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Siegel-Whittaker Functions on $Sp(2,\mathbf{R})$ *for Principal Series Representations*

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Abstract. We study a kind of generalized spherical functions on $Sp(2,\mathbf{R})$ for principal series representations, which are related to archimedean theory of automorphic forms (Siegel wave forms). By solving some differential equations, we obtain explicit formulas for boundary values of these functions in terms of Meijer's G-functions.

Introduction

In this paper we investigate Siegel-Whittaker functions on the real symplectic group of degree two for the principal series representations. Before we discuss our problem, let us recall the general setting in the theory of spherical functions.

Let G be a real reductive Lie group and $\mathfrak{g}$ the Lie algebra of G. Fix a maximal compact subgroup K of G and a closed subgroup R of G. Take an irreducible smooth representation ξ of R and consider C^∞-induction $C^\infty\mathrm{Ind}_R^G(\xi)$. For an irreducible admissible representation (π, H_π) of G with $H_{\pi,K}$ the subspace of K-finite vectors, consider the following problems:

(1) Is $\mathrm{Hom}_{(\mathfrak{g},K)}(H_{\pi,K}, C^\infty\mathrm{Ind}_R^G(\xi))$ finite dimensional? Moreover, we want to know whether the dimension is at most one under some growth conditions (Multiplicity Free Theorem).

(2) For nonzero $\Phi \in \mathrm{Hom}_{(\mathfrak{g},K)}(H_{\pi,K}, C^\infty\mathrm{Ind}_R^G(\xi))$, what is the realization $\mathrm{Im}(\Phi)$ of π in $C^\infty\mathrm{Ind}_R^G(\xi)$? Or, equivalently, give explicit formulas for its elements (as functions on G).

Depending on the choices of R, these problems correspond to various fundamental questions in the local (archimedean) theory of automorphic forms. For example, they are related to the construction of automorphic L-function

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and Fourier expansions of automorphic forms. When R is a maximal unipotent subgroup of G and ξ is a unitary character of R, the space of intertwining operators $\mathrm{Hom}_{(\mathfrak{g},K)}(H_{\pi,K}, C^{\infty}\mathrm{Ind}_R^G(\xi))$ of $(\mathfrak{g},K)$-modules is called the space of Whittaker functionals and has been studied for two or three decades (cf. [16], [13], [14]).

Now we explain our situation in this paper. Let $G = Sp(2,\mathbf{R})$, P_s the Siegel parabolic subgroup of G with the Levi subgroup L_s and the abelian unipotent radical N_s, η a definite character of N_s and $SO(\eta)$ the identity component of the stabilizer of η in L_s. Take $R = SO(\eta) \ltimes N_s$ and ξ as the semidirect product of η and a unitary character χ of $SO(\eta)$. In this case, the intertwining space is usually called the space of the generalized Whittaker functionals. Since there are many possible notions of generalized Whittaker functionals, we call it the space of Siegel-Whittaker functionals.

Consider the restriction of elements in this space to a specific K-type as follows. Let (τ^*, V_{τ^*}) be the multiplicity one K-type of π, where τ^* means the contragradient representation of τ, and let $\iota : V_{\tau^*} \to H_\pi$ be a K-equivariant map. By $\Phi(\iota(v^*))(g) = \langle v^*, \phi_{\pi,\tau}(g)\rangle$, we can define the function $\phi_{\pi,\tau}$ contained in the space $C^{\infty}_{\chi\cdot\eta,\tau}(R\backslash G/K)$ of V_τ-valued smooth functions on G satisfying $f(rgk) = (\chi\cdot\eta)(r)\tau(k)^{-1}f(g)$ for all $(r,g,k) \in R\times G\times K$. Here $\langle\ ,\ \rangle$ is the canonical pairing on $V_\tau \times V_{\tau^*}$. We call the function $\phi_{\pi,\tau}$ the *Siegel-Whittaker function with K-type τ^* for π* and denote by $\mathcal{W}_{\chi\cdot\eta,\,\tau^*}(\pi)$ the space of these functions. By a decomposition $G = RAK$ with $A = \{\mathrm{diag}(a_1,a_2,a_1^{-1},a_2^{-1}) \mid a_1,a_2>0\}$, we see that $\phi_{\pi,\tau}$ is determined by its restriction to A. We call this restriction $\phi_{\pi,\tau}|_A$ the A-radial part of $\phi_{\pi,\tau}$. Our aim is to find an explicit formula for this function.

Miyazaki obtained a system of differential equations satisfied by Siegel-Whittaker functions for the principal series representation in the paper [12] (this part is omitted in published version), whose purpose was to show the multiplicity one theorem and to give explicit formulas in the case when π is the P_J-principal series representation or the large discrete series representation. In the present paper, we study this system and obtain the following theorem.

Multiplicity Free Theorem (Theorems 10.1 and 15.1). *Let $\pi_{\sigma,\nu}$ be the principal series representation of $G = Sp(2,\mathbf{R})$, and assume that the parameter $\nu = (\nu_1,\nu_2)$ of $\pi_{\sigma,\nu}$ satisfies $\nu_1,\nu_2,\nu_1\pm\nu_2 \notin \mathbf{Z}$. We also assume the character η of N_s is 'positive definite.' We define the subspace*

$C^\infty\mathrm{Ind}_R^G(\chi\cdot\eta)^{\mathrm{rap}}$ of $C^\infty\mathrm{Ind}_R^G(\chi\cdot\eta)$, whose elements are rapidly decreasing functions on G. Then

$$\dim_{\mathbf{C}} \mathrm{Hom}_{(\mathfrak{g},K)}(H_{\pi_{\sigma,\nu},K}, C^\infty\mathrm{Ind}_R^G(\chi\cdot\eta)^{\mathrm{rap}}) \leq 1.$$

Moreover the boundary values of the radial parts of the elements in $\mathcal{W}^{\mathrm{rap}}_{\chi\cdot\eta,\tau^*}(\pi)$ (see §10 and §15 for the precise definition) are written explicitly in terms of Meijer's G-function $G^{4,0}_{2,4}$ if

(1) *$\pi_{\sigma,\nu}$ is in the 'even' principal series and $\tau^* = \tau_{(0,0)}$ or $\tau_{(1,1)}$,*

(2) *$\pi_{\sigma,\nu}$ is in the 'odd' principal series and $\tau^* = \tau_{(0,-1)}$.*

Here $\tau_{(\lambda_1,\lambda_2)}$ is the irreducible representation of K with highest weight (λ_1,λ_2) and the definitions of 'even' and 'odd' are given in §5.

This result seems to cover only particular K-types, but it includes explicit formulas for all minimal K-types τ^* of π.

As far as we know, the previous results related to our investigations are only due to Niwa and Hori. Niwa ([15]) obtained the integral representation of the Siegel-Whittaker function with trivial K-type, i.e, π is even and $\tau^* = \tau_{(0,0)}$, and by using this result, Hori ([10]) computed the gamma factor of L-function of Siegel wave forms of degree two.

Let us outline the contents of this paper. From §1 to §6, we recall the basic facts of representation theory and review the system of differential equations following [12]. We also recall the basic properties of Meijer's G-functions. From §7 to §11, we treat the even principal series. We calculate characteristic indices at the singularity of the system (§8), and find the explicit formulas for the holomorphic solutions along the singular divisor and deduce the multiplicity free theorem (§9, §10). In §11 we see relations between the results of [15], [10] and ours. We give an integral representation of a Meijer's G-function and a simpler proof for the computation of the gamma factor of L-function. From §12 to §15, we treat the odd principal series in a way similar to the even case.

We want to comment on some other related works. Hirano ([8], [9]) investigated the case where R is the Jacobi subgroup of the Jacobi parabolic subgroup of $Sp(2,\mathbf{R})$ and π are P_J-principal series and the large discrete series, and obtained explicit formulas in terms of Meijer's G-function $G^{3,0}_{2,3}$.

These results are related to Fourier-Jacobi expansions of non-holmorphic automorphic forms. In the case $G = SU(2,2)$, which has the same root system of type C_2, Hayata ([6], [7]) studied the Whittaker functions and Gon ([5]) studied the generalized Whittaker functions with respect to Siegel parabolic subgroup (i.e., Siegel-Whittaker functions).

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§1. Definition of Siegel-Whittaker Functions

We first define the space of algebraic Siegel-Whittaker functionals for an irreducible admissible representation (π, H_π) of the real symplectic group of degree two. This is the space of intertwining operators from π to an induced representation, which is called the reduced generalized Gelfand-Graev representation by Yamashita ([20]).

(1.1) Definition of the space of algebraic Siegel-Whittaker functionals

Let G be the real symplectic group $Sp(2, \mathbf{R})$,

$$G = Sp(2, \mathbf{R}) = \left\{ g \in SL(4, \mathbf{R}) \,\middle|\, {}^t g J_2 g = J_2 = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} \right\}.$$

Here we denote by 1_2 the unit matrix of degree two. The Siegel parabolic subgroup P_s of G is a maximal parabolic subgroup corresponding to the short root with abelian unipotent radical N_s. A Levi decomposition is given by $P_s = L_s \ltimes N_s$, where $L_s = \left\{ \left(\begin{smallmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{smallmatrix} \right) \mid A \in GL(2, \mathbf{R}) \right\}$ and $N_s = \left\{ n(T) = \left(\begin{smallmatrix} 1_2 & T \\ 0 & 1_2 \end{smallmatrix} \right) \mid {}^t T = T \in M_2(\mathbf{R}) \right\}$. Fix a unitary character η of N_s defined by

$$\eta(n(T)) = \exp\bigl(2\pi\sqrt{-1}\,\mathrm{tr}(H_\eta T)\bigr)$$

with $H_\eta = \left(\begin{smallmatrix} h_1 & h_3/2 \\ h_3/2 & h_2 \end{smallmatrix} \right) \in M_2(\mathbf{R})$. In this paper we assume that H_η is nondegenerate, that is, $h_1 h_2 - h_3^2/4 \neq 0$.

Consider the action of L_s on N_s by conjugation, and also the induced action of L_s on the character group $\hat{N}_s$. Let $SO(\eta)$ be the identity component of the stabilizer subgroup of the character η. Then

$$SO(\eta) = \left\{ \left(\begin{array}{cc} A & 0 \\ 0 & {}^tA^{-1} \end{array} \right) \middle| A \in GL(2, \mathbf{R}),\ {}^tAH_\eta A = H_\eta \right\}^\circ$$

and it is isomorphic to $SO(2)$ for definite H_η and to $SO_o(1,1)$ for indefinite H_η. In this paper we only treat the case of definite H_η. Take a unitary character χ of $SO(\eta) \cong SO(2)$ and define the subgroup R of P_s by $R = SO(\eta) \ltimes N_s$. Then we can construct a well-defined character $\chi\cdot\eta$ of R by $(\chi\cdot\eta)(r) = \chi(l)\eta(n)$ for $r = (l, n) \in SO(\eta) \ltimes N_s$. We consider the C^∞-induced representation from R to G,

$$C^\infty\mathrm{Ind}_R^G(\chi\cdot\eta) = \{f : G \to \mathbf{C},\ C^\infty \mid f(rg) = (\chi\cdot\eta)(r)f(g),\ \forall r \in R,\ \forall g \in G\}$$

with the action of G via right translation. This is called the reduced generalized Gelfand-Graev representation in [20].

For an irreducible admissible representation (π, H_π) of G, take the subspace $H_{\pi,K}$ of K-finite vectors in H_π where K is a maximal compact subgroup of G. Then the Lie algebra $\mathfrak{g}$ of G acts on $H_{\pi,K}$.

Definition 1.1. The space $W_{\chi\cdot\eta}(\pi) := \mathrm{Hom}_{(\mathfrak{g},K)}(H_{\pi,K}, C^\infty\mathrm{Ind}_R^G(\chi\cdot\eta))$ of intertwining operators of $(\mathfrak{g}, K)$-modules is called the space of *algebraic Siegel-Whittaker functionals for the representation* (π, H_π) *of* G.

(1.2) Siegel-Whittaker functions with a fixed K-type

For $\Phi \in W_{\chi\cdot\eta}(\pi)$ and $v \in H_{\pi,K}$ we want to study the $\mathbf{C}$-valued functions $\Phi(v) \in C^\infty\mathrm{Ind}_R^G(\chi\cdot\eta)$ on G. More precisely, we consider those $\Phi(v)$ with v belonging to a specific K-type in π. Let (τ^*, V_{τ^*}) be a multiplicity one K-type of (π, H_π), where τ^* is the contragradient representation of τ and $\iota : V_{\tau^*} \to H_\pi$ a K-equivariant map. By $\Phi(\iota(v^*))(g) = \langle v^*, \phi_{\pi,\tau}(g)\rangle$, we can define the function $\phi_{\pi,\tau}$ contained in the space

$$C^\infty_{\chi\cdot\eta,\tau}(R\backslash G/K) = \{f : G \to V_\tau,\ C^\infty \mid f(rgk) = (\chi\cdot\eta)(r)\tau(k)^{-1}f(g),\ \forall (r, g, k) \in R \times G \times K\}$$

Here $\langle\ ,\ \rangle$ is the canonical pairing on $V_\tau \times V_{\tau^*}$.

Definition 1.2. We call the above function $\phi_{\pi,\tau}$ the *Siegel-Whittaker function with K-type (τ^*, V_{τ^*}) for (π, H_π)* and denote by $\mathcal{W}_{\chi\cdot\eta,\tau^*}(\pi)$ the space of Siegel-Whittaker functions with K-type τ^* for π.

Considering a decomposition $G = RAK$ with $A = \{\mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \mid a_1, a_2 > 0\}$, we see that $\phi_{\pi,\tau}$ is determined by its restriction to A. We call this restriction $\phi_{\pi,\tau}|_A$ the *A-radial part* of $\phi_{\pi,\tau}$. Our aim is to give an explicit formula for this function.

Remark 1.3. We give some remarks on the unitary character χ of $SO(\eta) \cong SO(2)$. Let $d\chi : \mathfrak{so}(\eta) \to \sqrt{-1}\mathbf{R}$ be its differential and $\phi : \mathfrak{so}(\eta) \cong \mathfrak{so}(2)$ be an isomorphism of Lie algebras given by $X \longmapsto HXH^{-1}$ with $H = \mathrm{diag}(\sqrt{h_1}, \sqrt{h_2})$. Let χ_0 be the character of $SO(2)$ defined by $\chi_0(r_\theta) = e^{\sqrt{-1}m_0\theta}$, where r_θ is the rotation with angle θ and $m_0 \in \mathbf{Z}$, and $d\chi_0$ differential of χ_0. Then we have $d\chi_0(Y) = \sqrt{-1}m_0$ with $Y = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and define χ to satisfy $d\chi = d\chi_0 \circ \phi$.

Our notation is related to that of [12] as follows. We may assume $h_1, h_2 > 0$ and $h_3 = 0$. Since a generator of $\mathfrak{so}(\eta)$ is taken as $Y_\eta = H_\eta^{-1}Y$ in [12], we have $\chi(Y_\eta)\sqrt{h_1h_2} = \sqrt{-1}m_0$.

§2. Lie Groups and Algebras

(2.1) Maximal compact subgroup

A maximal compact subgroup K of $G = Sp(2, \mathbf{R})$ is given by

$$K = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in G \,\middle|\, A,\, B \in M_2(\mathbf{R}) \right\}$$

and is isomorphic to the unitary group $U(2)$ via the homomorphism

$$K \ni \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \longmapsto A + \sqrt{-1}B \in U(2).$$

(2.2) Lie algebras

The Lie algebra $\mathfrak{g}$ of G is given by $\mathfrak{g} = \{X \in M_4(\mathbf{R}) \mid JX + {}^tXJ = 0\}$. Let θ be the Cartan involution of $\mathfrak{g}$ defined by $\theta(X) = -{}^tX$ for $X \in \mathfrak{g}$. Then the subspaces

$$\begin{aligned}
\mathfrak{k} &= \{X \in \mathfrak{g} \mid \theta(X) = X\} \\
&= \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \,\middle|\, A, B \in M_2(\mathbf{R}),\ {}^tA = -A,\ {}^tB = B \right\}, \\
\mathfrak{p} &= \{X \in \mathfrak{g} \,|\, \theta(X) = -X\} \\
&= \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \,\middle|\, A, B \in M_2(\mathbf{R}),\ {}^tA = A,\ {}^tB = B \right\}
\end{aligned}$$

give a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Notice that $\mathfrak{k}$ is the Lie algebra of K and is isomorphic to the unitary Lie algebra $\mathfrak{u}(2)$ via the linear map

$$\mathfrak{k} \ni \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \longmapsto A + \sqrt{-1}B \in \mathfrak{u}(2).$$

(2.3) Root system of $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$

Let us consider the complexification $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$ of $\mathfrak{g}$ and $\mathfrak{u}(2)_{\mathbf{C}} = \mathfrak{u}(2) \otimes_{\mathbf{R}} \mathbf{C}$. Take a basis of $\mathfrak{u}(2)_{\mathbf{C}}$ by

$$\begin{aligned}
Z &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
Y &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Y' = \sqrt{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\end{aligned}$$

Note that $\{H',\, X = (Y - \sqrt{-1}\bar{Y})/2,\, \bar{X} = (-Y - \sqrt{-1}\bar{Y})/2\}$ is an $\mathfrak{sl}(2)$-triple, that is, $[H', X] = 2X,\ [H', \bar{X}] = -2\bar{X},\ [X, \bar{X}] = H'$. Via the isomorphism $\mathfrak{k}_{\mathbf{C}} \simeq \mathfrak{u}(2)_{\mathbf{C}}$, the preimage of the above basis of $\mathfrak{u}(2)_{\mathbf{C}}$ is given by

$$\begin{aligned}
&Z = -\sqrt{-1} \left(\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline -1 & & & \\ & -1 & & \end{array} \right), \quad H' = -\sqrt{-1} \left(\begin{array}{cc|cc} & & 1 & \\ & & & -1 \\ \hline -1 & & & \\ & 1 & & \end{array} \right), \\
&Y = \left(\begin{array}{cc|cc} & 1 & & \\ -1 & & & \\ \hline & & & 1 \\ & & -1 & \end{array} \right), \qquad Y' = \left(\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & -1 & & \\ -1 & & & \end{array} \right).
\end{aligned}$$

Set $T_1 = \frac{1}{2}\sqrt{-1}(Z + H')$, $T_2 = \frac{1}{2}\sqrt{-1}(Z - H')$ and fix a compact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ by $\mathfrak{h} = \mathbf{R}T_1 \oplus \mathbf{R}T_2$. For linear form $\beta : \mathfrak{h}_{\mathbf{C}} \to \mathbf{C}$ we write $\beta(T_i) = -\sqrt{-1}\beta_i \in \mathbf{C}$ and put $\mathfrak{g}_\beta = \{X \in \mathfrak{g}_{\mathbf{C}} \mid [H, X] = \beta(H)X,\ \forall X \in \mathfrak{h}_{\mathbf{C}}\}$. Then the set of roots Δ of $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ is given by $\Delta = \{\beta = (\beta_1, \beta_2) \mid \mathfrak{g}_\beta \neq 0, \beta \neq (0,0)\} = \{\pm(2,0), \pm(0,2), \pm(1,1), \pm(1,-1)\}$. Fix the positive system $\Delta^+ = \{(2,0),\ (0,2),\ (1,1),\ (1,-1)\}$ and choose a root vector X_β in $\mathfrak{g}_\beta$ as follows:

$$X_{(2,0)} = \left(\begin{array}{cc|cc} 1 & & \sqrt{-1} & \\ & 0 & & 0 \\ \hline \sqrt{-1} & & -1 & \\ & 0 & & 0 \end{array}\right),$$

$$X_{(1,1)} = \left(\begin{array}{cc|cc} & 1 & & \sqrt{-1} \\ 1 & & \sqrt{-1} & \\ \hline & \sqrt{-1} & & -1 \\ \sqrt{-1} & & -1 & \end{array}\right),$$

$$X_{(0,2)} = \left(\begin{array}{cc|cc} 0 & & 0 & \\ & 1 & & \sqrt{-1} \\ \hline 0 & & 0 & \\ & \sqrt{-1} & & -1 \end{array}\right),$$

$$X_{(1,-1)} = \left(\begin{array}{cc|cc} & 1 & & -\sqrt{-1} \\ -1 & & -\sqrt{-1} & \\ \hline & \sqrt{-1} & & 1 \\ \sqrt{-1} & & -1 & \end{array}\right),$$

and $X_{-\beta} = \bar{X}_\beta$ for $\beta \in \Delta^+$. Then $\Delta_c^+ = \{(1,-1)\}$ and $\Delta_n^+ = \{(2,0), (0,2), (1,1)\}$ are the sets of compact and non-compact positive roots, respectively. We have a decomposition $\mathfrak{p}_{\mathbf{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ with $\mathfrak{p}_\pm = \sum_{\beta \in \Delta_n^\pm} \mathfrak{g}_\beta$. If we put $\|\beta\| = \sqrt{\beta_1^2 + \beta_2^2}$ then the set

$$\left\{c\|\beta\|(X_\beta + X_{-\beta}),\ c\sqrt{-1}\|\beta\|(X_\beta - X_{-\beta})\ \ (\beta \in \Delta_n^+)\right\}$$

forms an orthonormal basis of $\mathfrak{p}$ with respect to the Killing form for some constant c.

(2.4) Root system of $(\mathfrak{g}, \mathfrak{a})$

Put $H_1 = \operatorname{diag}(1, 0, -1, 0)$, $H_2 = \operatorname{diag}(0, 1, 0, -1)$ and $\mathfrak{a} = \mathbf{R}H_1 \oplus \mathbf{R}H_2$. Then $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p}$. Define linear forms e_i on

$\mathfrak{a}$ by $e_i(a_1H_1 + a_2H_2) = a_i$ $(i = 1, 2)$ and put $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X,\ \forall H \in \mathfrak{a}\}$. Then the restricted root system $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ is given by $\Sigma = \{\pm 2e_1, \pm 2e_2, \pm e_1 \pm e_2\}$, and we fix a positive system Σ^+ by $\Sigma^+ = \{2e_1, 2e_2, e_1 \pm e_2\}$. We choose a root vector $E_\alpha \in \mathfrak{g}_\alpha$ as

$$E_{2e_1} = E_{1,3},\ E_{2e_2} = E_{2,4},\ E_{e_1+e_2} = E_{1,4} + E_{2,3},\ E_{e_1-e_2} = E_{1,2} - E_{4,3},$$

and $E_{-\alpha} = -{}^tE_\alpha$ for $\alpha \in \Sigma^+$. Here $E_{i,j}$ is a matrix with 1 in the (i,j)-entry and 0 elsewhere. If we put $\mathfrak{n} = \sum_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$, we have an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

§3. Representations of the Maximal Compact Subgroup

In this section we recall some basic facts about the representations of the maximal compact subgroup K of G. Since $K \cong U(2)$, its complexification $K_{\mathbf{C}}$ is isomorphic to $GL(2, \mathbf{C})$. The set $\{\lambda = (\lambda_1, \lambda_2) \in \mathbf{Z} \oplus \mathbf{Z} \mid \lambda_1 \geq \lambda_2\}$ parametrizes the set of irreducible representations of K. For each dominant weight λ, put $d_\lambda = \lambda_1 - \lambda_2$. Then the dimension of the representation space V_λ associated to λ is $d_\lambda + 1$. We can choose a basis $\{v_j^\lambda \mid 0 \leq j \leq d_\lambda\}$ for V_λ so that the associated representation τ_λ is given as

$$\begin{cases} \tau_\lambda(Z)v_j^\lambda = (\lambda_1 + \lambda_2)v_j^\lambda, & \tau_\lambda(H')v_j^\lambda = (2j - d_\lambda)v_j^\lambda, \\ \tau_\lambda(X)v_j^\lambda = (j+1)v_{j+1}^\lambda, & \tau_\lambda(\bar{X})v_j^\lambda = (d_\lambda + 1 - j)v_{j-1}^\lambda. \end{cases}$$

Then we have isomorphisms $\mathfrak{p}_+ \cong V_{(2,0)}$ and $\mathfrak{p}_- \cong V_{(0,-2)}$, where K acts on $\mathfrak{p}_\pm$ via the adjoint representation. The correspondences of the basis are

$$(X_{(0,2)}, X_{(1,1)}, X_{(2,0)}) \longmapsto (v_0^{(2,0)}, v_1^{(2,0)}, v_2^{(2,0)}),$$
$$(X_{(-2,0)}, X_{(-1,-1)}, X_{(0,-2)}) \longmapsto (v_0^{(0,-2)}, -v_1^{(0,-2)}, v_2^{(0,-2)}).$$

Let us consider the tensor product $V_\lambda \otimes \mathfrak{p}_+$. We have the decomposition $V_\lambda \otimes \mathfrak{p}_+ \cong \sum_{\beta\in\Delta_n^+} V_{\lambda+\beta}$. Let P^{up}, P^{even} and P^{down} be the projectors from $V_\lambda \otimes \mathfrak{p}_+$ into the irreducible factors $V_{(\lambda_1+2,\lambda_2)}$, $V_{(\lambda_1+1,\lambda_2+1)}$ and $V_{(\lambda_1,\lambda_2+2)}$, respectively. Then

Lemma 3.1.

(1)
$$P^{up}(v_j^\lambda \otimes X_{(2,0)}) = \tfrac{1}{2}(j+1)(j+2)v_{j+2}^{\lambda+(2,0)},$$
$$P^{up}(v_j^\lambda \otimes X_{(1,1)}) = (j+1)(d_\lambda+1-j)v_{j+1}^{\lambda+(2,0)},$$
$$P^{up}(v_j^\lambda \otimes X_{(0,2)}) = \tfrac{1}{2}(d_\lambda+1-j)(d_\lambda+2-j)v_j^{\lambda+(2,0)},$$

for $0 \le j \le d_\lambda$.

(2)
$$P^{even}(v_j^\lambda \otimes X_{(2,0)}) = (j+1)v_{j+1}^{\lambda+(1,1)} \qquad (0 \le j \le d_\lambda - 1),$$
$$P^{even}(v_j^\lambda \otimes X_{(1,1)}) = (d_\lambda - 2j)v_j^{\lambda+(1,1)} \qquad (0 \le j \le d_\lambda),$$
$$P^{even}(v_j^\lambda \otimes X_{(0,2)}) = -(d_\lambda+1-j)v_{j-1}^{\lambda+(1,1)} \qquad (1 \le j \le d_\lambda),$$

and the others are 0.

(3)
$$P^{down}(v_j^\lambda \otimes X_{(2,0)}) = v_j^{\lambda+(0,2)} \qquad (0 \le j \le d_\lambda - 2),$$
$$P^{down}(v_j^\lambda \otimes X_{(1,1)}) = -2v_{j-1}^{\lambda+(0,2)} \qquad (1 \le j \le d_\lambda - 1),$$
$$P^{down}(v_j^\lambda \otimes X_{(0,2)}) = v_{j-2}^{\lambda+(0,2)} \qquad (2 \le j \le d_\lambda),$$

and the others are 0.

§4. Gradient Operators, Shift Operators and Casimir Operator

In this section we define the gradient operators, the shift operators and the Casimir operator which characterize the Siegel-Whittaker functions, and give the radial part of the gradient operators and the Casimir operator. We denote by $C_\tau^\infty(G/K)$ the space of smooth functions $f : G \to V_\tau$ satisfying $f(gk) = \tau(k)^{-1}f(g)$ for $(k,g) \in K \times G$.

(4.1) Gradient operators and shift operators

Definition 4.1. Let (τ, V_τ) be a finite dimensional irreducible representation of K, $\{X_i\,(i \in I)\}$ be an orthonormal basis of $\mathfrak{p}$ with respect to the Killing form of $\mathfrak{g}$ and $\mathrm{Ad}_{\mathfrak{p}_{\mathbf{C}}}$ be the adjoint representation of K on $\mathfrak{p}_{\mathbf{C}}$. We define the *gradient operator* $\nabla : C_\tau^\infty(G/K) \to C_{\tau\otimes\mathrm{Ad}_{\mathfrak{p}_{\mathbf{C}}}}^\infty(G/K)$ as $\nabla\phi = \sum_{i\in I} R_{X_i}\phi(\cdot) \otimes X_i$ for $\phi \in C_\tau^\infty(G/K)$. Here R_X is the right differential, $R_X\phi(g) = \frac{d}{dt}\phi(g\exp tX)\big|_{t=0}$.

We remark that the gradient operator is independent of the choice of the orthonormal basis.

DEFINITION 4.2. Under the same setting as above, let τ' be an irreducible component of $\tau \otimes \mathrm{Ad}_{\mathfrak{p}_{\mathbf{C}}}$ and $P_{\tau'}$ be the projection to τ'. We call the composition of operators $P_{\tau'} \circ \nabla$ a *shift operator*.

The shift operators move the K-type parameter of the Siegel-Whittaker functions. If we apply the (up and down) shift operators a number of times to Siegel-Whittaker function with multiplicity one K-type such that the result lies in the original K-type, then the composed operator acts on this function as multiplication by a scalar, whose value is calculated in [13].

(4.2) Radial parts of the gradient operators

We first define the radial part of an operator. Let $C^\infty(A, V_\tau) = \{f : A \to V_\tau, C^\infty\}$ and $\mathrm{res}A_\tau : C^\infty_{\chi\cdot\eta,\tau}(R\backslash G/K) \to C^\infty(A, V_\tau)$ be the restriction map. For K-modules $(\tau_1, V_{\tau_1}), (\tau_2, V_{\tau_2})$ and a linear map $D : C^\infty_{\chi\cdot\eta,\tau_1}(R\backslash G/K) \to C^\infty_{\chi\cdot\eta,\tau_2}(R\backslash G/K)$, we have a unique linear map $R(D) : C^\infty(A, V_\tau) \to C^\infty(A, V_\tau)$ satisfying $R(D) \circ \mathrm{res}A_{\tau_1} = \mathrm{res}A_{\tau_2} \circ D$. We call this linear map $R(D)$ the *A-radial part* of D.

Let us take an orthonormal basis of $\mathfrak{p}$ as $\{c\|\beta\|(X_\beta + X_{-\beta}), c\sqrt{-1}\|\beta\|(X_\beta - X_{-\beta})\ (\beta \in \Delta_n^+)\}$ for some nonnegative constant c (§2). By using this basis, the gradient operator ∇ is described as

$$\nabla F = 2c^2 \textstyle\sum_{\beta\in\Sigma_n^+} \|\beta\|^2 R_{X_{-\beta}} F \otimes X_\beta + 2c^2 \textstyle\sum_{\beta\in\Sigma_n^+} \|\beta\|^2 R_{X_\beta} F \otimes X_{-\beta},$$

for $F \in C^\infty_\tau(G/K)$. Corresponding to the decomposition $\mathfrak{p}_{\mathbf{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$, we can decompose the gradient operator as $\nabla = \nabla^+ \oplus \nabla^-$. Here $\nabla^\pm : C^\infty_\tau(G/K) \to C^\infty_{\tau\otimes\mathrm{Ad}_{\mathfrak{p}_\pm}}(G/K)$ are defined by $\nabla^+ F = \frac{1}{4}\sum_{\beta\in\Sigma_n^+} \|\beta\|^2 R_{X_{-\beta}} F \otimes X_\beta$, $\nabla^- F = \frac{1}{4}\sum_{\beta\in\Sigma_n^+} \|\beta\|^2 R_{X_\beta} F \otimes X_{-\beta}$. By using the Iwasawa decomposition, we obtain

PROPOSITION 4.3 ([12, Proposition 5.3]). *Assume that H_η is nondegenerate and $h_3 = 0$. Then the A-radial parts $R(\nabla^\pm_{\chi\cdot\eta,\tau_\lambda}) : C^\infty(A, V_{\tau_\lambda}) \to$*

$C^\infty(A, V_{\tau_\lambda} \otimes \mathfrak{p}_\pm)$ *of the gradient operators are given as*

(1) $R(\nabla^+_{\chi\cdot\eta,\tau_\lambda})f(a)$

$$= \big[\partial_1 + 4\pi h_1 a_1^2 + (\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_+})(H_1') + 2\tfrac{h_2 a_2^2}{\mathcal{L}} - 2\big](f(a) \otimes X_{(2,0)})$$
$$+ \big[I - \tfrac{h_2 a_2^2}{\mathcal{L}}(\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_+})(X) + \tfrac{h_1 a_1^2}{\mathcal{L}}(\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_+})(\bar{X})\big](f(a) \otimes X_{(1,1)})$$
$$+ \big[\partial_2 + 4\pi h_2 a_2^2 + (\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_+})(H_2') - 2\tfrac{h_1 a_1^2}{\mathcal{L}} - 2\big](f(a) \otimes X_{(0,2)}),$$

(2) $R(\nabla^-_{\chi\cdot\eta,\tau_\lambda})f(a)$

$$= \big[\partial_1 - 4\pi h_1 a_1^2 + (\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_-})(H_1') + 2\tfrac{h_2 a_2^2}{\mathcal{L}} - 2\big](f(a) \otimes X_{(-2,0)})$$
$$+ \big[I - \tfrac{h_1 a_1^2}{\mathcal{L}}(\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_-})(X) + \tfrac{h_2 a_2^2}{\mathcal{L}}(\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_-})(\bar{X})\big](f(a) \otimes X_{(-1,-1)})$$
$$+ \big[\partial_2 - 4\pi h_2 a_2^2 + (\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_-})(H_2') - 2\tfrac{h_1 a_1^2}{\mathcal{L}} - 2\big](f(a) \otimes X_{(0,-2)}).$$

Here we use the symbols $\partial_i = a_i \frac{\partial}{\partial a_i}$ $(i = 1, 2), \mathcal{L} = h_1 a_1^2 - h_2 a_2^2$ and $I = -\chi(Y_\eta)\frac{\det H_\eta a_1 a_2}{\mathcal{L}}$.

(4.3) Casimir operator

The Casimir operator L in the center $Z(\mathfrak{g}_{\mathbf{C}})$ of the universal enveloping algebra of $\mathfrak{g}_{\mathbf{C}}$ is given as

$$\begin{aligned} L = {} & H_1^2 + H_2^2 - 4H_1 - 2H_2 \\ & + 2E_{e_1-e_2} \cdot E_{-e_1+e_2} + 4E_{2e_1} \cdot E_{-2e_1} \\ & + 2E_{e_1+e_2} \cdot E_{-e_1-e_2} + 4E_{2e_2} \cdot E_{-2e_2}, \end{aligned}$$

up to scalar ([13, §7]).

Proposition 4.4 ([12, Proposition 5.6]). *Assume that H_η is nondegenerate and $h_3 = 0$. Then the A-radial part $R(L_{\chi\cdot\eta,\tau_\lambda})$ of the Casimir operator L is given as*

$$\begin{aligned} R(L_{\chi\cdot\eta,\tau_\lambda}) = {} & \partial_1^2 + \partial_2^2 + 2(\tfrac{h_2 a_2^2}{\mathcal{L}} - 1)\partial_1 - 2(\tfrac{h_1 a_1^2}{\mathcal{L}} + 1)\partial_2 - 16\pi^2(h_1^2 a_1^4 + h_2^2 a_2^4) \\ & + 2S^2 - 8\pi h_1 a_1^2 \tau_\lambda(H_1') - 8\pi h_2 a_2^2 \tau_\lambda(H_2') \\ & + 2S\tfrac{h_1 a_1^2 + h_2 a_2^2}{\mathcal{L}}\tau_\lambda(W) + 2\tfrac{h_1 a_1^2 h_2 a_2^2}{\mathcal{L}^2}\{\tau_\lambda(W)\}^2 \end{aligned}$$

with $W = X - \bar{X} \in \mathfrak{k}_{\mathbf{C}}$, $S = \chi(Y_\eta)\frac{h_1 a_1 h_2 a_2}{\mathcal{L}}$.

§5. Principal Series Representations and a System of Differential Equations

In this section we recall the principal series representations of G and review the system of differential equations of Siegel-Whittaker functions obtained by Miyazaki.

(5.1) Principal series representation and its K-types

Let P_0 be the standard minimal parabolic subgroup of G. Its Langlands decomposition is $P_0 = M_0A_0N_0$, where $M_0 = \{\mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2) \mid \varepsilon_i \in \{\pm 1\}\ (i = 1, 2)\}$, $A_0 = A = \{\mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \mid a_i > 0\ (i = 1, 2)\}$ and

$$N_0 = \left\{ \left(\begin{array}{cc|cc} 1 & n_0 & & \\ 0 & 1 & & \\ \hline & & 1 & 0 \\ & & -n_0 & 1 \end{array} \right) \cdot \left(\begin{array}{cc|cc} 1 & 0 & n_1 & n_2 \\ 0 & 1 & n_2 & n_3 \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array} \right) \middle|\, n_0, n_1, n_2, n_3 \in \mathbf{R} \right\}.$$

Take a character $\exp\nu$ of A_0 given by $\exp\nu(a) = \exp(\nu_1(\log a_1) + \nu_2(\log a_2))$ for $a \in A_0$, with $\nu = (\nu_1, \nu_2) \in \mathfrak{a}_{\mathbf{C}}^*$ and a character σ of M_0, $\sigma : M_0 \to \{\pm 1\}$.

Definition 5.1. We call the induced representation $\pi_{\sigma,\nu} := \mathrm{Ind}_{P_0}^{G}(\sigma \otimes \exp(\nu + \rho_0) \otimes 1_{N_0})$ the *principal series representation* of G, where ρ_0 is the half-sum of the restricted positive roots fixed in §2, that is, $\rho_0 = \frac{1}{2}\{(2, 0) + (0, 2) + (1, 1) + (1, -1)\} = (2, 1)$.

Now we see the K-type decomposition of the principal series, which depends only on σ. Put $\gamma_{2e_1} = \mathrm{diag}(-1, 1, -1, 1)$ and $\gamma_{2e_2} = \mathrm{diag}(1, -1, 1, -1)$. By the Frobenius reciprocity, we have

Proposition 5.2 ([13, Proposition 3.2]). *The multiplicity of $\tau_{(\lambda_1,\lambda_2)} \in \hat{K}$ in the restriction of $\pi_{\sigma,\nu}$ to K is the cardinality of the set*

$$\{m \in \mathbf{Z} \mid \lambda_2 \le m \le \lambda_1,\ (-1)^m = \sigma(\gamma_{2e_1}),\ (-1)^{\lambda_1+\lambda_2-m} = \sigma(\gamma_{2e_2})\}.$$

In particular, the following $\tau_{(\lambda_1,\lambda_2)}$ occurs in $\pi_{\sigma,\nu}|_K$ with multiplicity one.

(1) (i) $\tau_{(\lambda,\lambda)}$ *with* $\lambda \in 2\mathbf{Z}$ *for* $\sigma(\gamma_{2e_1}) = \sigma(\gamma_{2e_2}) = 1$,

(ii) $\tau_{(\lambda,\lambda)}$ *with* $\lambda \in 2\mathbf{Z} + 1$ *for* $\sigma(\gamma_{2e_1}) = \sigma(\gamma_{2e_2}) = -1$,

(2) $\tau_{(\lambda+1,\lambda)}$ *with* $\lambda \in \mathbf{Z}$ *for* $\sigma(\gamma_{2e_1}) = -\sigma(\gamma_{2e_2})$.

We say that $\pi_{\sigma,\nu}$ is *even* if $\sigma(\gamma_{2e_1}) = \sigma(\gamma_{2e_2})$ and *odd* if $\sigma(\gamma_{2e_1}) = -\sigma(\gamma_{2e_2})$.

Lemma 5.3. *If* $\mathcal{W}_{\chi\cdot\eta,\tau^*}(\pi_{\sigma,\nu}) \neq \{0\}$ *then the parameter* m_0 *of the character* χ *of* $SO(\eta)$ *is even if* $\pi_{\sigma,\nu}$ *is even and odd if* $\pi_{\sigma,\nu}$ *is odd.*

Proof. Let $\phi_{\pi_{\sigma,\nu},\tau_\lambda} \in C^\infty_{\chi\cdot\eta,\tau_\lambda}(R\backslash G/K)$ be a Siegel-Whittaker function for $\pi_{\sigma,\nu}$ with K-type τ^*_λ. Since the centralizer $Z_K(A)$ of A in K is M_0, $SO(\eta) \cap Z_K(A) = \{\pm 1_4\}$, where 1_4 is the unit matrix of size four. If we denote $m = -1_4$ then

$$\phi_{\pi_{\sigma,\nu},\tau_\lambda}(a) = \phi_{\pi_{\sigma,\nu},\tau_\lambda}(mam^{-1}) = (\chi\cdot\eta)(m)\tau_\lambda(m)\phi_{\pi_{\sigma,\nu},\tau_\lambda}(a).$$

By using $\tau_{(\lambda,\lambda)}(m) = \det(-1_2)^\lambda$, $\tau_{(\lambda+1,\lambda)}(m) = \det(-1_2)^\lambda \otimes \mathrm{Sym}^1(-1_2) = -1_2$ and $(\chi\cdot\eta)(m) = \chi(m) = \exp(\pi\sqrt{-1}m_0)$, we have the lemma. $\square$

(5.2) System of differential equations (even case)

Proposition 5.4 (Miyazaki). *Assume that* h_1, $h_2 > 0$ *and* $h_3 = 0$ *for* H_η. *Let* $\phi(a_1,a_2) = b_0(a_1,a_2)v_0$ *be a Siegel-Whittaker function with* K-*type* $\tau_{(\lambda,\lambda)}$ *for the even principal series representation. Then the* A-*radial coefficient* $b_0(a_1,a_2)$ *satisfies*

$$\begin{aligned}(1)\ &\Big\{\partial_1\partial_2 - \tfrac{h_2a_2^2}{\mathcal{L}}\partial_1 + \tfrac{h_1a_1^2}{\mathcal{L}}\partial_2 - 4\pi h_2a_2^2\partial_1 - 4\pi h_1a_1^2\partial_2 + 16\pi^2h_1a_1^2h_2a_2^2 - S^2\\ &\qquad + (\lambda-3)(\partial_1+\partial_2) - 4\pi(\lambda-3)(h_1a_1^2+h_2a_2^2) + (\lambda-2)(\lambda-3)\Big\}\\ &\Big\{\partial_1\partial_2 - \tfrac{h_2a_2^2}{\mathcal{L}}\partial_1 + \tfrac{h_1a_1^2}{\mathcal{L}}\partial_2 + 4\pi h_2a_2^2\partial_1 + 4\pi h_1a_1^2\partial_2 + 16\pi^2h_1a_1^2h_2a_2^2 - S^2\\ &\qquad - (\lambda+1)(\partial_1+\partial_2) - 4\pi(\lambda+1)(h_1a_1^2+h_2a_2^2) + \lambda(\lambda+1)\Big\}\, b_0(a_1,a_2)\\ &= \{\nu_1^2 - (\lambda-1)^2\}\{\nu_2^2 - (\lambda-1)^2\}\, b_0(a_1,a_2),\end{aligned}$$

$$\begin{aligned}(2)\ &\Big\{\partial_1^2 + \partial_2^2 - 2(\partial_1+\partial_2) + 2\tfrac{h_2a_2^2}{\mathcal{L}}\partial_1 - 2\tfrac{h_1a_1^2}{\mathcal{L}}\partial_2 - 16\pi^2(h_1^2a_1^4 + h_2^2a_2^4)\\ &\qquad + 8\pi\lambda(h_1a_1^2+h_2a_2^2) + 2S^2\Big\}\, b_0(a_1,a_2) = (\nu_1^2+\nu_2^2-5)\, b_0(a_1,a_2).\end{aligned}$$

Proof. (1) is obtained from the action of the composition of four shift operators, $P^{down} \circ R(\nabla^{-}_{\chi\cdot\eta,\,\tau_{-\lambda+2,-\lambda}}) \circ P^{up} \circ R(\nabla^{-}_{\chi\cdot\eta,\,\tau_{-\lambda+2,-\lambda+2}}) \circ P^{down} \circ R(\nabla^{+}_{\chi\cdot\eta,\,\tau_{-\lambda+2,-\lambda}}) \circ P^{up} \circ R(\nabla^{+}_{\chi\cdot\eta,\,\tau_{-\lambda,-\lambda}})$, which acts by the scalar multiplication with value $4\{\nu_1^2 - (-\lambda+1)^2\}\{\nu_2^2 - (-\lambda+1)^2\}$, while (2) is obtained from the action of the Casimir operator. □

(5.3) System of differential equations (odd case)

Proposition 5.5 (Miyazaki). *Assume that h_1, $h_2 > 0$ and $h_3 = 0$ for H_η. Let $\phi(a_1, a_2) = b_0(a_1, a_2)v_0 + b_1(a_1, a_2)v_1$ be a Siegel-Whittaker function with K-type $\tau_{(\lambda,\lambda-1)}$ for the odd principal series representation. Then the A-radial coefficients $b_0(a_1, a_2)$ and $b_1(a_1, a_2)$ satisfy*

$$(1)\quad \begin{pmatrix} P_1 + S^2 & S(\partial_1 + \partial_2 - 4\pi\mathcal{L} - 2\frac{h_1 a_1^2}{\mathcal{L}} - 2) \\ S(\partial_1 + \partial_2 + 4\pi\mathcal{L} + 2\frac{h_2 a_2^2}{\mathcal{L}} - 2) & P_2 + S^2 \end{pmatrix} \times \begin{pmatrix} b_0(a_1, a_2) \\ b_1(a_1, a_2) \end{pmatrix} = \{\nu^2 - (\lambda-1)^2\} \begin{pmatrix} b_0(a_1, a_2) \\ b_1(a_1, a_2) \end{pmatrix},$$

where

$$\begin{aligned} P_1 &= \partial_1^2 + 2(\tfrac{h_2 a_2^2}{\mathcal{L}} - 1)\partial_1 + 8\pi\lambda h_1 a_1^2 - 16\pi^2 h_1^2 a_1^4 \\ &\quad - (\tfrac{h_1 a_1^2}{\mathcal{L}} + 3)\tfrac{h_2 a_2^2}{\mathcal{L}} - \lambda(\lambda-2), \\ P_2 &= \partial_2^2 + 2(-\tfrac{h_1 a_1^2}{\mathcal{L}} - 1)\partial_2 + 8\pi\lambda h_2 a_2^2 - 16\pi^2 h_2^2 a_2^4 \\ &\quad - (\tfrac{h_2 a_2^2}{\mathcal{L}} - 3)\tfrac{h_1 a_1^2}{\mathcal{L}} - \lambda(\lambda-2), \\ \nu &= \begin{cases} \nu_1 & \text{if } \lambda \text{ is even,} \\ \nu_2 & \text{if } \lambda \text{ is odd,} \end{cases} \end{aligned}$$

$$(2)\quad \begin{pmatrix} P + 8\pi\lambda h_1 a_1^2 + 8\pi(\lambda-1)h_2 a_2^2 & -2\frac{h_1 a_1^2 + h_2 a_2^2}{\mathcal{L}} S \\ 2\frac{h_1 a_1^2 + h_2 a_2^2}{\mathcal{L}} S & P + 8\pi(\lambda-1)h_1 a_1^2 + 8\pi\lambda h_2 a_2^2 \end{pmatrix} \times \begin{pmatrix} b_0(a_1, a_2) \\ b_1(a_1, a_2) \end{pmatrix} = (\nu_1^2 + \nu_2^2 - 5) \begin{pmatrix} b_0(a_1, a_2) \\ b_1(a_1, a_2) \end{pmatrix},$$

where

$$P = \partial_1^2 + \partial_2^2 + 2(\tfrac{h_2 a_2^2}{\mathcal{L}} - 1)\partial_1 - 2(\tfrac{h_1 a_1^2}{\mathcal{L}} + 1)\partial_2 \\ - 16\pi^2(h_1^2 a_1^4 + h_2^2 a_2^4) - 2\tfrac{h_1 a_1^2}{\mathcal{L}}\tfrac{h_2 a_2^2}{\mathcal{L}} + 2S^2.$$

Proof. (1) is obtained from the action of the composition of two shift operators, $P^{even} \circ R(\nabla^-_{\chi\cdot\eta,\,\tau_{-\lambda+2,-\lambda+1}}) \circ P^{even} \circ R(\nabla^+_{\chi\cdot\eta,\,\tau_{-\lambda+1,-\lambda}})$, and (2) from the Casimir operator. □

§6. Meijer's G-Functions

In this section we recall some basic facts on Meijer's G-functions. The main references are [11] and [3].

(6.1) Definition and basic properties

Definition 6.1. Suppose that m, n, p and q are integers with $1 \le q$, $0 \le n \le p \le q$ and $0 \le m \le q$, suppose further that the number x satisfies $0 < |x| < 1$ if $q = p$ and $x \neq 0$ if $p < q$, moreover that the numbers $a_1, \dots, a_n$ and $b_1, \dots, b_m$ fulfill the condition $a_i - b_j \neq 1, 2, 3, \dots$ $(1 \le i \le n,\ 1 \le j \le m)$. Then *Meijer's G-function with parameters* a_i, b_j is defined as

$$\text{(1)} \quad G^{m,n}_{p,q}\left(x \,\middle|\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) \\ = \frac{1}{2\pi\sqrt{-1}} \int_C \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{i=1}^n \Gamma(1 - a_i + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{i=n+1}^p \Gamma(a_i - s)} x^s ds.$$

The contour C is a loop starting and ending at $+\infty$ and encircling all poles of $\Gamma(b_j - s)$ $(1 \le j \le m)$ once in the negative direction, but none of the poles of $\Gamma(1 - a_i + s)$ $(1 \le i \le n)$. This integral converges if $p < q$ or $p = q$ and $|x| < 1$.

Assume that $b_j - b_h \notin \mathbf{Z}$ $(1 \le j \neq h \le m)$. Then the integral (1) can be evaluated as a sum of residues. Thus $G^{m,n}_{p,q}$ is expressed by the generalized hypergeometric function ${}_pF_{q-1}$:

LEMMA 6.2. *Under the above conditions*

$$G^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix} a_1,\,\ldots\,,\,a_p \\ b_1,\,\ldots\,,\,b_q\end{matrix}\right)$$
$$= \sum_{h=1}^{m} \frac{\prod_{j=1,j\neq h}^{m}\Gamma(b_j-b_h)\prod_{i=1}^{n}\Gamma(1+b_h-a_i)}{\prod_{j=m+1}^{q}\Gamma(1+b_h-b_j)\prod_{i=n+1}^{p}\Gamma(a_i-b_h)}\,x^{b_h}$$
$$\times\ {}_pF_{q-1}\left(\begin{matrix} 1+b_h-a_1, & \ldots, & 1+b_h-a_p \\ 1+b_h-b_1, & \overset{*}{\ldots}, & 1+b_h-b_q\end{matrix}\,\middle|\,(-1)^{p-m-n}x\right),$$

where

$$ {}_pF_{q-1}\left(\begin{matrix} \alpha_1, & \ldots, & \alpha_p \\ \beta_1, & \ldots, & \beta_{q-1}\end{matrix}\,\middle|\,x\right) = \sum_{n=0}^{\infty}\left(\prod_{i=1}^{p}\frac{\Gamma(\alpha_i+n)}{\Gamma(\alpha_i)}\prod_{j=1}^{q-1}\frac{\Gamma(\beta_j)}{\Gamma(\beta_j+n)}\right)\frac{x^n}{n!}$$

and the asterisk means that the number $1+b_h-b_h$ *is to be omitted in the sequence* $1+b_h-b_1,\ldots,1+b_h-b_q$.

The following formulas are deduced from the definition.

$$\text{(2)}\quad x^{\sigma}G^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix} a_1,\,\ldots\,,\,a_p \\ b_1,\,\ldots\,,\,b_q\end{matrix}\right) = G^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix} a_1+\sigma,\,\ldots\,,\,a_p+\sigma \\ b_1+\sigma,\,\ldots\,,\,b_q+\sigma\end{matrix}\right),$$

$$\text{(3)}\quad x\frac{d}{dx}G^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix} a_1,\,\ldots\,,\,a_p \\ b_1,\,\ldots\,,\,b_q\end{matrix}\right) = G^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix} a_1-1,\,\ldots\,,\,a_p \\ b_1,\,\ldots\,,\,b_q\end{matrix}\right)$$
$$+\,(a_1-1)\,G^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix} a_1,\,\ldots\,,\,a_p \\ b_1,\,\ldots\,,\,b_q\end{matrix}\right),\quad n\geq 1.$$

(6.2) Asymptotic expansions

We review some asymptotic expansions of $G^{m,n}_{p,q}$ proved by Barnes ([1], [11]).

LEMMA 6.3. *Suppose that* t, p, q *are integers with* $1\leq t\leq p<q$ *and the parameters satisfy* $a_t-b_j\neq 1,2,3,\ldots\,(1\leq j\leq q)$. *Then*

$$G^{q,1}_{p,q}\left(x\,\middle|\,\begin{matrix} a_t,\,a_1,\,\overset{*}{\ldots},\,a_p \\ b_1,\,\ldots\,,\,b_q\end{matrix}\right) \sim \frac{\prod_{j=1}^{q}\Gamma(1+b_j-a_t)}{\prod_{j=1,\,j\neq t}^{p}\Gamma(1+a_j-a_t)}\,x^{-1+a_t}$$
$$\times\ {}_pF_{q-1}\left(\begin{matrix} 1+b_1-a_t,\,\ldots\,,\,1+b_q-a_p \\ 1+a_1-a_t,\,\overset{*}{\ldots},\,1+a_p-a_t\end{matrix}\,\middle|\,-\frac{1}{x}\right),$$

as $|x| \to \infty$ *in* $|\arg x| < \frac{q-p+2}{2}\pi$*, where the asterisk means that the number* $1+a_t-a_t$ *is to be omitted in the sequence* $1+a_1-a_t, \ldots, 1+a_p-a_t$.

Lemma 6.4. *Let* ϵ *be* $1/2$ *if* $q=p+1$ *and* 1 *if* $q \geq p+2$*. Then*

$$G^{q,0}_{p,q}\left(x \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) \sim \exp\bigl((p-q)x^{\frac{1}{q-p}}\bigr)x^{\vartheta}\left\{\frac{(2\pi)^{\frac{q-p-1}{2}}}{\sqrt{q-p}} + \frac{M_1}{x^{\frac{1}{q-p}}} + \frac{M_2}{x^{\frac{2}{q-p}}} + \cdots\right\}$$

as $|x| \to \infty$ *in* $|\arg x| < (q-p+\epsilon)\pi$*, where* $\vartheta = \frac{1}{q-p}\bigl(\frac{p-q+1}{2} + \sum_{h=1}^{q} b_h - \sum_{h=1}^{p} a_h\bigr)$ *and the coefficients* $M_1, M_2, \ldots$ *do not depend on* x *but on the parameters* a_h, b_h.

Lemma 6.5. *Suppose that* $0 \leq p \leq q-1$, $\alpha_j \neq 0, -1, -2, \ldots$ $(1 \leq j \leq p)$ *and* $\alpha_j - \alpha_t \notin \mathbf{Z}$ $(1 \leq j \neq t \leq p)$*. Then*

$${}_pF_q\left(\begin{matrix} \alpha_1, & \ldots, & \alpha_p \\ \beta_1, & \ldots, & \beta_q \end{matrix} \,\middle|\, x\right) \sim \frac{\prod_{j=1}^{q}\Gamma(\beta_j)}{\prod_{j=1}^{p}\Gamma(\alpha_j)} \exp\Bigl((q-p+1)x^{\frac{1}{q-p+1}}\Bigr)\,x^{\gamma} \times\left\{\frac{(2\pi)^{\frac{p-q}{2}}}{\sqrt{q-p+1}} + \frac{N_1}{x^{\frac{1}{q-p+1}}} + \frac{N_2}{x^{\frac{2}{q-p+1}}} + \cdots\right\}$$

as $|x| \to \infty$ *in* $|\arg x| < \pi$*, where* $\gamma = \frac{1}{q-p+1}\bigl(\frac{q-p}{2} + \sum_{j=1}^{p}\alpha_h + \sum_{j=1}^{q}\beta_h\bigr)$ *and the coefficients* N_1, $N_2, \ldots$ *do not depend on* x *but on the parameters* a_j, b_j.

§7. Reduction of the System of Differential Equations

In this section we reduce the system of differential equations of even principal series to simpler ones. From now on we use the notation $D_i = \frac{\partial}{\partial y_i}$ $(i=1,2)$.

(7.1) Reduction of the system of differential equations

Proposition 7.1. *Under the same assumptions as in Proposition 5.4, if we write*

$$b_0(a_1,a_2) = (\sqrt{h_1}a_1)^{\lambda+1}(\sqrt{h_2}a_2)^{\lambda+1}\exp\{-2\pi(h_1a_1^2+h_2a_2^2)\}c(a_1,a_2),$$

and introduce new variables $y = (y_1, y_2) = (2\pi h_1 a_1^2, 2\pi h_2 a_2^2)$, *then we have*

$$(1)\quad y_1y_2\big[D_1D_2 + (\tfrac{\lambda-1}{y_2} - 2 - \tfrac{1}{2}\tfrac{1}{y_1-y_2})D_1 + (\tfrac{\lambda-1}{y_1} - 2 - \tfrac{1}{2}\tfrac{1}{y_1-y_2})D_2 - 2(\lambda-1)(\tfrac{1}{y_1} + \tfrac{1}{y_2}) + 4 + \tfrac{m_0^2}{4}\tfrac{1}{(y_1-y_2)^2} + (\lambda - \tfrac{1}{2})(\lambda-1)\tfrac{1}{y_1y_2}\big]$$
$$y_1y_2\big[D_1D_2 - \tfrac{1}{2}\tfrac{1}{y_1-y_2}D_1 + \tfrac{1}{2}\tfrac{1}{y_1-y_2}D_2 + \tfrac{m_0^2}{4}\tfrac{1}{(y_1-y_2)^2}\big]\,c(y) = \tfrac{1}{16}\{\nu_1^2 - (\lambda-1)^2\}\{\nu_2^2 - (\lambda-1)^2\}\,c(y),$$

$$(2)\quad \big[y_1^2D_1^2 + y_2^2D_2^2 + \{(\lambda+1)y_1 - 2y_1^2 + \tfrac{y_1y_2}{y_1-y_2}\}D_1 + \{(\lambda+1)y_2 - 2y_2^2 + \tfrac{y_1y_2}{y_1-y_2}\}D_2 - (y_1+y_2) + \tfrac{1}{2}(\lambda+1)(\lambda-2) - \tfrac{1}{4}(\nu_1^2+\nu_2^2-5) - \tfrac{m_0^2}{2}\tfrac{y_1y_2}{(y_1-y_2)^2}\big]\,c(y) = 0.$$

PROOF. Straightforward computations. □

Now we eliminate the terms $(y_1 - y_2)^{-2} = (2\pi\mathcal{L})^{-2}$.

PROPOSITION 7.2. *If we put* $c(y) = \mathcal{L}^{|m_0|/2}\,f(y)$, *the differential equations in Proposition 7.1 are rewritten as*

$$P_Af(y) = [P_{A_1}\circ P_{A_2} - \tfrac{y_1-y_2}{16y_1^2y_2^2}\{\nu_1^2 - (\lambda-1)^2\}\{\nu_2^2 - (\lambda-1)^2\}]\,f(y) = 0,$$
$$P_Bf(y) = 0,$$

with

$$P_{A_1} = D_1D_2 + \lambda(y_1y_2)^{-1}E_y - \tfrac{|m_0|-1}{2}(y_1-y_2)^{-1}(D_1-D_2) - 2(D_1+D_2) + (|m_0|-1)(y_1-y_2)^{-2} - 2\lambda(y_1^{-1}+y_2^{-1}) + \lambda(\lambda + \tfrac{1}{2}(|m_0|-1))(y_1y_2)^{-1} + 4,$$
$$P_{A_2} = (y_1-y_2)D_1D_2 - \tfrac{1}{2}(|m_0|+1)D_1 + \tfrac{1}{2}(|m_0|+1)D_2,$$
$$P_B = E_y^2 + (\lambda+|m_0|)E_y - 2y_1y_2D_1D_2 - 2(y_1^2D_1 + y_2^2D_2) + (|m_0|+1)y_1y_2(y_1-y_2)^{-1}(D_1-D_2) - (|m_0|+1)(y_1+y_2) + C.$$

Here we use the symbols

$$E_y = y_1D_1 + y_2D_2,\quad C = \tfrac{1}{4}(|m_0|+1)^2 + \tfrac{1}{2}(\lambda-1)(\lambda+|m_0|) - \tfrac{1}{4}(\nu_1^2+\nu_2^2).$$

Proof. Note that

$$\mathcal{L}^{-\frac{|m_0|}{2}} D_1(\mathcal{L}^{-\frac{|m_0|}{2}} f) = [D_1 + \tfrac{|m_0|}{2} \tfrac{1}{y_1 - y_2}] f,$$

$$\mathcal{L}^{-\frac{|m_0|}{2}} D_1 D_2(\mathcal{L}^{-\frac{|m_0|}{2}} f)$$

$$= [D_1 D_2 - \tfrac{|m_0|}{2} \tfrac{1}{y_1 - y_2}(D_1 - D_2) - \tfrac{|m_0|}{2}(\tfrac{|m_0|}{2} - 1)\tfrac{1}{(y_1 - y_2)^2}] f,$$

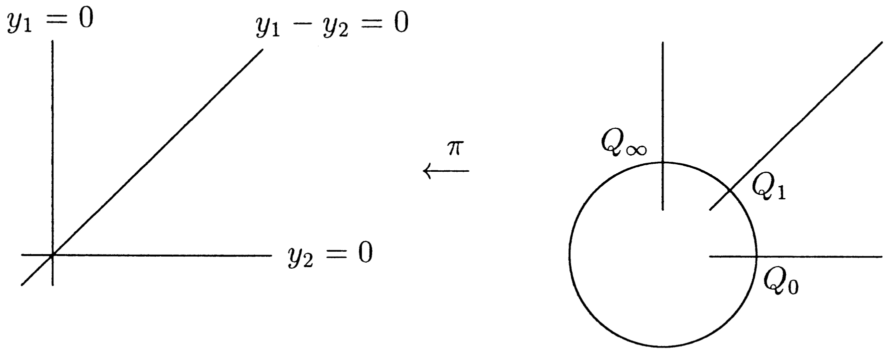
$$\mathcal{L}^{-\frac{|m_0|}{2}} D_1^2(\mathcal{L}^{-\frac{|m_0|}{2}} f) = [D_1^2 + |m_0| \tfrac{1}{y_1 - y_2} D_1 + \tfrac{|m_0|}{2}(\tfrac{|m_0|}{2} - 1)\tfrac{1}{(y_1 - y_2)^2}] f. \square$$

(7.2) Singularities of the system of differential equations

Let us define 3 divisors in the affine space $\mathbf{C}^2$ by $Y_0 = \{(y_1, y_2) \mid y_2 = 0\}$, $Y_\infty = \{(y_1, y_2) \mid y_1 = 0\}$ and $Y_1 = \{(y_1, y_2) \mid y_1 - y_2 = 0\}$. These divisors are regular singularities of our system in the sense of [17]. Since they are not normal crossing at the point $y_1 = y_2 = 0$, it is natural to blow up $\mathbf{C}^2$ at the origin. Put $Y = \{(y_1, y_2) \times [\xi_1 : \xi_2] \in \mathbf{C}^2 \times P^1_{\mathbf{C}} \mid y_1\xi_2 - y_2\xi_1 = 0\}$. Here $[\xi_1 : \xi_2]$ is a system of homogeneous coordinates on $P^1_{\mathbf{C}}$. We have an embedding $i : Y \hookrightarrow \mathbf{C}^2 \times P^1_{\mathbf{C}}$ and a projection $pr_1 :\ \mathbf{C}^2 \times P^1_{\mathbf{C}} \to \mathbf{C}^2$. The map $\pi = i \circ pr_1$ is a blow up of $\mathbf{C}^2$ at $(0,0)$. If $(y_1, y_2) \neq (0,0)$, $\pi^{-1}((y_1, y_2)) = (y_1, y_2) \times [y_1 : y_2]$, and $\pi^{-1}((0,0)) \simeq P^1_{\mathbf{C}}$. We write nonhomogeneous local coordinates of $P^1_{\mathbf{C}}$ as $u = \xi_2/\xi_1$, $v = \xi_1/\xi_2$ and set

$$Q_0 := (0,0) \times [1:0], \quad Q_\infty := (0,0) \times [0:1] \ \text{ and } \ Q_1 := (0,0) \times [1:1].$$

At these points in Y, we can take local coordinates (y_1, u), (v, y_2) and $(y_1, u-1)$ respectively.



§8. Characteristic Indices

We consider the pull back of the equations to Y from now on. To consider the formal solutions of the differential equations in Proposition 7.2, we calculate the characteristic indices at Q_0, Q_∞ and Q_1.

(8.1) Characteristic indices at Q_0

Since local coordinate at Q_0 is (y_1, u) $(u = y_2/y_1)$, we can write the formal solution at Q_0 as $f(y) = \sum_{m,n\geq 0} c_{m,n} y_1^{\sigma_1+m} u^{\sigma_2+n}$ with $c_{0,0} \neq 0$. We give recurrence relations of $c_{m,n}$ to determine the characteristic indices (σ_1, σ_2).

Lemma 8.1. *If we write $M_0 = c_{m,n}\, y_1^{\sigma_1+m} u^{\sigma_2+n}$, we have*

(1) $D_1(M_0) = \{(\sigma_1+m) - (\sigma_2+n)\}\, c_{m,n}\, y_1^{\sigma_1+m-1} u^{\sigma_2+n}$,

(2) $D_2(M_0) = (\sigma_2+n)\, c_{m,n}\, y_1^{\sigma_1+m-1} u^{\sigma_2+n-1}$,

(3) $D_1 D_2(M_0) = \{(\sigma_1+m) - (\sigma_2+n)\}(\sigma_2+n)\, c_{m,n}\, y_1^{\sigma_1+m-2} u^{\sigma_2+n-1}$,

(4) $E_y(M_0) = (\sigma_1+m) M_0, \quad E_y^2(M_0) = (\sigma_1+m)^2 M_0$.

Proof. Direct computations. $\square$

We put $P_{A_1}^0 = (y_1-y_2)^2 P_{A_1}$, $P_{A_2}^0 = P_{A_2}$, $P_B^0 = (y_1-y_2) P_B$ and consider the equation $P_A^0 f(y) = 0$. If we denote by $s_1 = \sigma_1 + m$ and $s_2 = \sigma_2 + n$ for short, then

$$\begin{aligned} P_{A_2}^0 f(y) = \textstyle\sum_{m,n\geq 0} \big[& s_2\{s_1 - s_2 + \tfrac{1}{2}(|m_0|+1)\}\, c_{m,n} \\ & - (s_1 - s_2 + 1)\{s_2 + \tfrac{1}{2}(|m_0|-1)\}\, c_{m,n-1} \big]\, y_1^{s_1-1} u^{s_2-1}, \end{aligned}$$

by Lemma 8.1. If we denote by $\sum_{m,n\geq 0} d_{m,n}\, y_1^{s_1-1} u^{s_2-1}$ the above equation and use Lemma 8.1 again, then have

$$\begin{aligned} & P_{A_1}^0 \circ P_{A_2}^0 f(y) = P_{A_1}^0 \big(\textstyle\sum_{m,n\geq 0} d_{m,n}\, y_1^{s_1-1} u^{s_2-1} \big) \\ & = \textstyle\sum_{m,n\geq 0} \big[\big\{ s_2\big(s_1 - s_2 - \tfrac{1}{2}(|m_0|+1)\big) \\ & \quad + (\lambda-1)\big(s_1 + \lambda + \tfrac{1}{2}(|m_0|+1)\big) \big\}\, d_{m+1,n+2} \\ & \quad + \big\{ -2s_2(s_1-s_2) - \tfrac{1}{2}(|m_0|+3)s_1 \\ & \qquad - 2 + 2(\lambda-1)\big(s_1 - \lambda - \tfrac{1}{2}(|m_0|+1)\big) \big\}\, d_{m+1,n+1} \end{aligned}$$

$$
\begin{aligned}
&+ \bigl\{(s_1 - s_2 - 2)\bigl(s_2 + \tfrac{1}{2}(|m_0| - 1)\bigr) \\
&\quad + (\lambda - 1)\bigl(s_1 + \lambda + \tfrac{1}{2}(|m_0| + 1)\bigr)\bigr\}\, d_{m+1,n} \\
&- 2(s_2 + \lambda + 1)\, d_{m,n+2} - 2(s_1 - 3s_2 - \lambda - 1)\, d_{m,n+1} \\
&\quad + 2(2s_1 - 3s_2 - \lambda + 1)\, d_{m,n} \\
&- 2(s_1 - s_2 - \lambda)\, d_{m,n-1} + 4d_{m-1,n+1} - 8d_{m-1,n} + 4d_{m-1,n-1}\bigr]\, y_1^{s_1} u^{s_2}.
\end{aligned}
$$

Then we get

$$
\begin{aligned}
&\bigl[\bigl\{\bigl(s_1 - s_2 - \tfrac{1}{2}(|m_0| + 1)\bigr)(s_2 + n + 2) + (\lambda - 1)\bigl(s_1 + \lambda + \tfrac{1}{2}(|m_0| + 1)\bigr)\bigr\} \\
&\quad \times (s_2 + 2)\bigl(s_1 - s_2 + \tfrac{1}{2}(|m_0| - 1)\bigr) \\
&\quad - \tfrac{1}{16}\{\nu_1^2 - (\lambda - 1)^2\}\{\nu_2^2 - (\lambda - 1)^2\}\bigr]\, c_{m+1,n+2} \\
&\quad + (\text{`lower order terms'}) = 0.
\end{aligned}
$$

Here 'lower order terms' means terms which contain $c_{i,j}$ with $i \le m+1$ and $j \le n+1$. Since $c_{0,0} \ne 0$, we have

$$
\begin{aligned}
\text{(4)} \qquad \sigma_2(\sigma_2 + \lambda - 1)&\bigl(\sigma_1 - \sigma_2 + \tfrac{1}{2}(|m_0| - 1)\bigr) \\
&\times \bigl(\sigma_1 - \sigma_2 + \tfrac{1}{2}(|m_0| - 1) + \lambda - 1\bigr) \\
&= \tfrac{1}{16}(\nu_1 - \lambda + 1)(\nu_1 + \lambda - 1)(\nu_2 - \lambda + 1)(\nu_2 + \lambda - 1).
\end{aligned}
$$

Now we treat the Casimir equation $P_B^0 f(y) = 0$. In the same way as above we get

$$
\begin{aligned}
\{s_1(s_2 + \lambda + |m_0|) - s_2(2s_1 - 2s_2 + |m_0| + 1) + C\}\, c_{m,n}& \\
+ (\text{`lower order terms'}) = 0.&
\end{aligned}
$$

Then

$$
\begin{aligned}
\text{(5)} \qquad &\sigma_2(\sigma_2 + \lambda - 1) \\
&\quad + \bigl\{\sigma_1 - \sigma_2 + \tfrac{1}{2}(|m_0| - 1)\bigr\}\bigl\{\sigma_1 - \sigma_2 + \tfrac{1}{2}(|m_0| - 1) + \lambda - 1\bigr\} \\
&= \tfrac{1}{4}\{(\nu_1 - \lambda + 1)(\nu_1 + \lambda - 1) + (\nu_2 - \lambda + 1)(\nu_2 + \lambda - 1)\}.
\end{aligned}
$$

From (4) and (5) we obtain

Proposition 8.2. *The characteristic indices (σ_1, σ_2) at Q_0 are*

$$
\begin{aligned}
&\bigl(\tfrac{1}{2}(\varepsilon\nu_1 \pm \nu_2 - |m_0| - 2\lambda + 1), \tfrac{1}{2}(\varepsilon\nu_1 - \lambda + 1)\bigr), \\
&\bigl(\tfrac{1}{2}(\pm\nu_1 + \varepsilon\nu_2 - |m_0| - 2\lambda + 1), \tfrac{1}{2}(\varepsilon\nu_2 - \lambda + 1)\bigr), \qquad \varepsilon \in \{\pm 1\}.
\end{aligned}
$$

REMARK 8.3. The above set is invariant under the action of Weyl group of $Sp(2,\mathbf{R})$, $\mathfrak{S}_2 \ltimes (\mathbf{Z}/2\mathbf{Z})^2$. Throughout this paper we assume that ν_1, ν_2 and $\nu_1 \pm \nu_2$ are not integers. This implies that the principal series representation $\pi_{\sigma,\nu}$ is irreducible. Therefore we have an 8-dimensional space of meromorphic solutions at Q_0.

(8.2) Characteristic indices at Q_∞

Since local coordinate at Q_∞ is (v, y_2) $(v = y_1/y_2)$, we can write the formal solution at Q_∞ as $f(y) = \sum_{m,n\geq 0} c_{m,n} v^{\rho_1+m} y_2^{\rho_2+n}$ with $c_{0,0} \neq 0$. Since our system in Proposition 7.2 is symmetric with respect to y_1 and y_2, we get the following from Proposition 8.2.

PROPOSITION 8.4. *The characteristic indices (ρ_1, ρ_2) at Q_∞ are*

$$\begin{aligned}
&\left(\tfrac{1}{2}(\varepsilon\nu_1 - \lambda + 1), \tfrac{1}{2}(\varepsilon\nu_1 \pm \nu_2 - |m_0| - 2\lambda + 1)\right),\\
&\left(\tfrac{1}{2}(\varepsilon\nu_2 - \lambda + 1), \tfrac{1}{2}(\pm\nu_1 + \varepsilon\nu_2 - |m_0| - 2\lambda + 1)\right), \qquad \varepsilon \in \{\pm 1\}.
\end{aligned}$$

(8.3) Characteristic indices at Q_1

We take the local coordinate $(y_1, u-1)$ at Q_1 and write the formal solution as $f(y) = \sum_{m,n\geq 0} c_{m,n}\, y_1^{\tau_1+m}(u-1)^{\tau_2+n}$ with $c_{0,0} \neq 0$. The way to determine the characteristic indices is the same as in 8.1, but we need a little more complicated calculation.

Put $P_A^1 = y_1^2 y_2^2 P_A,\ P_B^1 = P_B$ and begin with $P_A^1 f(y) = 0$. In a similar way to 8.1, if we denote $t_1 = \tau_1 + m$ and $t_2 = \tau_2 + n$ for short, then we obtain the following recurrence relation.

$$\begin{aligned}
&4(t_2+1)(t_2+|m_0|+1)(t_2+3)(t_2+|m_0|+3)\, c_{m+1,n+3}\\
&\ -[2(t_2+1)(t_2+|m_0|+1)(t_1-t_2-1)(2t_2+|m_0|+5)\\
&\qquad +\{2t_1(2t_2+|m_0|+1) - 12t_2^2 - 2(5|m_0|+3)t_2 - 2(|m_0|+1)\}\\
&\qquad \times (t_2+2)(t_2+|m_0|+2)]\, c_{m+1,n+2}\\
&\ +[\{t_1(2t_2+|m_0|+1) - 6t_2^2 - (5|m_0|+3)t_2 - (|m_0|+1)\}\\
&\qquad \times (t_1-t_2)(2t_2+|m_0|+3)\\
&\qquad -2\{2t_1(2t_2+\lambda+|m_0|-1) - 6t_2^2 - 2(2|m_0|-3)t_2 + 3|m_0| - 1\\
&\qquad\quad +(\lambda-1)(2\lambda+|m_0|+1)\}(t_2+1)(t_2+|m_0|+1)]\, c_{m+1,n+1}
\end{aligned}$$

$$
\begin{aligned}
&+ [\{2t_1(2t_2 + \lambda + |m_0| - 1) - 6t_2^2 - 2(2|m_0| - 3)t_2 + 3|m_0| - 1 \\
&\qquad\quad + (\lambda - 1)(2\lambda + |m_0| + 1)\}(t_1 - t_2 + 1)(2t_2 + |m_0| + 1) \\
&\quad - 2\{t_1(2t_2 + 2\lambda + |m_0| - 3) - 2t_2^2 - (|m_0| - 5)t_2 + 2(|m_0| - 1) \\
&\qquad\quad + (\lambda - 1)(2\lambda + |m_0| + 1)\}t_2(t_2 + |m_0|)]\, c_{m+1,n} \\
&+ [\{t_1(2t_2 + 2\lambda + |m_0| - 3) - 2t_2^2 - (|m_0| - 5)t_2 + 2(|m_0| - 1) \\
&\qquad\quad + (\lambda - 1)(2\lambda + |m_0| + 1)\}(t_1 - t_2 + 2)(2t_2 + |m_0| - 1) \\
&\qquad\quad + \tfrac{1}{4}\{\nu_1^2 - (\lambda - 1)^2\}\{\nu_2^2 - (\lambda - 1)^2\}]\, c_{m+1,n-1} \\
&+ (\text{‘lower order terms’}) = 0.
\end{aligned}
$$

Here ‘lower order terms’ are terms which contain $c_{i,j}$ with $i \leq m$. We denote by (A-1), … ,(A-5) the recurrence relations which are obtained by substituting $m = -1$ and $n = -3, -2, -1, 0, 1$ in the above.

Now we treat the Casimir equation $P_B^1 f(y) = 0$. We have

$$
\begin{aligned}
&2(t_2 + 2)(t_2 + |m_0| + 2)\, c_{m,n+2} \\
&+ \{(-t_1 + t_2 + 1)(2t_2 + |m_0| + 3) + 2(t_2 + 1)(t_2 + |m_0| + 1)\}\, c_{m,n+1} \\
&+ \{t_1(t_1 + \lambda + |m_0|) - (t_1 - t_2)(2t_2 + |m_0| + 1) + C\}\, c_{m,n} \\
&- 2(t_1 + t_2 + |m_0|)\, c_{m-1,n} - (2t_2 + |m_0| - 1)\, c_{m-1,n-1} = 0.
\end{aligned}
$$

We also denote by (B-1), … ,(B-5) the recurrence relations obtained by substituting $m = 0$ and $n = -2, -1, 0, 1, 2$.

Since (A-1), (A-2), (A-3) and (A-4) are deduced from (B-1), (B-2), (B-3) and (B-4), they give no new information. First we substitute (B-2), (B-3), (B-4) and (B-5) into (A-5) and get a new recurrence relation among $c_{0,2}, c_{0,1}$ and $c_{0,0}$. Secondly by using (B-1), (B-2) and (B-3), we eliminate $c_{0,2}, c_{0,1}$ and reach

$$
\begin{aligned}
&\{\tau_1(\tau_1 + \lambda + |m_0|) + C\} \\
&\quad \times \{\tau_1(\tau_1 + 3\lambda + |m_0| - 2) + (\lambda - 1)(2\lambda + |m_0| - 1) + C\} \\
&\quad\quad = \tfrac{1}{4}\{\nu_1^2 - (\lambda - 1)^2\}\{\nu_2^2 - (\lambda - 1)^2\}.
\end{aligned}
$$

Combined with $\tau_2(\tau_2 + |m_0|) = 0$ ((B-1)), we get

Proposition 8.5. *The characteristic indices (τ_1, τ_2) at Q_1 are*

$$
\begin{aligned}
&\big(\tfrac{1}{2}\{\pm(\nu_1 \pm \nu_2) - |m_0| - 2\lambda + 1\},\ 0\big), \\
&\big(\tfrac{1}{2}\{\pm(\nu_1 \pm \nu_2) - |m_0| - 2\lambda + 1\},\ -|m_0|\big).
\end{aligned}
$$

REMARK 8.6. Since we have assumed that ν_1, ν_2 and $\nu_1 \pm \nu_2$ are not integers, there exists a 4-dimensional space of holomorphic solutions along Y_1 corresponding to $\tau_2 = 0$ and another 4-dimensional space of solutions (possibly with logarithmic branch) corresponding to $\tau_2 = -|m_0|$. Thus we have $\dim_{\mathbf{C}} \mathrm{Hom}_{(\mathfrak{g},K)}(H_{\pi_{\sigma,\nu},K}, C^\infty \mathrm{Ind}_R^G(\chi \cdot \eta)) \leq 4$.

REMARK 8.7. Part of our computation is suggested by a similar one by Oda for P_J-principal series (private note). But our case is more complicated because the rank of holonomic system becomes 8 for 4 in the P_J case.

§9. Holomorphic Solutions Along the Singular Divisor Y_1

In this section we study holomorphic solutions along Y_1. For each non-negative integer n, if we put $\varphi_n(y_1) = \sum_{m \geq 0} c_{m,n} y_1^{\tau_1 + m}$, then we have

$$f(y) = f(y_1, u-1) = \textstyle\sum_{n \geq 0} \varphi_n(y_1)(u-1)^n \qquad (u = y_2/y_1).$$

The even principal series Siegel-Whittaker function with K-type $\tau_{(\lambda,\lambda)}$ is of the form

$$b_0(y_1, y_2)v_0 = \left(\frac{y_1 y_2}{2\pi}\right)^{\frac{\lambda+1}{2}} \left(\frac{y_1 - y_2}{2\pi}\right)^{\frac{|m_0|}{2}} e^{-(y_1+y_2)} \sum_{n \geq 0} \varphi_n(y_1)(u-1)^n \, v_0.$$

Here v_0 is a basis of $\tau^*_{(\lambda,\lambda)}$. Our concern in this section is the boundary value of $b_0(y_1, y_2)$ along $y_1 - y_2 = 0$. We first review the definition of boundary values.

(9.1) Definition of boundary values

DEFINITION 9.1 ([19, Definition 0.1]). Let U be a neighborhood of the point $x = y = 0$ in $\mathbf{C}^2$ and Ω a simply connected domain whose closure $\bar{\Omega}$ is also simply connected, satisfying the conditions, $\Omega \subseteq \{(x,y) \mid x \neq 0\}$ and $U \cap \{(x,y) \mid x = 0\} \subseteq \bar{\Omega}$. Let $g(x,y)$ be a holomorphic function on the domain Ω admitting the expression, $x^{\alpha_0} g_0(x,y) + x^{\alpha_1} g_1(x,y) + \cdots + x^{\alpha_p} g_p(x,y)$ with $\alpha_i - \alpha_j \notin \mathbf{Z}$ in $U \cap \Omega$ where the functions g_k are holomorphic at $x = y = 0$ and $g_k(0,y) \not\equiv 0$. We call the function $g_k(0,y)$ *the boundary value of $g(x,y)$ with respect to the characteristic exponent α_k along $x = 0$.*

Note that this is compatible with the more general definition of boundary values given in [17, §3]. Since

$$b_0(y_1,y_2) = (u-1)^{\frac{|m_0|}{2}} \sum_{n\geq 0} (-1)^{\frac{|m_0|}{2}} \times \left(\frac{y_1}{2\pi}\right)^{\lambda+1+\frac{|m_0|}{2}} e^{-2y_1} \varphi_n(y_1)\, e^{-y_1(u-1)} u^{\frac{\lambda+1}{2}} (u-1)^n,$$

the boundary value of $b_0(y_1,y_2)$ with respect to the characteristic exponent $|m_0|/2$ along $u-1=0$ is

$$(-1)^{\frac{|m_0|}{2}} \left(\frac{y_1}{2\pi}\right)^{\lambda+1+\frac{|m_0|}{2}} e^{-2y_1} \varphi_0(y_1).$$

To determine $\varphi_0(y_1)$, we deduce a differential equation for $\varphi_0(y_1)$ from difference-differential equations for $\varphi_n(y_1)$. We construct space of solutions by using Meijer's G-functions and show that the dimension of the subspace of rapidly decreasing solutions is at most one (Multiplicity Free Theorem).

(9.2) Differential equation for $\varphi_0(y_1)$

We first find difference-differential equations for $\varphi_n(y_1)$ by similar calculations as in the previous section.

Lemma 9.2. *For $f(y_1,u-1) = \sum_{n\geq 0} \varphi_n(y_1)(u-1)^n$, we have*

(1) $D_1(f) = \sum_{n\geq 0} \left(\varphi_n'(y_1) - \frac{n}{y_1}\varphi_n(y_1)\right)(u-1)^n - \frac{n}{y_1}\varphi_n(y_1)(u-1)^{n-1}$,

(2) $D_2(f) = \sum_{n\geq 0} \frac{n}{y_1}\varphi_n(y_1)(u-1)^{n-1}$,

(3) $D_1D_2(f) = \sum_{n\geq 0}\left(\frac{n}{y_1}\varphi_n'(y_1) - \frac{n^2}{y_1^2}\varphi(y_1)\right)(u-1)^{n-1} - \frac{n(n-1)}{y_1^2}\varphi_n(y_1)(u-1)^{n-2}$,

(4) $E_y(f) = \sum_{n\geq 0} y_1\varphi_n'(y_1)(u-1)^n$,
$E_y^2(f) = \sum_{n\geq 0}(y_1\varphi_n'(y_1) + y_1^2\varphi_n''(y_1))(u-1)^n$.

Proof. Note that $D_1 u = -u/y_1,\ D_2 u = u/y_1$. $\square$

We begin with $P_A^1 f = 0$. For short we write $\varphi_n(y_1)$ as φ_n. If we denote $P_{A_2}^1 f(y) = \sum_{n\geq -1} \psi_n(y_1)(u-1)^n$ then

$$\psi_n = -\bigl(n+\tfrac{1}{2}(|m_0|+1)\bigr)(\varphi_n' - ny_1^{-1}\varphi_n) + (n+1)(n+|m_0|+1)y_1^{-1}\varphi_{n+1}, \tag{6}$$

by Lemma 9.2. We use Lemma 9.2 again and obtain

$$\begin{aligned}
(7)\quad & P_{A_1}^1 \circ P_{A_2}^1 f(y) = P_{A_1}^1\bigl(\textstyle\sum_{n\geq -1}\psi_n(y_1)(u-1)^n\bigr)\\
&= \textstyle\sum_{n\geq -2}\bigl[-(n+1)(n+|m_0|+1)y_1^2\psi_{n+2} + \bigl(n+\tfrac{1}{2}(|m_0|+1)\bigr)y_1^3\psi_{n+1}'\\
&\quad - \bigl(3n^2+\tfrac{1}{2}(5|m_0|+3)n+\tfrac{1}{2}(|m_0|+1)\bigr)y_1^2\psi_{n+1}\\
&\quad + \bigl(-2y_1^4+(2n+|m_0|+\lambda-1)y_1^3\bigr)\psi_n'\\
&\quad + \bigl\{4y_1^4+(2n-4\lambda)y_1^3+\bigl(-3n^2+(-2|m_0|+3)n+\lambda^2\\
&\qquad\quad + \tfrac{1}{2}(|m_0|+1)\lambda+|m_0|-1\bigr)y_1^2\bigr\}\psi_n\\
&\quad + \bigl\{-4y_1^4+\bigl(n+\lambda-\tfrac{1}{2}(|m_0|-3)\bigr)y_1^3\bigr\}\psi_{n-1}'\\
&\quad + \bigl\{8y_1^4+(4n-6\lambda-4)y_1^3+\bigl(-n^2-\tfrac{1}{2}(|m_0|-5)n+\lambda^2\\
&\qquad\quad + \tfrac{1}{2}(|m_0|-1)\lambda+\tfrac{1}{2}(|m_0|-3)\bigr)y_1^2\bigr\}\psi_{n-1}\\
&\quad - 2y_1^4\psi_{n-2}' + \bigl(4y_1^4+(2n-2\lambda-4)y_1^3\bigr)\psi_{n-2}\bigr](u-1)^n
\end{aligned}$$

Substituting (6) (and its differential) into (7), we get a recurrence relation for φ_n. Then

$$\begin{aligned}
&\text{(C-1)}\quad 2\varphi_1 - y_1\varphi_0' = 0,\\
&\text{(C-2)}\quad 12\varphi_3 - 6y_1\varphi_2' + 12\varphi_2 + y_1^2\varphi_1'' - 2y_1\varphi_1' + 2\varphi_1 = 0,\\
&\text{(C-3)}\quad 32(|m_0|+2)(|m_0|+4)\varphi_4 - 2(|m_0|+5)(5|m_0|+11)y_1\varphi_3'\\
&\qquad + 6(9|m_0|^2+52|m_0|+67)\varphi_3 + (|m_0|+3)(|m_0|+5)y_1^2\varphi_2''\\
&\qquad + 8(|m_0|+2)\{2y_1^2+(2|m_0|+\lambda-6)y_1\}\varphi_2'\\
&\qquad - \{32(|m_0|+2)y_1^2 - 32(\lambda-1)(|m_0|+2)y_1\\
&\qquad\qquad + 8(|m_0|+2)\lambda^2 + 4(|m_0|+2)(|m_0|-3)\lambda\\
&\qquad - 6(5|m_0|+9)(|m_0|+3)\}\varphi_2\\
&\qquad - (|m_0|+3)\{2y_1^3-(|m_0|+\lambda+1)y_1^2\}\varphi_1''\\
&\qquad + (|m_0|+3)\{8y_1^3-8(\lambda-1)y_1^2\\
&\qquad\qquad + (2\lambda^2+(|m_0|-3)\lambda-2(|m_0|+2))y_1\}\varphi_1'
\end{aligned}$$

$$
\begin{aligned}
&- (|m_0|+3)\{8y_1^2 - 8(\lambda-1)y_1 \\
&\qquad + 2\lambda^2 + (|m_0|-3)\lambda - 2(|m_0|+2)\}\varphi_1 \\
&- \tfrac{1}{4}\{\nu_1^2 - (\lambda-1)^2\}\{\nu_2^2 - (\lambda-1)^2\}\varphi_0 = 0.
\end{aligned}
$$

Here we use that $|m_0|$ is even, in particular $|m_0| \neq 1$ (Lemma 5.3).

Next we treat the Casimir equation $P_B^1 f(y) = 0$. By Lemma 9.2, we have

$$
\begin{aligned}
&2(n+2)(n+|m_0|+2)\varphi_{n+2} - (2n+|m_0|+3)y_1\varphi'_{n+1} \\
&+ (n+1)(4n+3|m_0|+5)\varphi_{n+1} + y_1^2\varphi''_n + \{-2y_1^2 + (-2n+\lambda)y_1\}\varphi'_n \\
&+ \{-2(n+|m_0|+1)y_1 + 2n^2 + (|m_0|+1)n + C\}\varphi_n \\
&- (2n+|m_0|-1)y_1\varphi_{n-1} = 0.
\end{aligned}
$$

So we have

$$
\begin{aligned}
&\text{(D-1)}\ \ 2\varphi_1 - y_1\varphi'_0 = 0, \\
&\text{(D-2)}\ \ 4(|m_0|+2)\varphi_2 - (|m_0|+3)y_1\varphi'_1 + (3|m_0|+5)\varphi_1 \\
&\qquad\quad + y_1^2\varphi''_0 - (2y_1^2 - \lambda y_1)\varphi'_0 - \{2(|m_0|+1)y_1 - C\}\varphi_0 = 0, \\
&\text{(D-3)}\ \ 6(|m_0|+3)\varphi_3 - (|m_0|+5)y_1\varphi'_2 + 6(|m_0|+3)\varphi_2 + y_1^2\varphi''_1 \\
&\qquad\quad - \{2y_1^2 + (2-\lambda)y_1\}\varphi'_1 + \{-2(|m_0|+2)y_1 + |m_0| + 3 + C\}\varphi_1 \\
&\qquad\quad - (|m_0|+1)y_1\varphi_0 = 0, \\
&\text{(D-4)}\ \ 8(|m_0|+4)\varphi_4 - (|m_0|+7)y_1\varphi'_3 + 3(3|m_0|+13)\varphi_3 \\
&\qquad\quad + y_1^2\varphi''_2 - \{2y_1^2 + (4-\lambda)y_1\}\varphi'_2 \\
&\qquad\quad + \{-2(|m_0|+3)y_1 + 2(|m_0|+5) + C\}\varphi_2 - (|m_0|+3)y_1\varphi_1 = 0.
\end{aligned}
$$

By using (C-1), ... , (D-4), we eliminate $\varphi_4, \varphi_3, \varphi_2$ and φ_1 step by step and reach a differential equation satisfied by φ_0.

Proposition 9.3. *If we write* $\Phi(y_1) = e^{-2y_1}\varphi_0(y_1)$ *and* $\theta = y_1\frac{d}{dy_1}$, *then*

$$
\begin{aligned}
(8)\qquad &\bigl[-4y_1^2(\theta+\lambda)(\theta+\lambda+1) + 4\lambda y_1(\theta+\lambda)(\theta+\lambda+\tfrac{1}{2}|m_0|) \\
&\quad + \bigl(\theta+\lambda+\tfrac{1}{2}(m_0-1+\nu_1+\nu_2)\bigr)\bigl(\theta+\lambda+\tfrac{1}{2}(m_0-1+\nu_1-\nu_2)\bigr) \\
&\qquad \times \bigl(\theta+\lambda+\tfrac{1}{2}(m_0-1-\nu_1+\nu_2)\bigr) \\
&\qquad \times \bigl(\theta+\lambda+\tfrac{1}{2}(m_0-1-\nu_1-\nu_2)\bigr)\bigr]\Phi(y_1) = 0.
\end{aligned}
$$

Since the shift operators (which are second-order partial differential operators in this situation) move the K-type parameter (λ, λ) to $(\lambda \pm 2, \lambda \pm 2)$, we need only solve the above differential equation the case where $\lambda = 0, 1$. We remark that integral representations of the solutions were obtained in case of $\lambda = 0$ in [15].

(9.3) Solution of the differential equation

We construct the space of solutions of (8) when $\lambda = 0, \pm 1$. Let us write the formal solution at the origin as $\Phi(y_1) = \sum_{m \geq 0} c_m y_1^{\sigma+m}$ $(c_0 \neq 0)$. Since $\sigma = \tau_1 = \frac{1}{2}(\varepsilon_1\nu_1 + \varepsilon_2\nu_2 - |m_0| - 2\lambda + 1)$, $\varepsilon_i \in \{\pm 1\}$ $(i = 1, 2)$, we have

$$
\begin{aligned}
(9) \qquad & m(m+\varepsilon_1\nu_1)(m+\varepsilon_2\nu_2)(m+\varepsilon_1\nu_1+\varepsilon_2\nu_2)\, c_m \\
& + \lambda(2m+\varepsilon_1\nu_1+\varepsilon_2\nu_2-1)(2m+\varepsilon_1\nu_1+\varepsilon_2\nu_2-|m_0|-1)\, c_{m-1} \\
& - (2m+\varepsilon_1\nu_1+\varepsilon_2\nu_2-|m_0|-1) \\
& \times (2m+\varepsilon_1\nu_1+\varepsilon_2\nu_2-|m_0|-3)\, c_{m-2} = 0.
\end{aligned}
$$

In case of $\lambda = 0$ we can easily deduce the following.

PROPOSITION 9.4. *Let $y_1 > 0$. The following four functions are linearly independent solutions of* (8) *for $\lambda = 0$.*

$$
\begin{aligned}
&\Phi^0_{\varepsilon_1,\varepsilon_2}(y_1) \\
&= y_1^{(-|m_0|+1+\varepsilon_1\nu_1+\varepsilon_2\nu_2)/2}\, {}_2F_3\left(\left. \begin{matrix} \frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2-|m_0|+1}{4}, \quad \frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2-|m_0|+3}{4} \\ \frac{\varepsilon_1\nu_1+2}{2}, \quad \frac{\varepsilon_2\nu_2+2}{2}, \quad \frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2+2}{2} \end{matrix} \, \right| \, y_1^2 \right).
\end{aligned}
$$

These basic solutions are not convenient to investigate the asymptotic behavior, since they are all rapidly increasing as the absolute value of y_1 goes to infinity (See Lemma 6.5). To deduce a multiplicity free theorem, we must construct another basis containing a rapidly decreasing solution by taking a suitable linear combination of $\Phi^0_{\varepsilon_1,\varepsilon_2}$.

PROPOSITION 9.5. *Under the same assumption as in Proposition 9.4,* *we get the following new basis.*

$$\Phi_1^0(y_1) = y_1^{(-|m_0|+1)/2}\, G_{2,4}^{4,0}\left(y_1^2 \,\middle|\, \begin{matrix} a_1,\ a_2 \\ b_1,\ b_2,\ b_3,\ b_4 \end{matrix} \right),$$

$$\Phi_2^0(y_1) = y_1^{(-|m_0|+1)/2}\, G_{2,4}^{4,0}\left(y_1^2 e^{-2\pi\sqrt{-1}} \,\middle|\, \begin{matrix} a_1,\ a_2 \\ b_1,\ b_2,\ b_3,\ b_4 \end{matrix} \right),$$

$$\Phi_3^0(y_1) = y_1^{(-|m_0|+1)/2}\, G_{2,4}^{4,1}\left(y_1^2 e^{\pi\sqrt{-1}} \,\middle|\, \begin{matrix} a_1,\ a_2 \\ b_1,\ b_2,\ b_3,\ b_4 \end{matrix} \right),$$

$$\Phi_4^0(y_1) = y_1^{(-|m_0|+1)/2}\, G_{2,4}^{4,1}\left(y_1^2 e^{\pi\sqrt{-1}} \,\middle|\, \begin{matrix} a_2,\ a_1 \\ b_1,\ b_2,\ b_3,\ b_4 \end{matrix} \right),$$

with $a_1 = \frac{|m_0|+3}{4}$, $a_2 = \frac{|m_0|+1}{4}$, $b_1 = \frac{\nu_1+\nu_2}{4}$, $b_2 = \frac{-\nu_1+\nu_2}{4}$, $b_3 = \frac{\nu_1-\nu_2}{4}$, $b_4 = \frac{-\nu_1-\nu_2}{4}$. *Moreover as* $|y_1| \to \infty$, $\Phi_1^0(y_1) \sim e^{-2y_1} y_1^{-|m_0|-1}$, $\Phi_2^0(y_1) \sim e^{2y_1} y_1^{-|m_0|-1}$ *and* $\Phi_3^0(y_1)$ *and* $\Phi_4^0(y_1)$ *are moderate growth.*

Proof. If we write $\Phi_i^0(y_1) = \sum_{\varepsilon_1,\varepsilon_2 \in \{\pm 1\}} \Gamma_i^{\varepsilon_1,\varepsilon_2} \Phi_{\varepsilon_1,\varepsilon_2}^0(y_1)$ $(1 \le i \le 4)$, $\Gamma_i^{\varepsilon_1,\varepsilon_2}$ are determined by Lemma 6.2. (They are fractional of Γ-functions.) In each case, the asymptotic behavior is obtained from Lemma 6.3 and 6.4. $\square$

Secondly we treat the case where $\lambda = \varepsilon \in \{\pm 1\}$. From (9) we obtain

Proposition 9.6. *Let* $y_1 > 0$. *The following four functions are linear independent solutions of* (8) *for* $\lambda = \varepsilon$.

$$\begin{aligned}
\Phi_{\varepsilon_1,\varepsilon_2}^{\varepsilon}(y_1) = y_1^{(-|m_0|+1-2\varepsilon+\varepsilon_1\nu_1+\varepsilon_2\nu_2)/2} \\
\times \Bigg[{}_2F_3\left(\begin{matrix} \frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2-|m_0|+1}{4}, & \frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2-|m_0|+3}{4} \\ \frac{\varepsilon_1\nu_1+1}{2},\ \frac{\varepsilon_2\nu_2+1}{2}, & \frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2+2}{2} \end{matrix} \,\middle|\, y_1^2 \right) \\
- \frac{\varepsilon(\varepsilon_1\nu_1+\varepsilon_2\nu_2-|m_0|+1)y_1}{(\varepsilon_1\nu_1+1)(\varepsilon_2\nu_2+1)} \\
\times {}_2F_3\left(\begin{matrix} \frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2-|m_0|+3}{4}, & \frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2-|m_0|+5}{4} \\ \frac{\varepsilon_1\nu_1+3}{2},\ \frac{\varepsilon_2\nu_2+3}{2}, & \frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2+2}{2} \end{matrix} \,\middle|\, y_1^2 \right) \Bigg].
\end{aligned}$$

As in the case with $\lambda = 0$, we take a linear combination of them.

PROPOSITION 9.7. *Under the same assumption as in Proposition 9.6, we get the following new basis.*

$$\begin{aligned}
\Phi_1^{\varepsilon}(y_1) = y_1^{-\frac{|m_0|}{2}+1-\varepsilon}\Bigg[& G_{2,4}^{4,0}\left(y_1^2 \,\middle|\, \begin{matrix} c_1,\ c_2 \\ d_1,\ d_2,\ d_3,\ d_4 \end{matrix}\right) \\
& + \varepsilon\, G_{2,4}^{4,0}\left(y_1^2 \,\middle|\, \begin{matrix} c_1,\ c_2 \\ d_1',\ d_2',\ d_3',\ d_4' \end{matrix}\right)\Bigg], \\
\Phi_2^{\varepsilon}(y_1) = y_1^{-\frac{|m_0|}{2}+1-\varepsilon}\Bigg[& G_{2,4}^{4,0}\left(y_1^2 e^{-2\pi\sqrt{-1}} \,\middle|\, \begin{matrix} c_1,\ c_2 \\ d_1,\ d_2,\ d_3,\ d_4 \end{matrix}\right) \\
& - \varepsilon\, G_{2,4}^{4,0}\left(y_1^2 e^{-2\pi\sqrt{-1}} \,\middle|\, \begin{matrix} c_1,\ c_2 \\ d_1',\ d_2',\ d_3',\ d_4' \end{matrix}\right)\Bigg], \\
\Phi_3^{\varepsilon}(y_1) = y_1^{-\frac{|m_0|}{2}+1-\varepsilon}\Bigg[& G_{2,4}^{4,1}\left(y_1^2 e^{\pi\sqrt{-1}} \,\middle|\, \begin{matrix} c_1,\ c_2 \\ d_1,\ d_2,\ d_3,\ d_4 \end{matrix}\right) \\
& + \varepsilon\sqrt{-1}\, G_{2,4}^{4,1}\left(y_1^2 e^{\pi\sqrt{-1}} \,\middle|\, \begin{matrix} c_2,\ c_1 \\ d_1',\ d_2',\ d_3',\ d_4' \end{matrix}\right)\Bigg], \\
\Phi_4^{\varepsilon}(y_1) = y_1^{-\frac{|m_0|}{2}+1-\varepsilon}\Bigg[& G_{2,4}^{4,1}\left(y_1^2 e^{\pi\sqrt{-1}} \,\middle|\, \begin{matrix} c_1,\ c_2 \\ d_1',\ d_2',\ d_3',\ d_4' \end{matrix}\right) \\
& - \varepsilon\sqrt{-1}\, G_{2,4}^{4,1}\left(y_1^2 e^{\pi\sqrt{-1}} \,\middle|\, \begin{matrix} c_2,\ c_1 \\ d_1,\ d_2,\ d_3,\ d_4 \end{matrix}\right)\Bigg],
\end{aligned}$$

with

$$\begin{gathered}
c_1 = \tfrac{|m_0|+2}{4},\ c_2 = \tfrac{|m_0|}{4}, \\
d_1 = \tfrac{\nu_1+\nu_2-1}{4},\ d_2 = \tfrac{-\nu_1+\nu_2+1}{4},\ d_3 = \tfrac{\nu_1-\nu_2+1}{4},\ d_4 = \tfrac{-\nu_1-\nu_2-1}{4}, \\
d_1' = \tfrac{\nu_1+\nu_2+1}{4},\ d_2' = \tfrac{-\nu_1+\nu_2-1}{4},\ d_3' = \tfrac{\nu_1-\nu_2-1}{4},\ d_4' = \tfrac{-\nu_1-\nu_2+1}{4}.
\end{gathered}$$

Moreover as $|y_1| \to \infty$, $\Phi_1^{\varepsilon}(y_1) \sim e^{-2y_1} y_1^{-|m_0|-1}$, $\Phi_2^{\varepsilon}(y_1)$ *are rapidly increasing and* $\Phi_3^{\varepsilon}(y_1)$ *and* $\Phi_4^{\varepsilon}(y_1)$ *are moderate growth.*

PROOF. The proof is similar to Proposition 9.5. Here we notice that since $\Phi_i^{\varepsilon}(y_1)$ are 'sum' of ${}_2F_3$, $\Gamma_i^{\varepsilon_1,\varepsilon_2}$ must be compatible between these two functions. $\square$

§10. Multiplicity One Theorem and Explicit Formulas for Siegel-Whittaker Functions (Even Case)

Now we can state the main theorem for the even principal series. We have defined the space $\mathcal{W}_{\chi\cdot\eta,\tau_\lambda}(\pi)$ in §1. For the even principal series representation $\pi_{\sigma,\nu}$ we define the subspace $\mathcal{W}^{\mathrm{rap}}_{\chi\cdot\eta,\tau_\lambda}(\pi_{\sigma,\nu})$ of $\mathcal{W}_{\chi\cdot\eta,\tau_\lambda}(\pi_{\sigma,\nu})$ as the set of functions $\phi_{\pi_{\sigma,\nu},\tau_\lambda} \in \mathcal{W}_{\chi\cdot\eta,\tau_\lambda}(\pi_{\sigma,\nu})$ such that $\phi_{\pi_{\sigma,\nu},\tau_\lambda}|_A$ is rapidly decreasing along $h_1a_1^2 - h_2a_2^2 = 0$. Since

$$\dim_{\mathbf{C}} \mathrm{Hom}_{(\mathfrak{g},K)}(H_{\pi_{\sigma,\nu},K}, C^\infty\mathrm{Ind}_R^G(\chi\cdot\eta)^{\mathrm{rap}}) \leq \dim_{\mathbf{C}} \mathcal{W}^{\mathrm{rap}}_{\chi\cdot\eta,\tau_\lambda}(\pi_{\sigma,\nu}),$$

the contents of the previous sections can be summarized as follows.

Theorem 10.1. *We assume that H_η satisfies $h_1, h_2 > 0, h_3 = 0$ and the parameters ν_1, ν_2* of $\pi_{\sigma,\nu}$ *satisfy $\nu_1, \nu_2, \nu_1 \pm \nu_2 \notin \mathbf{Z}$. Then*

$$\dim_{\mathbf{C}} \mathrm{Hom}_{(\mathfrak{g},K)}(H_{\pi_{\sigma,\nu},K}, C^\infty\mathrm{Ind}_R^G(\chi\cdot\eta)^{\mathrm{rap}}) \leq 1.$$

Moreover

(1) *When $\lambda = (0,0)$, $\dim_{\mathbf{C}} \mathcal{W}^{\mathrm{rap}}_{\chi\cdot\eta,\tau_\lambda}(\pi_{\sigma,\nu}) \leq 1$ and the boundary value $\phi(y_1)$ of $\phi_{\pi_{\sigma,\nu},\tau_\lambda}|_A$ with respect to the characteristic exponent $|m_0|/2$ is of the form*

$$c_0\, G^{4,0}_{2,4}\left(y_1^2 \,\middle|\, \begin{matrix} \frac{|m_0|}{4}+\frac{3}{2},\ \frac{|m_0|}{4}+1 \\ \frac{\nu_1+\nu_2+3}{4},\ \frac{-\nu_1+\nu_2+3}{4},\ \frac{\nu_1-\nu_2+3}{4},\ \frac{-\nu_1-\nu_2+3}{4} \end{matrix}\right)$$

for some constant c_0.

(2) *When $\lambda = (\varepsilon,\varepsilon)$ $(\varepsilon \in \{\pm 1\})$, $\dim_{\mathbf{C}} \mathcal{W}^{\mathrm{rap}}_{\chi\cdot\eta,\tau_\lambda}(\pi_{\sigma,\nu}) \leq 1$ and the boundary value $\phi(y_1)$ of $\phi_{\pi_{\sigma,\nu},\tau_\lambda}|_A$ with respect to the characteristic exponent $|m_0|/2$ is of the form*

$$c_1\left[G^{4,0}_{2,4}\left(y_1^2 \,\middle|\, \begin{matrix} \frac{|m_0|}{4}+\frac{3}{2},\ \frac{|m_0|}{4}+1 \\ \frac{\nu_1+\nu_2+3}{4},\ \frac{-\nu_1+\nu_2+5}{4},\ \frac{\nu_1-\nu_2+5}{4},\ \frac{-\nu_1-\nu_2+3}{4} \end{matrix}\right)\right.$$
$$\left.+\ \varepsilon\, G^{4,0}_{2,4}\left(y_1^2 \,\middle|\, \begin{matrix} \frac{|m_0|}{4}+\frac{3}{2},\ \frac{|m_0|}{4}+1 \\ \frac{\nu_1+\nu_2+5}{4},\ \frac{-\nu_1+\nu_2+3}{4},\ \frac{\nu_1-\nu_2+3}{4},\ \frac{-\nu_1-\nu_2+5}{4} \end{matrix}\right)\right]$$

for some constant c_1.

Remark 10.2. We use (2) for the parameter of Meijer's G-function. We can find the boundary value of Siegel-Whittaker functions with K-type $\tau_{(\lambda,\lambda)}$ by applying compositions of some second-ordered differential operators to the ones with K-type $\tau_{(0,0)}$ or $\tau_{(1,1)}$ (cf. remark after Proposition 9.3). Since the differential of Meijer's G-function is a sum of G-functions of the same shape (cf.(3)), the boundary value can be written as a linear combination of $G^{4,0}_{2,4}$. However the explicit formula seems to be complicated. We also remark that Niwa([14]) obtained explicit formula of Siegel-Whittaker function (not only the boundary value) and multiplicity one property in the case (1).

§11. A Formula for a Meijer's G-Function and the Gamma Factor of L-Function of Siegel Wave Forms of Degree Two

As we remarked, Niwa obtained an integral representation of class one generalized Whittaker function, that is, Siegel-Whittaker function with trivial K-type, and Hori found the gamma factor of L-function of Siegel wave forms of degree two by calculating the Mellin transform of Niwa's formula.

In this section we give an integral representation of a Meijer's G-function and a simpler proof of Hori's result. We remark that our method can remove the unessential assumptions on the parameters of the principal series provided in [10].

(11.1) A formula for a Meijer's G-function

We first review Niwa's result. The parameters $\tilde{\nu_1}, \tilde{\nu_2}$ of the principal series in [15] are $\tilde{\nu_1} = (\nu_1 + \nu_2 - 1)/2$ and $\tilde{\nu_2} = (\nu_1 - \nu_2 - 1)/2$ in our notation.

Proposition 11.1 ([15, Proposition 1]). *Put* $\tilde{\nu_1} = (\sqrt{1 + 4\lambda_1} - 1)/2$, $\tilde{\nu_2} = (\sqrt{1 + 4\lambda_2} - 1)/2$ *and assume that* $-1 < \mathrm{Re}(\tilde{\nu_1}) < 0$, $-1 < \mathrm{Re}(\tilde{\nu_2}) < 0$ *and* $\lambda_1, \lambda_2 \notin \mathbf{Z}$. *For a nonnegative integer n the differential equation*

$$\begin{aligned}&\big[y^4 \tfrac{d^4}{dy^4} + 4(n+2) y^3 \tfrac{d^3}{dy^3} + \{-4\pi^2 y^2 + 6n^2 + 18n + 14 - (\lambda_1 + \lambda_2)\} y^2 \tfrac{d^2}{dy^2} \\ &\quad + \{-16\pi^2 y^2 + 4(n+1)^3 - 2(n+1)(\lambda_1 + \lambda_2)\} y \tfrac{d}{dy} \\ &\quad + \{-8\pi^2 y^2 + n^2(n+1)^2 - n(n+1)(\lambda_1 + \lambda_2) + \lambda_1 \lambda_2\}\big]\, a(y) = 0\end{aligned}$$

has unique rapidly decreasing solution as $y \to \infty$*, which is given by*

$$a(y) = \int_1^\infty \int_1^\infty P_{\tilde{\nu_1}}^n(z_1) P_{\tilde{\nu_2}}^n(z_2)(z_1^2-1)^{\frac{n}{2}}(z_2^2-1)^{\frac{n}{2}} e^{-2\pi z_1 z_2 y}\, dz_1 dz_2,$$

up to constant. Here $P_\nu^n(z)$ *is the associated Legendre function of the first kind.*

If we put $\lambda = 0$, $m_0 = 2m\,(m \in \mathbf{Z})$, $((\nu_1+\nu_2)^2-1)/4 = \lambda_1$, $((\nu_1-\nu_2)^2-1)/4 = \lambda_2$, $y_1 = \pi y$ and $\Phi(y_1) = y_1 \tilde{\Phi}(y_1)$ in (8), then we can see $\tilde{\Phi}(y_1)$ satisfies the same differential equation as in Proposition 11.1. By the multiplicity free theorem, we have an integral representation of $G_{2,4}^{4,0}(y_1^2)$. But more generally we can prove the following formula.

Proposition 11.2. *For nonnegative integers* $m_1, \ldots, m_N$ *and* $y_1 > 0$,

$$\begin{aligned}\int_1^\infty \cdots \int_1^\infty & P_{\mu_1}^{m_1}(z_1) \cdots P_{\mu_N}^{m_N}(z_N)(z_1^2-1)^{\frac{m_1}{2}} \cdots (z_N^2-1)^{\frac{m_N}{2}} \\ & \times e^{-2z_1 \cdots z_N y_1}\, dz_1 \cdots dz_N \\ = c\, G_{2N-2,2N}^{2N,0} & \left(y_1^2 \left| \begin{matrix} \frac{1}{2}, & 0, & \ldots, & \frac{1}{2}, & 0 \\ \frac{\mu_1-m_1}{2}, & \frac{-\mu_1-m_1-1}{2}, & \ldots, & \frac{\mu_N-m_N}{2}, & \frac{-\mu_N-m_N-1}{2} \end{matrix} \right. \right).\end{aligned}$$

Here the constant c *can be given explicitly.*

Proof. We first prove the case where $N=1$. By [18, 2.17.7.5],

$$\int_1^\infty P_{\mu_1}^{-m_1}(z_1)(z_1^2-1)^{\frac{m_1}{2}} e^{-2z_1 y_1} dz_1 = c'\,(2y_1)^{-m_1-\frac{1}{2}} K_{\mu_1+\frac{1}{2}}(2y_1),$$

where $K_\nu(x)$ is the modified Bessel function and c' is some constant. By using

$$P_\mu^m(z) = \frac{\Gamma(\mu+m+1)}{\Gamma(\mu-m+1)} P_\mu^{-m}(z) \qquad \text{for } m \in \mathbf{Z}_{\geq 0},$$

$$y^\nu K_\mu(y) = 2^{\nu-1}\, G_{0,2}^{2,0}\left(\frac{1}{4}y^2 \left| \begin{matrix} \\ \frac{\nu+\mu}{2}, & \frac{\nu-\mu}{2} \end{matrix} \right. \right) \qquad ([3,\ \text{p.219}]),$$

we get the assertion in this case. If we use

$$\begin{aligned}\int_1^\infty & (x^2-1)^{-\frac{\lambda}{2}} P_\nu^\lambda(x) G_{p,q}^{m,n}\left(yx^2 \left| \begin{matrix} (a_p) \\ (b_q) \end{matrix} \right. \right) dx \\ & = c''\, G_{p+2,q+2}^{m+2,n}\left(y \left| \begin{matrix} (a_p), & 0, & \frac{1}{2} \\ \frac{\lambda-1-\nu}{2}, & \frac{\lambda+\nu}{2}, & (b_q) \end{matrix} \right. \right),\end{aligned}$$

([18, 2.24.6.2]) repeatedly, we get the formula for general N. $\square$

REMARK 11.3. If we put $N = 2^{n-1}$ in the above formula, this G-function seems to be related to the boundary value of principal series Siegel-Whittaker functions on $Sp(n, \mathbf{R})$ with trivial K-type. (In case of $N = 1, 2$, this is true.)

(11.2) The gamma factor of L-function of Siegel wave forms of degree two

Let $F(Z)$ be a Siegel wave form on the Siegel upper half space H_2 of degree two ([10, Definition 1.1]). Consider the integral transform

$$\tilde{R}_F(s) = \int_0^\infty \int_{X_{1_2}(\mathbf{R})/X_{1_2}(\mathbf{Z})} F(X + \sqrt{-1}v1_2)v^{s-1}\, dX dv,$$

with $Z = X + \sqrt{-1}Y \in H_2$, $X_{1_2}(\mathbf{R}) = \{X \in M_2(\mathbf{R}) \mid {}^tX = X,\ \mathrm{tr}(X) = 0\}$ and $X_{1_2}(\mathbf{Z}) = X_{1_2}(\mathbf{R}) \cap M_2(\mathbf{Z})$. Let

$$F(Z) = \sum_{N \in \mathfrak{N}_2} a_{F,N}(Y) \exp(2\pi\sqrt{-1}\mathrm{tr}(NX))$$

be the Fourier expansion of F along P_s ([10, §1]) with $\mathfrak{N}_2 = \{N \in M_2(\mathbf{Q}) \mid {}^tN = N,\ \text{semi-integral}\}$. For definite N, we have further expansion

$$a_{F,N}(Y) = \sum_{n \in \mathbf{Z}} a_{F,N,n} W_{N,n}(Y).$$

Here $W_{N,n}(Y)$ is the class one generalized Whittaker function in [15] and in particular, $W_{1_2,0}(v1_2)$ which corresponds to $\Phi(y_1)$ in §9 and can be written as follows (up to constant).

(1) If $-1 < \mathrm{Re}((-1 + \nu_1 \pm \nu_2)/2) < 0$ and $((\nu_1 \pm \nu_2)^2 - 1)/4$ are not integers,

$$W_{1_2,0}(v1_2) = v^2 \int_1^\infty \int_1^\infty P^0_{\frac{-1+\nu_1+\nu_2}{2}}(z_1) P^0_{\frac{-1+\nu_1-\nu_2}{2}}(z_2) e^{-4\pi z_1 z_2 v}\, dz_1 dz_2,$$

(2) If ν_1, ν_2 and $\nu_1 \pm \nu_2$ are not integers,

$$W_{1_2,0}(v1_2) = G^{4,0}_{2,4}\left((2\pi v)^2 \,\middle|\, \begin{matrix} & \frac{3}{2}, & 1 & \\ \frac{\nu_1+\nu_2+3}{4}, & \frac{-\nu_1+\nu_2+3}{4}, & \frac{\nu_1-\nu_2+3}{4}, & \frac{-\nu_1-\nu_2-3}{4} \end{matrix} \right),$$

Moreover,

$$\tilde{R}_F(s) = \sum_{m\in\mathbf{N}} (a_{F,m1_2,0} + a_{F,-m1_2,0})m^{-s} \int_0^\infty W_{1_2,0}(v1_2)v^{s-1}\,dv$$

([10, §4]). Then the Mellin transformation of $W_{1_2,0}$ gives the gamma factor of the L-function.

Proposition 11.4 ([10, Proposition 4.1]). *Assume that ν_1, ν_2 and $\nu_1 \pm \nu_2$ are not integers. For* $\mathrm{Re}(s) > \max \mathrm{Re}((\pm\nu_1 \pm \nu_2 + 1)/2)$,

$$\int_0^\infty W_{1_2,0}(v1_2)v^{s-1}\,dv = c\,\pi^{-s}\Gamma\left[\begin{array}{c} \frac{s}{2}+\frac{\nu_1+\nu_2+3}{4},\ \frac{s}{2}+\frac{-\nu_1+\nu_2+3}{4},\ \frac{s}{2}+\frac{\nu_1-\nu_2+3}{4},\ \frac{s}{2}+\frac{-\nu_1-\nu_2+3}{4} \\ s+2 \end{array}\right].$$

for some constant c. Here we use the notation

$$\Gamma\left[\begin{array}{c} a_1, \ldots, a_n \\ b_1, \ldots, b_m \end{array}\right] = \prod_{i=1}^n \Gamma(a_i) \Big/ \prod_{i=1}^m \Gamma(b_i).$$

Proof. This formula can be easily shown by using

$$\int_0^\infty x^{s-1} G^{m,n}_{p,q}\left(x \,\middle|\, \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array}\right) dx = \frac{\prod_{i=1}^m \Gamma(b_i+s)\prod_{i=1}^n \Gamma(1-a_i-s)}{\prod_{j=n+1}^p \Gamma(a_j+s)\prod_{j=m+1}^q \Gamma(1-b_j-s)},$$

for $0 \le m \le q$, $0 \le n \le p$, $p+q < 2(m+n)$ and $-\min_{1\le j\le m}\mathrm{Re}(b_j) < \mathrm{Re}(s) < 1 - \max_{1\le j\le n}\mathrm{Re}(a_j)$ ([4, p.337]), and $\Gamma(2z) = 2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma(z+\frac{1}{2})$. For our purpose we must extend the domain of convergence. Since $G^{4,0}_{2,4}(x) = O(|x|^{\max_{1\le j\le m}\mathrm{Re}(b_j)})$ (as $x \to 0$) and $G^{4,0}_{2,4}(x) = O(e^{-2\sqrt{x}})$ (as $x \to \infty$) ([3, p.212]), we get the assertion for $\mathrm{Re}(s) > \max\mathrm{Re}((\pm\nu_1\pm\nu_2+1)/2)$. □

Remark 11.5. We give the Mellin transformation of the boundary value in the case where the K-type is $\tau_{(\varepsilon,\varepsilon)}$ ($\varepsilon \in \{\pm 1\}$) and $m_0 = 0$. For

$\mathrm{Re}(s) > \max \mathrm{Re}((\pm\nu_1 \pm \nu_2 + 1)/2)$,

$$\int_0^\infty \phi((2\pi v)^2) v^{s-1}\,dv$$
$$= c\,\pi^{-s}\Bigg(\Gamma\left[\begin{matrix} \frac{s}{2}+\frac{\nu_1+\nu_2+3}{4}, & \frac{s}{2}+\frac{-\nu_1+\nu_2+5}{4}, \frac{s}{2}+\frac{\nu_1-\nu_2+5}{4}, & \frac{s}{2}+\frac{-\nu_1-\nu_2+3}{4} \\ & s+2 & \end{matrix}\right]$$
$$+\varepsilon\,\Gamma\left[\begin{matrix} \frac{s}{2}+\frac{\nu_1+\nu_2+5}{4}, & \frac{s}{2}+\frac{-\nu_1+\nu_2+3}{4}, \frac{s}{2}+\frac{\nu_1-\nu_2+3}{4}, & \frac{s}{2}+\frac{-\nu_1-\nu_2+5}{4} \\ & s+2 & \end{matrix}\right]\Bigg)$$

This is compatible with the gamma factor of spinor L-function in terms of Langlands parameter of principal series representations (cf.[2]).

From now on we study the Siegel-Whittaker function for the odd principal series representation with two dimensional K-type. Though it is vector-valued, the procedure is similar to that in the even case.

§12. Reduction of the System of Differential Equations

PROPOSITION 12.1. *Under the same assumptions as in Proposition 5.5, we write*

$$b_0(a_1,a_2) = (\sqrt{h_1}a_1)^{\lambda+1}(\sqrt{h_2}a_2)^{\lambda}\exp\{-2\pi(h_1a_1^2+h_2a_2^2)\}\,c_0(a_1,a_2)$$
$$b_1(a_1,a_2) = (\sqrt{h_1}a_1)^{\lambda}(\sqrt{h_2}a_2)^{\lambda+1}\exp\{-2\pi(h_1a_1^2+h_2a_2^2)\}\,c_1(a_1,a_2)$$

and introduce new variables $y=(y_1,y_2)=(2\pi h_1a_1^2, 2\pi h_2a_2^2)$. *Further put* $c_i(y)=\mathcal{L}^{(|m_0|-1)/2}f_i(y)$ $(i=1,2)$. *Then we have*

$$\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}\begin{pmatrix} f_0(y) \\ f_1(y) \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}\begin{pmatrix} f_0(y) \\ f_1(y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$\begin{aligned} P_{11} = {} & y_1^2D_1^2 + \{\lambda - 2y_1 + |m_0|y_1(y_1-y_2)^{-1}\}y_1D_1 - \tfrac{1}{2}|m_0|y_1y_2(y_1-y_2)^{-2} \\ & + \{-|m_0|y_1 + \tfrac{1}{4}|m_0|(|m_0|+2\lambda-2)\}y_1(y_1-y_2)^{-1} \\ & + \tfrac{1}{4}\{(\lambda-1)^2-\nu^2\}, \\ P_{12} = {} & \tfrac{\sqrt{-1}}{2}|m_0|y_2(y_1-y_2)^{-1}\{E_y - 2y_1 - y_1(y_1-y_2)^{-1} + \lambda + \tfrac{1}{2}|m_0| - 1\}, \end{aligned}$$

$$P_{21} = \tfrac{\sqrt{-1}}{2}|m_0|y_1(y_1-y_2)^{-1}\{E_y - 2y_2 + y_2(y_1-y_2)^{-1} + \lambda + \tfrac{1}{2}|m_0| - 1\},$$
$$\begin{aligned}P_{22} = {} & y_2^2D_2^2 + \{\lambda - 2y_2 - |m_0|y_2(y_1-y_2)^{-1}\}y_2D_2 - \tfrac{1}{2}|m_0|y_1y_2(y_1-y_2)^{-2} \\ & + \{|m_0|y_2 - \tfrac{1}{4}|m_0|(|m_0| + 2\lambda - 2)\}y_2(y_1-y_2)^{-1} \\ & + \tfrac{1}{4}\{(\lambda-1)^2 - \nu^2\},\end{aligned}$$
$$Q_{11} = Q + y_1D_1 + \tfrac{1}{2}|m_0|y_2(y_1-y_2)^{-1},$$
$$Q_{12} = \tfrac{\sqrt{-1}}{2}|m_0|(y_1+y_2)y_2(y_1-y_2)^{-2},$$
$$Q_{21} = \tfrac{\sqrt{-1}}{2}|m_0|(y_1+y_2)y_1(y_1-y_2)^{-2},$$
$$Q_{22} = Q + y_2D_2 - \tfrac{1}{2}|m_0|y_1(y_1-y_2)^{-1},$$

with

$$\begin{aligned}Q = {} & E_y^2 - 2y_1y_2D_1D_2 + (\lambda-2)E_y - 2(y_1^2D_1 + y_2^2D_2) \\ & + |m_0|(y_1-y_2)^{-1}(y_1^2D_1 - y_2^2D_2) \\ & - |m_0|y_1y_2(y_1-y_2)^{-2} - |m_0|(y_1+y_2) \\ & + \tfrac{1}{4}|m_0|^2 + \tfrac{1}{2}(\lambda-1)|m_0| + \tfrac{1}{2}\lambda^2 - \tfrac{3}{2}\lambda + \tfrac{1}{4}(\nu_1^2 + \nu_2^2 - 5).\end{aligned}$$

§13. Characteristic Indices

As in the even case, we write the formal solution at Q_0, Q_∞ and Q_1 as

$$\begin{pmatrix} f_0(y) \\ f_1(y) \end{pmatrix} = \sum_{m,n\geq 0} \begin{pmatrix} c^0_{m,n} \\ d^0_{m,n} \end{pmatrix} y_1^{\sigma_1+m}u^{\sigma_2+n} \text{ with } \begin{pmatrix} c^0_{0,0} \\ d^0_{0,0} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} f_0(y) \\ f_1(y) \end{pmatrix} = \sum_{m,n\geq 0} \begin{pmatrix} c^\infty_{m,n} \\ d^\infty_{m,n} \end{pmatrix} v^{\rho_1+m}y_2^{\rho_2+n} \text{ with } \begin{pmatrix} c^\infty_{0,0} \\ d^\infty_{0,0} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} f_0(y) \\ f_1(y) \end{pmatrix} = \sum_{m,n\geq 0} \begin{pmatrix} c^1_{m,n} \\ d^1_{m,n} \end{pmatrix} y_1^{\tau_1+m}(u-1)^{\tau_2+n} \text{ with } \begin{pmatrix} c^1_{0,0} \\ d^1_{0,0} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In a similar way to 8.1, 8.2 and 8.3, we have

Proposition 13.1. *The characteristic indices at Q_0, Q_∞ and Q_1 are*

$$\begin{aligned}(\sigma_1, \sigma_2) = {} & \big(\tfrac{1}{2}(\varepsilon\nu_1 \pm \nu_2 - |m_0| - 2\lambda + 1),\ \tfrac{1}{2}(\varepsilon\nu - \lambda + 1)\big), \\ & \big(\tfrac{1}{2}(\varepsilon\nu_1 \pm \nu_2 - |m_0| - 2\lambda + 3),\ \tfrac{1}{2}(\varepsilon\tilde{\nu} - \lambda + 2)\big),\end{aligned}$$

$$\begin{aligned}(\rho_1,\rho_2) = &\big(\tfrac{1}{2}(\varepsilon\nu-\lambda+2),\ \tfrac{1}{2}(\pm\nu_1+\varepsilon\nu_2-|m_0|-2\lambda+3)\big),\\ &\big(\tfrac{1}{2}(\varepsilon\tilde{\nu}-\lambda+1),\ \tfrac{1}{2}(\pm\nu_1+\varepsilon\nu_2-|m_0|-2\lambda+1)\big),\end{aligned}$$

and

$$\begin{aligned}(\tau_1,\tau_2) = &\big(\tfrac{1}{2}(\pm\nu_1\pm\nu_2-|m_0|-2\lambda+3),\ 0\big),\\ &\big(\tfrac{1}{2}(\pm\nu_1\pm\nu_2-|m_0|-2\lambda+3),\ -|m_0|\big).\end{aligned}$$

Here $\varepsilon\in\{\pm1\}$ *and* $\tilde{\nu}=\nu_1+\nu_2-\nu$.

§14. Holomorphic Solutions Along the Singular Divisor Y_1

As in the even case, we study the 4-dimensional space of holomorphic solutions along Y_1. For each nonnegative integer n, if we put $\varphi_n(y_1) = \sum_{m\geq 0} c_{m,n}\, y_1^{\tau_1+m}$ and $\psi_n(y_1) = \sum_{m\geq 0} d_{m,n}\, y_1^{\tau_1+m}$ then we have

$$\begin{pmatrix} f_0(y)\\ f_1(y)\end{pmatrix} = \begin{pmatrix} \varphi(y_1,u-1)\\ \psi(y_1,u-1)\end{pmatrix} = \sum_{n\geq 0}\begin{pmatrix} \varphi_n(y_1)\\ \psi_n(y_1)\end{pmatrix}(u-1)^n.$$

Since the odd principal series Siegel-Whittaker function with K-type $\tau_{(\lambda,\lambda-1)}$ is written by $b_0(y_1,y_2)v_0 + b_1(y_1,y_2)v_1$ with

$$\begin{aligned} b_0(y_1,y_2) &= \big(\tfrac{y_1}{2\pi}\big)^{\frac{\lambda+1}{2}}\big(\tfrac{y_2}{2\pi}\big)^{\frac{\lambda}{2}} e^{-(y_1+y_2)}\big(\tfrac{y_1-y_2}{2\pi}\big)^{\frac{|m_0|-1}{2}}\textstyle\sum_{n\geq 0}\varphi_n(y_1)(u-1)^n,\\ b_1(y_1,y_2) &= \big(\tfrac{y_1}{2\pi}\big)^{\frac{\lambda}{2}}\big(\tfrac{y_2}{2\pi}\big)^{\frac{\lambda+1}{2}} e^{-(y_1+y_2)}\big(\tfrac{y_1-y_2}{2\pi}\big)^{\frac{|m_0|-1}{2}}\textstyle\sum_{n\geq 0}\psi_n(y_1)(u-1)^n,\end{aligned}$$

(see §12), the boundary value of $\begin{pmatrix} b_0(y)\\ b_1(y)\end{pmatrix}$ with respect to the characteristic exponent $(|m_0|-1)/2$ along $u-1=0$ is

$$(-1)^{\frac{|m_0|-1}{2}}\Big(\frac{y_1}{2\pi}\Big)^{\lambda+\frac{|m_0|}{2}} e^{-2y_1}\begin{pmatrix}\varphi_0(y_1)\\ \psi_0(y_1)\end{pmatrix}.$$

We shall give an explicit formula for $\varphi_0(y_1)$ and $\psi_0(y_1)$ and prove a multiplicity free theorem in a way similar to the even case.

(14.1) Differential equation for $\varphi_0(y_1)$

Proposition 14.1. *Put* $\Phi(y_1) = e^{-2y_1}\varphi_0(y_1)$. *Then*

$$\psi_0(y_1) = \sqrt{-1}\,\mathrm{sgn}(m_0)\,\varphi_0(y_1),$$

$$
\begin{aligned}
(10)\quad \Big[&256y_1^3(|m_0|-1)(\theta+\lambda)(\theta+\lambda-1) \\
&-64y_1^2(\theta+\lambda-1)\big[\{2(|m_0|-1)(2\lambda-1)+(\nu_1^2+\nu_2^2-2\nu^2)\}\theta \\
&\qquad +(\lambda+1)(\nu_1^2+\nu_2^2-2\nu^2)+(|m_0|-1)(2\lambda-1)(2\lambda+|m_0|-3)\big] \\
&+4y_1\big[-16(|m_0|-1)\theta^4-16(|m_0|-1)(4\lambda+2|m_0|-7)\theta^3 \\
&\quad +8[(2\lambda+|m_0|-2)(\nu_1^2+\nu_2^2-2\nu^2) \\
&\qquad +(|m_0|-1)\{2\nu^2+3(2\lambda+|m_0|-3)(2\lambda+|m_0|-4)\}]\theta^2 \\
&\quad +4[(8\lambda^2+6|m_0|\lambda-14\lambda+2|m_0|^2-11|m_0|+11)(\nu_1^2+\nu_2^2-2\nu^2) \\
&\qquad +(|m_0|-1)\{4(2\lambda+|m_0|-4)\nu^2 \\
&\qquad +(2\lambda+|m_0|-3)^2(4\lambda+2|m_0|-9)\}]\theta \\
&\quad -[(|m_0|-3)(\nu_1^2+\nu_2^2-2\nu^2)^2-2\{8\lambda^3+4(2|m_0|-5)\lambda^2 \\
&\qquad +2(2|m_0|^2-13|m_0|+13)\lambda+(|m_0|-1)(|m_0|^2-8|m_0|+17)\} \\
&\qquad \times(\nu_1^2+\nu_2^2-2\nu^2)+(|m_0|-1)(2\lambda+|m_0|-3)(2\lambda+m_0-5) \\
&\qquad\qquad \times(2\lambda+|m_0|-3-2\nu)(2\lambda+m_0-5+2\nu)]\big] \\
&+(\nu_1^2+\nu_2^2-2\nu^2)\{(2\theta+2\lambda+|m_0|-3+\nu)^2-(\nu_1^2+\nu_2^2-\nu^2)\} \\
&\qquad \times\{(2\theta+2\lambda+|m_0|-3-\nu)^2-(\nu_1^2+\nu_2^2-\nu^2)\}\Big]\,\Phi(y_1)=0,
\end{aligned}
$$

with $\mathrm{sgn}(m_0)$ *is* $+1$ *for* $m_0>0$ *and* -1 *for* $m_0<0$.

Since the shift operators move the K-type parameter $(\lambda,\lambda-1)$ to $(\lambda-1,\lambda-2)$ or $(\lambda+1,\lambda)$, we need only solve (10) in the case where $\lambda=0$.

(14.2) Solutions of the differential equation

We solve (10) when $\lambda=0$, $\nu=\nu_1$. Let us write the formal solution at the origin, $\Phi(y_1)=\sum_{m\ge 0}c_m\,y_1^{\sigma+m}$ with $c_0\neq 0$. By using $\sigma=\tau_1=(3-m_0+\varepsilon_1\nu_1+\varepsilon_2\nu_2)/2$, $\varepsilon_i\in\{\pm 1\}$ $(i=1,2)$, we obtain

Proposition 14.2. *Let* $y_1>0$. *The following four functions are linear independent solutions of* (10) *for* $\lambda=0,\nu=\nu_1$.

$$
\begin{aligned}
\Phi_{\varepsilon_1,\varepsilon_2}(y_1)&=y_1^{(-m_0+3+\varepsilon_1\nu_1+\varepsilon_2\nu_2)/2} \\
&\quad\times\left[{}_2F_3\left(\begin{matrix}\frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2-|m_0|+1}{4},\ \frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2-|m_0|+3}{4} \\ \frac{\varepsilon_1\nu_1+2}{2},\ \frac{\varepsilon_2\nu_2+1}{2},\ \frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2+1}{2}\end{matrix}\ \middle|\ y_1^2\right)\right.
\end{aligned}
$$

$$+\frac{(\varepsilon_1\nu_1+\varepsilon_2\nu_2-|m_0|+1)y_1}{(\varepsilon_2\nu_2+1)(\varepsilon_1\nu_1+\varepsilon_2\nu_2+1)}$$
$$\times\, {}_2F_3\left(\left.\begin{matrix} \frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2-|m_0|+3}{4}, & \frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2-|m_0|+5}{4} \\ \frac{\varepsilon_1\nu_1+2}{2},\ \frac{\varepsilon_2\nu_2+3}{2}, & \frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2+3}{2}\end{matrix}\ \right|\ y_1^2\right)\Bigg].$$

As in the even case we take a linear combination.

PROPOSITION 14.3. *Under the same assumptions in Proposition 14.2, we get the following new basis.*

$$\begin{aligned}
\Phi_1(y_1) &= y_1^{-|m_0|/2+2}\left[G^{4,0}_{2,4}\left(y_1^2\,\middle|\,\begin{matrix} a_1,\ a_2 \\ b_1,\ b_2,\ b_3,\ b_4\end{matrix}\right)\right.\\
&\qquad\left. - G^{4,0}_{2,4}\left(y_1^2\,\middle|\,\begin{matrix} a_1,\ a_2 \\ b_1',\ b_2',\ b_3',\ b_4'\end{matrix}\right)\right],\\
\Phi_2(y_1) &= y_1^{-|m_0|/2+2}\left[G^{4,0}_{2,4}\left(y_1^2e^{-2\pi\sqrt{-1}}\,\middle|\,\begin{matrix} a_1,\ a_2 \\ b_1,\ b_2,\ b_3,\ b_4\end{matrix}\right)\right.\\
&\qquad\left. + G^{4,0}_{2,4}\left(y_1^2e^{-2\pi\sqrt{-1}}\,\middle|\,\begin{matrix} a_1,\ a_2 \\ b_1',\ b_2',\ b_3',\ b_4'\end{matrix}\right)\right],\\
\Phi_3(y_1) &= y_1^{-|m_0|/2+2}\left[G^{4,1}_{2,4}\left(y_1^2e^{\pi\sqrt{-1}}\,\middle|\,\begin{matrix} a_1,\ a_2 \\ b_1,\ b_2,\ b_3,\ b_4\end{matrix}\right)\right.\\
&\qquad\left. + \sqrt{-1}\,G^{4,1}_{2,4}\left(y_1^2e^{\pi\sqrt{-1}}\,\middle|\,\begin{matrix} a_2,\ a_1 \\ b_1',\ b_2',\ b_3',\ b_4'\end{matrix}\right)\right],\\
\Phi_4(y_1) &= y_1^{-|m_0|/2+2}\left[G^{4,1}_{2,4}\left(y_1^2e^{\pi\sqrt{-1}}\,\middle|\,\begin{matrix} a_1,\ a_2 \\ b_1',\ b_2',\ b_3',\ b_4'\end{matrix}\right)\right.\\
&\qquad\left. + \sqrt{-1}\,G^{4,1}_{2,4}\left(y_1^2e^{\pi\sqrt{-1}}\,\middle|\,\begin{matrix} a_2,\ a_1 \\ b_1,\ b_2,\ b_3,\ b_4\end{matrix}\right)\right],
\end{aligned}$$

where

$$a_1=\tfrac{|m_0|+2}{4},\ a_2=\tfrac{|m_0|}{4},$$
$$b_1=\tfrac{\nu_1+\nu_2-1}{4},\ b_2=\tfrac{-\nu_1+\nu_2-1}{4},\ b_3=\tfrac{\nu_1-\nu_2+1}{4},\ b_4=\tfrac{-\nu_1-\nu_2+1}{4},$$
$$b_1'=\tfrac{\nu_1+\nu_2+1}{4},\ b_2'=\tfrac{-\nu_1+\nu_2+1}{4},\ b_3'=\tfrac{\nu_1-\nu_2-1}{4},\ b_4'=\tfrac{-\nu_1-\nu_2-1}{4}.$$

Moreover as $|y_1|\to\infty$, $\Phi_1(y_1)\sim e^{-2y_1}y_1^{-|m_0|-1}$, $\Phi_2(y_1)\sim e^{2y_1}y_1^{-|m_0|-3}$ *and* $\Phi_3(y_1)$ *and* $\Phi_4(y_1)$ *are moderate growth.*

§15. Multiplicity One Theorem and Explicit Formulas for Siegel-Whittaker Functions (Odd Case)

We state the main theorem for odd principal series $\pi_{\sigma,\nu}$. Define the subspace $\mathcal{W}^{\mathrm{rap}}_{\chi\cdot\eta,\tau_\lambda}(\pi_{\sigma,\nu})$ of $\mathcal{W}_{\chi\cdot\eta,\tau_\lambda}(\pi_{\sigma,\nu})$ as the set of functions $\phi_{\pi_{\sigma,\nu},\tau_\lambda} \in \mathcal{W}_{\chi\cdot\eta,\tau_\lambda}(\pi_{\sigma,\nu})$ such that $\phi_{\pi_{\sigma,\nu},\tau_\lambda}|_A$ is rapidly decreasing along $h_1a_1^2-h_2a_2^2=0$. Since

$$\dim_{\mathbf{C}} \mathrm{Hom}_{(\mathfrak{g},K)}(H_{\pi_{\sigma,\nu},K}, C^\infty\mathrm{Ind}_R^G(\chi\cdot\eta)^{\mathrm{rap}}) \le \dim_{\mathbf{C}} \mathcal{W}^{\mathrm{rap}}_{\chi\cdot\eta,\tau_\lambda}(\pi_{\sigma,\nu}),$$

by summarizing the previous sections we obtain the main theorem.

Theorem 15.1. *We assume that H_η satisfies $h_1, h_2 > 0, h_3 = 0$ and the parameters ν_1,ν_2* of $\pi_{\sigma,\nu}$ *satisfy $\nu_1,\nu_2,\nu_1\pm\nu_2 \notin \mathbf{Z}$. Then*

$$\dim_{\mathbf{C}} \mathrm{Hom}_{(\mathfrak{g},K)}(H_{\pi_{\sigma,\nu},K}, C^\infty\mathrm{Ind}_R^G(\chi\cdot\eta)^{\mathrm{rap}}) \le 1.$$

Moreover if $\lambda = (0,-1)$ then $\dim_{\mathbf{C}} \mathcal{W}^{\mathrm{rap}}_{\chi\cdot\eta,\tau_\lambda}(\pi_{\sigma,\nu}) \le 1$ and the boundary value $\phi(y_1) = \begin{pmatrix} \phi_0(y_1) \\ \sqrt{-1}\mathrm{sgn}(m_0)\cdot\phi_0(y_1)\end{pmatrix}$ of $\phi_{\pi_{\sigma,\nu},\tau_\lambda}|_A$ with respect to the characteristic exponent $(|m_0|-1)/2$ is of the form

$$\phi_0(y_1) = c\left[G^{4,0}_{2,4}\left(y_1^2 \,\middle|\, \begin{matrix} \frac{|m_0|}{4}+\frac{3}{2},\ \frac{|m_0|}{4}+1 \\ \frac{\nu_1+\nu_2+3}{4},\ \frac{-\nu_1+\nu_2+3}{4},\ \frac{\nu_1-\nu_2+5}{4},\ \frac{-\nu_1-\nu_2+5}{4}\end{matrix}\right) - G^{4,0}_{2,4}\left(y_1^2 \,\middle|\, \begin{matrix} \frac{|m_0|}{4}+\frac{3}{2},\ \frac{|m_0|}{4}+1 \\ \frac{\nu_1+\nu_2+5}{4},\ \frac{-\nu_1+\nu_2+5}{4},\ \frac{\nu_1-\nu_2+3}{4},\ \frac{-\nu_1-\nu_2+3}{4}\end{matrix}\right)\right]$$

for some constant c.

Remark 15.2. As in the even case, we give the Mellin transform of the boundary value. For $\mathrm{Re}(s) > \max \mathrm{Re}(\pm\nu_1\pm\nu_2+1)/2$,

$$\begin{aligned}&\int_0^\infty \phi_1((2\pi v)^2)v^{s-1}\,dv \\ &\quad = c\,\pi^{-s}\left(\Gamma\left[\begin{matrix} \frac{s}{2}+\frac{\nu_1+\nu_2+3}{4},\ \frac{s}{2}+\frac{-\nu_1+\nu_2+3}{4},\ \frac{s}{2}+\frac{\nu_1-\nu_2+5}{4},\ \frac{s}{2}+\frac{-\nu_1-\nu_2+5}{4} \\ s+\frac{5}{2}\end{matrix}\right]\right. \\ &\qquad \left. + \Gamma\left[\begin{matrix} \frac{s}{2}+\frac{\nu_1+\nu_2+5}{4},\ \frac{s}{2}+\frac{-\nu_1+\nu_2+5}{4},\ \frac{s}{2}+\frac{\nu_1-\nu_2+3}{4},\ \frac{s}{2}+\frac{-\nu_1-\nu_2+3}{4} \\ s+\frac{5}{2}\end{matrix}\right]\right)\end{aligned}$$

This is compatible with the gamma factor of spinor L-function in terms of Langlands parameter of odd principal series representations.

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