

Twining Character Formula of Borel-Weil-Bott Type

By Satoshi NAITO

Abstract. We prove a twining character formula of Borel-Weil-Bott type for a connected, simply connected, simple affine algebraic group G over \mathbb{C} , by combining a Lefschetz type fixed point formula for the flag variety G/B and a Kostant type twining character formula for the complex simple Lie algebra \mathfrak{g} of G . Our proof is analogous to the well-known “geometric” proof of the Weyl character formula for finite-dimensional irreducible G -modules.

1. Introduction

The geometric representation theory of a semi-simple affine algebraic group has been extensively studied, and nowadays it plays a central role in the representation theory. However, until recently, the main object of the study is a connected one, though there are various phenomena peculiar to the representation theory of a non-connected one (for example, see [M]).

The Borel-Weil-Bott theorem can surely be regarded as one of the most fundamental results in the geometric representation theory of a semi-simple affine algebraic group. In this paper, we prove a theorem of Borel-Weil-Bott type for a typical non-connected, simple affine algebraic group $\langle \omega \rangle \ltimes G$, which is the semi-direct product of a connected, simply connected, simple affine algebraic group G and the cyclic subgroup $\langle \omega \rangle$ of $\text{Aut}(G)$ generated by (a lift of) a Dynkin diagram automorphism ω .

Let us explain our result more precisely. Let G be a connected, simply connected, simple affine algebraic group over \mathbb{C} with maximal torus T and Borel subgroup $B \supset T$. We denote by \mathfrak{g} , \mathfrak{h} , and \mathfrak{b} the Lie algebras of G , T , and B , respectively. The (rational) character group $X(T) := \text{Hom}(T, \mathbb{C}^*)$ of T can be identified with the (additive) integral weight lattice $\mathfrak{h}_{\mathbb{Z}}^* \subset \mathfrak{h}^*$ by taking the differential at the identity element, i.e., by the map $\lambda \mapsto d\lambda$.

1991 *Mathematics Subject Classification.* Primary 20G05; Secondary 17B10, 20G10, 14M15.

Key words: Twining character, Borel-Weil-Bott theorem.

We choose a set of positive roots $\Delta_+ \subset X(T) \simeq \mathfrak{h}_{\mathbb{Z}}^*$ in such a way that the roots of B (hence of \mathfrak{b}) are the negative roots $\Delta_- = -\Delta_+$.

The graph automorphism ω of the Dynkin diagram of \mathfrak{g} induces (non-canonically) a certain automorphism ω of the Lie algebra \mathfrak{g} , which further lifts to an automorphism ω of the algebraic group G stabilizing both the subgroups T and B . We call these automorphisms ω (Dynkin) diagram automorphisms. Denote the naturally induced action of $\omega \in \text{Aut}(G)$ on $X(T)$ again by ω , and set

$$X(T)^\omega := \{\lambda \in X(T) \mid \omega \cdot \lambda = \lambda\}.$$

Notice that, under the identification $X(T) \simeq \mathfrak{h}_{\mathbb{Z}}^*$, $X(T)^\omega$ is identified with $(\mathfrak{h}_{\mathbb{Z}}^*)^0 := \{\lambda \in \mathfrak{h}_{\mathbb{Z}}^* \mid \omega^*(\lambda) = \lambda\}$, where $\omega^* \in GL(\mathfrak{h}^*)$ is the transposed map of the restriction of ω to \mathfrak{h} . Then the subgroup W^ω of the Weyl group $W \simeq N_G(T)/T$ consisting of all the elements fixed by the naturally induced action of $\langle \omega \rangle$ is given by:

$$\begin{aligned} W^\omega &= \{w \in W \mid (\omega^*)^{-1}w\omega^* = w\} \\ &= \{nT/T \in N_G(T)/T \mid n^{-1}\omega(n) \in T\}. \end{aligned}$$

It is known that W^ω is a Coxeter group. Furthermore, we form the semi-direct product $\langle \omega \rangle \ltimes G$ of G and the cyclic subgroup $\langle \omega \rangle$ of $\text{Aut}(G)$ generated by ω , and then form its closed subgroups $\langle \omega \rangle \ltimes T$ and $\langle \omega \rangle \ltimes B$. It is clear that all these groups are closed subgroups of the affine algebraic group $\text{Aut}(G) \ltimes G$, and hence are affine algebraic groups.

Let $\mathcal{B} := G/B$ be the flag variety, which is an $\langle \omega \rangle \ltimes G$ -variety as well as a G -variety since $\omega \in \text{Aut}(G)$ stabilizes B . For a weight $\lambda \in X(T)$, we denote by $\mathcal{L}_{\mathcal{B}}(\mathbb{C}_\lambda)$ the (G -equivariant) sheaf of $\mathcal{O}_{\mathcal{B}}$ -modules on \mathcal{B} associated to the one-dimensional B -module \mathbb{C}_λ on which B acts by the weight λ through the quotient $B \twoheadrightarrow T$. Then, for each $j \in \mathbb{Z}_{\geq 0}$, the cohomology group $H^j(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_\lambda))$ of the associated sheaf $\mathcal{L}_{\mathcal{B}}(\mathbb{C}_\lambda)$ on X is naturally equipped with a structure of rational G -module. If $\Lambda \in X(T)$ is a dominant weight and $v \in W \simeq N_G(T)/T$ is an element of the Weyl group, then the Borel-Weil-Bott theorem gives an identity of ordinary characters:

$$\text{ch } H^{\ell(v)}(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{v(\Lambda+\rho)-\rho})) = \text{ch } L(\Lambda),$$

where $L(\Lambda)$ is the irreducible highest weight G -module of highest weight Λ , $\rho := (1/2) \cdot \sum_{\alpha \in \Delta_+} \alpha \in X(T)^\omega \simeq (\mathfrak{h}_{\mathbb{Z}}^*)^0$ is the Weyl vector, and $\ell: W \rightarrow \mathbb{Z}_{\geq 0}$ is the length function of the Coxeter group W .

Now assume that $\lambda \in X(T)^\omega$ and fix an $\text{ord}(\omega)$ -th root of unity $\zeta \in \mathbb{C}^*$. We denote by $\mathbb{C}_{\lambda, \zeta}$ the one-dimensional $\langle \omega \rangle \rtimes B$ -module on which B acts by the weight λ through the quotient $B \rightarrow T$ and ω by the scalar ζ . Then the sheaf $\mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda, \zeta})$ of $\mathcal{O}_{\mathcal{B}}$ -modules on \mathcal{B} associated to $\mathbb{C}_{\lambda, \zeta}$ becomes $\langle \omega \rangle \rtimes G$ -equivariant, and hence each cohomology group $H^j(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda, \zeta}))$, $j \geq 0$, naturally comes equipped with a structure of rational $\langle \omega \rangle \rtimes G$ -module. Here we note that $\mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda, \zeta})$ is the (locally free) sheaf of local sections of the algebraic line bundle

$$p: G \times^B \mathbb{C}_{\lambda, \zeta} \rightarrow G/B = \mathcal{B},$$

and that the following isomorphism of line bundles gives $\mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda, \zeta})$ a natural structure of $\langle \omega \rangle \rtimes G$ -equivariant sheaf on \mathcal{B} (see §4.1 for details):

$$\begin{array}{ccc} G \times^B \mathbb{C}_{\lambda, \zeta} & \xrightarrow{\Phi} & (\langle \omega \rangle \rtimes G) \times^{(\langle \omega \rangle \rtimes B)} \mathbb{C}_{\lambda, \zeta} \\ p \downarrow & & \downarrow q \\ G/B & \xrightarrow{\Psi} & (\langle \omega \rangle \rtimes G) / (\langle \omega \rangle \rtimes B), \end{array}$$

where $\Phi([g, z]) := [(1, g), z]$ and $\Psi(gB) := (1, g)(\langle \omega \rangle \rtimes B)$ for $g \in G$ and $z \in \mathbb{C}_{\lambda, \zeta}$. Thus, establishing a Borel-Weil-Bott theorem for the non-connected, simple affine algebraic group $\langle \omega \rangle \rtimes G$ is equivalent to determining the $\langle \omega \rangle \rtimes G$ -module structure of $H^j(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda, \zeta}))$ for each $j \geq 0$.

For a rational $\langle \omega \rangle \rtimes T$ -module V with the T -weight space decomposition $V = \bigoplus_{\mu \in X(T)} V_\mu$, the twining character $\text{ch}^\omega(V)$ of V is defined by

$$\text{ch}^\omega(V) := \sum_{\mu \in X(T)^\omega} \text{Tr}(\omega|_{V_\mu}) e(\mu)$$

as an element of the group algebra $\mathbb{C}[X(T)^\omega]$ of $X(T)^\omega$ over \mathbb{C} with the canonical basis $e(\mu)$, $\mu \in X(T)^\omega \simeq (\mathfrak{h}_{\mathbb{Z}}^*)^0$. Note that the twining character $\text{ch}^\omega(V) \in \mathbb{C}[X(T)^\omega]$ can be viewed as the trace function:

$$T \ni t \mapsto \text{Tr}((\omega, t); V) \in \mathbb{C}.$$

In fact, we have for each $t \in T$,

$$\text{Tr}((\omega, t) ; V) = \sum_{\mu \in X(T)^\omega} \text{Tr}(\omega|_{V_\mu}) e(\mu)(t) \in \mathbb{C},$$

where $e(\mu)(t) := \mu(t) \in \mathbb{C}^*$ for $\mu \in X(T)^\omega$ and $t \in T$. We know from [M, Ch. 2] that the character (i.e., the trace function) of a rational $\langle \omega \rangle \rtimes G$ -module is completely determined by its values on $\langle \omega \rangle \rtimes T^\omega = \langle \omega \rangle \times T^\omega$, where $T^\omega := \{t \in T \mid \omega(t) = t\}$. Therefore, the study of the $\langle \omega \rangle \rtimes G$ -module structure of $H^j(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda, \xi}))$ is reduced to the study of the twining character $\text{ch}^\omega(H^j(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda, \xi})))$ for $j \geq 0$. In fact, we obtain the following theorem, which may be thought of as a Borel-Weil-Bott theorem for the non-connected, simple affine algebraic group $\langle \omega \rangle \rtimes G$.

THEOREM. *Let $\Lambda \in X(T)^\omega$ be a dominant weight fixed by ω and $v \in W^\omega$ an element of the Weyl group $W \simeq N_G(T)/T$ fixed by ω . Then we have an identity of twining characters*

$$\text{ch}^\omega(H^{\ell(v)}(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{v(\Lambda+\rho)-\rho, \zeta}))) = (-1)^{\ell(v)-\widehat{\ell}(v)} \times \zeta \times \text{ch}^\omega(L(\Lambda)),$$

where $\widehat{\ell}: W^\omega \rightarrow \mathbb{Z}_{\geq 0}$ is the length function of the Coxeter group W^ω . Hence we have an isomorphism of $\langle \omega \rangle \rtimes G$ -modules

$$H^{\ell(v)}(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{v(\Lambda+\rho)-\rho, \zeta})) \simeq \mathbb{C}_{(-1)^{\ell(v)-\widehat{\ell}(v)}\zeta} \otimes_{\mathbb{C}} L(\Lambda),$$

where $\mathbb{C}_{(-1)^{\ell(v)-\widehat{\ell}(v)}\zeta}$ denotes a one-dimensional rational $\langle \omega \rangle \rtimes G$ -module on which ω acts by the scalar $(-1)^{\ell(v)-\widehat{\ell}(v)}\zeta$ and G trivially.

REMARK. The scalar $(-1)^{\ell(v)-\widehat{\ell}(v)}$ in the theorem above, which does not appear in the ordinary Borel-Weil-Bott theorem, seems to come from the character of the cyclic group $\langle \omega \rangle$.

Our proof of this theorem is analogous to the well-known “geometric” proof in [AB, §5] (see also [CG, Ch. 6.1]) of the Weyl character formula for the irreducible highest weight G -module $L(\Lambda)$ of dominant highest weight

$\Lambda \in X(T)$. But, unlike the case of an ordinary character, it is not so easy to determine the alternating sum

$$\sum_{j \geq 0} (-1)^j \text{ch}^\omega(\bigwedge^j(\mathfrak{g}/\mathfrak{b})^*)$$

of the twining characters $\text{ch}^\omega(\bigwedge^j(\mathfrak{g}/\mathfrak{b})^*)$, $j \geq 0$, whose description can be given by a Kostant type twining character formula obtained in [Na3].

This paper is organized as follows. In Section 2, following [FSS], [FRS], and [Na3], we recall the definition of a twining character, the twining character formula for $L(\Lambda)$ with $\Lambda \in X(T)$ dominant, and a Kostant type twining character formula. In Section 3, following [Ni] (and also [CG]), we briefly review a Lefschetz type fixed point formula. In Section 4, we prove our main theorem above, by combining a Lefschetz type fixed point formula and a Kostant type twining character formula.

Acknowledgments. I am very grateful to the referee for some invaluable comments on an earlier version of this paper.

2. Twining Characters

2.1. Diagram automorphisms

Let \mathfrak{g} be a (finite-dimensional) complex simple Lie algebra with Cartan subalgebra \mathfrak{h} and Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$. We choose a set of positive roots $\Delta_+ \subset \mathfrak{h}^* := \text{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ such that the roots of \mathfrak{b} are the negative roots $\Delta_- = -\Delta_+$. Let $\{\alpha_i \mid i \in I\}$ be the set of simple roots in Δ_+ , $\{h_i \mid i \in I\}$ the set of simple coroots in \mathfrak{h} , $A = (a_{ij})_{i,j \in I}$ the Cartan matrix with $a_{ij} = \alpha_j(h_i)$, and $W = \langle r_i \mid i \in I \rangle \subset GL(\mathfrak{h}^*)$ the Weyl group with r_i a simple reflection. We take and fix a Chevalley basis $\{e_\alpha, f_\alpha \mid \alpha \in \Delta_+\} \cup \{h_i \mid i \in I\}$ of \mathfrak{g} so that $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_{\alpha_i}$, $\mathfrak{g}_{-\alpha_i} = \mathbb{C}f_{\alpha_i}$, and $[e_{\alpha_i}, f_{\alpha_i}] = h_i$ for each $i \in I$.

A graph automorphism of the Dynkin diagram of \mathfrak{g} is a bijection $\omega: I \rightarrow I$ of the index set I of the simple roots α_i such that

$$a_{\omega(i), \omega(j)} = a_{ij} \quad \text{for all } i, j \in I.$$

Let N be the order of ω , and N_i the number of elements of the ω -orbit I_i of $i \in I$:

$$I_i := \{\omega^k(i) \mid 0 \leq k \leq N_i - 1\}.$$

This ω can be extended (non-canonically) to an automorphism ω , called a diagram automorphism, of order N of the Lie algebra \mathfrak{g} in such a way that

$$\begin{cases} \omega(e_{\alpha_i}) := e_{\alpha_{\omega(i)}} & \text{for } i \in I, \\ \omega(f_{\alpha_i}) := f_{\alpha_{\omega(i)}} & \text{for } i \in I, \\ \omega(h_i) := h_{\omega(i)} & \text{for } i \in I. \end{cases}$$

Note that we have

$$(\omega(x)|\omega(y)) = (x|y) \quad \text{for all } x, y \in \mathfrak{g},$$

where $(\cdot|\cdot)$ is the Killing form on \mathfrak{g} .

The restriction of ω to the Cartan subalgebra \mathfrak{h} induces a transposed map $\omega^*: \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ by $\omega^*(\lambda)(h) := \lambda(\omega(h))$ for $\lambda \in \mathfrak{h}^*$ and $h \in \mathfrak{h}$. We set

$$(\mathfrak{h}^*)^0 := \{\lambda \in \mathfrak{h}^* \mid \omega^*(\lambda) = \lambda\},$$

and call an element of $(\mathfrak{h}^*)^0$ a symmetric weight. Notice that the Weyl vector $\rho := (1/2) \cdot \sum_{\alpha \in \Delta_+} \alpha$ is a symmetric weight.

2.2. Fixed point Weyl group

We define the fixed point subgroup W^ω of the Weyl group W by

$$W^\omega := \{w \in W \mid (\omega^*)^{-1}w\omega^* = w\},$$

and call it the fixed point Weyl group. Notice that W^ω stabilizes the subspace $(\mathfrak{h}^*)^0$ of \mathfrak{h}^* . Choose and fix a complete set \widehat{I} of representatives of the ω -orbits in I , and set for each $i \in \widehat{I}$,

$$w_i := \begin{cases} \prod_{k=0}^{N_i/2-1} (r_{\omega^k(i)} r_{\omega^{k+N_i/2}(i)} r_{\omega^k(i)}) & \text{if } \sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 1, \\ \prod_{k=0}^{N_i-1} r_{\omega^k(i)} & \text{if } \sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 2. \end{cases}$$

Note that for each $i \in \widehat{I}$ we have $\sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 1$ or 2 , and that if $\sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 1$ then N_i is an even integer. It is also known (see, for

example, [FRS]) that the fixed point Weyl group W^ω is a Coxeter group with the canonical generator system $\{w_i \mid i \in \widehat{I}\}$. We denote the length function of the Coxeter system $(W, \{r_i \mid i \in I\})$ (resp. $(W^\omega, \{w_i \mid i \in \widehat{I}\})$) by $\ell: W \rightarrow \mathbb{Z}_{\geq 0}$ (resp. $\widehat{\ell}: W^\omega \rightarrow \mathbb{Z}_{\geq 0}$). Recall from [KN, Lemma 1.3.1] that if $w = w_{i_1}w_{i_2} \cdots w_{i_n} \in W^\omega$ is a reduced expression of $w \in W^\omega$ in the Coxeter system $(W^\omega, \{w_i \mid i \in \widehat{I}\})$, i.e., $\widehat{\ell}(w) = n$, then

$$(2.2.1) \quad \ell(w) = \ell(w_{i_1}) + \ell(w_{i_2}) + \cdots + \ell(w_{i_n}).$$

2.3. Twining character formulas

We denote by $\langle \omega \rangle$ the cyclic subgroup of $\text{Aut}(\mathfrak{g})$ generated by the diagram automorphism $\omega \in \text{Aut}(\mathfrak{g})$. A finite-dimensional vector space V over \mathbb{C} is called an \mathfrak{h} -module if V admits a weight space decomposition with respect to \mathfrak{h} :

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu.$$

Further, an \mathfrak{h} -module V is called an $(\mathfrak{h}, \langle \omega \rangle)$ -module if V admits a \mathbb{C} -linear $\langle \omega \rangle$ -action such that

$$\omega \cdot (xv) = \omega(x)(\omega \cdot v) \quad \text{for all } x \in \mathfrak{g} \text{ and } v \in V.$$

Notice that $\omega \cdot V_\mu = V_{(\omega^*)^{-1}(\mu)}$ for $\mu \in \mathfrak{h}^*$. In view of this fact, we define the twining character $\text{ch}^\omega(V)$ of V by

$$\text{ch}^\omega(V) := \sum_{\mu \in (\mathfrak{h}^*)^0} \text{Tr}(\omega|_{V_\mu}) e(\mu)$$

as an element of the group algebra $\mathbb{C}[(\mathfrak{h}^*)^0]$ of $(\mathfrak{h}^*)^0$ over \mathbb{C} with the canonical basis $\{e(\mu) \mid \mu \in (\mathfrak{h}^*)^0\}$.

Here we recall two kinds of formulas for twining characters, which have been proved more generally for Kac-Moody algebras.

FORMULA 1. Let $\Lambda \in \mathfrak{h}^*$ be a symmetric dominant integral weight. Then, as in [FSS], [FRS], and [Na3], the irreducible highest weight \mathfrak{g} -module $L(\Lambda)$ of highest weight Λ is equipped with a structure of $(\mathfrak{h}, \langle \omega \rangle)$ -module

such that $\omega \cdot v_\Lambda = v_\Lambda$, where v_Λ is a (nonzero) highest weight vector of $L(\Lambda)$. It is shown in [FRS] that

$$\text{ch}^\omega(L(\Lambda)) = \frac{\sum_{w \in W^\omega} (-1)^{\widehat{\ell}(w)} e(w(\Lambda + \rho))}{\sum_{w \in W^\omega} (-1)^{\widehat{\ell}(w)} e(w(\rho))}.$$

FORMULA 2. Let \mathfrak{n}_- (resp. \mathfrak{n}_+) be the sum of all negative (resp. positive) root spaces \mathfrak{g}_α , $\alpha \in \Delta_-$ (resp. $\alpha \in \Delta_+$). Then $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}_+$, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_-$, and \mathfrak{n}_+ all become $(\mathfrak{h}, \langle \omega \rangle)$ -modules under the adjoint action of \mathfrak{h} and the natural action of $\omega \in \text{Aut}(\mathfrak{g})$ since ω stabilizes \mathfrak{n}_- , \mathfrak{h} , and \mathfrak{n}_+ . Notice that the quotient module $\mathfrak{g}/\mathfrak{b}$ is isomorphic to \mathfrak{n}_+ as an $(\mathfrak{h}, \langle \omega \rangle)$ -module. Moreover, since the Killing form $(\cdot | \cdot)$ on \mathfrak{g} is nondegenerate and $\langle \omega \rangle$ -invariant, we have an isomorphism of $(\mathfrak{h}, \langle \omega \rangle)$ -modules

$$(\mathfrak{g}/\mathfrak{b})^* \simeq \mathfrak{n}_-,$$

where $(\mathfrak{g}/\mathfrak{b})^*$ is the dual $(\mathfrak{h}, \langle \omega \rangle)$ -module of $\mathfrak{g}/\mathfrak{b}$. Hence, by taking the exterior power, we obtain that

$$\text{ch}^\omega(\bigwedge^j (\mathfrak{g}/\mathfrak{b})^*) = \text{ch}^\omega(\bigwedge^j \mathfrak{n}_-)$$

for each $j \geq 0$. It is shown in [Na3] that in $\mathbb{C}[(\mathfrak{h}^*)^0]$,

$$\sum_{j \geq 0} (-1)^j \text{ch}^\omega(\bigwedge^j \mathfrak{n}_-) = \sum_{j \geq 0} (-1)^j \text{ch}^\omega(H_j(\mathfrak{n}_-, \mathbb{C}))$$

and

$$\text{ch}^\omega(H_j(\mathfrak{n}_-, \mathbb{C})) = \sum_{\substack{w \in W^\omega \\ \widehat{\ell}(w) = j}} (-1)^{\widehat{\ell}(w) - j} e(w(\rho) - \rho)$$

for each $j \geq 0$, where $H_j(\mathfrak{n}_-, \mathbb{C})$ is the usual Lie algebra homology module of \mathfrak{n}_- with coefficients in the trivial module \mathbb{C} .

3. Lefschetz Type Fixed Point Formula

In this section, following [Ni, §4] (see also [CG, Ch. 5]), we review briefly a special case of (a K-theoretic version of) the Lefschetz fixed point formula that suffices for our purpose. Here we often identify an algebraic vector bundle over a smooth algebraic variety with the locally free coherent sheaf of local sections of it.

Let A be a (not necessarily connected) diagonalizable algebraic group over \mathbb{C} . We denote by $X(A)$ the (rational) character group $\text{Hom}(A, \mathbb{C}^*)$ of A , by $R(A)$ the group ring $\mathbb{Z}[X(A)]$ of $X(A)$ with the canonical basis $\{e(\chi) \mid \chi \in X(A)\}$, and by $S^{-1}R(A)$ the localization of $R(A)$ with respect to the multiplicative subset of $R(A)$ generated by elements of the form $1 - e(\chi)$ for nontrivial $\chi \in X(A)$. Note that the group algebra $\mathbb{C}[X(A)] = \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}[X(A)]$ is identified with the algebra $\text{Mor}(A, \mathbb{C})$ of regular algebraic functions on A by: $e(\chi)(a) = \chi(a) \in \mathbb{C}^*$ for $\chi \in X(A)$ and $a \in A$.

For a (finite-dimensional) rational A -module V , we set

$$\text{Tr}(V) := \sum_{\chi \in X(A)} (\dim_{\mathbb{C}} V_{\chi}) e(\chi) \in R(A),$$

where $V_{\chi} := \{v \in V \mid av = \chi(a)v \text{ for all } a \in A\}$ for $\chi \in X(A)$. Note that the evaluation map (at $a \in A$) $\text{ev}_a: X(A) \rightarrow \mathbb{C}$ given by $\chi \mapsto \chi(a)$ gives rise to a ring homomorphism $\text{ev}_a: R(A) \rightarrow \mathbb{C}$ such that $\text{ev}_a(\text{Tr}(V))$ is equal to the usual trace $\text{Tr}(a; V) \in \mathbb{C}$ for the operation of $a \in A$ on V . Also, for an algebraic vector bundle E over Z and a point $z \in Z$, we denote by $E|_{\{z\}}$ the fiber of E at the point z .

The following is a special case of the Lefschetz fixed point formula (see [Ni, §4], and also [CG, Ch. 5]).

THEOREM 3.1. *Let Z be a smooth projective A -variety (i.e., a variety with an algebraic action of A) such that all the A -fixed points Z^A of Z form a variety of finite set. For an A -equivariant algebraic vector bundle E over Z (viewed as a locally free coherent sheaf on Z), we have in $S^{-1}R(A)$,*

$$(\heartsuit) \quad \sum_{j \geq 0} (-1)^j \text{Tr}(H^j(Z, E)) = \sum_{z \in Z^A} \text{Tr}(E|_{\{z\}}) \times \left(\text{Tr}(\lambda_A|_{\{z\}}) \right)^{-1}.$$

Here

$$\mathrm{Tr}(\lambda_A|_{\{z\}}) := \sum_{j \geq 0} (-1)^j \mathrm{Tr}(\bigwedge^j T_z^* Z),$$

where $\bigwedge^j T_z^* Z$ is the j -th exterior power of the (Zariski) cotangent space $T_z^* Z$ to Z at the point z .

4. Twining Character Formula of Borel-Weil-Bott Type

4.1. Geometric setting

Let G be a connected, simply connected, simple affine algebraic group over \mathbb{C} with maximal torus T and Borel subgroup $B \supset T$, so that the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ of G is a complex simple Lie algebra with Cartan subalgebra $\mathfrak{h} = \mathrm{Lie}(T)$ and Borel subalgebra $\mathfrak{b} = \mathrm{Lie}(B) \supset \mathfrak{h}$. Thus we can use the (algebraic) setting in Section 2. Recall that the (rational) character group $X(T) = \mathrm{Hom}(T, \mathbb{C}^*)$ of T is identified with the (additive) integral weight lattice $\mathfrak{h}_{\mathbb{Z}}^* \subset \mathfrak{h}^*$ by taking the differential at the identity element, i.e., by the map $\lambda \mapsto d\lambda$. Recall also that the Weyl group $W \subset GL(\mathfrak{h}^*)$ is identified with the quotient group $N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G . For each $w \in W \simeq N_G(T)/T$, we denote by $\dot{w} \in N_G(T)$ a right coset representative of w . Notice that if V is a rational G -module and V_{μ} is a T -weight space corresponding to $\mu \in X(T)$, then we have $\dot{w}V_{\mu} = V_{w(\mu)}$.

Since the affine algebraic group G is simply connected, there exists an automorphism (of order N) of G whose differential at the identity element coincides with the diagram automorphism $\omega \in \mathrm{Aut}(\mathfrak{g})$ in §2.1. By abuse of notation, we denote by ω this automorphism of G and by $\langle \omega \rangle$ the cyclic subgroup of $\mathrm{Aut}(G)$ generated by the $\omega \in \mathrm{Aut}(G)$. Note that, under the isomorphism of algebraic groups $\mathrm{Aut}(G) \xrightarrow{\sim} \mathrm{Aut}(\mathfrak{g})$ given by taking the differential at the identity element, the two cyclic groups $\langle \omega \rangle$ are isomorphic. We define the fixed point subgroup

$$X(T)^{\omega} := \{ \lambda \in X(T) \mid \omega \cdot \lambda = \lambda \}$$

of $X(T)$ by the naturally induced action of ω , which coincides with the restriction of $(\omega^{-1})^* = (\omega^*)^{-1} \in GL(\mathfrak{h}^*)$ under the identification $X(T) \simeq \mathfrak{h}_{\mathbb{Z}}^*$. Hence we have an identification:

$$X(T)^{\omega} \simeq (\mathfrak{h}_{\mathbb{Z}}^*)^0 := \{ \lambda \in \mathfrak{h}_{\mathbb{Z}}^* \mid \omega^*(\lambda) = \lambda \}.$$

Furthermore, it is easy to check that for $w \in W$, the element $\omega(\dot{w}) \in N_G(T)$ is a right coset representative of $(\omega^*)^{-1}w\omega^* \in W \simeq N_G(T)/T$. Hence we see that, under the identification $W \simeq N_G(T)/T = \{\dot{w}T/T \mid w \in W\}$, W^ω is identified with $\{\dot{w}T/T \mid \dot{w}^{-1}\omega(\dot{w}) \in T\}$. In fact, for each $w \in W^\omega$, we can (and will) take a right coset representative $\dot{w} \in N_G(T)$ of w such that $\omega(\dot{w}) = \dot{w}$ (see [Sp, Ch. 9.3]).

We form the semi-direct product $\langle \omega \rangle \rtimes G$ of G and $\langle \omega \rangle \subset \text{Aut}(G)$, and then its closed subgroups $\langle \omega \rangle \rtimes B$ and $\langle \omega \rangle \rtimes T$, all of which are closed subgroups of the affine algebraic group $\text{Aut}(G) \rtimes G$, and hence are affine algebraic groups. Notice that a (finite-dimensional) rational $\langle \omega \rangle \rtimes T$ -module obviously becomes an $(\mathfrak{h}, \langle \omega \rangle)$ -module in §2.3.

REMARK 4.1.1. Let V be a rational $\langle \omega \rangle \rtimes T$ -module with the T -weight space decomposition $V = \bigoplus_{\mu \in X(T)} V_\mu$. Then, for each $t \in T$, we have

$$\text{Tr}((\omega, t) ; V) = \sum_{\mu \in X(T)^\omega} \text{Tr}(\omega|_{V_\mu}) e(\mu)(t) \in \mathbb{C},$$

where $e(\mu)(t) := \mu(t) \in \mathbb{C}^*$ for $\mu \in X(T)^\omega$ and $t \in T$. Hence the twining character $\text{ch}^\omega(V) \in \mathbb{C}[(\mathfrak{h}_\mathbb{Z}^*)^0] \simeq \mathbb{C}[X(T)^\omega]$ can be thought of as the trace function:

$$T \ni t \mapsto \text{Tr}((\omega, t) ; V) \in \mathbb{C}.$$

Let $\mathcal{B} := G/B$ be the flag variety and $\pi: G \rightarrow G/B = \mathcal{B}$ the quotient morphism. It is well-known that \mathcal{B} is a smooth projective G -variety and $\pi: G \rightarrow \mathcal{B}$ has local sections. To each (finite-dimensional) rational B -module M , we can associate a G -equivariant vector bundle $p: L_{\mathcal{B}}(M) \rightarrow \mathcal{B}$ over \mathcal{B} by setting

$$L_{\mathcal{B}}(M) := G \times^B M = (G \times M)/B \quad \text{and} \quad p([g, m]) := gB/B \in \mathcal{B},$$

where B acts from the right on the direct product $G \times M$ by

$$(g, m)b := (gb, b^{-1}m) \quad \text{for } g \in G, m \in M, \text{ and } b \in B,$$

and where G acts from the left on $G \times^B M$ via left multiplication on the first factor. (The quotient $(G \times M)/B$ actually is a smooth algebraic variety since $\pi: G \rightarrow \mathcal{B}$ has local sections.) Note that the sheaf $\mathcal{L}_{\mathcal{B}}(M)$, called the

associated sheaf to M on \mathcal{B} , of local sections of the algebraic vector bundle $L_{\mathcal{B}}(M)$ is a locally free G -equivariant (or G -linearized) sheaf of $\mathcal{O}_{\mathcal{B}}$ -modules (see [J, Part I. Ch. 5] and [CG, Ch. 5]).

Furthermore, if M is a (finite-dimensional) rational $\langle \omega \rangle \times B$ -module, then the vector bundle $p: L_{\mathcal{B}}(M) \rightarrow \mathcal{B}$ comes equipped with a structure of $\langle \omega \rangle \times G$ -equivariant algebraic vector bundle by the rational $\langle \omega \rangle$ -action:

$$\omega \cdot ([g, m]) := [\omega(g), \omega \cdot m] \quad \text{for } g \in G \text{ and } m \in M$$

on $L_{\mathcal{B}}(M)$ and the natural algebraic action of $\langle \omega \rangle$ on \mathcal{B} . Hence the associated sheaf $\mathcal{L}_{\mathcal{B}}(M)$ of $\mathcal{O}_{\mathcal{B}}$ -modules becomes $\langle \omega \rangle \times G$ -equivariant (cf. [KN, §2.3]).

REMARK 4.1.2. To each (finite-dimensional) rational $\langle \omega \rangle \times B$ -module M , we can associate an $\langle \omega \rangle \times G$ -equivariant algebraic vector bundle

$$q: (\langle \omega \rangle \times G) \times^{(\langle \omega \rangle \times B)} M \rightarrow (\langle \omega \rangle \times G)/(\langle \omega \rangle \times B)$$

by replacing G and B in the definition of $L_{\mathcal{B}}(M)$ above with $\langle \omega \rangle \times G$ and $\langle \omega \rangle \times B$, respectively. (Note that the canonical morphism $\langle \omega \rangle \times G \rightarrow (\langle \omega \rangle \times G)/(\langle \omega \rangle \times B)$ has local sections since $\pi: G \rightarrow G/B$ does.) Then we have the following isomorphism of $\langle \omega \rangle \times G$ -equivariant vector bundles:

$$\begin{array}{ccc} G \times^B M & \xrightarrow{\Phi} & (\langle \omega \rangle \times G) \times^{(\langle \omega \rangle \times B)} M \\ p \downarrow & & \downarrow q \\ G/B & \xrightarrow{\Psi} & (\langle \omega \rangle \times G)/(\langle \omega \rangle \times B), \end{array}$$

where $\Phi([g, m]) := [(1, g), m]$ and $\Psi(gB) := (1, g)(\langle \omega \rangle \times B)$ for $g \in G$ and $m \in M$.

The quotient $\mathfrak{g}/\mathfrak{b}$ is viewed as a rational B -module by the adjoint representation $\text{Ad}: B \rightarrow GL(\mathfrak{g}/\mathfrak{b})$. In addition, since $\omega \in \text{Aut}(\mathfrak{g})$ and hence $\omega \in \text{Aut}(G)$ stabilize $\mathfrak{b} \subset \mathfrak{g}$, the quotient $\mathfrak{g}/\mathfrak{b}$ can be made into a rational $\langle \omega \rangle \times B$ -module. Moreover, we can check the following lemma.

LEMMA 4.1.3. *The cotangent bundle $T^*\mathcal{B}$ over the flag variety \mathcal{B} is isomorphic to the vector bundle $L_{\mathcal{B}}((\mathfrak{g}/\mathfrak{b})^*)$ over \mathcal{B} associated to the dual*

$\langle \omega \rangle \rtimes B$ -module $(\mathfrak{g}/\mathfrak{b})^*$ of $\mathfrak{g}/\mathfrak{b}$ as $\langle \omega \rangle \rtimes G$ -equivariant vector bundles over \mathcal{B} .

PROOF. Cf. the proof of [CG, Lemma 1.4.9]. \square

4.2. Proof of the formula

Let $\lambda \in X(T)^\omega$ and $\zeta \in \mathbb{C}^*$ an N -th root of unity. We denote by $\mathbb{C}_{\lambda, \zeta}$ the one-dimensional rational $\langle \omega \rangle \rtimes B$ -module on which B acts by the weight λ through the quotient $B \twoheadrightarrow T$ and ω by the scalar ζ . Then, for each $j \geq 0$, the cohomology group $H^j(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda, \zeta}))$ of the associated sheaf $\mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda, \zeta})$ on \mathcal{B} naturally comes equipped with a rational $\langle \omega \rangle \rtimes G$ -module structure, since $\mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda, \zeta})$ is a locally free $\langle \omega \rangle \rtimes G$ -equivariant sheaf of $\mathcal{O}_{\mathcal{B}}$ -modules (cf. [J, Part I. Ch. 5] and also [KN, §2.3]).

Let $\Lambda \in X(T)^\omega$ be dominant and $v \in W^\omega$. We set

$$\lambda := v \circ \Lambda = v(\Lambda + \rho) - \rho \in X(T)^\omega,$$

where $\rho = (1/2) \cdot \sum_{\alpha \in \Delta_+} \alpha \in X(T)^\omega$ is the Weyl vector. The Borel-Weil-Bott theorem tells us that $H^j(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda, \zeta})) = 0$ unless $j \neq \ell(v)$, and that $H^{\ell(v)}(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda, \zeta}))$ is isomorphic, as a G -module, to the irreducible highest weight module $L(\Lambda)$ of highest weight Λ .

Now we are ready to state our main result, which, in view of Remarks 4.1.1 and 4.1.2, may be thought of as a Borel-Weil-Bott theorem for the non-connected, simple affine algebraic group $\langle \omega \rangle \rtimes G$.

THEOREM 4.2.1. *Let $\Lambda \in X(T)^\omega$ be a dominant weight fixed by the diagram automorphism ω , $v \in W^\omega$, and $\zeta \in \mathbb{C}^*$ an N -th root of unity. Set $\lambda := v \circ \Lambda = v(\Lambda + \rho) - \rho$. Then we have the following identity of twining characters:*

$$\text{ch}^\omega(H^{\ell(v)}(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda, \zeta}))) = (-1)^{\ell(v) - \widehat{\ell}(v)} \times \zeta \times \text{ch}^\omega(L(\Lambda))$$

in the group algebra $\mathbb{C}[X(T)^\omega]$ of $X(T)^\omega$ over \mathbb{C} with the canonical basis $\{e(\mu) \mid \mu \in X(T)^\omega\}$. Here $L(\Lambda)$ is the irreducible highest weight G -module of highest weight Λ . Hence we have an isomorphism of $\langle \omega \rangle \rtimes G$ -modules

$$H^{\ell(v)}(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda, \zeta})) \simeq \mathbb{C}_{(-1)^{\ell(v) - \widehat{\ell}(v)} \zeta} \otimes_{\mathbb{C}} L(\Lambda),$$

where $\mathbb{C}_{(-1)^{\ell(v)-\widehat{\ell}(v)}\zeta}$ denotes a one-dimensional rational $\langle \omega \rangle \rtimes G$ -module on which ω acts by the scalar $(-1)^{\ell(v)-\widehat{\ell}(v)}\zeta$ and G trivially.

REMARK 4.2.2. We know from the Borel-Weil-Bott theorem (see, for example, [J, Part II. Ch. 5]) that each $\lambda \in X(T)$ such that $H^j(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda})) \neq 0$ for some $j \in \mathbb{Z}_{\geq 0}$ can be written uniquely in the form $\lambda = v \circ \Lambda$ for some $v \in W$ and dominant $\Lambda \in X(T)$. It is obvious that if this $\lambda = v \circ \Lambda$ is an element of $X(T)^\omega$, then $v \in W^\omega$.

REMARK 4.2.3. It follows from Equality (2.2.1) that for $v \in W^\omega$, the scalar $(-1)^{\ell(v)-\widehat{\ell}(v)}$ is not equal to 1 in general. Hence Theorem 4.2.1 implies that, as $\langle \omega \rangle \rtimes G$ -modules, $H^{\ell(v)}(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda,1}))$ and $L(\Lambda)$ are not necessarily isomorphic. This scalar $(-1)^{\ell(v)-\widehat{\ell}(v)}$ seems to come from the character of the cyclic group $\langle \omega \rangle$. In fact, when G is of type D_4 and ω is of order 3, we see by (2.2.1) that $(-1)^{\ell(v)-\widehat{\ell}(v)} = 1$ for all $v \in W^\omega$.

The rest of this subsection is devoted to the proof of Theorem 4.2.1. Now we set $V := H^{\ell(v)}(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda,\zeta}))$ and compute the twining character

$$\text{ch}^\omega(V) = \sum_{\mu \in X(T)^\omega} \text{Tr}(\omega|_{V_\mu}) e(\mu)$$

of the rational $\langle \omega \rangle \rtimes T$ -module V with the T -weight space decomposition $V = \bigoplus_{\mu \in X(T)} V_\mu$. Because we already know from the Borel-Weil-Bott theorem that $V = H^{\ell(v)}(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda,\zeta}))$ is a simple module of highest weight Λ as a G -module (and hence simple as an $\langle \omega \rangle \rtimes G$ -module), it suffices to prove the equality

$$(4.2.1) \quad \text{ch}^\omega(V) = (-1)^{\ell(v)-\widehat{\ell}(v)} \times \zeta \times \text{ch}^\omega(L(\Lambda))$$

in the group algebra $\mathbb{C}[X(T)^\omega] \subset \mathbb{C}[X(T)]$. (Here we recall that $\mathbb{C}[X(T)] = \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}[X(T)]$ is identified with the algebra $\text{Mor}(T, \mathbb{C})$ of regular algebraic functions on T by: $e(\mu)(t) = \mu(t) \in \mathbb{C}^*$ for $\mu \in X(T)$ and $t \in T$.)

We define the fixed point torus $T^\omega \subset T$ by

$$T^\omega := \{t \in T \mid \omega(t) = t\}.$$

The Lie algebra $\text{Lie}(T^\omega)$ of $T^\omega \subset T$ is

$$\mathfrak{h}^0 := \{h \in \mathfrak{h} \mid \omega(h) = h\}.$$

It is known (see [St, §8]) that T^ω is, in fact, connected. Hence T^ω is a subtorus of T . Furthermore, it easily follows from the canonical isomorphism $(\mathfrak{h}^*)^0 \simeq (\mathfrak{h}^0)^*$ that the group homomorphism

$$\text{Res}: X(T)^\omega \rightarrow X(T^\omega) := \text{Hom}(T^\omega, \mathbb{C}^*)$$

given by the restriction to $T^\omega \subset T$ is injective. So the induced algebra homomorphism

$$\text{Res}: \mathbb{C}[X(T)^\omega] \rightarrow \mathbb{C}[X(T^\omega)] \simeq \text{Mor}(T^\omega, \mathbb{C})$$

is also injective. Thus it suffices for us to show Equality (4.2.1) in the group algebra $\mathbb{C}[X(T^\omega)]$.

Here we remark that the semi-direct product $\langle \omega \rangle \rtimes T$ itself is not abelian, but its closed subgroup $\langle \omega \rangle \rtimes T^\omega$ splits into the direct product $\langle \omega \rangle \times T^\omega$, and hence that $\langle \omega \rangle \rtimes T^\omega = \langle \omega \rangle \times T^\omega$ is a (not necessarily connected) diagonalizable algebraic group. Thus the (closed) embedding $T^\omega \xrightarrow{\sim} \{\omega\} \times T^\omega \subset \langle \omega \rangle \times T^\omega$ induces the evaluation map (at $\omega \in \langle \omega \rangle$)

$$\text{ev}_\omega: \mathbb{Z}[X(\langle \omega \rangle \times T^\omega)] \rightarrow \mathbb{C}[X(T^\omega)].$$

By Remark 4.1.1, we see that for a (finite-dimensional) rational $\langle \omega \rangle \rtimes T$ -module U ,

$$(4.2.2) \quad \text{ev}_\omega(\text{Tr}(U)) = \text{ch}^\omega(U)$$

in the group algebra $\mathbb{C}[X(T^\omega)] \simeq \text{Mor}(T^\omega, \mathbb{C})$, where the rational $\langle \omega \rangle \rtimes T$ -module U is regarded as a rational $\langle \omega \rangle \times T^\omega$ -module by restriction. Note that $\text{ch}^\omega(U)$ is, in fact, an element of the group algebra $\mathbb{C}[X(T)^\omega] \hookrightarrow \mathbb{C}[X(T^\omega)]$, since U is a rational $\langle \omega \rangle \rtimes T$ -module.

In order to compute $\text{ev}_\omega(\text{Tr}(V)) \in \mathbb{C}[X(T^\omega)]$, we want to apply Theorem 3.1 to the flag variety $\mathcal{B} = G/B$, which is a smooth projective $\langle \omega \rangle \rtimes T$ -variety.

LEMMA 4.2.4. *We have*

$$\mathcal{B}^{\langle\omega\rangle \times T^\omega} = (\mathcal{B}^T)^{\langle\omega\rangle}.$$

PROOF. Take a regular semi-simple element $t \in T^\omega$ such that $t^N \in T^\omega$ is also regular. Let $x \in \mathcal{B}$ be a point fixed by $(\omega, t) \in \langle\omega\rangle \times T^\omega$. Then we have

$$x = (\omega, t)^N x = (\omega^N, t^N) x = (1, t^N) x = t^N x.$$

Thus it suffices to show that if $t \in T$ is a regular semi-simple element, then we have $\mathcal{B}^t = \mathcal{B}^T$. Let $t \in T$ be a regular semi-simple element and $gB/B \in \mathcal{B}^t$ with $g \in G$, i.e., $tgB = gB$. Then we have $g^{-1}tg \in B$, so that $t \in gBg^{-1} \cap T$. Since gBg^{-1} is also a Borel (i.e., maximal closed connected solvable) subgroup of G , there exists a maximal torus (i.e., closed connected diagonalizable subgroup) T' of gBg^{-1} such that $t \in T' \subset gBg^{-1}$. So we get $t \in T \cap T'$. Note that T' is also a maximal torus of G . Since $t \in T$ is a regular semi-simple element, it belongs to a unique maximal torus. Therefore, it follows that $T' = T$, and hence $T \subset gBg^{-1}$, i.e., $g^{-1}Tg \subset B$. Consequently, we see that $(g^{-1}Tg)B = B$, so that $TgB = gB$. Hence we have shown that $gB/B \in \mathcal{B}^T$, which proves the lemma. \square

REMARK 4.2.5. The set of regular elements t such that t^N is also regular are clearly dense in $T^\omega \subset T$, since the restriction of any positive root $\alpha \in \Delta_+ \subset X(T)$ to $T^\omega \subset T$ is not identically equal to 1.

Put $A := \langle\omega\rangle \times T^\omega$ and $E := L_{\mathcal{B}}(\mathbb{C}_{\lambda, \zeta})$, where $\lambda = v \circ \Lambda$ with $\Lambda \in X(T)^\omega$ dominant and $v \in W^\omega$. We recall the well-known fact that $\mathcal{B}^T = \{\dot{w}B/B \mid w \in W\}$. Hence we see by Lemma 4.2.4 that

$$\begin{aligned} \mathcal{B}^A &= (\mathcal{B}^T)^{\langle\omega\rangle} = \{\dot{w}B/B \mid \dot{\omega}^{-1}\omega(\dot{\omega}) \in T, w \in W\} \\ &= \{\dot{w}B/B \mid w \in W^\omega\}. \end{aligned}$$

In particular, \mathcal{B}^A is a variety of finite set. Therefore, we can apply Theorem 3.1 to our setting.

Let S be the multiplicative subset of $\mathbb{Z}[X(\langle\omega\rangle \times T^\omega)]$ generated by elements of the form $1 - e(\chi)$ for nontrivial $\chi \in X(\langle\omega\rangle \times T^\omega)$ as in §3. Then an

easy consideration shows that $0 \notin \text{ev}_\omega(S) \subset \mathbb{C}[X(T^\omega)]$. Hence the evaluation map $\text{ev}_\omega: \mathbb{Z}[X(\langle \omega \rangle \times T^\omega)] \rightarrow \mathbb{C}[X(T^\omega)]$ lifts to a ring homomorphism

$$S^{-1}\text{ev}_\omega: S^{-1}\mathbb{Z}[X(\langle \omega \rangle \times T^\omega)] \rightarrow \text{Frac}(\mathbb{C}[X(T^\omega)]),$$

where $\text{Frac}(\mathbb{C}[X(T^\omega)])$ is the fraction field of $\mathbb{C}[X(T^\omega)]$. Therefore, by applying $S^{-1}\text{ev}_\omega$ to both sides of (\heartsuit) in Theorem 3.1, we obtain the following formula in the fraction field $\text{Frac}(\mathbb{C}[X(T^\omega)]) \hookrightarrow \text{Frac}(\mathbb{C}[X(T^\omega)])$,

$$(4.2.3) \quad \sum_{j \geq 0} (-1)^j \text{ch}^\omega(H^j(\mathcal{B}, \mathcal{L}_\mathcal{B}(\mathbb{C}_{\lambda, \zeta}))) = \sum_{w \in W^\omega} \text{ch}^\omega(L_\mathcal{B}(\mathbb{C}_{\lambda, \zeta})|_{\{\dot{w}B/B\}}) \times \left(\sum_{j \geq 0} (-1)^j \text{ch}^\omega(\bigwedge^j T_{\dot{w}B/B}^* \mathcal{B}) \right)^{-1}.$$

Since $H^j(\mathcal{B}, \mathcal{L}_\mathcal{B}(\mathbb{C}_{\lambda, \zeta})) = 0$ unless $j = \ell(v)$ by the Borel-Weil-Bott theorem, the left-hand side of (4.2.3) becomes

$$(-1)^{\ell(v)} \times \text{ch}^\omega(H^{\ell(v)}(\mathcal{B}, \mathcal{L}_\mathcal{B}(\mathbb{C}_{\lambda, \zeta}))).$$

We now compute the right-hand side of (4.2.3).

LEMMA 4.2.6. *Let $\lambda \in X(T)^\omega$ and $w \in W^\omega$. Then the fiber $L_\mathcal{B}(\mathbb{C}_{\lambda, \zeta})|_{\{\dot{w}B/B\}}$ of the line bundle $p: L_\mathcal{B}(\mathbb{C}_{\lambda, \zeta}) \rightarrow \mathcal{B}$ at the point $\dot{w}B/B \in \mathcal{B}$ is isomorphic to $\mathbb{C}_{w(\lambda), \zeta}$ as an $\langle \omega \rangle \rtimes T$ -module.*

PROOF. Let $[\dot{w}, z] \in G \times^B \mathbb{C}_{\lambda, \zeta}$ with $z \in \mathbb{C}_{\lambda, \zeta}$ be an element of the fiber $L_\mathcal{B}(\mathbb{C}_{\lambda, \zeta})|_{\{\dot{w}B/B\}}$ of the line bundle $L_\mathcal{B}(\mathbb{C}_{\lambda, \zeta})$ at the point $\dot{w}B/B \in \mathcal{B}$. For $t \in T$, we have

$$\begin{aligned} t \cdot [\dot{w}, z] &= [t\dot{w}, z] \\ &= [\dot{w}(\dot{w}^{-1}t\dot{w}), z] = [\dot{w}, (\dot{w}^{-1}t\dot{w})z] \quad \text{since } \dot{w}^{-1}t\dot{w} \in T \\ &= [\dot{w}, \lambda(\dot{w}^{-1}t\dot{w}) \times z] = [\dot{w}, (w(\lambda))(t) \times z]. \end{aligned}$$

Also, we have

$$\begin{aligned} \omega([\dot{w}, z]) &= [\omega(\dot{w}), \omega \cdot z] \\ &= [\dot{w}, \zeta \times z] \end{aligned}$$

since the right coset representative $\dot{w} \in N_G(T)$ of $w \in W^\omega$ is chosen in such a way that $\omega(\dot{w}) = \dot{w}$. This proves the lemma. \square

By this lemma, we get

$$\text{ch}^\omega \left(L_{\mathcal{B}}(\mathbb{C}_{\lambda, \zeta})|_{\{\dot{w}B/B\}} \right) = \zeta \times e(w(\lambda)) \in \mathbb{C}[X(T)^\omega].$$

On the other hand, we see by Lemma 4.1.3 that the (Zariski) cotangent space $T_{\dot{w}B/B}^* \mathcal{B}$ to \mathcal{B} at the point $\dot{w}B/B$ for $w \in W^\omega$ is isomorphic to the dual $\langle \omega \rangle \rtimes T$ -module $(\mathfrak{g}/\text{Ad}(\dot{w})\mathfrak{b})^*$ of $\mathfrak{g}/\text{Ad}(\dot{w})\mathfrak{b}$ as an $\langle \omega \rangle \rtimes T$ -module, and hence is isomorphic to $\text{Ad}(\dot{w})\mathfrak{n}_-$ as an $\langle \omega \rangle \rtimes T$ -module. Therefore, we deduce that for each $w \in W^\omega$ and $j \geq 0$,

$$\text{ch}^\omega \left(\bigwedge^j T_{\dot{w}B/B}^* \mathcal{B} \right) = \text{ch}^\omega \left(\bigwedge^j (\text{Ad}(\dot{w})\mathfrak{n}_-) \right).$$

Here we note that the following diagram commutes for each $w \in W^\omega$:

$$\begin{array}{ccc} \mathfrak{n}_- & \xrightarrow{\omega} & \mathfrak{n}_- \\ \text{Ad}(\dot{w}) \downarrow & & \downarrow \text{Ad}(\dot{w}) \\ \text{Ad}(\dot{w})\mathfrak{n}_- & \xrightarrow{\omega} & \text{Ad}(\dot{w})\mathfrak{n}_-, \end{array}$$

since we have

$$\omega(\text{Ad}(\dot{w})x) = \text{Ad}(\omega(\dot{w}))\omega(x)$$

for all $y \in W$ and $x \in \mathfrak{g}$. From this commutative diagram, we see that for each $j \geq 0$,

$$\text{ch}^\omega \left(\bigwedge^j (\text{Ad}(\dot{w})\mathfrak{n}_-) \right) = w \left(\text{ch}^\omega \left(\bigwedge^j \mathfrak{n}_- \right) \right).$$

Consequently, we obtain in $\mathbb{C}[X(T)^\omega]$,

$$\begin{aligned} \sum_{j \geq 0} (-1)^j \text{ch}^\omega \left(\bigwedge^j T_{\dot{w}B/B}^* \mathcal{B} \right) &= \sum_{j \geq 0} (-1)^j \text{ch}^\omega \left(\bigwedge^j (\text{Ad}(\dot{w})\mathfrak{n}_-) \right) \\ &= \sum_{j \geq 0} (-1)^j \left(w \left(\text{ch}^\omega \left(\bigwedge^j \mathfrak{n}_- \right) \right) \right) \\ &= w \left(\sum_{j \geq 0} (-1)^j \text{ch}^\omega \left(\bigwedge^j \mathfrak{n}_- \right) \right). \end{aligned}$$

We put

$$H := \sum_{j \geq 0} (-1)^j \text{ch}^\omega(\bigwedge^j \mathfrak{n}_-)$$

as an element in $\mathbb{C}[X(T)^\omega] \simeq \mathbb{C}[(\mathfrak{h}_\mathbb{Z}^*)^0]$. It follows from Formula 2 that

$$H = e(-\rho) \cdot \left(\sum_{y \in W^\omega} (-1)^{\widehat{\ell}(y)} e(y(\rho)) \right).$$

So we have for each $w \in W^\omega$,

$$w(H) = e(-w(\rho)) \cdot w \left(\sum_{y \in W^\omega} (-1)^{\widehat{\ell}(y)} e(y(\rho)) \right).$$

Here, since for each $y \in W^\omega$

$$(-1)^{\widehat{\ell}(w)} (-1)^{\widehat{\ell}(y)} = (-1)^{\widehat{\ell}(w) + \widehat{\ell}(y)} = (-1)^{\widehat{\ell}(wy)},$$

we deduce that

$$\begin{aligned} (4.2.4) \quad w \left(\sum_{y \in W^\omega} (-1)^{\widehat{\ell}(y)} e(y(\rho)) \right) &= \sum_{y \in W^\omega} (-1)^{\widehat{\ell}(y)} e(wy(\rho)) \\ &= (-1)^{\widehat{\ell}(w)} \times \sum_{y \in W^\omega} (-1)^{\widehat{\ell}(w)} (-1)^{\widehat{\ell}(y)} e(wy(\rho)) \\ &= (-1)^{\widehat{\ell}(w)} \times \sum_{y \in W^\omega} (-1)^{\widehat{\ell}(wy)} e(wy(\rho)) \\ &= (-1)^{\widehat{\ell}(w)} \times \sum_{y \in W^\omega} (-1)^{\widehat{\ell}(y)} e(y(\rho)). \end{aligned}$$

Hence we get

$$w(H) = (-1)^{\widehat{\ell}(w)} \times e(-w(\rho)) \cdot \left(\sum_{y \in W^\omega} (-1)^{\widehat{\ell}(y)} e(y(\rho)) \right).$$

To sum up, the right-hand side of (4.2.3) is

$$F := \sum_{w \in W^\omega} \frac{\zeta \times e(w(\lambda))}{(-1)^{\widehat{\ell}(w)} \times e(-w(\rho)) \cdot \left(\sum_{y \in W^\omega} (-1)^{\widehat{\ell}(y)} e(y(\rho)) \right)} \in \text{Frac}(\mathbb{C}[X(T)^\omega]).$$

Since $\lambda = v \circ \Lambda$, we have $w(\lambda) = wv(\Lambda + \rho) - w(\rho)$ for each $w \in W^\omega$. Hence we get

$$\begin{aligned} F &= \left(\sum_{y \in W^\omega} (-1)^{\widehat{\ell}(y)} e(y(\rho)) \right)^{-1} \times \\ &\quad \times \left(\zeta \times \sum_{w \in W^\omega} (-1)^{\widehat{\ell}(w)} e(w(\rho)) \cdot e(wv(\Lambda + \rho) - w(\rho)) \right) \\ &= \left(\sum_{y \in W^\omega} (-1)^{\widehat{\ell}(y)} e(y(\rho)) \right)^{-1} \cdot \left(\zeta \times \sum_{w \in W^\omega} (-1)^{\widehat{\ell}(w)} e(wv(\Lambda + \rho)) \right). \end{aligned}$$

As in (4.2.4), we deduce that

$$\sum_{w \in W^\omega} (-1)^{\widehat{\ell}(w)} e(wv(\Lambda + \rho)) = (-1)^{\widehat{\ell}(v)} \times \sum_{w \in W^\omega} (-1)^{\widehat{\ell}(w)} e(w(\Lambda + \rho)).$$

Hence we have in $\text{Frac}(\mathbb{C}[X(T)^\omega])$,

$$\begin{aligned} F &= \zeta \times (-1)^{\widehat{\ell}(v)} \times \frac{\sum_{w \in W^\omega} (-1)^{\widehat{\ell}(w)} e(w(\Lambda + \rho))}{\sum_{y \in W^\omega} (-1)^{\widehat{\ell}(y)} e(y(\rho))} \\ &= \zeta \times (-1)^{\widehat{\ell}(v)} \times \text{ch}^\omega(L(\Lambda)) \quad \text{by Formula 1.} \end{aligned}$$

Thus, by Equality (4.2.3), we obtain Equality (4.2.1):

$$\text{ch}^\omega(V) = (-1)^{\ell(v) - \widehat{\ell}(v)} \times \zeta \times \text{ch}^\omega(L(\Lambda)) \in \mathbb{C}[X(T)^\omega].$$

Recall that $V = H^{\ell(v)}(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{v \circ \Lambda, \zeta}))$. Thus we have proved that

$$\text{ch}^\omega(H^{\ell(v)}(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{v \circ \Lambda, \zeta}))) = (-1)^{\ell(v) - \widehat{\ell}(v)} \times \zeta \times \text{ch}^\omega(L(\Lambda)),$$

as desired.

References

- [AB] Atiyah, M. F. and R. Bott, A Lefschetz fixed point theorem for elliptic complexes: II. Applications, *Ann. of Math.* **87** (1968), 451–491.
- [B] Borel, A., *Linear Algebraic Groups* (2nd enlarged edition), Graduate Texts in Math. Vol. 126, Springer-Verlag, Berlin, 1991.
- [CG] Chriss, N. and V. Ginzburg, *Representation Theory and Complex Geometry*, Birkhäuser, Boston, 1997.
- [FRS] Fuchs, J., Ray, U. and C. Schweigert, Some automorphisms of generalized Kac-Moody algebras, *J. Algebra* **191** (1997), 518–540.
- [FSS] Fuchs, J., Schellekens, B. and C. Schweigert, From Dynkin diagram symmetries to fixed point structures, *Comm. Math. Phys.* **180** (1996), 39–97.
- [H] Humphreys, J. E., *Linear Algebraic Groups*, Graduate Texts in Math. Vol. 21, Springer-Verlag, Berlin, 1987.
- [J] Jantzen, J. C., *Representations of Algebraic Groups*, Pure Appl. Math. Vol. 131, Academic Press, Boston, 1987.
- [K] Kac, V. G., *Infinite Dimensional Lie Algebras* (3rd edition), Cambridge Univ. Press, Cambridge, 1990.
- [KN] Kaneda, M. and S. Naito, A twining character formula for Demazure modules, *Transform. Groups* **7** (2002), 321–341.
- [KK] Kang, S.-J. and J.-H. Kwon, Graded Lie superalgebras, supertrace formula, and orbit Lie superalgebras, *Proc. London Math. Soc.* **81** (2000), 675–724.
- [M] Mohrdieck, S., Conjugacy classes of non-connected semisimple algebraic groups, Ph.D. dissertation, Hamburg University, 2000.
- [MFK] Mumford, D., Fogarty, J. and F. Kirwan, *Geometric Invariant Theory* (3rd enlarged edition), Springer-Verlag, Berlin, 1994.
- [Na1] Naito, S., Embedding into Kac-Moody algebras and construction of folding subalgebras for generalized Kac-Moody algebras, *Japan. J. Math. (New Series)* **18** (1992), 155–171.
- [Na2] Naito, S., Twining character formula of Kac-Wakimoto type for affine Lie algebras, *Represent. Theory* **6** (2002), 70–100 (electronic).
- [Na3] Naito, S., Twining characters and Kostant’s homology formula, to appear in *Tôhoku Math. J. (Second Series)*.
- [Ni] Nielsen, H. A., Diagonalizably linearized coherent sheaves, *Bull. Soc. Math. France* **102** (1974), 85–97.
- [Sp] Springer, T. A., *Linear Algebraic Groups* (2nd edition), Progr. Math. Vol. 9, Birkhäuser, Boston, 1998.
- [St] Steinberg, R., Endomorphisms of linear algebraic groups, *Mem. Amer. Math. Soc.* **80** (1968).

(Received July 2, 2001)

Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki 305-8571
Japan
E-mail: naito@math.tsukuba.ac.jp