases} \dot{x}(t) = -\alpha x(t)y(t) - px(t), \\ \dot{y}(t) = \alpha\theta x(t)y(t) - \beta\{y(t)\}^2 - \gamma y(t)z(t) - qy(t), \\ \dot{z}(t) = \alpha(1-\theta)x(t)y(t) + \beta\{y(t)\}^2 + \gamma y(t)z(t) + px(t) + qy(t), \end{cases} \tag{3.4.2}$$

It is easy to show that system (3.4.2) has a unique solution on $(-\infty,\, \infty)$ in

$$\Omega := \{(x,\, y,\, z) \in {\mathbb{R}_+}^3 \mid x + y + z = 1\}$$

for any initial data in Ω. Moreover, since $(x(t),\, y(t),\, z(t)) \in \Omega$, $(0,\, 0,\, 1)$ is the only equilibrium of system (3.4.2), $t \mapsto x(t)$ is monotone decreasing and $t \mapsto z(t)$ is monotone increasing. Therefore, we have the following result:

Theorem 3.4.1. *System (3.4.2) has only one equilibrium* $(0,\, 0,\, 1)$, *which is GAS in* Ω.

3.4.2 Variable rumor with rumor-suppressing mass media

Next, in the case of the transmission of a variable rumor in a closed population, if the dynamics of the population without mass media is governed by system (1.3.1), its dynamics taking the effect of mass media into consideration is governed by the following system:

$$\begin{cases} \dot{X}(t) = -\alpha X(t)\dfrac{Y(t)}{N(t)} + \eta Z(t) - pX(t), \\ \dot{Y}(t) = \alpha\theta X(t)\dfrac{Y(t)}{N(t)} - \beta Y(t)\dfrac{Y(t)}{N(t)} - \gamma Y(t)\dfrac{Z(t)}{N(t)} - qY(t), \\ \dot{Z}(t) = \alpha(1-\theta)X(t)\dfrac{Y(t)}{N(t)} + \beta Y(t)\dfrac{Y(t)}{N(t)} + \gamma Y(t)\dfrac{Z(t)}{N(t)} - \eta Z(t) + pX(t) + qY(t). \end{cases} \tag{3.4.3}$$

Taking into consideration that $N(t)$ is constant, we can rewrite it in the terms of

$$x(t) := \frac{X(t)}{N(t)},\ y(t) := \frac{Y(t)}{N(t)},\ z(t) := \frac{Z(t)}{N(t)}$$

as follows:

$$\begin{cases} \dot{x}(t) = -\alpha x(t)y(t) + \eta z(t) - px(t), \\ \dot{y}(t) = \alpha\theta x(t)y(t) - \beta\{y(t)\}^2 - \gamma y(t)z(t) - qy(t), \\ \dot{z}(t) = \alpha(1-\theta)x(t)y(t) + \beta\{y(t)\}^2 + \gamma y(t)z(t) - \eta z(t) + px(t) + qy(t), \end{cases} \tag{3.4.4}$$

It is easy to show that system (3.4.4) has a unique solution on $(-\infty, \infty)$ in

$$\Omega := \{(x, y, z) \in \mathbb{R}_+{}^3 \mid x + y + z = 1\}$$

for any initial data in Ω.

Substituting $x(t) = 1 - y(t) - z(t)$ into (3.4.4), we can obtain the equations for y and z only:

$$\begin{cases} \dot{y} = \alpha\theta(1 - y - z)y - \beta y^2 - \gamma yz - qy, \\ \dot{z} = \alpha(1 - \theta)(1 - y - z)y + \beta y^2 + \gamma yz - \eta z + p(1 - y - z) + qy. \end{cases} \tag{3.4.5}$$

We define a Dulac function $\rho(y, z) := (yz)^{-1}$ on the domain $\{(y, z) \in \mathbb{R}_+{}^2 \mid y > 0,\ z > 0,\ y + z < 1\}$. Then we have

$$\begin{aligned} &\frac{\partial}{\partial y}(\rho(y, z)\{\alpha\theta(1 - y - z)y - \beta y^2 - \gamma yz - qy\}) \\ &\quad + \frac{\partial}{\partial z}(\rho(y, z)\{\alpha(1 - \theta)(1 - y - z)y + \beta y^2 + \gamma yz - \eta z + p(1 - y - z) + qy\}) \\ &= -\frac{\alpha\theta + \beta}{z} - \frac{1}{z^2}\left\{\alpha(1 - \theta)(1 - y) + \beta y + p \cdot \frac{1 - y}{y} + q\right\}, \end{aligned}$$

which is strictly negative on the domain. Hence we can apply Dulac–Bendixson Criterion to exclude the possibility of a periodic orbit or a cyclic chain of equilibria.

Next, let us investigate the equilibria of system (3.4.4). (x^*, y^*, z^*) is one of them if and only if it satisfies $(x^*, y^*, z^*) \in \Omega$ and

$$-\alpha x^* y^* + \eta z^* - p x^* = 0, \tag{3.4.6}$$

$$\alpha\theta x^* y^* - \beta(y^*)^2 - \gamma y^* z^* - q y^* = 0. \tag{3.4.7}$$

(3.4.7) implies

$$y^* = 0 \quad \text{or} \quad \alpha\theta x^* - \beta y^* - \gamma z^* - q = 0.$$

In the case $y^* = 0$, it follows from (3.4.6) and $(x^*, y^*, z^*) \in \Omega$ that

$$x^* = \frac{\eta}{p + \eta}, \quad z^* = \frac{p}{p + \eta}.$$

Hence,

$$E_0\left(\frac{\eta}{p + \eta},\ 0,\ \frac{p}{p + \eta}\right)$$

is a boundary equilibrium of system (3.4.4).

The Jacobian matrix of (3.4.5) at the equilibrium (x^*, y^*, z^*) is

$$M = \begin{pmatrix} \alpha\theta(x^* - y^*) - 2\beta y^* - \gamma z^* - q & -(\alpha\theta + \gamma)y^* \\ \alpha(1 - \theta)(x^* - y^*) + 2\beta y^* + \gamma z^* - p + q & \{-\alpha(1 - \theta) + \gamma\}y^* - \eta - p \end{pmatrix}.$$

Hence we have

$$M|_{E_0} = \begin{pmatrix} R & 0 \\ \alpha(1 - \theta)\frac{\eta}{p+\eta} + \gamma \cdot \frac{p}{p+\eta} - p + q & -\eta - p \end{pmatrix},$$

where

$$R := \alpha\theta\frac{\eta}{p + \eta} - \gamma \cdot \frac{p}{p + \eta} - q.$$

Therefore, we see that if $R < 0$ then E_0 is LAS, and if $R > 0$ then E_0 is a saddle, whose stable manifold is $\{(x, y, z) \in \Omega \mid y = 0\}$.

Next, we shall examine the case $\alpha\theta x^* - \beta y^* - \gamma z^* - q = 0$. Let

$$\Delta := \{(x, y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0,\ x + y \leq 1\}.$$

the equation x^* must satisfy is $f(x^*) = 0$, where

$$f(x) := \{(\alpha\theta + \gamma)x - \gamma - q\}(\alpha x + \eta) + (-\beta + \gamma)\{-(p + \eta)x + \eta\}.$$

Here, $f(x)$ is quadratic and the coefficient of its quadratic term is $\alpha(\alpha\theta + \gamma)$, which is strictly positive. In addition, we obtain

$$f(0) = -(\beta + q)\eta < 0,$$

$$f\left(\frac{\eta}{p+\eta}\right) = R \cdot \frac{\eta(\alpha+p+\eta)}{p+\eta}.$$

Observing that it follows from $(x^*,\, y^*,\, z^*) \in \Omega$ and (3.4.6) that

$$x^* \in \left[0,\, \frac{\eta}{p+\eta}\right],$$

if $R \leq 0$ then E_0 is the only equilibrium of system (3.4.4) and GAS in Ω. Moreover, if $R > 0$ then $f(x)$ has only one root x_1 in $(0,\, \frac{\eta}{p+\eta})$, which satisfies $f'(x_1) > 0$, and system (3.4.4) has just two equilibria E_0 and $E_1(x_1,\, y_1,\, z_1)$ (interior equilibrium), where

$$y_1 := \frac{-(p+\eta)x_1 + \eta}{\alpha x_1 + \eta}(> 0), \quad z_1 := 1 - x_1 - y_1(> 0).$$

The Jacobian matrix at E_1 is calculated as follows:

$$M|_{E_1} = \begin{pmatrix} -(\alpha\theta+\beta)y_1 & -(\alpha\theta+\gamma)y_1 \\ \alpha x_1 - \alpha(1-\theta)y_1 + \beta y_1 - p & \{-\alpha(1-\theta)+\gamma\}y_1 - \eta - p \end{pmatrix}.$$

Then a little calculation shows that

$$\begin{aligned} \det M|_{E_1} &= y_1 \cdot f'(x_1) > 0, \\ \operatorname{tr} M|_{E_1} &= (-\alpha-\beta)y_1 - p - \eta + \gamma y_1 \\ &= -(\alpha+\beta)y_1 - p - \frac{\alpha x_1 y_1 + p x_1}{z_1} + \frac{y_1(\alpha\theta x_1 - \beta y_1 - q)}{z_1} \\ &= -(\alpha+\beta)y_1 - p - \frac{\alpha(1-\theta)x_1 y_1 + p x_1 + \beta(y_1)^2 + q y_1}{z_1} < 0, \end{aligned}$$

which implies that $M|_{E_1}$ has two eigenvalues whose real parts are both strictly negative. Hence E_1 is GAS in $\{(x,\, y,\, z) \in \Omega \mid y \neq 0\}$.

The above argument is summarized as follows:

Theorem 3.4.2. *System (3.4.4) has two equilibria E_0, E_1 in Ω, while E_1 exists in Ω if and only if*

$$R := \alpha\theta\frac{\eta}{p+\eta} - \gamma \cdot \frac{p}{p+\eta} - q > 0.$$

If $R \leq 0$, then E_0 is GAS in Ω.

Otherwise, the segment $\{(x,\, y,\, z) \in \Omega \mid y = 0\}$ is the stable manifold of the saddle E_0, and E_1 is GAS in $\{(x,\, y,\, z) \in \Omega \mid y > 0\}$.

3.5 Discussion

In this chapter we have investigated the impact of mass media on rumor transmission, i.e., how it affects the system's dynamics to add the rumor-class transitions caused by mass media. For example, while the equilibria of system (1.2.1) are $(N_1,\, 0,\, N_0 - N_1)$ $(0 \leq N_1 \leq N_0)$, where N_0 is the total population which is constant, system (3.2.1) has a unique equilibrium with no susceptibles. This means that, although a rumor, if not discussed by mass media, is destined to fade out, almost all people get to know it and some of them actively try to spread it if mass media keeps discussing it. On the other hand, the analysis of system (3.4.1) implies that almost all people get to know it and never try to spread it if mass media keeps suppressing it. This interpretation seems to be convincing when we consider the transmission of a constant rumor in a closed population in a short period. However, if we consider the modification of rumor, i.e., compare systems (1.3.1), (3.2.3) and (3.4.3), the interpretation we can draw out, if any, is that the ratio of each rumor-class after long periods of time varies according to the effect of mass media. We have not obtained the variation in concrete form, which would be troublesome to calculate.

In Section 3.3 we have examined the dynamics of rumor transmission with active stiflers. Systems (3.3.1), (3.3.4) and (3.3.7) are substantially equivalent to generalized Lotka–Volterra 2-species competition system ([63]), so it is very natural that possible final situations are extinction, domination, coexistence and bistability. We have also proved that possible final situations in system (3.3.14) are coexistence and bistability. We should notice that our ODE systems describing rumor transmission with passive stiflers have no chance of bistability. Concerning generalized Lotka–Volterra 2-species competition systems it is well-known that they are bistable if the competition between the two species is fierce, so it can be safely said that our variable-rumor-transmission models with active stiflers are bistable if the "competition" between spreaders and active stiflers is fierce, where its meaning is harder to understand.

In Subsections 3.3.3, if we replace the role of the rumor with that of its rival rumor, i.e., we denotes by $Z(t)$ the population (density) of spreaders and by $Y(t)$ the population (density) of active stiflers, then we should change α, δ, γ, η_1, η_2 into δ, α, $-\gamma$, η_2, η_1 respectively. However, this transformation does not change the discussion in Subsection 3.3.3. If we transform so in Subsection 3.3.4, then the model becomes a model of the transmission of a variable rumor with active stiflers and rumor-suppressing mass media. Hence we have not treated such a model in Section 3.4.

We have implicitly assumed that mass media feed information about the rumor at a constant pace. However, the pace may undergo a lot of changes. For example, they alternate discussing it in a short time and putting aside it over a long time. We could formulate the influence of such kind of behavior as a periodic pulse $-\sum_{m=1}^{\infty} \delta(t-mt_0)pX(t)$ instead of $-pX(t)$ and so on, where $\delta(t)$ is the δ-function and t_0 is the cycle. Such a formulation is found in many models for infectious diseases with periodic vaccination, and so we could expect that we can mathematically analyze rumor-transmission models with periodic pulse-like influence of mass media and as a result we can show that some of them have periodic solutions corresponding to recursive rumors.

Chapter 4

Rumor transmission in a age-structured population with active stiflers

4.1 Model formulation

Let us consider the transmission of a constant rumor in a closed age-structured population under the demographic growth. While we regarded stiflers as passive in Chapter 1, here we regard them as so active that they voluntarily change susceptibles and spreaders into active stiflers by making them believe that the rumor is false.

Let $a \in [0, \omega]$, where $\omega(< \infty)$ denotes the life span of the population, and $X(t,a)$, $Y(t,a)$, $Z(t,a)$ be the age-density functions at time t of the susceptible class, the spreader class and the active stifler class respectively.

$$P(t,a) := X(t,a) + Y(t,a) + Z(t,a)$$

denotes the age-density of the total number of individuals, then the total size of the population is given by $N(t) := \int_0^\omega P(t,a)\,\mathrm{d}a$.

The system can be formulated as follows:

$$\begin{cases} (\partial_t + \partial_a)X(t,a) = -\{\mu(a) + \lambda_1(t,a) + \lambda_2(t,a)\}X(t,a), \\ (\partial_t + \partial_a)Y(t,a) = \lambda_1(t,a)X(t,a) - \{\mu(a) + \lambda_3(t,a)\}Y(t,a), \\ (\partial_t + \partial_a)Z(t,a) = \lambda_2(t,a)X(t,a) + \lambda_3(t,a)Y(t,a) - \mu(a)Z(t,a), \\ \qquad X(t,0) = \int_0^\omega m(a)P(t,a)\,\mathrm{d}a, \\ \qquad Y(t,0) = 0, \quad Z(t,0) = 0, \\ \qquad X(0,a) = X_0(a), \quad Y(0,a) = Y_0(a), \quad Z(0,a) = Z_0(a). \end{cases} \tag{4.1.1}$$

$\mu(a), m(a)$ stand for the age-specific natural death rate and fertility rate respectively. $(X_0(a), Y_0(a), Z_0(a))$ is a given initial data. $\lambda_1(t,a)$ is the force of transition into the spreader class on a susceptible individual aged a at time t and defined by

$$\lambda_1(t,a) := \frac{1}{N(t)}\int_0^\omega \alpha(a,\sigma)Y(t,\sigma)\,\mathrm{d}\sigma,$$

where $\alpha(a,\sigma)$ is the transmission rate between a susceptible individual aged a and a spreader aged σ. $\lambda_2(t,a)$ is the force of transition into the active stifler class on a susceptible individual aged a at time t and defined by

$$\lambda_2(t,a) := \frac{1}{N(t)}\int_0^\omega \delta(a,\sigma)Z(t,\sigma)\,\mathrm{d}\sigma,$$

where $\alpha(a,\sigma)$ is the transmission rate between a susceptible individual aged a and an active stifler aged σ.

$\lambda_3(t,a)$ is the force of transition into the active stifler class on a spreader aged a at time t and defined by

$$\lambda_3(t,a) := \frac{1}{N(t)}\int_0^\omega \gamma(a,\sigma)Z(t,\sigma)\,\mathrm{d}\sigma,$$

where $\gamma(a,\sigma)$ is the transmission rate between a spreader aged a and an active stifler aged σ.

It follows from (4.1.1) that $P(t,a)$ satisfies the McKendrick equation (1.4.2). Note that we implicitly assume that there is no true interaction between demography and the spread of the rumor. Hence, it is convenient to introduce the fractional age distribution for each rumor-class as follows:

$$x(t,a) := \frac{X(t,a)}{P(t,a)}, \quad y(t,a) := \frac{Y(t,a)}{P(t,a)}, \quad z(t,a) := \frac{Z(t,a)}{P(t,a)}.$$

Then the new system for the fractional age distributions is given as follows:

$$\begin{cases} (\partial_t + \partial_a)x(t,a) = -\lambda_1(t,a)x(t,a) - \lambda_2(t,a)x(t,a), \\ (\partial_t + \partial_a)y(t,a) = \lambda_1(t,a)x(t,a) - \lambda_3(t,a)y(t,a), \\ (\partial_t + \partial_a)z(t,a) = \lambda_2(t,a)x(t,a) + \lambda_3(t,a)y(t,a), \\ x(t,0) = 1, \quad y(t,0) = 0, \quad z(t,0) = 0, \\ x(0,a) = x_0(a), \quad y(0,a) = y_0(a), \quad z(0,a) = z_0(a), \\ \lambda_1(t,a) = \displaystyle\int_0^\omega \alpha(a,\sigma)\psi(t,\sigma)y(t,\sigma)\,\mathrm{d}\sigma, \\ \lambda_2(t,a) = \displaystyle\int_0^\omega \delta(a,\sigma)\psi(t,\sigma)z(t,\sigma)\,\mathrm{d}\sigma, \\ \lambda_3(t,a) = \displaystyle\int_0^\omega \gamma(t,\sigma)\psi(a,\sigma)z(t,\sigma)\,\mathrm{d}\sigma, \end{cases} \tag{4.1.2}$$

where $\psi(t,a)$ is defined by

$$\psi(t,a) := \frac{P(t,a)}{\int_0^\omega P(t,a)\,\mathrm{d}a}.$$

According to the stable population theory, ψ converges to the persistent normalized age distribution uniformly with respect to a as $t \to \infty$:

$$\lim_{t\to\infty} \psi(t,a) = c(a) := \frac{\mathrm{e}^{-\lambda_0 a}\mathcal{F}(a)}{\int_0^\omega \mathrm{e}^{-\lambda_0 a}\mathcal{F}(a)\,\mathrm{d}a},$$

where λ_0 denotes the intrinsic rate of natural increase, $\mathcal{F}(a)$ is the survival rate defined by

$$\mathcal{F}(a) := \exp\left(-\int_0^a \mu(\sigma)\,\mathrm{d}\sigma\right),$$

and $c(a)$ is called relatively stable age distribution. Note that $\int_0^\omega c(a)\,\mathrm{d}a = 1$.

In the following we assume that the stable age distribution is already attained. Then system (4.1.2) is rewritten as the autonomous system below:

$$\begin{cases} (\partial_t + \partial_a)x(t,a) = -\lambda_1(t,a)x(t,a) - \lambda_2(t,a)x(t,a), \\ (\partial_t + \partial_a)y(t,a) = \lambda_1(t,a)x(t,a) - \lambda_3(t,a)y(t,a), \\ (\partial_t + \partial_a)z(t,a) = \lambda_2(t,a)x(t,a) + \lambda_3(t,a)y(t,a), \\ x(t,0) = 1, \quad y(t,0) = 0, \quad z(t,0) = 0, \\ x(0,a) = x_0(a), \quad y(0,a) = y_0(a), \quad z(0,a) = z_0(a), \\ \lambda_1(t,a) = \displaystyle\int_0^\omega \alpha(a,\sigma)c(\sigma)y(t,\sigma)\,\mathrm{d}\sigma, \\ \lambda_2(t,a) = \displaystyle\int_0^\omega \delta(a,\sigma)c(\sigma)z(t,\sigma)\,\mathrm{d}\sigma, \\ \lambda_3(t,a) = \displaystyle\int_0^\omega \gamma(a,\sigma)c(\sigma)z(t,\sigma)\,\mathrm{d}\sigma. \end{cases} \tag{4.1.3}$$

We mainly consider the system (4.1.3) under the condition

$$x(t,a) \geq 0, \; y(t,a) \geq 0, \; z(t,a) \geq 0, \; x(t,a) + y(t,a) + z(t,a) = 1. \tag{4.1.4}$$

Under this condition, we can formally exclude the susceptible class from the basic system. That is, instead of the basic system (4.1.3), we can consider the following system with linear homogeneous boundary conditions, which is more convenient to consider the well-definedness of the time evolution problem:

$$\begin{cases} (\partial_t + \partial_a)y(t,a) = \lambda_1(t,a)\{1 - y(t,a) - z(t,a)\} - \lambda_3(t,a)y(t,a), \\ (\partial_t + \partial_a)z(t,a) = \lambda_2(t,a)\{1 - y(t,a) - z(t,a)\} + \lambda_3(t,a)y(t,a), \\ y(t,0) = 0, \quad z(t,0) = 0, \\ y(0,a) = y_0(a), \quad z(0,a) = z_0(a), \\ \lambda_1(t,a) = \displaystyle\int_0^\omega \alpha(a,\sigma)c(\sigma)y(t,\sigma)\,\mathrm{d}\sigma, \\ \lambda_2(t,a) = \displaystyle\int_0^\omega \delta(a,\sigma)c(\sigma)z(t,\sigma)\,\mathrm{d}\sigma, \\ \lambda_3(t,a) = \displaystyle\int_0^\omega \gamma(a,\sigma)c(\sigma)z(t,\sigma)\,\mathrm{d}\sigma. \end{cases} \tag{4.1.5}$$

The state space of this system is

$$\Omega = \{(y,z) \in (L^1_+(0,\omega))^2 \mid y + z \le 1\},$$

Let A be a differential operator on $(L^1(0,\omega))^2$ defined by (1.4.7) and (1.4.8). Let F be a nonlinear operator on $(L^1(0,\omega))^2$ defined by

$$F(\phi)(a) := \begin{pmatrix} \lambda_1[a \mid \phi_1]\{1 - \phi_1(a) - \phi_2(a)\} - \lambda_3[a \mid \phi_2]\phi_1(a) \\ \lambda_2[a \mid \phi_2]\{1 - \phi_1(a) - \phi_2(a)\} + \lambda_3[a \mid \phi_2]\phi_2(a) \end{pmatrix},$$

where $\lambda_1[a \mid \phi_1]$, $\lambda_2[a \mid \phi_2]$ and $\lambda_3[a \mid \phi_2]$ are defined by

$$\begin{aligned} \lambda_1[a \mid \phi] &:= \int_0^\omega \alpha(a,\sigma)c(\sigma)\phi(\sigma)\,\mathrm{d}\sigma, \\ \lambda_2[a \mid \phi] &:= \int_0^\omega \delta(a,\sigma)c(\sigma)\phi(\sigma)\,\mathrm{d}\sigma, \\ \lambda_3[a \mid \phi] &:= \int_0^\omega \gamma(a,\sigma)c(\sigma)\phi(\sigma)\,\mathrm{d}\sigma. \end{aligned}$$

System (4.1.5) can be formulated as a semilinear Cauchy problem on the Banach space $(L^1(0,\omega))^2$:

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t) + F(u(t)), \quad u(0) = u_0. \tag{4.1.6}$$

Observe that Lemma 1.4.1 holds here.

Assumption 4.1.1. *α, δ, $\gamma \in L^\infty_+((0,\omega) \times (0,\omega))$ and $c \in L^\infty_+(0,\omega)$.*

Under this assumption, set

$$\alpha^\infty := \operatorname{ess\,sup} \alpha, \;\; \delta^\infty := \operatorname{ess\,sup} \delta, \;\; \gamma^\infty := \operatorname{ess\,sup} \gamma, \;\; c^\infty := \operatorname{ess\,sup} c.$$

$L^1(0,\omega)$ is endowed with the usual norm, and $(L^1(0,\omega))^2$ is endowed with the following norm :

$$\|\phi\|_2 := \max\{\|\phi_1\|, \|\phi_2\|\}, \quad \phi = {}^t(\phi_1, \phi_2) \in (L^1(0,\omega))^2.$$

We can easily show as Lemma 1.4.3 that under Assumption 4.1.1 the map $F|_\Omega : \Omega \to (L^1(0,\omega))^2$ is Lipschitz continuous and $(I + kF)(\Omega) \subset \Omega$ holds for some $k > 0$.

Then, the same argument as Theorem 1.4.4 leads to the following theorem :

Theorem 4.1.2. *The Cauchy problem (4.1.6) has a unique mild solution $S(t)u_0$, and Ω is positively invariant with respect to the semiflow $\{S(t)u_0\}_{t \ge 0}$. If $u_0 \in \mathcal{D}(A)$, then $S(t)u_0$ gives a classical solution.*

4.2 Existence of REE

In this section, let us consider the existence of rumor-endemic equilibria of system (4.1.3), which has $(x(a), y(a), z(a)) = (1, 0, 0)$ as RFE. We denote the density vector at the REE by ${}^t(x^*, y^*, z^*)$ and the forces of rumor-class transition $\lambda_j{}^*$ $(j = 1, 2, 3)$. They must satisfy the following system :

$$\frac{\mathrm{d}}{\mathrm{d}a}x^*(a) = -\lambda_1{}^*(a)x^*(a) - \lambda_2{}^*(a)x^*(a), \tag{4.2.1a}$$

$$\frac{\mathrm{d}}{\mathrm{d}a}y^*(a) = \lambda_1{}^*(a)x^*(a) - \lambda_3{}^*(a)y^*(a), \tag{4.2.1b}$$

$$\frac{\mathrm{d}}{\mathrm{d}a}z^*(a) = \lambda_2{}^*(a)x^*(a) + \lambda_3{}^*(a)y^*(a), \tag{4.2.1c}$$

$$x^*(0) = 1, \quad y^*(0) = 0, \quad z^*(0) = 0, \tag{4.2.1d}$$

$$\lambda_1{}^*(a) = \int_0^\omega \alpha(a,\sigma)c(\sigma)y^*(\sigma)\,\mathrm{d}\sigma, \tag{4.2.1e}$$

$$\lambda_2{}^*(a) = \int_0^\omega \delta(a,\sigma)c(\sigma)z^*(\sigma)\,\mathrm{d}\sigma, \tag{4.2.1f}$$

$$\lambda_3{}^*(a) = \int_0^\omega \gamma(a,\sigma)c(\sigma)z^*(\sigma)\,\mathrm{d}\sigma. \tag{4.2.1g}$$

By formal integration, we obtain the following expressions:

$$x^*(a) = \mathrm{e}^{-\int_0^a\{\lambda_1{}^*(\sigma)+\lambda_2{}^*(\sigma)\}\,\mathrm{d}\sigma},$$

$$y^*(a) = \int_0^a \mathrm{e}^{-\int_\sigma^a \lambda_3{}^*(\tau)\,\mathrm{d}\tau}\lambda_1{}^*(\sigma)\mathrm{e}^{-\int_0^\sigma\{\lambda_1{}^*(\tau)+\lambda_2{}^*(\tau)\}\,\mathrm{d}\tau}\,\mathrm{d}\sigma,$$

$$z^*(a) = \int_0^a \left\{\lambda_2{}^*(b)\mathrm{e}^{-\int_0^b\{\lambda_1{}^*(\sigma)+\lambda_2{}^*(\sigma)\}\,\mathrm{d}\sigma} + \lambda_3{}^*(b)\int_0^b \mathrm{e}^{-\int_\sigma^b \lambda_3{}^*(\tau)\,\mathrm{d}\tau}\lambda_1{}^*(\sigma)\mathrm{e}^{-\int_0^\sigma\{\lambda_1{}^*(\tau)+\lambda_2{}^*(\tau)\}\,\mathrm{d}\tau}\,\mathrm{d}\sigma\right\}\mathrm{d}b.$$

Substituting them into (4.2.1e) (4.2.1f) and (4.2.1g) gives the following nonlinear integral equations for $\lambda^* = {}^t(\lambda_1{}^*,\, \lambda_2{}^*,\, \lambda_3{}^*)$:

$$\lambda^* = \Phi(\lambda^*),$$

where

$$\Phi(u) = {}^t(\Phi_1(u),\, \Phi_2(u),\, \Phi_3(u)),$$

$$\Phi_1(u)(a) := \int_0^\omega \alpha(a,\sigma)c(\sigma)\left\{\int_0^\sigma \mathrm{e}^{-\int_b^\sigma u_3(\tau)\,\mathrm{d}\tau}u_1(b)\mathrm{e}^{-\int_0^b\{u_1(\tau)+u_2(\tau)\}\,\mathrm{d}\tau}\,\mathrm{d}b\right\}\mathrm{d}\sigma,$$

$$\Phi_2(u)(a) := \int_0^\omega \delta(a,\sigma)c(\sigma)\left\{\int_0^\sigma u_2(b)\mathrm{e}^{-\int_0^b\{u_1(\tau)+u_2(\tau)\}\,\mathrm{d}\tau}\,\mathrm{d}b\right\}\mathrm{d}\sigma$$
$$+\int_0^\omega \delta(a,\sigma)c(\sigma)\left[\int_0^\sigma u_3(a')\left\{\int_0^{a'} \mathrm{e}^{-\int_b^{a'} u_3(\tau)\,\mathrm{d}\tau}u_1(b)\mathrm{e}^{-\int_0^b\{u_1(\tau)+u_2(\tau)\}\,\mathrm{d}\tau}\,\mathrm{d}b\right\}\mathrm{d}a'\right]\mathrm{d}\sigma,$$

$$\Phi_3(u)(a) := \int_0^\omega \gamma(a,\sigma)c(\sigma)\left\{\int_0^\sigma u_2(b)\mathrm{e}^{-\int_0^b\{u_1(\tau)+u_2(\tau)\}\,\mathrm{d}\tau}\,\mathrm{d}b\right\}\mathrm{d}\sigma$$
$$+\int_0^\omega \gamma(a,\sigma)c(\sigma)\left[\int_0^\sigma u_3(a')\left\{\int_0^{a'} \mathrm{e}^{-\int_b^{a'} u_3(\tau)\,\mathrm{d}\tau}u_1(b)\mathrm{e}^{-\int_0^b\{u_1(\tau)+u_2(\tau)\}\,\mathrm{d}\tau}\,\mathrm{d}b\right\}\mathrm{d}a'\right]\mathrm{d}\sigma$$

for $u = {}^t(u_1,\, u_2,\, u_3)$.

We find that Φ is a positive operator on $(L^1(0,\omega))^3$ and $\Phi(0) = 0$. Let $T : (L^1(0,\omega))^3 \to (L^1(0,\omega))^3$ be the Fréchet derivative of Φ at 0, then T is given as follows:

$$Tu = {}^t(T_1u,\, T_2u,\, T_3u), \quad u \in (L^1(0,\omega))^3,$$

$$(T_1u)(a) := \int_0^\omega \phi_\alpha(a,b)u_1(b)\,\mathrm{d}b, \quad (T_2u)(a) := \int_0^\omega \phi_\delta(a,b)u_2(b)\,\mathrm{d}b, \quad (T_3u)(a) := \int_0^\omega \phi_\gamma(a,b)u_2(b)\,\mathrm{d}b,$$

$$\phi_\alpha(a,b) := \int_b^\omega \alpha(a,\sigma)c(\sigma)\,\mathrm{d}\sigma, \quad \phi_\delta(a,b) := \int_b^\omega \delta(a,\sigma)c(\sigma)\,\mathrm{d}\sigma, \quad \phi_\gamma(a,b) := \int_b^\omega \gamma(a,\sigma)c(\sigma)\,\mathrm{d}\sigma.$$

Let us define linear operators $\tilde{T}_j$ $(j = 1,\, 2,\, 3)$ on $L^1(0,\omega)$ by

$$(\tilde{T}_1u)(a) := \int_0^\omega \phi_\alpha(a,b)u(b)\,\mathrm{d}b, \quad (\tilde{T}_2u)(a) := \int_0^\omega \phi_\delta(a,b)u(b)\,\mathrm{d}b, \quad (\tilde{T}_3u)(a) := \int_0^\omega \phi_\gamma(a,b)u(b)\,\mathrm{d}b.$$

If $v = {}^t(v_1,\, v_2,\, v_3)$ is an eigenvector of T corresponding to $\lambda(\neq 0)$, then we have

$$\tilde{T}_1v_1 = T_1v = \lambda v_1,$$

$$\tilde{T}_2 v_2 = T_2 v = \lambda v_2,$$
$$\tilde{T}_3 v_2 = T_3 v = \lambda v_3,$$

i.e., v_1 is an eigenvector of $\tilde{T}_1$ corresponding to λ, and v_2 is an eigenvector of $\tilde{T}_2$ corresponding to λ. On the contrary, for given $\lambda \neq 0$, if v_1 is an eigenvector of $\tilde{T}_1$ corresponding to λ and v_2 is an eigenvector of $\tilde{T}_2$ corresponding to λ, then

$$v = {}^t(v_1,\, v_2,\, \lambda^{-1}(\tilde{T}_3 v_2))$$

is an eigenvector of T corresponding to λ. In particular, $v = {}^t(v_1,\, v_2,\, v_3)$ is a positive eigenvector of T corresponding to $\lambda \neq 0$ if and only if v_1 is a positive eigenvector of $\tilde{T}_1$ corresponding to λ, v_2 is a positive eigenvector of $\tilde{T}_2$ corresponding to λ and $v_3 = \lambda^{-1}(\tilde{T}_3 v_2)$, where it should be noted that either v_1 or v_2 (but not both of them) can equal to 0.

The following technical assumption gives some important aspects of $\tilde{T}_j (j = 1,\, 2)$ and Φ.

Assumption 4.2.1. *(i) $c(a)$ is strictly positive for almost all $a \in (0, \omega)$.*

(ii) There exist nonnegative functions $\eta_\alpha(\sigma)$, $\eta_\delta(\sigma)$ such that they are strictly positive for a left neighborhood at $\sigma = \omega$ and

$$\alpha(a, \sigma) \geq \eta_\alpha(\sigma), \quad \delta(a, \sigma) \geq \eta_\delta(\sigma)$$

for almost all $(a,\, \sigma) \in (0, \omega) \times (0, \omega)$.

(iii) $c(a)$, $\alpha(a, \sigma)$, $\delta(a, \sigma)$, $\gamma(a, \sigma)$ are extended as 0 when a or σ is in $\mathbb{R} \setminus [0, \omega]$, then the following holds uniformly with respect to σ :

$$\lim_{h \to 0} \int_0^\omega |\alpha(a + h, \sigma) - \alpha(a, \sigma)|\, da = 0,$$
$$\lim_{h \to 0} \int_0^\omega |\delta(a + h, \sigma) - \delta(a, \sigma)|\, da = 0,$$
$$\lim_{h \to 0} \int_0^\omega |\gamma(a + h, \sigma) - \gamma(a, \sigma)|\, da = 0$$

Then, the same statement as Lemmas 1.5.6, 1.5.7 and 1.5.8 holds, i.e.,

Lemma 4.2.2. *(i) $\tilde{T}_1$ and $\tilde{T}_2$ are nonsupporting and compact.*

(ii) Φ is completely continuous and $\Phi((L^1_+(0, \omega))^3)$ is bounded.

Lemma 4.2.2 (i) and Theorems 1.5.2, 1.5.3 lead to the following lemma:

Lemma 4.2.3. *(i) The spectral radius $r(\tilde{T}_1)$ is the only positive eigenvalue of $\tilde{T}_1$ with a positive eigenvector $u_1 \in L^1_+(0, \omega)$ which is a nonsupporting point. It is an eigenvalue of $\tilde{T}_1{}^*$ with a strictly positive eigenfunctional F_1.*

(ii) $r(\tilde{T}_2)$ is the only positive eigenvalue of $\tilde{T}_2$ with a positive eigenvector $u_2 \in L^1_+(0, \omega)$ which is a nonsupporting point. It is an eigenvalue of $\tilde{T}_2{}^*$ with a strictly positive eigenfunctional F_2.

Theorem 4.2.4. *(i) If $r(\tilde{T}_1) \leq 1$ and $r(\tilde{T}_2) \leq 1$, then $u = 0$ is the only solution of $u = \Phi(u)$ in $(L^1_+(0, \omega))^3$, i.e., RFE is the only equilibrium of the system.*

(ii) If $r(\tilde{T}_1) > 1$, then $u = \Phi(u)$ has a solution $u^\diamond = {}^t(u_1{}^\diamond,\, 0,\, 0)$ in $(L^1_+(0, \omega))^3$, i.e., the system has ${}^t(x^\diamond,\, y^\diamond, 0)$ as an ASEE (active-stifler-extinct equilibrium).

(iii) If $r(\tilde{T}_2) > 1$, then $u = \Phi(u)$ has a solution $u^\dagger = {}^t(0,\, u_2{}^\dagger,\, u_3{}^\dagger)$ in $(L^1_+(0, \omega))^3$, i.e., the system has ${}^t(x^\dagger,\, 0,\, z^\dagger)$ as a SpEE (spreader-extinct equilibrium).

Proof. Suppose $r(\tilde{T}_1) \leq 1$ and $r(\tilde{T}_2) \leq 1$. Let us assume that $u = \Phi(u)$ holds for some $u = {}^t(u_1,\, u_2,\, u_3) \in (L^1_+(0, \omega))^3 \setminus \{0\}$.

If $u_1 \neq 0$, then we obtain

$$\tilde{T}_1 u_1 - u_1 = \tilde{T}_1 u_1 - \Phi_1(u) \in L^1_+(0, \omega) \setminus \{0\}.$$

Since F_1 is a strictly positive functional, we find that $\langle F_1,\, \tilde{T}_1 u_1 - u_1 \rangle > 0$. On the other hand, we have

$$\langle F_1,\, \tilde{T}_1 u_1 - u_1 \rangle = \langle \tilde{T}_1^* F_1,\, u_1 \rangle - \langle F_1,\, u_1 \rangle = \{r(\tilde{T}_1) - 1\} \langle F_1,\, u_1 \rangle.$$

Since $r(\tilde{T}_1) - 1 \le 0$ and $\langle F_1, u_1 \rangle > 0$, e see that $\langle F_1, \tilde{T}_1 u_1 - u_1 \rangle \le 0$, which is a contradiction.

Otherwise, if $u_1 = 0$, then $(u_2, u_3) \neq (0, 0)$. We find that $u_2 \neq 0$, because $u_1 = u_2 = 0$ implies

$$u_3 = \Phi_3(u) = 0.$$

Hence, a similar argument using $\tilde{T}_2$, $r(\tilde{T}_2)$ and F_2 as above leads to a contradiction. Therefore, $r(\tilde{T}_1) \le 1$ and $r(\tilde{T}_2) \le 1$, then $u = 0$ is the only solution of $u = \Phi(u)$ in $(L^1_+(0, \omega))^3$.

Next suppose $r(\tilde{T}_1) > 1$. We have

$$\Phi({}^t(u_1, 0, 0)) = {}^t(\tilde{\Phi}_1(u_1), 0, 0),$$

where

$$\tilde{\Phi}_1(u_1) := \int_0^\omega \alpha(a, \sigma)c(\sigma) \int_0^\sigma u_1(b) \mathrm{e}^{-\int_0^b u_1(\tau)\,\mathrm{d}\tau} \,\mathrm{d}b\,\mathrm{d}\sigma.$$

The Fréchet derivative of $\tilde{\Phi}_1$ at 0 is given by $\tilde{T}_1$. Since $\tilde{T}_1$ is nonsupporting and compact, and $\tilde{\Phi}_1$ is completely continuous and $\tilde{\Phi}_1(L^1_+(0, \omega))$ is bounded, which is easily checked, it follows from Theorem 1.5.4 that $\tilde{\Phi}_1$ has a non-zero positive fixed point, which we shall denote by $u_1{}^\diamond$. Then we see that $u^\diamond = {}^t(u_1{}^\diamond, 0, 0)$ is a non-zero positive fixed point of Φ and the corresponding REE $(x^\diamond, y^\diamond, z^\diamond)$ is given by

$$\begin{aligned}
x^\diamond(a) &= \mathrm{e}^{-\int_0^a u_1{}^\diamond(\sigma)\,\mathrm{d}\sigma}, \\
y^\diamond(a) &= \int_0^a u_1{}^\diamond(\sigma) \mathrm{e}^{-\int_0^\sigma u_1{}^\diamond(\tau)\,\mathrm{d}\tau} \,\mathrm{d}\sigma, \\
z^\diamond(a) &= 0.
\end{aligned}$$

Finally, suppose $r(\tilde{T}_2) > 1$. We have

$$\Phi({}^t(0, u_2, u_3)) = {}^t(0, \tilde{\Phi}_2(u_2), \tilde{\Phi}_3(u_2)),$$

where

$$\begin{aligned}
\tilde{\Phi}_2(u_2) &:= \int_0^\omega \delta(a, \sigma)c(\sigma) \int_0^\sigma u_2(b) \mathrm{e}^{-\int_0^b u_2(\tau)\,\mathrm{d}\tau} \,\mathrm{d}b\,\mathrm{d}\sigma, \\
\tilde{\Phi}_3(u_2) &:= \int_0^\omega \gamma(a, \sigma)c(\sigma) \int_0^\sigma u_2(b) \mathrm{e}^{-\int_0^b u_2(\tau)\,\mathrm{d}\tau} \,\mathrm{d}b\,\mathrm{d}\sigma.
\end{aligned}$$

A similar argument as above implies that $\tilde{\Phi}_2$ has a non-zero positive fixed point, which we shall denote by $u_2{}^\dagger$. Let us define

$$u_3{}^\dagger := \tilde{\Phi}_3(u_2{}^\dagger),$$

then we see that $u^\dagger = {}^t(0, u_2{}^\dagger, u_3{}^\dagger)$ is a non-zero positive fixed point of Φ and the corresponding REE $(x^\dagger, y^\dagger, z^\dagger)$ is given by

$$\begin{aligned}
x^\dagger(a) &= \mathrm{e}^{-\int_0^a u_2{}^\dagger(\sigma)\,\mathrm{d}\sigma}, \\
y^\dagger(a) &= 0, \\
z^\dagger(a) &= \int_0^a u_2{}^\dagger(b) \mathrm{e}^{-\int_0^b u_2{}^\dagger(\sigma)\,\mathrm{d}\sigma} \,\mathrm{d}b.
\end{aligned}$$

This completes the proof. □

4.3 Stability of RFE

In this section, we consider the stability of RFE $(x(a), y(a), z(a)) = (1, 0, 0)$.

The first element $y(t)$ of $u(t)$ in (4.1.6) satisfies the following abstract equation on $L^1(0, \omega)$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} y(t) &= By(t) + P_\alpha y(t) \cdot \{1 - y(t) - z(t)\} - \lambda_3[a \,|\, z] y(t), \\
y(0) &= y_0 \in L^1(0, \omega),
\end{aligned} \tag{4.3.1}$$

where $z(t)$ is regarded as given, B is given by (1.6.1)(1.6.2), and we define as follows :

$$(P_\alpha f)(a) := \int_0^\omega \alpha(a, \sigma)c(\sigma)f(\sigma)\,\mathrm{d}\sigma,$$

$$(P_\delta f)(a) := \int_0^\omega \delta(a,\sigma)c(\sigma)f(\sigma)\,\mathrm{d}\sigma,$$
$$(P_\gamma f)(a) := \int_0^\omega \gamma(a,\sigma)c(\sigma)f(\sigma)\,d\sigma.$$

Let $\{T(t)\}_{t\geq 0}$ be the nilpotent translation semigroup on $L^1(0,\omega)$ defined by (1.4.10), which is generated by B. Let $C_\alpha(t)$ be a bounded operator on $L^1(0,\omega)$ defined by

$$C_\alpha(t)u := (P_\alpha u)\cdot\{1-y(t)-z(t)\} - \lambda_3[a\,|\,z]y(t).$$

For any $u \in L^1_+(0,\omega)$ we have $C_\alpha(t)u \leq P_\alpha u$, because

$$x(t) = 1-y(t)-z(t) \leq 1, \quad P_\alpha u(a) \geq 0, \quad \lambda_3[a\,|\,z]y(t) \geq 0.$$

Since $T(t),\ t\geq 0$ is a positive operator on $L^1(0,\omega)$, we have

$$T(s)C_\alpha(t)u \leq T(s)P_\alpha u, \quad \text{for all } s,t\geq 0 \text{ and for all } u\in L^1_+(0,\omega).$$

Hence, if we denote the C_0-semigroup generated by $B+P_\alpha$ by $\{W_\alpha(t)\}_{t\geq 0}$, we have

$$0 \leq y(t) \leq W_\alpha(t)y_0, \quad \text{for all } t\in\mathbb{R}_+.$$

In addition, if $r(\tilde{T}_1) < 1$, then $\omega_0(B+P)$, which means the growth bound of $\{W_\alpha(t)\}_{t\geq 0}$, is strictly negative and $\|W_\alpha(t)\| \to 0$ as $t\to\infty$, which gives rise to $y(t)\to 0$ as $t\to\infty$. The above statement can be proved in the same manner as in Section 1.4 and we omit the proof here.

Moreover, we can prove that RFE is GAS if $r(\tilde{T}_1) < 1$ and $r(\tilde{T}_2) < 1$ hold.

Theorem 4.3.1. *If $r(\tilde{T}_1) < 1$ and $r(\tilde{T}_2) < 1$ hold, then $(x(t),\ y(t),\ z(t)) \to (1,\ 0,\ 0)$ as $t\to\infty$ for any initial condition $(x(0),\ y(0),\ z(0)) = (x_0,\ y_0,\ z_0)$.*

Proof. As mentioned above, $r(\tilde{T}_1) < 1$ implies $y(t)\to 0$ as $t\to\infty$. Let $f(t)(a) := \lambda_3(t,a)y(t,a)$, then we have $f(t)\in L^1_+(0,\omega)$, and $f(t)\to 0$ as $t\to\infty$. The latter is obtained because

$$\begin{aligned} 0 \leq \|f(t)\| &= \int_0^\omega y(t,a)\left\{\int_0^\omega \gamma(a,\sigma)c(\sigma)z(t,\sigma)\,\mathrm{d}\sigma\right\}\mathrm{d}a \\ &\leq \int_0^\omega y(t,a)\gamma^\infty\,\mathrm{d}a \\ &= \gamma^\infty\|y(t)\|. \end{aligned}$$

The second element $z(t)$ of $u(t)$ in (4.1.6) satisfies the following abstract equation on $L^1(0,\ \omega)$:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}z(t) = Bz(t) + P_\delta z(t)\cdot\{1-y(t)-z(t)\} + f(t), \\ &z(0) = z_0 \in L^1(0,\omega), \end{aligned} \tag{4.3.2}$$

where $y(t)$ and $f(t)$ are regarded as given. Let $C_\delta(t)$ be a bounded operator on $L^1(0,\omega)$ defined by

$$C_\delta(t)u := (P_\delta u)\cdot\{1-y(t)-z(t)\}.$$

For any $u\in L^1_+(0,\omega)$ we have $C_\delta(t)u \leq P_\delta u$, because

$$x(t) = 1-y(t)-z(t) \leq 1, \quad P_\delta u(a) \geq 0.$$

Since $T(t),\ t\geq 0$ is a positive operator on $L^1(0,\omega)$, we have

$$T(s)C_\delta(t)u \leq T(s)P_\delta u \quad \text{for all } s,\ t\geq 0 \text{ and for all } u\in L^1_+(0,\omega).$$

Hence, rewriting the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}z(t) = Bz(t) + C_\delta(t)z(t) + f(t)$$

with the variation-of-constants formula gives

$$z(t) = T(t)z_0 + \int_0^t T(t-s)\{C_\delta(s)z(s) + f(s)\}\,\mathrm{d}s$$

$$\leq T(t)z_0 + \int_0^t T(t-s)\{P_\delta z(s) + f(s)\}\,\mathrm{d}s$$

for all $t \geq 0$. Hence, if we denote by $\tilde{z}(t)$ the solution of the inhomogeneous initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{z}(t) = (B + P_\delta)\tilde{z}(t) + f(t),$$
$$\tilde{z}(0) = z_0,$$

we obtain

$$0 \leq z(t) \leq \tilde{z}(t) \quad \text{for all } t \geq 0.$$

Under the assumption $r(\tilde{T}_2) < 1$, we can obtain $\omega_0(B + P_\delta) < 0$. Let $\{W_\delta(t)\}_{t\geq 0}$ be the C_0-semigroup generated by $B + P_\delta$, then there exist some $M > 0$ and $\mu < 0$ such that $\|W_\delta(t)\| \leq M\mathrm{e}^{-\mu t}$ because of the fact $\omega_0(B + P_\delta) < 0$. In addition, f is bounded and measurable on $[0, \infty)$ and $f(t) \to 0$ as $t \to \infty$. Hence we can obtain $\tilde{z}(t) \to 0$ as $t \to \infty$ ([36, Theorem 4.4]).

Therefore, $z(t) \to 0$ and $x(t) = 1 - y(t) - z(t) \to 0$ as $t \to \infty$. □

On the other hand, in the case $r(\tilde{T}_1) > 1$ or $r(\tilde{T}_2) > 1$, the following holds :

Theorem 4.3.2. *If $r(\tilde{T}_1) > 1$ or $r(\tilde{T}_2) > 1$, then RFE is unstable.*

Proof. As for Equation (4.3.1), the linearization of its right-hand side at 0 gives $(B + P_\alpha)y(t)$. $r(\tilde{T}_1) > 1$ implies $s(B + P_\alpha) > 0$, which is derived in the same way as Theorem 1.6.8. Then it follows that $B + P_\alpha$ has an eigenvalue whose real part is positive and RFE is unstable.

As for Equation (4.3.2), the linearization of its right-hand side at 0 gives $(B + P_\delta)z(t)$. $r(\tilde{T}_1) > 1$ implies $s(B + P_\delta) > 0$, then it follows that $B + P_\delta$ has an eigenvalue whose real part is positive and RFE is unstable. □

4.4 Stability of REE

In this section let us discuss the local stability of REE $(x(a), y(a), z(a)) = (x^*(a), y^*(a), z^*(a))$ under the proportionate mixing assumption (PMA):

Assumption 4.4.1. *α, δ, γ are expressed as follows:*

$$\alpha(a,\sigma) = \alpha_1(a)\alpha_2(\sigma), \quad \delta(a,\sigma) = \delta_1(a)\delta_2(\sigma), \quad \gamma(a,\sigma) = \gamma_1(a)\gamma_2(\sigma).$$

We assume that α_1, α_2, δ_1, δ_2, γ_1, $\gamma_2 \in L^\infty_+(0,\omega)$.

In the following we denote $\operatorname{ess\,sup}\alpha_1$ by α_1^∞ and so on.

Notice that we can obtain the concrete forms of $r(\tilde{T}_1)$ and $r(\tilde{T}_2)$. Since

$$(\tilde{T}_1\alpha_1)(a) = \alpha_1(a)\int_0^\omega \alpha_1(b)\int_b^\omega \alpha_2(\sigma)c(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}b,$$
$$(\tilde{T}_2\delta_1)(a) = \delta_1(a)\int_0^\omega \delta_1(b)\int_b^\omega \delta_2(\sigma)c(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}b,$$

we find that

$$r(\tilde{T}_1) = \int_0^\omega \alpha_1(b)\int_b^\omega \alpha_2(\sigma)c(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}b = \int_0^\omega \alpha_2(\sigma)c(\sigma)\int_0^\sigma \alpha_1(b)\,\mathrm{d}b\,\mathrm{d}\sigma, \tag{4.4.1}$$
$$r(\tilde{T}_2) = \int_0^\omega \delta_1(b)\int_b^\omega \delta_2(\sigma)c(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}b = \int_0^\omega \delta_2(\sigma)c(\sigma)\int_0^\sigma \delta_1(b)\,\mathrm{d}b\,\mathrm{d}\sigma. \tag{4.4.2}$$

Let

$$x(t,a) = x^*(a) + \bar{x}(t,a), \; y(t,a) = y^*(a) + \bar{y}(t,a), \; z(t,a) = z^*(a) + \bar{z}(t,a)$$

be a solution of system (4.1.3). $(\bar{x}(t,a), \bar{y}(t,a), \bar{z}(t,a))$ denote the small perturbations from REE. Note that

$$\bar{x}(t,0) = \bar{y}(t,0) = \bar{z}(t,0) = 0, \tag{4.4.3}$$
$$\bar{x}(t,a) + \bar{y}(t,a) + \bar{z}(t,a) = 0. \tag{4.4.4}$$

The small perturbations satisfy the following equations:

$$\begin{cases} (\partial_t + \partial_a)\bar{x}(t,a) = -\bar{x}(t,a)(\lambda_1{}^*(a) + \bar{\lambda}_1(t,a)) - x^*(a)\bar{\lambda}_1(t,a) - \bar{x}(t,a)(\lambda_2{}^*(a) + \bar{\lambda}_2(t,a)) - x^*(a)\bar{\lambda}_2(t,a), \\ (\partial_t + \partial_a)\bar{y}(t,a) = \bar{x}(t,a)(\lambda_1{}^*(a) + \bar{\lambda}_1(t,a)) + x^*(a)\bar{\lambda}_1(t,a) - \bar{y}(t,a)(\lambda_3{}^*(a) + \bar{\lambda}_3(t,a)) - y^*(a)\bar{\lambda}_3(t,a), \\ (\partial_t + \partial_a)\bar{z}(t,a) = \bar{x}(t,a)(\lambda_2{}^*(a) + \bar{\lambda}_2(t,a)) + x^*(a)\bar{\lambda}_2(t,a) + \bar{y}(t,a)(\lambda_3{}^*(a) + \bar{\lambda}_3(t,a)) + y^*(a)\bar{\lambda}_3(t,a), \end{cases} \tag{4.4.5}$$

where

$$\bar{\lambda}_1(t,a) = \int_0^\omega \alpha(a,\sigma)c(\sigma)\bar{y}(t,\sigma)\,\mathrm{d}\sigma,$$
$$\bar{\lambda}_2(t,a) = \int_0^\omega \delta(a,\sigma)c(\sigma)\bar{z}(t,\sigma)\,\mathrm{d}\sigma,$$
$$\bar{\lambda}_3(t,a) = \int_0^\omega \gamma(a,\sigma)c(\sigma)\bar{z}(t,\sigma)\,\mathrm{d}\sigma.$$

We can formulate (4.4.5) as an abstract semilinear problem on the Banach space $(L^1(0,\omega))^3$:

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t) + G(u(t)), \quad u(t) = {}^t(\bar{x}(t,\cdot), \bar{y}(t,\cdot), \bar{z}(t,\cdot)). \tag{4.4.6}$$

The generator A is defined by (1.7.7) and (1.7.8). The nonlinear term G is defined by

$$\begin{aligned} G(u) &:= {}^t(G_1(u), G_2(u), G_3(u)), \\ G_1(u) &:= -u_1(\lambda_1{}^* + P_\alpha u_2) - x^* P_\alpha u_2 - u_1(\lambda_2{}^* + P_\delta u_3) - x^* P_\delta u_3, \\ G_2(u) &:= u_1(\lambda_1{}^* + P_\alpha u_2) + x^* P_\alpha u_2 - u_2(\lambda_3{}^* + P_\gamma u_3) - y^* P_\gamma u_3, \\ G_3(u) &:= u_1(\lambda_2{}^* + P_\delta u_3) + x^* P_\delta u_3 + u_2(\lambda_3{}^* + P_\gamma u_3) + y^* P_\gamma u_3. \end{aligned}$$

The linearized equation of (4.4.6) around $u = 0$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = (A + C)u(t),$$

where the bounded linear operator C is the Fréchet derivative of $G(u)$ at $u = 0$ given by

$$Cu := \begin{pmatrix} -u_1\lambda_1{}^* - x^* P_\alpha u_2 - u_1\lambda_2^* - x^* P_\delta u_2 \\ u_1\lambda_1{}^* + x^* P_\alpha u_2 - u_2\lambda_3{}^* - y^* P_\gamma u_3 \\ u_1\lambda_2{}^* + x^* P_\delta u_3 + u_2\lambda_3{}^* + y^* P_\gamma u_3 \end{pmatrix}.$$

Now let us consider the resolvent equation for $A + C$:

$$(\zeta - (A + C))v = u, \quad v \in \mathcal{D}(A), \ u \in (L^1(0,\omega))^3, \ \zeta \in \mathbb{C}. \tag{4.4.7}$$

Then we have

$$v_1'(a) = -\zeta v_1(a) - \lambda_1{}^*(a)v_1(a) - x^*(a)(P_\alpha v_2)(a) - \lambda_2{}^*(a)v_1(a) - x^*(a)(P_\delta v_3)(a) + u_1(a), \tag{4.4.8a}$$
$$v_2'(a) = -\zeta v_2(a) + \lambda_1{}^*(a)v_1(a) + x^*(a)(P_\alpha v_2)(a) - \lambda_3{}^*(a)v_2(a) - y^*(a)(P_\delta v_3)(a) + u_2(a), \tag{4.4.8b}$$
$$v_3'(a) = -\zeta v_3(a) + \lambda_2{}^*(a)v_1(a) + x^*(a)(P_\delta v_3)(a) + \lambda_3{}^*(a)v_2(a) + y^*(a)(P_\gamma v_3)(a) + u_3(a). \tag{4.4.8c}$$

From (4.4.8a) and $v_1(0) = 0$, we obtain

$$v_1(a) = \int_0^a \{-x^*(\tau)(P_\alpha v_2)(\tau) - x^*(\tau)(P_\delta v_3)(\tau) + u_1(\tau)\} \mathrm{e}^{-\int_\tau^a (\zeta + \lambda_1{}^*(r') + \lambda_2{}^*(r'))\,\mathrm{d}r'}\,\mathrm{d}\tau. \tag{4.4.9}$$

From (4.4.8b) (4.4.9) and $v_2(0) = 0$, we have

$$\begin{aligned} v_2(a) &= \int_0^a \{\lambda_1{}^*(\sigma)v_1(\sigma) + x^*(\sigma)(P_\alpha v_2)(\sigma) - y^*(\sigma)(P_\gamma v_3)(\sigma) + u_2(\sigma)\}\mathrm{e}^{-\zeta(a-\sigma)}\mathrm{e}^{-\int_\sigma^a \lambda_3{}^*(r')\,\mathrm{d}r'}\,\mathrm{d}\sigma \\ &= \int_0^a \left\{\lambda_1{}^*(\sigma)\left(\int_0^\sigma \{-x^*(\tau)(P_\alpha v_2)(\tau) - x^*(\tau)(P_\delta v_3)(\tau) + u_1(\tau)\}\mathrm{e}^{-\int_\tau^\sigma (\zeta + \lambda_1{}^*(r') + \lambda_2{}^*(r'))\,\mathrm{d}r'}\,\mathrm{d}\tau\right)\right. \\ &\quad \left. + x^*(\sigma)(P_\alpha v_2)(\sigma) - y^*(\sigma)(P_\gamma v_3)(\sigma) + u_2(\sigma)\right\}\mathrm{e}^{-\int_\sigma^a (\zeta + \lambda_3{}^*(r'))\,\mathrm{d}r'}\,\mathrm{d}\sigma. \end{aligned} \tag{4.4.10}$$

From (4.4.8c) (4.4.9) (4.4.10) and $v_3(0) = 0$, we have

$$
\begin{aligned}
v_3(a) &= \int_0^a \{\lambda_2{}^*(\sigma)v_1(\sigma) + \lambda_3{}^*(\sigma)v_2(\sigma) + x^*(\sigma)(P_\delta v_3)(\sigma) + y^*(\sigma)(P_\gamma v_3)(\sigma) + u_3(\sigma)\}\mathrm{e}^{-(a-\sigma)\zeta}\,\mathrm{d}\sigma \\
&= \int_0^a \Big\{\lambda_2{}^*(\sigma)\Big(\int_0^\sigma \{-x^*(\tau)(P_\alpha v_2)(\tau) - x^*(\tau)(P_\delta v_3)(\tau) + u_1(\tau)\}\mathrm{e}^{-\int_\tau^\sigma(\zeta+\lambda_1{}^*(r')+\lambda_2{}^*(r'))\,\mathrm{d}r'}\,\mathrm{d}\tau\Big) \\
&\quad + \lambda_3{}^*(\sigma)\Big(\int_0^\sigma \Big(\lambda_1{}^*(r)\Big(\int_0^r \{-x^*(\tau)(P_\alpha v_2)(\tau) - x^*(\tau)(P_\delta v_3)(\tau) + u_1(\tau)\} \\
&\qquad\qquad \times \mathrm{e}^{-\int_\tau^r(\zeta+\lambda_1{}^*(r')+\lambda_2{}^*(r'))\,\mathrm{d}r'}\,\mathrm{d}\tau\Big) \\
&\quad + x^*(r)(P_\alpha v_2)(r) - y^*(r)(P_\gamma v_3)(r) + u_2(r)\Big)\mathrm{e}^{-\int_r^a(\zeta+\lambda_3{}^*(r'))\,\mathrm{d}r'}\,\mathrm{d}r\Big) \\
&\quad + x^*(\sigma)(P_\delta v_3)(\sigma) + y^*(\sigma)(P_\gamma v_3)(\sigma) + u_3(\sigma)\Big\}\mathrm{e}^{-(a-\sigma)\zeta}\,\mathrm{d}\sigma.
\end{aligned} \tag{4.4.11}
$$

Let

$$
\xi_1 := \int_0^\omega \alpha_2(\sigma)c(\sigma)v_2(\sigma)\,\mathrm{d}\sigma, \quad \xi_2 := \int_0^\omega \delta_2(\sigma)c(\sigma)v_3(\sigma)\,\mathrm{d}\sigma, \quad \xi_3 := \int_0^\omega \gamma_2(\sigma)c(\sigma)v_3(\sigma)\,\mathrm{d}\sigma, \tag{4.4.12}
$$

Assumption 4.4.1 implies

$$
(P_\alpha v_2)(a) = \xi_1\alpha_1(a), \quad (P_\delta v_3)(a) = \xi_2\delta_1(a), \quad (P_\gamma v_3)(a) = \xi_3\gamma_1(a).
$$

Inserting (4.4.9) and (4.4.10) into (4.4.12) yields a three-dimensional system as

$$
(I - \Phi(\zeta))\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix} = \begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix}, \tag{4.4.13}
$$

where I is the 3×3 unit matrix.

4.4.1 Stability of ASEE

In this subsection we consider the local stability of ASEE

$$
(x^*(a),\, y^*(a),\, z^*(a)) = (x^\diamond(a),\, y^\diamond(a),\, 0)
$$

under the condition $r(\tilde{T}_1) > 1$. Here we fix the coefficients α_1, α_2, δ_1, δ_2, γ_1, γ_2 so that $r(\tilde{T}_1) = 1$ holds, and we rewrite this α_1 as $\alpha_1^\star$. By definition we have

$$
\int_0^\omega \alpha_2(\sigma)c(\sigma)\int_0^\sigma \alpha_1^\star(b)\,\mathrm{d}b\,\mathrm{d}\sigma = 1. \tag{4.4.14}
$$

Let $\alpha_1(a) = \varepsilon\alpha_1^\star(a)$, where ε is the bifurcation parameter here and $r(\tilde{T}_1) = \varepsilon$. Assumption 4.4.1 implies that

$$
\begin{aligned}
&\lambda_1{}^*(a) = \varepsilon\alpha_1^\star(a)c_1, \quad \text{where } c_1 := \int_0^\omega \alpha_2(\sigma)c(\sigma)y^\diamond(\sigma)\,\mathrm{d}\sigma, \\
&\lambda_2{}^*(a) = \lambda_3{}^*(a) = 0, \\
&x^\diamond(a) = x^\diamond(a;\varepsilon) = \mathrm{e}^{-\varepsilon c_1\int_0^a \alpha_1^\star(r')\,\mathrm{d}r'}, \\
&y^\diamond(a) = y^\diamond(a;\varepsilon) = \int_0^a \varepsilon c_1\alpha_1^\star(\sigma)\mathrm{e}^{-\varepsilon c_1\int_0^\sigma \alpha_1^\star(r')\,\mathrm{d}r'}\,\mathrm{d}\sigma.
\end{aligned}
$$

Observe that $c_1 = 0$ corresponds to RFE and $c_1 > 0$ ASEE. Then we can obtain a nonlinear integral equation for c_1 corresponding to ASEE:

$$
\Theta_1(c_1;\varepsilon) = 0,
$$

where

$$
\Theta_1(c_1;\varepsilon) := \varepsilon\int_0^\omega \alpha_2(\sigma)c(\sigma)\Big(\int_0^\sigma \alpha_1^\star(b)\mathrm{e}^{-\int_0^b \varepsilon c_1\alpha_1^\star(r')\,\mathrm{d}r'}\,\mathrm{d}b\Big)\,\mathrm{d}\sigma - 1.
$$

Since
$$\frac{\partial \Theta_1}{\partial c_1}(0;1) = -\int_0^{\omega} \alpha_2(\sigma)c(\sigma)\int_0^{\sigma} \alpha_1^\star(b)\int_0^{b} \alpha_1^\star(r')\,\mathrm{d}r'\,\mathrm{d}b\,\mathrm{d}\sigma < 0,$$
we can apply the implicit function theorem to find a branching solution $c_1 = c_1(\varepsilon)$ with $c_1(0) = 0$ bifurcated at the point $\varepsilon = 1$ from the trivial solution $c_1 = 0$ when ε is sufficiently close to 1. In addition, we see that
$$c_1'(1) = -\left(\frac{\partial \Theta_1}{\partial c_1}(0;1)\right)^{-1}\frac{\partial \Theta_1}{\partial \varepsilon}(0;1) = -\left(\frac{\partial \Theta_1}{\partial c_1}(0;1)\right)^{-1}.$$

Now, let us return to (4.4.13). $\Phi(\zeta) = \Phi(\zeta,\varepsilon)$ takes the following form:
$$\Phi(\zeta,\varepsilon) = \begin{pmatrix} \phi_{11}(\zeta,\varepsilon) & * & * \\ 0 & \phi_{22}(\zeta,\varepsilon) & \phi_{23}(\varepsilon,\zeta) \\ 0 & \phi_{32}(\zeta,\varepsilon) & \phi_{33}(\varepsilon,\zeta) \end{pmatrix},$$
where
$$\begin{aligned}\phi_{11}(\zeta,\varepsilon) :=& \int_0^{\omega} \alpha_2(r)c(r)\int_0^{r}\left\{\varepsilon c_1(\varepsilon)\alpha_1^\star(\sigma)\left(\int_0^{\sigma} -x^\diamond(\tau;\varepsilon)\varepsilon\alpha_1^\star(\tau)\mathrm{e}^{-\int_\tau^\sigma(\zeta+\varepsilon c_1(\varepsilon)\alpha_1^\star(r'))\,\mathrm{d}r'}\,\mathrm{d}\tau\right)\right.\\ &\left.+ x^\diamond(\sigma;\varepsilon)\varepsilon\alpha_1^\star(\sigma)\right\}\mathrm{e}^{-(r-\sigma)\zeta}\,\mathrm{d}\sigma\,\mathrm{d}r,\\
\phi_{22}(\zeta,\varepsilon) =& \int_0^{\omega} \delta_2(r)c(r)\int_0^{r} x^\diamond(\sigma;\varepsilon)\delta_1(\sigma)\mathrm{e}^{-\zeta(r-\sigma)}\,\mathrm{d}\sigma\,\mathrm{d}r,\\
\phi_{23}(\zeta,\varepsilon) =& \int_0^{\omega} \delta_2(r)c(r)\int_0^{r} y^\diamond(\sigma;\varepsilon)\gamma_1(\sigma)\mathrm{e}^{-\zeta(r-\sigma)}\,\mathrm{d}\sigma\,\mathrm{d}r,\\
\phi_{32}(\zeta,\varepsilon) =& \int_0^{\omega} \gamma_2(r)c(r)\int_0^{r} x^\diamond(\sigma;\varepsilon)\delta_1(\sigma)\mathrm{e}^{-\zeta(r-\sigma)}\,\mathrm{d}\sigma\,\mathrm{d}r,\\
\phi_{33}(\zeta,\varepsilon) =& \int_0^{\omega} \gamma_2(r)c(r)\int_0^{r} y^\diamond(\sigma;\varepsilon)\gamma_1(\sigma)\mathrm{e}^{-\zeta(r-\sigma)}\,\mathrm{d}\sigma\,\mathrm{d}r.\end{aligned}$$

It can be easily seen that
$$\begin{aligned}\phi_{11}(0,1) &= \int_0^{\omega} \alpha_2(r)c(r)\int_0^{r}\alpha_1^\star(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}r = 1,\\
\phi_{22}(0,1) &= \int_0^{\omega} \delta_2(r)c(r)\int_0^{r}\delta_1(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}r = r(\tilde{T}_2),\\
\phi_{23}(0,1) &= \phi_{33}(0;1) = 0,\\
\frac{\partial\phi_{11}}{\partial\zeta}(0,1) &= \int_0^{\omega} \alpha_2(r)c(r)\int_0^{r}\alpha_1^\star(\sigma)(\sigma-r)\,\mathrm{d}\sigma\,\mathrm{d}r < 0,\\
\frac{\partial x^\diamond}{\partial\varepsilon}(a;1) &= -c_1'(1)\int_0^{a}\alpha_1^\star(r')\,\mathrm{d}r',\\
\frac{\partial\phi_{11}}{\partial\varepsilon}(0,1) &= \int_0^{\omega} \alpha_2(r)c(r)\int_0^{r}\left(-2\alpha_1^\star(\sigma)c_1'(1)\int_0^{\sigma}\alpha_1^\star(\tau)\,\mathrm{d}\tau + \alpha_1^\star(\sigma)\right)\mathrm{d}\sigma\,\mathrm{d}r = -1\end{aligned}$$

If we define $f(\zeta,\varepsilon) := \det(I - \Phi(\zeta,\varepsilon))$, then the roots of $f(\zeta,1)$ give the eigenvalues of the linearized system at RFE. We have $f(0,1) = 0$ and
$$\frac{\partial f}{\partial\zeta}(0,1) = \frac{\partial\phi_{11}}{\partial\zeta}(0,1)\{r(\tilde{T}_2) - 1\}.$$
Hence, under the condition $r(\tilde{T}_2) \neq 1$, we can apply the implicit function theorem that $f(\zeta,\varepsilon) = 0$ can be solved locally as $\zeta = \zeta(\varepsilon)$ with $\zeta(1) = 0$. At the same time, we find that
$$\frac{\partial f}{\partial\varepsilon}(0,1) = -\{r(\tilde{T}_2) - 1\},$$
which means that the dominant eigenvalue goes to the left half complex plane as ε increases small enough from 1. The well-known technique based on the Rouché's theorem yields that $\zeta(\varepsilon)$ is the dominant root of f as long as $|\varepsilon - 1|$ is small enough.

Then, we can derive the same lemma as Lemma 1.7.2. From this and the principle of linearized stability, we conclude :

Theorem 4.4.2. *Under Assumption 4.4.1 and the condition $r(\tilde{T}_2) \neq 1$, if $r(\tilde{T}_1) > 1$ and $r(\tilde{T}_1)$ is sufficiently close to 1, then ASEE bifurcates forward from RFE and is LAS.*

4.4.2 Stability of SpEE

In this subsection we consider the local stability of SpEE

$$(x^*(a),\, y^*(a),\, z^*(a)) = (x^\dagger(a),\, 0,\, z^\dagger(a))$$

under the condition $r(\tilde{T}_2) > 1$. Here we fix the coefficients α_1, α_2, δ_1, δ_2, γ_1, γ_2 so that $r(\tilde{T}_2) = 1$ holds, and we rewrite this δ_1 as $\delta_1^\star$. By definition we have

$$\int_0^\omega \delta_2(\sigma)c(\sigma)\int_0^\sigma \delta_1^\star(b)\,\mathrm{d}b\,\mathrm{d}\sigma = 1. \tag{4.4.15}$$

Let $\delta_1(a) = \varepsilon\delta_1^\star(a)$, where ε is the bifurcation parameter here and $r(\tilde{T}_2) = \varepsilon$. Assumption 4.4.1 implies that

$$\begin{aligned}
&\lambda_1{}^*(a) = 0,\\
&\lambda_2{}^*(a) = \varepsilon\delta_1^\star(a)c_2, \quad \text{where } c_1 := \int_0^\omega \delta_2(\sigma)c(\sigma)z^\dagger(\sigma)\,\mathrm{d}\sigma,\\
&\lambda_3{}^*(a) = \gamma_1(a)c_3, \quad \text{where } c_3 := \int_0^\omega \gamma_2(\sigma)c(\sigma)z^\dagger(\sigma)\,\mathrm{d}\sigma,\\
&x^\dagger(a) = x^\dagger(a;\varepsilon) = \mathrm{e}^{-\varepsilon c_2\int_0^a \delta_1^\star(r')\,\mathrm{d}r'},\\
&z^\dagger(a) = z^\dagger(a;\varepsilon) = \int_0^a \varepsilon c_2\delta_1^\star(\sigma)\mathrm{e}^{-\varepsilon c_2\int_0^\sigma \delta_1^\star(r')\,\mathrm{d}r'}\,\mathrm{d}\sigma.
\end{aligned}$$

Observe that $c_2 = 0$ corresponds to RFE and $c_2 > 0$ ASEE. Then we can obtain a nonlinear integral equation for c_2 corresponding to ASEE:

$$\Theta_2(c_2;\varepsilon) = 0,$$

where

$$\Theta_2(c_2;\varepsilon) := \varepsilon\int_0^\omega \delta_2(\sigma)c(\sigma)\left(\int_0^\sigma \delta_1^\star(b)\mathrm{e}^{-\int_0^b \varepsilon c_2\delta_1^\star(r')\,\mathrm{d}r'}\,\mathrm{d}b\right)\mathrm{d}\sigma - 1.$$

Since

$$\frac{\partial\Theta_2}{\partial c_2}(0;1) = -\int_0^\omega \delta_2(\sigma)c(\sigma)\int_0^\sigma \delta_1^\star(b)\int_0^b \delta_1^\star(r')\,\mathrm{d}r'\,\mathrm{d}b\,\mathrm{d}\sigma < 0,$$

we can apply the implicit function theorem to find a branching solution $c_2 = c_2(\varepsilon)$ with $c_2(1) = 0$ bifurcated at the point $\varepsilon = 1$ from the trivial solution $c_2 = 0$ when ε is sufficiently close to 1. In addition, we see that

$$c_2'(1) = -\left(\frac{\partial\Theta_2}{\partial c_2}(0;1)\right)^{-1}\frac{\partial\Theta_2}{\partial\varepsilon}(0;1) = -\left(\frac{\partial\Theta_2}{\partial c_2}(0;1)\right)^{-1}.$$

Notice that c_3 depends on c_2 in the following way:

$$c_3(\varepsilon) = \int_0^\omega \gamma_2(\sigma)c(\sigma)\int_0^\sigma \varepsilon c_2(\varepsilon)\delta_1^\star(b)\mathrm{e}^{-\int_0^b \varepsilon c_2(\varepsilon)\delta_1^\star(\tau)\,\mathrm{d}\tau}\,\mathrm{d}b\,\mathrm{d}\sigma.$$

In particular, it follows that $c_3(1) = 0$.

Now, let us return to (4.4.13). $\Phi(\zeta) = \Phi(\zeta,\varepsilon)$ takes the following form:

$$\Phi(\zeta,\varepsilon) = \begin{pmatrix} \phi_{11}(\zeta,\varepsilon) & 0 & 0\\ * & \phi_{22}(\zeta,\varepsilon) & 0\\ * & * & 0\end{pmatrix},$$

where

$$\begin{aligned}
\phi_{11}(\zeta,\varepsilon) &:= \int_0^\omega \alpha_2(r)c(r)\int_0^r x^\dagger(\sigma;\varepsilon)\alpha_1(\sigma)\mathrm{e}^{-\int_\sigma^r(\zeta+\gamma_1(r')c_3(\varepsilon))\,\mathrm{d}r'}\,\mathrm{d}\sigma\,\mathrm{d}r,\\
\phi_{22}(\zeta,\varepsilon) &= \int_0^\omega \delta_2(r)c(r)\int_0^r \Big(x^\dagger(\sigma;\varepsilon)\varepsilon\delta_1^\star(\sigma)\\
&\quad - \varepsilon c_2(\varepsilon)\delta_1^\star(\sigma)\int_0^\sigma x^\dagger(\tau;\varepsilon)\varepsilon\delta_1^\star(\tau)\mathrm{e}^{-\int_\tau^\sigma(\zeta+\varepsilon c_2(\varepsilon)\delta_1^\star(r'))\,\mathrm{d}r'}\,\mathrm{d}\tau\Big)\mathrm{e}^{-\zeta(r-\sigma)}\,\mathrm{d}\sigma\,\mathrm{d}r.
\end{aligned}$$

It can be easily seen that

$$\begin{aligned}
\phi_{11}(0,1) &= \int_0^\omega \alpha_2(r)c(r)\int_0^r \alpha_1(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}r = r(\tilde{T}_1),\\
\phi_{22}(0,1) &= \int_0^\omega \delta_2(r)c(r)\int_0^r \delta_1^\star(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}r = 1,\\
\frac{\partial\phi_{22}}{\partial\zeta}(0,1) &= \int_0^\omega \delta_2(r)c(r)\int_0^r \delta_1^\star(\sigma)(\sigma-r)\,\mathrm{d}\sigma\,\mathrm{d}r < 0,\\
\frac{\partial x^\dagger}{\partial\varepsilon}(a;1) &= -c_2'(1)\int_0^a \delta_1^\star(r')\,\mathrm{d}r',\\
\frac{\partial\phi_{22}}{\partial\varepsilon}(0,1) &= \int_0^\omega \delta_2(r)c(r)\int_0^r \left(-2\delta_1^\star(\sigma)c_2'(1)\int_0^\sigma \delta_1^\star(\tau)\,\mathrm{d}\tau + \delta_1^\star(\sigma)\right)\mathrm{d}\sigma\,\mathrm{d}r = -1.
\end{aligned}$$

If we define $f(\zeta,\varepsilon) := \det(I-\Phi(\zeta,\varepsilon))$, then the roots of $f(\zeta,1)$ give the eigenvalues of the linearized system at RFE. We have $f(0,1)=0$ and

$$\frac{\partial f}{\partial\zeta}(0,1) = \frac{\partial\phi_{22}}{\partial\zeta}(0,1)\{r(\tilde{T}_1)-1\}.$$

Hence, under the condition $r(\tilde{T}_1)\neq 1$, we can apply the implicit function theorem that $f(\zeta,\varepsilon)=0$ can be solved locally as $\zeta=\zeta(\varepsilon)$ with $\zeta(1)=0$. At the same time, we find that

$$\frac{\partial f}{\partial\varepsilon}(0,1) = -\{r(\tilde{T}_2)-1\},$$

which means that the dominant eigenvalue goes to the left half complex plane as ε increases small enough from 1. The well-known technique based on the Rouché's theorem yields that $\zeta(\varepsilon)$ is the dominant root of f as long as $|\varepsilon-1|$ is small enough.

Then, we can derive the same lemma as Lemma 1.7.2. From this and the principle of linearized stability, we conclude :

Theorem 4.4.3. *Under Assumption 4.4.1 and the condition $r(\tilde{T}_1)\neq 1$, if $r(\tilde{T}_2)>1$ and $r(\tilde{T}_2)$ is sufficiently close to 1, then SpEE bifurcates forward from RFE and is LAS.*

4.5 Rumor persistence

In this section we show that $r(\tilde{T}_1)>1$ and $r(\tilde{T}_2)>1$ implies uniform strong rumor persistence under PMA.

PMA with Assumption 4.2.1 implies that $\alpha_1(a)>0$ for almost every $a\in(0,\omega)$, that there exists some $b_\alpha>0$ such that $\alpha_2(\sigma)>0$ for almost every $\sigma\in(\omega-b_0,\omega)$, that $\delta_1(a)>0$ for almost every $a\in(0,\omega)$ and that there exists some $b_\delta>0$ such that $\delta_2(\sigma)>0$ for almost every $\sigma\in(\omega-b_1,\omega)$. In addition, it follows that

$$\lambda_1(t,a)=\alpha_1(a)\psi_\alpha(t),\quad \text{where}\quad \psi_\alpha(t):=\int_0^\omega \alpha_2(s)c(s)y(t,s)\,\mathrm{d}s, \tag{4.5.1}$$

$$\lambda_2(t,a)=\delta_1(a)\psi_\delta(t),\quad \text{where}\quad \psi_\delta(t):=\int_0^\omega \delta_2(s)c(s)z(t,s)\,\mathrm{d}s, \tag{4.5.2}$$

$$\lambda_3(t,a)=\alpha_1(a)\psi_\gamma(t),\quad \text{where}\quad \psi_\gamma(t):=\int_0^\omega \gamma_2(s)c(s)z(t,s)\,\mathrm{d}s. \tag{4.5.3}$$

By integrating along characteristics, from we have from (4.1.3) that

$$y(t,a)=\int_0^a \alpha_1(\sigma)\psi_\alpha(t-a+\sigma)x(t-a+\sigma,\sigma)\mathrm{e}^{-\int_\sigma^a \gamma_1(\tau)\psi_\gamma(t-a+\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma, \tag{4.5.4}$$

$$z(t,a)=\int_0^a \{\delta_1(\sigma)\psi_\delta(t-a+\sigma)x(t-a+\sigma,\sigma)+\gamma_1(\sigma)\psi_\gamma(t-a+\sigma)y(t-a+\sigma,\sigma)\}\,\mathrm{d}\sigma \tag{4.5.5}$$

if $t>a$, and

$$y(t,a)=y_0(a-t)\mathrm{e}^{-\int_0^t \gamma_1(\tau+a-t)\psi_\gamma(\tau)\,\mathrm{d}\tau}$$

$$+\int_0^t \alpha_1(a-t+\sigma)\psi_\alpha(\sigma)x(\sigma,\sigma+a-t)\mathrm{e}^{-\int_\sigma^t \gamma_1(\tau+a-t)\psi_\gamma(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma,$$

$$z(t,a)=z_0(a-t)+\int_0^t \{\delta_1(a-t+\sigma)\psi_\delta(\sigma)x(\sigma,\sigma+a-t)+\gamma_1(a-t+\sigma)\psi_\gamma(\sigma)y(\sigma,\sigma+a-t)\}\,\mathrm{d}\sigma$$

if $t<a$.

If $t>\omega$, then we substitute (4.5.4) into (4.5.1) and change the integral variable σ into $r:=s-\sigma$ to obtain

$$\begin{aligned}\psi_\alpha(t)&=\int_0^\omega \alpha_2(s)c(s)\int_0^s \alpha_1(s-r)\psi_\alpha(t-r)x(t-r,s-r)\mathrm{e}^{-\int_{s-r}^s \gamma_1(\tau)\psi_\gamma(t-s+\tau)\,\mathrm{d}\tau}\,\mathrm{d}r\,\mathrm{d}s\\&=\int_0^\omega \psi_\alpha(t-r)\int_r^\omega \alpha_2(s)c(s)\alpha_1(s-r)x(t-r,s-r)\mathrm{e}^{-\int_{s-r}^s \gamma_1(\tau)\psi_\gamma(t-s+\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\,\mathrm{d}r\\&=\int_0^t \psi_\alpha(t-r)\int_r^t \alpha_2(s)c(s)\alpha_1(s-r)x(t-r,s-r)\mathrm{e}^{-\int_{s-r}^s \gamma_1(\tau)\psi_\gamma(t-s+\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\,\mathrm{d}r.\end{aligned}\tag{4.5.6}$$

Here, in the last line we used the assumption $c(a)=0$ for $a>\omega$ (Assumption 4.2.1 (iii)). Let $\psi_{\alpha,b}(t):=\phi_\alpha(b+t)$ for $b\geq 0$, then we see that

$$\psi_{\alpha,b}(t)=\int_0^t \psi_{\alpha,b}(t-r)\int_r^t \alpha_2(s)c(s)\alpha_1(s-r)x(t+b-r,s-r)\mathrm{e}^{-\int_{s-r}^s \gamma_1(\tau)\psi_\gamma(t-s+\tau+b)\,\mathrm{d}\tau}\,\mathrm{d}s\,\mathrm{d}r.\tag{4.5.7}$$

for sufficiently large t.

In the same way, if $t>\omega$, then we have

$$\begin{aligned}\psi_\delta(t)&=\int_0^\omega \delta_2(s)c(s)\int_0^s \{\delta_1(s-r)\psi_\delta(t-r)x(t-r,s-r)+\gamma_1(s-r)\psi_\gamma(t-r)y(t-r,s-r)\}\,\mathrm{d}r\,\mathrm{d}s\\&\geq\int_0^t \psi_\delta(t-r)\int_r^t \delta_2(s)c(s)\delta_1(s-r)x(t-r,s-r)\,\mathrm{d}s\,\mathrm{d}r.\end{aligned}$$

Let $\psi_{\delta,b}(t):=\phi_\delta(b+t)$ for $b\geq 0$, then we see that

$$\psi_{\delta,b}(t)\geq\int_0^t \psi_{\delta,b}(t-r)\int_r^t \delta_2(s)c(s)\delta_1(s-r)x(t+b-r,s-r)\,\mathrm{d}s\,\mathrm{d}r.\tag{4.5.8}$$

for sufficiently large t.

Now, let us consider the semiflow induced by system (4.1.3). Theorem 4.1.2 yields the following proposition:

Proposition 4.5.1. *System (4.1.3) induces a continuous semiflow Φ on the state space*

$$\tilde{\Omega}:=\{{}^t(x,y,z)\in(L^1_+(0,\omega))^3 \mid x+y+z\leq 1\}.$$

Let

$$\begin{aligned}\rho({}^t(x,y,z))&:=\int_0^\omega c(a)\{y(a)+z(a)\}\,\mathrm{d}a,\\ \sigma(t,{}^t(x,y,z))&:=\rho(\Phi(t,{}^t(x,y,z)))\end{aligned}$$

for ${}^t(x,y,z)\in\tilde{\Omega}$ and $t\geq 0$. It is obvious that $\rho:\tilde{\Omega}\to\mathbb{R}$ is nonnegative and uniformly continuous. The following lemma can be proved in the same manner as Lemmas 1.8.1, 1.8.2 and 2.3.6.

Lemma 4.5.2. *(i) If $t>a$, then $x(t,a)>0$. And if $t<a$ and $x_0(a-t)>0$, then $x(t,a)>0$.*

(ii) If $y(t,a)>0$, then $y(t+\sigma,a+\sigma)>0$ for all $\sigma\in(0,\omega-a)$.

(iii) The function $\sigma\mapsto z(t+\sigma,a+\sigma)$ is nondecreasing. In particular, if $z(t,a)>0$, then $z(t+\sigma,a+\sigma)>0$ for all $\sigma\in(0,\omega-a)$.

(iv) If $\|y_0\|>0$, then $\|y(t,\cdot)\|>0$ for all $t>0$.

(v) $\sigma(t,{}^t(x_0,y_0,z_0))>0$ for all $t\geq 0$ whenever $\rho({}^t(x_0,y_0,z_0))>0$.

In addition, the same argument as Lemma 1.8.4 gives rise to the following lemma:

Lemma 4.5.3. *System (4.1.3) satisfies the compactness condition (C) in Chapter 2 with $\varepsilon_0 = 1$ and B the closure of $\Phi_{T_0}(\tilde{\Omega})$, where $T_0 > \omega$ is a sufficiently large positive number.*

Then, let us prove the results about rumor persistence.

Theorem 4.5.4. *If $r(\tilde{T}_1) > 1$ and $r(\tilde{T}_2) > 1$, then system (4.1.3) is uniformly weakly ρ-persistent.*

Proof. Assume that for any $\varepsilon > 0$ there exist some $T_0 > 0$ and some appropriate initial condition such that

$$\int_0^\omega c(a)\{y(t,a) + z(t,a)\}\,\mathrm{d}a \le \varepsilon \quad \text{for all } t \ge T_0.$$

We can choose T_0 to be so large that for all $t \ge T_0$ (4.5.7) holds. By the positivity of c, y, z and the definition of $\psi_\alpha(t)$, $\psi_\delta(t)$, $\psi_\gamma(t)$, it is easily seen that

$$\psi_\alpha(t) \le \alpha_2^\infty \varepsilon, \quad \psi_\delta(t) \le \delta_2^\infty \varepsilon, \quad \psi_\gamma(t) \le \gamma_2^\infty \varepsilon$$

for any $t \ge T_0$.

Integrating along characteristics the partial derivative equation for $x(t,a)$ in system (4.1.3) gives

$$\begin{aligned} x(t+b-r, s-r) &= \mathrm{e}^{-\int_0^{s-r}\{\psi_\alpha(t+b-s+\tau)\alpha_1(\tau) + \psi_\delta(t+b-s+\tau)\delta_1(\tau)\}\,\mathrm{d}\tau} \\ &\ge \mathrm{e}^{-\varepsilon\alpha_2^\infty \int_0^{s-r}\alpha_1(\tau)\,\mathrm{d}\tau - \varepsilon\delta_2^\infty\int_0^{s-r}\delta_1(\tau)\,\mathrm{d}\tau} \end{aligned} \tag{4.5.9}$$

for $t > T_0$, $0 \le r \le s \le t$ and $b \ge 0$. Hence it follows from (4.5.7) and (4.5.8) that

$$\psi_{\alpha,b}(t) \ge \int_0^t \psi_{\alpha,b}(t-r)\int_r^t \alpha_2(s)c(s)\alpha_1(s-r)\mathrm{e}^{-\varepsilon\alpha_2^\infty\int_0^{s-r}\alpha_1(\tau)\,\mathrm{d}\tau - \varepsilon\delta_2^\infty\int_0^{s-r}\delta_1(\tau)\,\mathrm{d}\tau}\mathrm{e}^{-\varepsilon\gamma_2^\infty\int_{s-r}^{s}\gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\,\mathrm{d}r,$$

$$\psi_{\delta,b}(t) \ge \int_0^t \psi_{\delta,b}(t-r)\int_r^t \delta_2(s)c(s)\delta_1(s-r)\mathrm{e}^{-\varepsilon\alpha_2^\infty\int_0^{s-r}\alpha_1(\tau)\,\mathrm{d}\tau - \varepsilon\delta_2^\infty\int_0^{s-r}\delta_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\,\mathrm{d}r.$$

Hence, for sufficiently large $T > 0$ and all $t \ge 0$, we have

$$\begin{aligned} &\psi_{\alpha,b+T}(t) \\ &= \psi_{\alpha,b}(t+T) \\ &\ge \int_0^{t+T}\psi_{\alpha,b+T}(t-r)\int_r^{t+T}\alpha_2(s)c(s)\alpha_1(s-r)\mathrm{e}^{-\varepsilon\alpha_2^\infty\int_0^{s-r}\alpha_1(\tau)\,\mathrm{d}\tau - \varepsilon\delta_2^\infty\int_0^{s-r}\delta_1(\tau)\,\mathrm{d}\tau}\mathrm{e}^{-\varepsilon\gamma_2^\infty\int_{s-r}^{s}\gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\,\mathrm{d}r \\ &\ge \int_0^{t}\psi_{\alpha,b+T}(t-r)\int_r^{r+T}\alpha_2(s)c(s)\alpha_1(s-r)\mathrm{e}^{-\varepsilon\alpha_2^\infty\int_0^{s-r}\alpha_1(\tau)\,\mathrm{d}\tau - \varepsilon\delta_2^\infty\int_0^{s-r}\delta_1(\tau)\,\mathrm{d}\tau}\mathrm{e}^{-\varepsilon\gamma_2^\infty\int_{s-r}^{s}\gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\,\mathrm{d}r \end{aligned}$$

and

$$\psi_{\delta,b+T}(t) \ge \int_0^t \psi_{\delta,b+T}(t-r)\int_r^{r+T}\delta_2(s)c(s)\delta_1(s-r)\mathrm{e}^{-\varepsilon\alpha_2^\infty\int_0^{s-r}\alpha_1(\tau)\,\mathrm{d}\tau - \varepsilon\delta_2^\infty\int_0^{s-r}\delta_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\,\mathrm{d}r.$$

Taking Laplace transforms leads to

$$\widehat{\psi_{\alpha,b+T}}(\lambda) \ge \widehat{\psi_{\alpha,b+T}}(\lambda)F_\alpha(\varepsilon,\lambda,T),, \tag{4.5.10}$$

$$\widehat{\psi_{\delta,b+T}}(\lambda) \ge \widehat{\psi_{\delta,b+T}}(\lambda)F_\delta(\varepsilon,\lambda,T), \tag{4.5.11}$$

where

$$\begin{aligned} F_\alpha(\varepsilon,\lambda,T) := \int_0^\infty \mathrm{e}^{-\lambda r}\Bigg\{&\int_r^{r+T}\alpha_2(s)c(s)\alpha_1(s-r) \\ &\times \mathrm{e}^{-\varepsilon\alpha_2^\infty\int_0^{s-r}\alpha_1(\tau)\,\mathrm{d}\tau - \varepsilon\delta_2^\infty\int_0^{s-r}\delta_1(\tau)\,\mathrm{d}\tau - \varepsilon\gamma_2^\infty\int_{s-r}^{s}\gamma_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\Bigg\}\,\mathrm{d}r. \end{aligned}$$

$$F_\delta(\varepsilon,\lambda,T) := \int_0^\infty \mathrm{e}^{-\lambda r}\left\{\int_r^{r+T}\delta_2(s)c(s)\delta_1(s-r)\mathrm{e}^{-\varepsilon\alpha_2^\infty\int_0^{s-r}\alpha_1(\tau)\,\mathrm{d}\tau - \varepsilon\delta_2^\infty\int_0^{s-r}\delta_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\right\}\mathrm{d}r.$$

Since ψ_α and ψ_δ is bounded, the Laplace transform of $\psi_{\alpha,b+T}(\lambda)$, $\psi_{\delta,b+T}(\lambda)$ is defined for all $\lambda \ge 0$. Moreover,

$$\lim_{T\to\infty} F_\alpha(0,0,T) = \int_0^\infty\int_r^\infty \alpha_2(s)c(s)\alpha_1(s-r)\,\mathrm{d}s\,\mathrm{d}r$$

$$\begin{aligned}
&= \int_0^\omega \int_r^\omega \alpha_2(s)c(s)\alpha_1(s-r)\,\mathrm{d}s\,\mathrm{d}r \\
&= r(\tilde{T}_1), \\
\lim_{T\to\infty} F_\delta(0,0,T) &= \int_0^\infty \int_r^\infty \delta_2(s)c(s)\delta_1(s-r)\,\mathrm{d}s\,\mathrm{d}r \\
&= \int_0^\omega \int_r^\omega \delta_2(s)c(s)\delta_1(s-r)\,\mathrm{d}s\,\mathrm{d}r \\
&= r(\tilde{T}_2).
\end{aligned}$$

Under the assumption $r(\tilde{T}_1) > 1$ and $r(\tilde{T}_2) > 1$, the continuity of F_α, F_δ implies that $F_\alpha(\varepsilon, \lambda, T) > 1$ and $F_\delta(\varepsilon, \lambda, T) > 1$ if $\varepsilon, \lambda > 0$ are chosen small enough and T large enough. (4.5.10) implies that $\widehat{\psi_{\alpha,b+T}}(\lambda) = 0$, which means $\psi_{\alpha,b+T} = 0$ a.e. on $[0, \infty)$, i.e., $\psi_\alpha(t) = 0$ for almost every $t > b + T$. The same argument as in the proof of Theorem 1.8.3 gives rise to $\int_0^\omega c(a)y(T_1, a)\,\mathrm{d}a = 0$. We also obtain from $F_\delta(\varepsilon, \lambda, T) > 1$ that $\int_0^\omega c(a)z(T_1, a)\,\mathrm{d}a = 0$. Therefore, we have

$$\int_0^\omega c(a)\{y(T_1, a) + z(T_1, a)\}\,\mathrm{d}a = 0.$$

However, this contradicts Lemma 4.5.2 (v). □

The following final theorem is concluded from Theorem 2.2.3, Lemma 4.5.2 (v), Lemma 4.5.3 and Theorem 4.5.4.

Theorem 4.5.5. *If $r(\tilde{T}_1) > 1$ and $r(\tilde{T}_2) > 1$, then system (4.1.3) is uniformly strongly ρ-persistent.*

4.6 Discussion

In this chapter we have examined an age-structured rumor transmission model with active stiflers. We have found that it has two thresholds $r(\tilde{T}_1)$ and $r(\tilde{T}_2)$. If $r(\tilde{T}_1) < 1$ and $r(\tilde{T}_2) < 1$ hold, then RFE is the only equilibrium and globally asymptotically stable. If $r(\tilde{T}_1) > 1$, then RFE is unstable and there exists an ASEE. If $r(\tilde{T}_2) > 1$, then RFE is also unstable and there exists a SpEE. Moreover, under PMA, an ASEE bifurcates forward from RFE and is locally asymptotically stable if $r(\tilde{T}_1) > 1$, $r(\tilde{T}_2) \neq 1$ and $|r(\tilde{T}_1) - 1|$ is small enough, a SpEE bifurcates forward from RFE and is locally asymptotically stable if $r(\tilde{T}_2) > 1$, $r(\tilde{T}_1) \neq 1$ and $|r(\tilde{T}_2) - 1|$ is small enough, and the rumor is uniformly strongly persistent whenever $r(\tilde{T}_1) > 1$ and $r(\tilde{T}_2) > 1$.

We can find the same kind of open problems as the age-structured rumor transmission model in Chapter 1: analysis without PMA, the number of REEs, their local or global stability far from the bifurcation point, the case of variable rumor and/or with mass-media, and so on. And there are another open problems left for our future consideration: for example, whether there exists any REE other than ASEE or SpEE when $r(\tilde{T}_1) > 1$ or $r(\tilde{T}_2) > 1$, and whether we can loosen the sufficient condition for the rumor's uniform strong persistence.

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