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Real Shintani Functions on $\mathsf{U}(n,1)$ *II, Computation of Zeta Integrals*

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Abstract. We explicitly evaluate the archimedian local zeta integral arising from a certain Rankin-Selberg integral considered by Murase-Sugano associated with cusp forms on real-rank-one unitary groups.

1. Introduction and Basic Notations

1.1. Introduction

The aim of this paper is to compute explicitly the archimedian local zeta integrals of the real Shintani functions arising from a certain Rankin-Selberg integral considered by Murase-Sugano for unitary groups. Let G be a classical group defined over $\mathbb{Q}$ which acts on a vector space preserving a non-degenerate ϵ-hermitian form and G_0 the stabilizer in G of a non-zero vector. For a pair of cusp forms f and F respectively on $\mathsf{G}_0(\mathbb{A})$ and $\mathsf{G}(\mathbb{A})$, Murase-Sugano introduced a generalized spherical function on $\mathsf{G}(\mathbb{A})$, say $\mathcal{W}_{f,F}$, which they call *global Shintani function* ([2]). Using it, they study Rankin-Selberg type integrals attached to f and F, to evaluate them when f and F are holomorphic Hecke eigen cusp forms in terms of the standard L-functions of f and F in many cases ([2], [3], [4] and [5]). We can remove this assumption of holomorphy at least when $\mathsf{G}_0(\mathbb{R})$ and $\mathsf{G}(\mathbb{R})$ are both real rank one unitary groups and calculate the local zeta integrals in a general situation (Theorem 7.2.1). This is because our knowlegde of the real Shintani functions is developed enough for such groups ([7]).

Now we shall explain contents of this paper briefly. We recall a few standard concepts concerning automorphic forms on unitary groups to fix notations in the next section. Sections 3 and 4 are preliminary in nature, where we first recall the basic settings in the theory of Murase and Sugano, and then

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introduce a vector-valued Eisenstein series which is involved in the Rankin-Selberg integral (6.1.1). Through a standard procedure of unwinding the integrals, we get the so called basic identity, that relates the Rankin-Selberg integral to a certain integral transform of the global Shintani function, as is proved by Murase-Sugano. We reproduce the proof of it in a 'vector-valued' situation, for the sake of completeness of this article.

We can use the multiplicity one theorem for the real Shintani functional (Theorem 5.1.1) to know that the global Shintani function $\mathcal{W}_{f,F}$ defined on $\mathsf{G}(\mathbb{A})$ is decompsed into a product of two functions, say $\mathcal{W}^{\mathrm{fin}}$ and $\mathcal{W}^{\infty}$, such that for any $g \in \mathsf{G}(\mathbb{A})$ the values $\mathcal{W}^{\mathrm{fin}}(g)$ and $\mathcal{W}^{\infty}(g)$ depend only on the finite part of g and the infinite part of g respectively (Proposition 5.2.1). Thus the necessary calculation of the integral transform is reduced to the computation of the zeta integrals attached to $\mathcal{W}^{\mathrm{fin}}$ and $\mathcal{W}^{\infty}$, which we can consider purely locally. The caluculation of zeta integrals for $\mathcal{W}^{\mathrm{fin}}$ is carried out by Murase-Sugano. The main body of this article is section 7, which is devoted to the evaluation of the zeta integrals for $\mathcal{W}^{\infty}$ without any assumption on the representation of $\mathsf{G}_0(\mathbb{R}) \times \mathsf{G}(\mathbb{R})$ generated by $\mathcal{W}^{\infty}$.

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1.2. Notations

For any number field F, let $\mathbb{A}_F$ be the ring of adeles of F and $\mathbb{A}_{F,\mathrm{f}}$ the ring of finite adeles. Put $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ and $\mathbb{A}_{\mathrm{f}} = \mathbb{A}_{\mathbb{Q},\mathrm{f}}$.

The unit group of a ring R is denoted by R^*.

Let R be a locally compact topological ring. For any $x \in R^*$, the modulus of the automorphism $a \mapsto xa$ of the underlying additive topological group R is denoted by $|x|_R$. For a number field F, we put $|x|_F = |x|_{\mathbb{A}_F},\ x \in \mathbb{A}_F^*$.

For a vector space V over a field F, $V^\vee$ denotes the dual space of V, $\langle\ ,\ \rangle : V \times V^\vee \to F$ the natural F-bi-linear form and $\mathtt{I}_V$ the identity map of V. For finite dimensional F-vector spaces V and W, we always identify $(V \otimes_F W)^\vee$ with $V^\vee \otimes_F W^\vee$, and $V^{\vee\vee}$ with V by means of the canonical isomorphism.

Let F be a commutative ring. For a given positive integer n, let $F^n = M_{n,1}(F)$ be the space of all column vectors with n entries. We naturally identify the space $\mathrm{End}_F(F^n)$ with $M_n(F)$ by letting a matrix $A = (a_{ij})_{1 \leqslant i,j \leqslant n} \in M_n(F)$ operate on $x = (x_i)_{1 \leqslant i \leqslant n} \in F^n$ as $Ax =$

$(\sum_{j=1}^{n} a_{ij}x_j)_{1\leqslant i\leqslant n} \in F^n$. We write $\mathtt{I}_n$ for $\mathtt{I}_{F^n}$. For positive integers p and q, we write $\mathtt{O}_{p,q}$ the $p \times q$-matrix whose entries are all zero.

Let G be a real reductive group, K a maximal compact subgroup and $\mathfrak{g}$ the complexified Lie algebra of G. Given a $(\mathfrak{g}, K)$-module (π, H_π), we write $(\pi^\vee, H_\pi^\vee)$ for the contragredient $(\mathfrak{g}, K)$-module of π and $(\pi^\infty, H_\pi^\infty)$ for the smooth Fréchet G-module of moderate growth which is the canonical extention of π in the sense of [1].

2. Automorphic Forms on Unitary Groups

2.1. Unitary groups

Let E be an imaginary quadratic extension of $\mathbb{Q}$, $\mathcal{O}_E$ the ring of integers of E. The non-trivial automorphism of E over $\mathbb{Q}$ is denoted by $x \mapsto \bar{x}$.

Let V be a finite dimensional E-vector space and $\mathtt{S} : \mathsf{V} \times \mathsf{V} \to E$ a non-degenerate skew Hermitian form. Let $\mathsf{U}(\mathtt{S})$ be the automorphism group of the skew Hermitian space $(\mathsf{V}, \mathtt{S})$, i.e., $\mathsf{U}(\mathtt{S})$ is the algebraic group over $\mathbb{Q}$ whose set of R-valued points is

$$\mathsf{U}(\mathtt{S}\,; R) = \{g \in \mathsf{GL}(\mathsf{V} \otimes_{\mathbb{Q}} R) |\ \mathtt{S}(g(v), g(w)) = \mathtt{S}(v, w),\ \forall v, w \in \mathsf{V} \otimes_{\mathbb{Q}} R\}$$

for any $\mathbb{Q}$-algebra R; $\mathsf{U}(\mathtt{S})$ is a connected reductive algebraic group.

2.2. Automorphic forms

For a reductive algebraic group G over $\mathbb{Q}$ and a maximal compact subgroup K_∞ of $\mathsf{G}(\mathbb{R})$, let $\mathfrak{S}(\mathsf{G})$ denote the space of cusp forms (with respect to K_∞) on $\mathsf{G}(\mathbb{A})$ in the sense of [6].

The space $\mathfrak{S}(\mathsf{G})$ carries a $(\mathfrak{g}, K_\infty) \times \mathsf{G}(\mathbb{A}_\mathrm{f})$-module structure naturally, that is induced from the right translation on the space of functions on $\mathsf{G}(\mathbb{A})$. Here we put $\mathfrak{g} = \mathrm{Lie}(\mathsf{G}(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}$.

Let (τ, W_τ) be a finite dimensional unitary representation of K_∞ and K_0 an open compact subgroup of $\mathsf{G}(\mathbb{A}_\mathrm{f})$. We put

$$\mathfrak{S}_\tau(\mathsf{G}) = \mathrm{Hom}_{K_\infty}(W_\tau, \mathfrak{S}(\mathsf{G})) = (W_\tau^\vee \otimes \mathfrak{S}(\mathsf{G}))^{K_\infty},$$

which we consider to be a subspace of $W_\tau^\vee$-valued smooth functions on $\mathsf{G}(\mathbb{A})$ naturally. The space $\mathfrak{S}_\tau(\mathsf{G})$ inherits a smooth $\mathsf{G}(\mathbb{A}_\mathrm{f})$-module structure from that of $\mathfrak{S}(\mathsf{G})$. Let $\mathfrak{S}_\tau(\mathsf{G}\,; K_0)$ be the K_0-invariant part of $\mathfrak{S}_\tau(\mathsf{G})$.

Let (π, H_π) be an irreducible $(\mathfrak{g}, K_\infty)$-module. We define $\mathfrak{S}(\mathsf{G})_\pi$, the space of π-*cusp forms* on $\mathsf{G}(\mathbb{A})$, to be the subspace of $\mathfrak{S}(\mathsf{G})$ generated by $\mathrm{Im}(\psi)$'s with $\psi : H_\pi \to \mathfrak{S}(\mathsf{G})$ ranging over $(\mathfrak{g}, K_\infty)$-intertwining operators. Put

$$\mathfrak{S}_\tau(\mathsf{G})_\pi = \mathrm{Hom}_{K_\infty}(W_\tau, \mathfrak{S}(\mathsf{G})_\pi),$$
$$\mathfrak{S}_\tau(\mathsf{G}\,; K_0)_\pi = \mathfrak{S}_\tau(\mathsf{G})_\pi \cap \mathfrak{S}_\tau(\mathsf{G}\,; K_0).$$

3. Preliminary Constructions

3.1. Embeddings of vector spaces

Let $\mathsf{V}_0 = E^n$ and S_0 a non-degenerate skew Hermitian matrix of size $n \geqslant 2$. Put $\mathsf{S}_0(\mathsf{v}_0, \mathsf{v}_0') = {}^t\bar{\mathsf{v}}_0 \mathsf{S}_0 \mathsf{v}_0'$ for $\mathsf{v}_0, \mathsf{v}_0' \in \mathsf{V}_0$.

We put

$$\mathsf{V}_1 = \begin{pmatrix} E \\ \mathsf{V}_0 \\ E \end{pmatrix} = E^{n+2}, \quad \mathsf{e}^+ = \begin{pmatrix} 1 \\ \mathsf{0}_{n,1} \\ 0 \end{pmatrix}, \ \mathsf{e}^- = \begin{pmatrix} 0 \\ \mathsf{0}_{n,1} \\ 1 \end{pmatrix}$$

and define the skew Hermitian form $\mathsf{S}_1 : \mathsf{V}_1 \times \mathsf{V}_1 \to E$ by

$$\mathsf{S}_1(\mathsf{v}_1, \mathsf{v}_1') = {}^t\bar{\mathsf{v}}_1 \begin{pmatrix} & & -1 \\ & \mathsf{S}_0 & \\ 1 & & \end{pmatrix} \mathsf{v}_1', \quad \mathsf{v}_1, \mathsf{v}_1' \in \mathsf{V}_1;$$

the vectors e^+ and e^- are isotropic vectors in V_1 satisfying $\mathsf{S}_1(\mathsf{e}^+, \mathsf{e}^-) = 1$.

Let η be an anisotropic vector in V_1 of the form $\eta = {}^t(a, {}^t\mathsf{a}, 1)$ with $a \in E$ and $\mathsf{a} \in \mathsf{V}_0$. Put $\Delta = \mathsf{S}_1(\eta, \eta) = a - \bar{a} + \mathsf{S}_0(\mathsf{a}, \mathsf{a})$, a non-zero element in E. We put

$$\mathsf{V} = \begin{pmatrix} \mathsf{V}_0 \\ E \end{pmatrix} = E^{n+1}$$

and define the E-linear inclusion $j_\eta : \mathsf{V} \to \mathsf{V}_1$ by

$$j_\eta : \begin{pmatrix} \mathsf{v}_0 \\ z \end{pmatrix} \mapsto \begin{pmatrix} \bar{a}z - \mathsf{S}_0(\mathsf{a}, \mathsf{v}_0) \\ \mathsf{v}_0 \\ z \end{pmatrix}, \quad \mathsf{v}_0 \in \mathsf{V}_0, z \in E.$$

Then $\mathrm{Im}(j_\eta)$ coincides with the orthogonal complement of the anisotropic line $E\eta$ in V_1 with respect to S_1. Now we introduce the skew Hermitian form S_η on V so that the map j_η becomes an isometry. From the remark above the skew Hermitian space $(\mathsf{V}, \mathsf{S}_\eta)$ so obtained is non-degenerate. We explicitly have

$$\mathsf{S}_\eta(\mathsf{v}, \mathsf{v}') = {}^t\bar{\mathsf{v}} \begin{pmatrix} \mathsf{S}_0 & -\mathsf{S}_0\mathsf{a} \\ -{}^t\bar{\mathsf{a}}\mathsf{S}_0 & \bar{a} - a \end{pmatrix} \mathsf{v}', \quad \mathsf{v}, \mathsf{v}' \in \mathsf{V}.$$

Let $j_0 : \mathsf{V}_0 \to \mathsf{V}$ denote the E-linear inclusion given by

$$j_0(\mathsf{v}_0) = \begin{pmatrix} \mathsf{v}_0 \\ 0 \end{pmatrix}, \quad \mathsf{v}_0 \in \mathsf{V}_0.$$

Then $j_0 : \mathsf{V}_0 \to \mathsf{V}$ is an isometry and its image coincides with the orthogonal complement of the anisotropic line $E\xi$ in V with respect to S_η, where $\xi = \Delta^{-1t}({}^t\mathsf{a}, 1) \in \mathsf{V}$.

3.2. Embeddings of groups

Put $\mathsf{G}_0 = \mathsf{U}(\mathsf{S}_0)$, $\mathsf{G}_1 = \mathsf{U}(\mathsf{S}_1)$ and $\mathsf{G} = \mathsf{U}(\mathsf{S}_\eta)$. Then we have a sequence of inclusions of algebraic groups

$$\mathsf{G}_0 \xrightarrow{\iota_0} \mathsf{G} \xrightarrow{\iota} \mathsf{G}_1, \tag{3.2.1}$$

where ι_0 and ι are homomorphisms defined by

$$\iota_0(g_0)(j_0(\mathsf{v}_0) + t\xi) = j_0(g_0\mathsf{v}_0) + t\xi, \quad g_0 \in \mathsf{G}_0,\ \mathsf{v}_0 \in \mathsf{V}_0,\ t \in E,$$
$$\iota(g)(j_\eta(\mathsf{v}) + t\eta) = j_\eta(g\mathsf{v}) + t\eta, \quad g \in \mathsf{G},\ \mathsf{v} \in \mathsf{V},\ t \in E.$$

Note that $\iota_0(\mathsf{G}_0)$ coincides with the stabilizer of the vector ξ in G and $\iota(\mathsf{G})$ that of the vector η in G_1.

3.3. A parabolic subgroup

Let P_1 be the maximal parabolic $\mathbb{Q}$-subgroup of G_1 defined as the stabilizer of the isotropic line $E\mathsf{e}^+$ of V_1. Let R be a $\mathbb{Q}$-algebra. For $(g_0, t) \in \mathsf{G}_0(R) \times (E \otimes_\mathbb{Q} R)^*$, put

$$\mathsf{m}_1(g_0\ ; t) = \mathrm{diag}(t, g_0, \bar{t}^{-1})$$

For $\mathsf{y} \in (E \otimes_\mathbb{Q} R)^n$ and $z \in E \otimes_\mathbb{Q} R$ with $z - \overline{z} + \mathsf{S}_0(\mathsf{y}, \mathsf{y}) = 0$, we put

$$\mathsf{n}_1(\mathsf{y}\ ; z) = \begin{pmatrix} 1 & -{}^t\bar{\mathsf{y}}\mathsf{S}_0 & z \\ \mathsf{0}_{n,1} & \mathtt{I}_n & \mathsf{y} \\ 0 & \mathsf{0}_{1,n} & 1 \end{pmatrix}.$$

Then the elements $\mathsf{m}_1(g_0\,;t)$ (resp. $\mathsf{n}_1(\mathsf{y}\,;z)$) make up the set $\mathsf{M}_1(R)$ (resp. $\mathsf{N}_1(R)$) with M_1 (resp. N_1) a Levi $\mathbb{Q}$-subgroup of P_1 (resp. the unipotent radical of P_1). We quote the following lemma from [2].

Lemma 3.3.1.

(1) *Suppose that* G *is* $\mathbb{Q}$*-isotropic. Then there exists an element* $\mathtt{x}_0 \in \mathsf{G}_1(\mathbb{Q})$ *such that* $\{\mathtt{I}_{n+2}, \mathtt{x}_0\}$ *gives a complete set of representatives for* $\mathsf{P}_1(\mathbb{Q})\backslash\mathsf{G}_1(\mathbb{Q})/\iota(\mathsf{G}(\mathbb{Q}))$ *and* $\mathtt{x}_0^{-1}\mathsf{P}_1(\mathbb{Q})\mathtt{x}_0 \cap \iota(\mathsf{G}(\mathbb{Q})) = \iota(\mathsf{P}(\mathbb{Q}))$, $\iota(\mathsf{N}(\mathbb{Q})) \subset \mathtt{x}_0^{-1}\mathsf{N}_1(\mathbb{Q})\mathtt{x}_0$ *with* N *the unipotent radical of a parabolic* $\mathbb{Q}$*-subgroup* P *in* G*. We have* $\mathsf{P}_1(\mathbb{Q}) \cap \iota(\mathsf{G}(\mathbb{Q})) = \iota \circ \iota_0(\mathsf{G}_0(\mathbb{Q}))$.

(2) *Suppose that* G *is* $\mathbb{Q}$*-anisotropic. Then we have* $\mathsf{G}_1(\mathbb{Q}) = \mathsf{P}_1(\mathbb{Q})\iota(\mathsf{G}(\mathbb{Q}))$ *and* $\mathsf{P}_1(\mathbb{Q}) \cap \iota(\mathsf{G}(\mathbb{Q})) = \iota_0 \circ \iota(\mathsf{G}_0(\mathbb{Q}))$.

Proof. See [2, Proposition 2.4, Lemma 2.5, Lemma 2.6]. □

3.4. An assumption at the archimedian place

We assume that the signature of $(\mathsf{V}_0(\mathbb{R}), \mathsf{S}_0)$ is $((n-1)+, 1-)$ and that of $(\mathsf{V}(\mathbb{R}), \mathsf{S}_\eta)$ is $(n+, 1-)$. Then the skew Hermitian space $(\mathsf{V}_1(\mathbb{R}), \mathsf{S}_1)$ has a signature $(n+, 2-)$. Put $2d = -\sqrt{-1}\Delta$. We have $d > 0$ from the assumption.

3.5. Maximal compact subgroups at the archimedian place

Fix a negative line $V_{0,\infty}^-$ (through the origin) in $\mathsf{V}_0(\mathbb{R}) = \mathbb{C}^n$ and put

$$V_\infty^- = j_0(V_{0,\infty}^-), \qquad V_{1,\infty}^- = j_\eta \circ j_0(V_{0,\infty}^-) + \mathbb{C}\cdot\eta.$$

Let K_∞, $K_{0,\infty}$ and $K_{1,\infty}$ be the stabilizers of $V_{0,\infty}^-$, V_∞^- and $V_{1,\infty}^-$ in $\mathsf{G}_0(\mathbb{R})$, $\mathsf{G}(\mathbb{R})$ and $\mathsf{G}_1(\mathbb{R})$ respectively. From the assumption in 3.4, $K_{0,\infty}$, K_∞ and $K_{1,\infty}$ are maximal compact subgroups of $\mathsf{G}_0(\mathbb{R})$, $\mathsf{G}(\mathbb{R})$ and $\mathsf{G}_1(\mathbb{R})$ respectively. We fix these maximal compact subgroups throughout this article. When we speak of automorphic forms on $\mathsf{G}_0(\mathbb{A})$, $\mathsf{G}(\mathbb{A})$ and $\mathsf{G}_1(\mathbb{A})$, we always understand that they are required to be finite under the actions of the maximal compact subgroups $K_{0,\infty}$, K_∞ and $K_{1,\infty}$ respectively.

Now take a vector v_n^- in $V_{0,\infty}^-$ with $\sqrt{-1}\mathsf{S}_0(\mathsf{v}_n^-, \mathsf{v}_n^-) = -1$. Let $(\mathsf{v}_i^+)_{1\leqslant i\leqslant n-1}$ be an orthonormal basis of the orthogonal complement of $V_{0,\infty}^-$ in $\mathsf{V}_0(\mathbb{R})$, i.e., $\sqrt{-1}\mathsf{S}_0(\mathsf{v}_i^+, \mathsf{v}_j^+) = \delta_{ij}$, $\mathsf{S}_0(\mathsf{v}_i^+, \mathsf{v}_n^-) = 0$ for $1 \leqslant i, j \leqslant n-1$. Put

$$\begin{aligned} &\xi_i^+ = j_0(\mathsf{v}_i^+), \quad 1 \leqslant i \leqslant n-1, \\ &\xi_n^+ = (2d)^{-1/2}\xi, \qquad \xi_{n+1}^- = j_0(\mathsf{v}_n^-) \end{aligned}$$

and

$$\begin{aligned} &\eta_i^+ = j_\eta \circ j_0(\mathsf{v}_i^+), \quad 1 \leqslant i \leqslant n-1, \\ &\eta_n^+ = j_\eta(\xi_n^+), \quad \eta_{n+1}^- = j_\eta(\xi_{n+1}^-), \quad \eta_{n+2}^- = (2d)^{-1/2}\eta. \end{aligned}$$

Then $\{\mathsf{v}_1^+, \ldots, \mathsf{v}_{n-1}^+, \mathsf{v}_n^-\}$, $\{\xi_1^+, \ldots, \xi_{n-1}^+, \xi_n^+, \xi_{n+1}^-\}$ and $\{\eta_1^+, \ldots, \eta_{n-1}^+, \eta_n^+, \eta_{n+1}^-, \eta_{n+2}^-\}$ are pseudo-orthonormal basis of $\mathsf{V}_0(\mathbb{R}) = \mathbb{C}^n$, $\mathsf{V}(\mathbb{R}) = \mathbb{C}^{n+1}$ and $\mathsf{V}_1(\mathbb{R}) = \mathbb{C}^{n+2}$ respectively. We shall fix these basis in what follows. For positive integers p and q, let

$$\mathsf{U}(p,q) = \{g \in \mathsf{GL}_{p+q}(\mathbb{C}) \mid {}^t\overline{g}\mathrm{diag}(\mathtt{I}_p, -\mathtt{I}_q)g = \mathrm{diag}(\mathtt{I}_p, -\mathtt{I}_q)\}. \tag{3.5.1}$$

Put

$$\begin{aligned} &\mathsf{c}_0 = (\mathsf{v}_1^+ \ \mathsf{v}_2^+ \ \ldots \ \mathsf{v}_{n-1}^+ \ \mathsf{v}_n^-) \in \mathrm{M}_n(\mathbb{C}), \\ &\mathsf{c} = (\xi_1^+ \ \xi_2^+ \ \ldots \ \xi_n^+ \ \xi_{n+1}^-) \in \mathrm{M}_{n+1}(\mathbb{C}), \\ &\mathsf{c}_1 = (\eta_1^+ \ \eta_2^+ \ \ldots \ \eta_n^+ \ \eta_{n+1}^- \ \eta_{n+2}^-) \in \mathrm{M}_{n+2}(\mathbb{C}). \end{aligned}$$

Then the maps

$$\mathsf{G}_0(\mathbb{R}) \ni g_0 \mapsto {\mathsf{c}_0}^{-1} g_0 \mathsf{c}_0 \in \mathsf{U}(n-1,1), \tag{3.5.2}$$

$$\mathsf{G}(\mathbb{R}) \ni g \mapsto \mathsf{c}^{-1} g \mathsf{c} \in \mathsf{U}(n,1), \tag{3.5.3}$$

$$\mathsf{G}_1(\mathbb{R}) \ni g_1 \mapsto {\mathsf{c}_1}^{-1} g_1 \mathsf{c}_1 \in \mathsf{U}(n,2) \tag{3.5.4}$$

give isomorphisms of Lie groups.

Lemma 3.5.1. *The diagram*

$$\begin{array}{ccccc} \mathsf{G}_0(\mathbb{R}) & \xrightarrow{\iota_0} & \mathsf{G}(\mathbb{R}) & \xrightarrow{\iota} & \mathsf{G}_1(\mathbb{R}) \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ \mathsf{U}(n-1,1) & \xrightarrow{i_0} & \mathsf{U}(n,1) & \xrightarrow{i} & \mathsf{U}(n,2) \end{array}$$

is commutative, where the virtical arrows are maps defined by (3.5.2), (3.5.3) *and* (3.5.4), *and* i_0 *and* i *are given as*

$$\text{(3.5.5)} \qquad i_0 : \mathsf{U}(n-1,1) \ni \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \longrightarrow \begin{pmatrix} x_{11} & \mathsf{0}_{n-1,1} & x_{12} \\ \mathsf{0}_{1,n-1} & 1 & 0 \\ x_{21} & 0 & x_{22} \end{pmatrix} \in \mathsf{U}(n,1),$$

$$\text{(3.5.6)} \qquad i : \mathsf{U}(n,1) \ni y \longrightarrow \mathrm{diag}(y,1) \in \mathsf{U}(n,2).$$

Proof. Obvious. □

We put

$$\mathsf{k}_0[u_0\ ; x_0] = \mathsf{c}_0\ \mathrm{diag}(u_0, x_0)\ \mathsf{c}_0{}^{-1}, \quad u_0 \in \mathsf{U}(n-1),\ x_0 \in \mathsf{U}(1),$$
$$\mathsf{k}[u\ ; x] = \mathsf{c}\ \mathrm{diag}(u, x)\ \mathsf{c}^{-1}, \quad u \in \mathsf{U}(n),\ x \in \mathsf{U}(1),$$
$$\mathsf{k}_1[u_1\ ; u_2] = \mathsf{c}_1\ \mathrm{diag}(u_1, u_2)\ \mathsf{c}_1{}^{-1}, \quad u_1 \in \mathsf{U}(n),\ u_2 \in \mathsf{U}(2).$$

Then we obviously have

$$K_{0,\infty} = \{\mathsf{k}_0[u_0\ ; x_0] \mid u_0 \in \mathsf{U}(n-1),\ x_0 \in \mathsf{U}(1)\},$$
$$K_{\infty} = \{\mathsf{k}[u\ ; x] \mid u \in \mathsf{U}(n),\ x \in \mathsf{U}(1)\},$$
$$K_{1,\infty} = \{\mathsf{k}_1[u_1\ ; u_2] \mid u_1 \in \mathsf{U}(n),\ u_2 \in \mathsf{U}(2)\}.$$

We also have

$$\text{(3.5.7)} \quad \iota_0(\mathsf{k}_0[u_0\ ; x_0]) = \mathsf{k}[\mathrm{diag}(u_0, 1)\ ; x_0], \quad u_0 \in \mathsf{U}(n-1),\ x_0 \in \mathsf{U}(1),$$
$$\text{(3.5.8)} \quad \iota(\mathsf{k}[u\ ; x]) = \mathsf{k}_1[u\ ; \mathrm{diag}(x, 1)], \quad u \in \mathsf{U}(n),\ x \in \mathsf{U}(1)$$

from Lemma 3.5.1.

By the Iwasawa decomposition we have $\mathsf{G}_1(\mathbb{R}) = \mathsf{P}_1(\mathbb{R})K_{1,\infty}$. Hence for $g_1 \in \mathsf{G}_1(\mathbb{R})$, we can write

$$g_1 = \mathsf{m}_1(\beta(g_1)\ ; t(g_1))n_1(g_1)k_1(g_1)$$

with $\beta(g_1) \in \mathsf{G}_0(\mathbb{R})$, $t(g_1) \in \mathbb{C}^*$, $n_1(g_1) \in \mathsf{N}_1(\mathbb{R})$ and $k_1(g_1) \in K_{1,\infty}$. But this decomposition is not unique. We need the structure of the intersection $\mathsf{P}_1(\mathbb{R}) \cap K_{1,\infty}$ explicitly.

LEMMA 3.5.2. *For $t \in \mathbb{C}^{(1)}$ and $g_0 = \mathsf{k}_0[u_0\ ; x_0] \in K_{0,\infty}$ with $u_0 \in \mathsf{U}(n-1)$, $x_0 \in \mathbb{C}^{(1)}$, we have*

$$\begin{aligned}&\mathsf{m}_1(g_0\ ; t)\mathsf{n}_1(tg_0^{-1}\mathsf{a} - \mathsf{a}\ ; -t^{-1}\mathsf{S}_0(\mathsf{a}, tg_0^{-1}\mathsf{a} - \mathsf{a}))\\ &= \mathsf{k}_1[\mathrm{diag}(u_0, t)\ ; \mathrm{diag}(x_0, t)],\end{aligned}$$

and the group $\mathsf{P}_1(\mathbb{R}) \cap K_{1,\infty}$ consists of all the points of this form.

PROOF. A direct computation. □

3.6. Compact groups at finite places

Let $K_{0,\mathrm{f}}$, K_{f} and $K_{1,\mathrm{f}}$ be open compact subgroups of $\mathsf{G}_0(\mathbb{A}_{\mathrm{f}})$, $\mathsf{G}(\mathbb{A}_{\mathrm{f}})$ and $\mathsf{G}_1(\mathbb{A}_{\mathrm{f}})$ respectively with the following properties.

(a) $\iota_0(K_{0,\mathrm{f}}) \subset K_{\mathrm{f}}$ and $\iota(K_{\mathrm{f}}) \subset K_{1,\mathrm{f}}$.

(b) $\mathsf{G}_1(\mathbb{A}_{\mathrm{f}}) = \mathsf{P}_1(\mathbb{A}_{\mathrm{f}})K_{1,\mathrm{f}}$, i.e., for $g_1 \in \mathsf{G}_1(\mathbb{A}_{\mathrm{f}})$ there exist $\beta(g_1) \in \mathsf{G}_0(\mathbb{A}_{\mathrm{f}})$, $t(g_1) \in \mathbb{A}_{E,\mathrm{f}}^*$, $n_1(g_1) \in \mathsf{N}_1(\mathbb{A}_{\mathrm{f}})$ and $k_1(g_1) \in K_{1,\mathrm{f}}$ such that
$$g_1 = \mathsf{m}_1(\beta(g_1)\ ; t(g_1))n_1(g_1)k_1(g_1).$$

(c) If $\mathsf{m}_1(g_{0,\mathrm{f}}\ ; t_{\mathrm{f}}) \in K_{1,\mathrm{f}}\mathsf{N}_1(\mathbb{A}_{\mathrm{f}})$ with $g_{0,\mathrm{f}} \in \mathsf{G}_0(\mathbb{A}_{\mathrm{f}})$ and $t_{\mathrm{f}} \in \mathbb{A}_{E,\mathrm{f}}^*$, then $g_{0,\mathrm{f}} \in K_{0,\mathrm{f}}$ and $t_{\mathrm{f}} \in \prod_p \mathcal{O}_{E,p}^*$.

We fix $K_{0,\mathrm{f}}$, K_{f} and $K_{1,\mathrm{f}}$ with these properties for once and for all. By the property (b) and Lemma 3.5.2, we have Iwasawa decomposition $\mathsf{G}_1(\mathbb{A}) = \mathsf{P}_1(\mathbb{A})K_{1,\mathrm{f}}K_{1,\infty}$.

REMARK 3.6.1. In [2], [5], by means of maximal $\mathcal{O}_E$-integral lattices, a concrete choice of $(K_{0,\mathrm{f}}, K_{\mathrm{f}}, K_{1,\mathrm{f}})$ is made. Since our main concern in this paper is archimedian local theory, we refrain from recalling their construction but extract a necessary properties of K_{f} etc. above just to ensure the well-definedness of the function Ψ in Lemma 4.1.2.

4. Eisenstein Series

In this section we introduce a vector-valued Eisenstein series which enters in the definition of the Rankin-Selberg integrals that will be introduced in section 6.

4.1. Vector-valued Eisenstein series

Let (τ_0, W_0) and (τ, W) be irreducible unitary representations of $K_{0,\infty}$ and K_∞ respectively. Since the centers of $K_{0,\infty}$ and K_∞ respectively equal $\{\mathsf{k}_0[x_0^+ \mathtt{I}_{n-1}\,; x_0^-] \mid x_0^\pm \in \mathbb{C}^{(1)}\}$ and $\{\mathsf{k}[x^+ \mathtt{I}_n\,; x^-] \mid x^\pm \in \mathbb{C}^{(1)}\}$, Schur's lemma implies there exist pairs of integers (c_0^+, c_0^-) and (c^+, c^-) such that

$$\tau_0(\mathsf{k}_0[x_0^+ \mathtt{I}_{n-1}\,; x_0^-]) = (x_0^+)^{c_0^+}(x_0^-)^{c_0^-}\,\mathtt{I}_{W_0}, \quad x_0^+,\ x_0^- \in \mathbb{C}^{(1)}, \tag{4.1.1}$$

$$\tau(\mathsf{k}[x^+ \mathtt{I}_n\,; x^-]) = (x^+)^{c^+}(x^-)^{c^-}\,\mathtt{I}_W, \quad x^+,\ x^- \in \mathbb{C}^{(1)}. \tag{4.1.2}$$

We call (c^+, c^-) and (c_0^+, c_0^-) the *central characters* of τ and τ_0 respectively. We assume that τ_0 occurs in $\tau|K_{0,\infty} = \tau\circ(\iota_0|K_{0,\infty})$, and fix a $K_{0,\infty}$-inclusion $i_\tau^{\tau_0} : W_0 \to W$ once and for all. We then have $c^- = c_0^-$.

Lemma 4.1.1. *There exists a unique unitary representation τ_1 of $K_{1,\infty}$ on W, the representation space of τ, satisfying*

$$\tau_1(\mathsf{k}_1[u_1\,; u_2]) = \tau(\mathsf{k}[u_1\,; 1])\det(u_2)^{c_0^-}, \quad u_1 \in \mathsf{U}(n), u_2 \in \mathsf{U}(2). \tag{4.1.3}$$

Proof. Obvious. $\square$

We take an idele class character $\omega : \mathbb{A}_E^*/E^* \to \mathbb{C}^*$ of E. We assume that

$$\omega(t_{\mathrm{f}}) = 1, \quad t_{\mathrm{f}} \in \prod_p \mathcal{O}_{E,p}^*, \tag{4.1.4}$$

$$\omega(t_\infty) = t_\infty^{-c_0^- - c^+ + c_0^+}, \quad t_\infty \in \mathbb{C}^{(1)}. \tag{4.1.5}$$

Lemma 4.1.2. *Let $f \in \mathfrak{S}_{\tau_0^\vee}(\mathsf{G}_0\,; K_{0,\mathrm{f}})$. Then there exists a unique W-valued function $(g_1, s) \mapsto \Psi(f\otimes\omega\,; s\,; g_1)$ on $\mathsf{G}_1(\mathbb{A}) \times \mathbb{C}$ that is smooth with respect to the first variable and holomorphic with respect to the second one and satisfies*

$$\begin{aligned}&\Psi(f\otimes\omega\,; s\,; \mathsf{m}_1(g_0\,; t)n_1) = \omega(t)|t|_E^s \cdot i_\tau^{\tau_0}(f(g_0)),\\ &\qquad g_0 \in \mathsf{G}_0(\mathbb{A}),\ t \in \mathbb{A}_E^*,\ n_1 \in \mathsf{N}_1(\mathbb{A}),\end{aligned} \tag{4.1.6}$$

$$\begin{aligned}&\Psi(f\otimes\omega\,; s\,; g_1 k_{1,\mathrm{f}} k_{1,\infty}) = \tau_1(k_{1,\infty})^{-1}\Psi(f\otimes\omega\,; s\,; g_1),\\ &\qquad k_{1,\mathrm{f}} \in K_{1,\mathrm{f}},\ k_{1,\infty} \in K_{1,\infty}.\end{aligned} \tag{4.1.7}$$

Proof. For a $g_1 \in \mathsf{G}_1(\mathbb{A})$, we can write it of the form

$$g_1 = \mathsf{m}_1(g_0; t) n_1 k_{1,\infty} k_{1,\mathrm{f}},\\ g_0 \in \mathsf{G}_0(\mathbb{A}),\ t \in \mathbb{A}_E^*,\ n_1 \in \mathsf{N}_1(\mathbb{A}),\ k_{1,\mathrm{f}} \in K_{1,\mathrm{f}},\ k_{1,\infty} \in K_{1,\infty}.$$

along the decomposition $\mathsf{G}_1(\mathbb{A}) = \mathsf{P}_1(\mathbb{A}) K_{1,\mathrm{f}} K_{1,\infty}$ (see 3.6). The conditions (4.1.6) and (4.1.7) mean

$$\Psi(f \otimes \omega\ ; s\ ; g_1) = \omega(t)|t|_E^s \cdot \tau_1(k_{1,\infty}^{-1}) i_\tau^{\tau_0}(f(g_0)). \tag{4.1.8}$$

Thus we have only to show that the right-hand side of (4.1.8) does not depend on a choice of Iwasawa decompositions of g_1. If $g_1 = \mathsf{m}_1(g_0'\ ; t') n_1' k_{1,\mathrm{f}}' k_{1,\infty}'$ is another decomposition of g_1 similar to above, then we have

$$\mathsf{m}_1(g_0^{-1} g_0'\ ; t^{-1} t') n_1'' = (k_{1,\mathrm{f}} k_{1,\infty})(k_{1,\mathrm{f}}' k_{1,\infty}')^{-1}$$

with some $n_1'' \in \mathsf{N}_1(\mathbb{A})$. Hence Lemma 3.5.2 and (c) in 3.6 imply that there exist $k_{0,\infty} = \mathsf{k}_0[u_0\ ; x_0] \in K_{0,\infty}$ with $u_0 \in \mathsf{U}(n-1)$, $x_0 \in \mathbb{C}^{(1)}$, $y_\infty \in \mathbb{C}^{(1)}$ and $k_{0,\mathrm{f}} \in K_{0,\mathrm{f}}$, $y_\mathrm{f} \in \prod_p \mathcal{O}_{E,p}^*$ such that

$$g_0^{-1} g_0' = k_{0,\infty} k_{0,\mathrm{f}}, \quad t^{-1} t' = y_\infty y_\mathrm{f}, \quad k_{1,\infty} {k_{1,\infty}'}^{-1} = \mathsf{k}_1[u_1\ ; u_2]$$

with

$$u_1 = \mathrm{diag}(u_0, y_\infty) \in \mathsf{U}(n), \quad u_2 = \mathrm{diag}(x_0, y_\infty) \in \mathsf{U}(2).$$

Using the $K_{0,\mathrm{f}}$-invariance of f and (4.1.4), we have

$$\begin{aligned}
&\omega(t')|t'|_E^s \cdot \tau_1(k_{1,\infty}')^{-1} \circ i_\tau^{\tau_0} \circ f(g_0')\\
&= \omega(t y_\infty y_\mathrm{f})|t y_\infty y_\mathrm{f}|_E^s \cdot \tau_1(k_{1,\infty}^{-1} \mathsf{k}_1[u_1\ ; u_2]) \circ i_\tau^{\tau_0} \circ f(g_0 k_{0,\infty} k_{0,\mathrm{f}})\\
&= \omega(t y_\infty y_\mathrm{f})|t y_\infty y_\mathrm{f}|_E^s \cdot \tau_1(k_{1,\infty})^{-1} \circ \tau_1(\mathsf{k}_1[u_1\ ; u_2]) \circ i_\tau^{\tau_0} \circ \tau_0(k_{0,\infty})^{-1} \circ f(g_0)\\
&= \omega(y_\infty)\omega(t)|t|_E^s \cdot \tau_1(k_{1,\infty})^{-1} \circ \tau_1(\mathsf{k}_1[u_1\ ; u_2]) \circ \tau(k_{0,\infty})^{-1} \circ i_\tau^{\tau_0} \circ f(g_0)\\
&= \omega(y_\infty)\omega(t)|t|_E^s x_0^{-c^-}\\
&\quad \cdot \tau_1(k_{1,\infty})^{-1} \circ \tau_1(\mathsf{k}_1[u_1\ ; u_2]) \circ \tau(\mathsf{k}[\mathrm{diag}(u_0, 1)\ ; 1])^{-1} \circ i_\tau^{\tau_0} \circ f(g_0).
\end{aligned}$$

By Lemma 4.1.1 and the relation $c_0^- = c^-$, we have

$$\begin{aligned}
&x_0^{-c^-} \cdot \tau_1(\mathsf{k}_1[u_1\ ;u_2]) \circ \tau(\mathsf{k}[\mathrm{diag}(u_0,1)\ ;1])^{-1} \circ i_\tau^{\tau_0} \circ f(g_0) \\
&= x_0^{-c_0^-}(x_0 y_\infty)^{c_0^-} \\
&\quad \cdot \tau(\mathsf{k}[\mathrm{diag}(u_0,y_\infty)\ ;1]) \circ \tau(\mathsf{k}[\mathrm{diag}(u_0,1)\ ;1])^{-1} \circ i_\tau^{\tau_0} \circ f(g_0) \\
&= y_\infty^{c_0^-} \cdot \tau(\mathsf{k}[\mathrm{diag}(\mathtt{I}_{n-1},y_\infty)\ ;1]) \circ i_\tau^{\tau_0} \circ f(g_0) \\
&= y_\infty^{c^+ + c_0^-} \cdot \tau(\mathsf{k}[\mathrm{diag}(y_\infty^{-1}\mathtt{I}_{n-1},1)\ ;1]) \circ i_\tau^{\tau_0} \circ f(g_0) \\
&= y_\infty^{c^+ + c_0^-} \cdot (\tau|K_{0,\infty})(\mathsf{k}_0[y_\infty^{-1}\mathtt{I}_{n-1}\ ;1]) \circ i_\tau^{\tau_0} \circ f(g_0) \\
&= y_\infty^{c^+ + c_0^-} \cdot i_\tau^{\tau_0} \circ \tau_0(\mathsf{k}_0[y_\infty^{-1}\mathtt{I}_{n-1}\ ;1]) \circ f(g_0) \\
&= y_\infty^{c^+ + c_0^-} \cdot i_\tau^{\tau_0}(y_\infty^{-c_0^+} f(g_0)) \\
&= y_\infty^{c^+ - c_0^+ + c_0^-} \cdot i_\tau^{\tau_0}(f(g_0)).
\end{aligned}$$

Using (4.1.5), we finally have

$$\omega(t')|t'|_E^s \cdot \tau_1(k'_{1,\infty})^{-1} i_\tau^{\tau_0}(f(g'_0)) = \omega(t)|t|_E^s \cdot \tau_1(k_{1,\infty})^{-1} i_\tau^{\tau_0}(f(g_0))$$

as desired. □

Let $s \in \mathbb{C}$ and ω the idele class character as above. Let π_0 be an irreducible $(\mathfrak{g}_0, K_{0,\infty})$-module with $\mathfrak{g}_0 = \mathrm{Lie}(\mathsf{G}_0(\mathbb{R})) \otimes \mathbb{C}$. Let $f \in \mathfrak{S}_{\tau_0^\vee}(\mathsf{G}_0\ ;K_{0,\mathrm{f}})_{\pi_0^\vee}$. We introduce a vector-valued Eisenstein series as

$$\begin{aligned}
(4.1.9)\quad &\mathbf{E}(\mathsf{P}_1\ ;f\otimes\omega\ ;s\ ;g_1) \\
&\quad = \sum_{\gamma_1 \in \mathsf{P}_1(\mathbb{Q})\backslash \mathsf{G}_1(\mathbb{Q})} \Psi\left(f\otimes\omega\ ;s+\frac{n+1}{2}\ ;\gamma_1 g_1\right), \quad g_1 \in \mathsf{G}_1(\mathbb{A}).
\end{aligned}$$

The holomorphic function given by the absolutely convergent infinite series (4.1.9) on $\mathrm{Re}(s) > (n+1)/2$ has a meromorphic continuation to the whole $\mathbb{C}$ ([9]). We have

$$\begin{aligned}
&\mathbf{E}(\mathsf{P}_1\ ;f\otimes\omega\ ;s\ ;\gamma_1 g_1 k_{1,\infty} k_{1,\mathrm{f}}) = \tau_1(k_{1,\infty})^{-1}\mathbf{E}(\mathsf{P}_1\ ;f\otimes\omega\ ;s\ ;g_1), \\
&\quad \gamma_1 \in \mathsf{G}_1(\mathbb{Q}),\ k_{1,\infty} \in K_{1,\infty},\ k_{1,\mathrm{f}} \in K_{1,\mathrm{f}}.
\end{aligned}$$

5. Shintani Functions

In this section we first recall the definition of local Shintani functions briefly and then introduce the global Shintani function associated with a pair of cusp forms on $\mathsf{G}_0(\mathbb{A})$ and $\mathsf{G}(\mathbb{A})$. Using the multiplicity free theorem for the real Shintani functional (Theorem 5.1.1), we prove that the global Shintani function is a product of a real Shintani function and a function on the finite adeles (Proposition 5.2.1).

Here is a convention, that will be adopted hereafter. By the inclusions $\iota_0 : \mathsf{G}_0 \to \mathsf{G}$ and $\iota : \mathsf{G} \to \mathsf{G}_1$, we consider G_0 and G to be subgroups of G_1; correspondingly, for $g_0 \in \mathsf{G}_0$ and $g \in \mathsf{G}$, we simply write g_0 and g in place of $\iota_0(g_0)$ and $\iota(g)$ respectively. Let $\mathfrak{g}_0$ and $\mathfrak{g}$ be the complexified Lie algebras of $\mathsf{G}_0(\mathbb{R})$ and $\mathsf{G}(\mathbb{R})$ respectively.

5.1. Real Shintani functions

Let (π_0, H_{π_0}) and (π, H_π) be irreducible $(\mathfrak{g}_0, K_{0,\infty})$-module and irreducible $(\mathfrak{g}, K_\infty)$-module respectively. Let $\mathcal{S}(\mathsf{G}(\mathbb{R}))$ be the Schwartz space for $\mathsf{G}(\mathbb{R})$ in the sense of Casselman [1, page 392]. It is a smooth Fréchet $\mathsf{G}(\mathbb{R}) \times \mathsf{G}(\mathbb{R})$-module of moderate growth. By restricting the action to the subgroup $\mathsf{G}_0(\mathbb{R}) \times \mathsf{G}(\mathbb{R})$, $\mathcal{S}(\mathsf{G}(\mathbb{R}))$ is considered to be a $\mathsf{G}_0(\mathbb{R}) \times \mathsf{G}(\mathbb{R})$-module. Put

$$\mathcal{I}^{\mathrm{mod}}(\pi_0|\pi) = \mathrm{Hom}_{(\mathfrak{g}_0 \oplus \mathfrak{g}, K_{0,\infty} \times K_\infty)}(\pi_0^\vee \boxtimes \pi, \mathcal{S}(\mathsf{G}(\mathbb{R})))$$

and

$$\mathrm{Sh}(\pi_0, \pi) = \mathrm{Im}(\mathcal{I}^{\mathrm{mod}}(\pi_0|\pi) \otimes {H_{\pi_0}}^\vee \otimes H_\pi \to \mathcal{S}(\mathsf{G}(\mathbb{R}))),$$

where the arrow stands for the natural map; $\mathrm{Sh}(\pi_0, \pi)$ becomes a $\pi_0^\vee \boxtimes \pi$-isotypic $(\mathfrak{g}_0 \oplus \mathfrak{g}, K_{0,\infty} \times K_\infty)$-submodule of $\mathcal{S}(\mathsf{G}(\mathbb{R}))$. For irreducible finite dimensional continuous representations (τ_0, W_0) and (τ, W) of $K_{0,\infty}$ and K_∞ respectively, we set

$$\mathrm{Sh}_{\tau_0,\tau}(\pi_0, \pi) = \mathrm{Hom}_{K_{0,\infty} \times K_\infty}(W_0^\vee \otimes_{\mathbb{C}} W, \mathrm{Sh}(\pi_0, \pi)),$$

which we consider to be a subspace of the space of smooth $W_0 \otimes_{\mathbb{C}} W^\vee$-valued functions on $\mathsf{G}(\mathbb{R})$ in the obvious manner. Any function which belongs to the space $\mathrm{Sh}_{\tau_0,\tau}(\pi_0, \pi)$ is called a *Shintani function* with $K_{0,\infty} \times K_\infty$-type $(\tau_0^\vee, \tau)$ belonging to the representation ${\pi_0}^\vee \boxtimes \pi$.

Theorem 5.1.1 (Multiplicity free theorem). *Let π_0 and π be as above. Then*

$$\dim_{\mathbb{C}} \mathcal{I}^{\mathrm{mod}}(\pi_0|\pi) \leqslant 1.$$

Proof. Put $H = \mathsf{G}_0(\mathbb{R})Z$ with Z the center of $\mathsf{G}(\mathbb{R})$. We can extend the representation π_0^∞ of $\mathsf{G}_0(\mathbb{R})$ to H so that the extended representation η^∞ of H, when restricted to Z, corresponds to the same character as $\pi^\infty|Z$. Since the inclusion $\mathcal{S}(\mathsf{G}(\mathbb{R})) \hookrightarrow C^\infty(\mathsf{G}(\mathbb{R}))$ is continuous, by [7, Corollary 2.4.1, Theorem 8.3.1], we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{H\times \mathsf{G}(\mathbb{R})}((\eta^\infty)^\vee \boxtimes \pi^\infty, \mathcal{S}(\mathsf{G}(\mathbb{R})) \leqslant 1. \tag{5.2.1}$$

By [1, Corollary 10.5], any $\Phi \in \mathcal{I}^{\mathrm{mod}}(\pi_0|\pi)$ can be extended to a continuous intertwining operator $(\eta^\infty)^\vee \boxtimes \pi^\infty \to \mathcal{S}(\mathsf{G}(\mathbb{R}))$; hence $\dim_{\mathbb{C}} \mathcal{I}^{\mathrm{mod}}(\pi_0|\pi)$ is dominated by the left-hand side of (5.2.1). $\square$

5.2. Global Shintani functions arising from automorphic forms

Let (τ_0, W_0) and (τ, W) be finite dimensional unitary representations of $K_{0,\infty}$ and K_∞ respectively. Let $dg_0 = d_{\mathsf{G}_0(\mathbb{A})}(g_0)$ be a Haar measure of $\mathsf{G}_0(\mathbb{A})$ and $d\dot{g}_0$ the corresponding quotient measure on $\mathsf{G}_0(\mathbb{Q})\backslash \mathsf{G}_0(\mathbb{A})$. For automorphic forms $f \in \mathfrak{S}_{\tau_0^\vee}(\mathsf{G}_0)$ and $F \in \mathfrak{S}_\tau(\mathsf{G})$, we consider the integral

$$\mathcal{W}_{f,F}(g) = \int_{\mathsf{G}_0(\mathbb{Q})\backslash \mathsf{G}_0(\mathbb{A})} f(g_0) \otimes F(g_0 g) d\dot{g}_0, \quad g \in \mathsf{G}(\mathbb{A}). \tag{5.3.1}$$

It turns out that this defines a $W_0 \otimes W^\vee$-valued smooth function $\mathcal{W}_{f,F}$ on $\mathsf{G}(\mathbb{A})$ which we call *the global Shintani function* associated with f and F ([2], [3], [4], [5]).
From definition we have the equation

$$\begin{aligned} \mathcal{W}_{f,F}(k_{0,\infty} g k_\infty) &= (\tau_0(k_{0,\infty}) \otimes \tau^\vee(k_\infty)^{-1}) \mathcal{W}_{f,F}(g), \\ & k_{0,\infty} \in K_{0,\infty},\ k_\infty \in K_\infty. \end{aligned} \tag{5.3.2}$$

As a consequence of the multiplicity free theorem for the real Shintani functional (Theorem 5.1.1), we have the following.

Proposition 5.2.1. *Let (π_0, H_{π_0}) and (π, H_π) be irreducible $(\mathfrak{g}_0, K_{0,\infty})$-module and irreducible $(\mathfrak{g}, K_\infty)$-module respectively. Let $f \in$*

$\mathfrak{S}_{\tau_0^\vee}(\mathsf{G}_0)_{\pi_0^\vee}$ *and* $F \in \mathfrak{S}_\tau(\mathsf{G})_\pi$. *Suppose that* $\mathcal{W}_{f,F}|\mathsf{G}(\mathbb{R})$ *is not identically zero. Then there exists a unique real Shintani function* $\mathcal{W}^\infty_{f,F} \in \mathrm{Sh}_{\tau_0,\tau}(\pi_0,\pi)$ *and a unique smooth function* $\mathcal{W}^{\mathrm{f}}_{f,F} : \mathsf{G}(\mathbb{A}_{\mathrm{f}}) \to \mathbb{C}$ *such that* $\mathcal{W}^{\mathrm{f}}_{f,F}(\mathtt{I}_{n+1}) = 1$ *and*

$$\mathcal{W}_{f,F}(g_\infty g_{\mathrm{f}}) = \mathcal{W}^{\mathrm{f}}_{f,F}(g_{\mathrm{f}})\mathcal{W}^\infty_{f,F}(g_\infty), \quad g_{\mathrm{f}} \in \mathsf{G}(\mathbb{A}_{\mathrm{f}}),\ g_\infty \in \mathsf{G}(\mathbb{R}). \tag{5.3.4}$$

Proof. By assumption we can take a $K_{0,\infty}$-inclusion $\iota^{\pi_0^\vee}_{\tau_0^\vee} : W_0^\vee \to H^\vee_{\pi_0}$ and a K_∞-inclusion $\iota^\pi_\tau : W \to H_\pi$. There exists a $(\mathfrak{g}_0, K_{0,\infty})$-intertwining operator $\psi_0 : H_{\pi_0}{}^\vee \to \mathfrak{S}_{\tau_0^\vee}(\mathsf{G}_0)$ and a $(\mathfrak{g}, K_\infty)$-intertwining operator $\psi : H_\pi \to \mathfrak{S}_\tau(\mathsf{G})$ such that

$$\begin{aligned} &\langle f(g_0), w_0^\vee\rangle = \psi_0(\iota^{\pi_0^\vee}_{\tau_0^\vee}(w_0^\vee))(g_0), \quad w_0^\vee \in W_0^\vee,\ g_0 \in \mathsf{G}_0(\mathbb{A}), \\ &\langle F(g), w\rangle = \psi(\iota^\pi_\tau(w))(g), \quad w \in W,\ g \in \mathsf{G}(\mathbb{A}). \end{aligned} \tag{5.3.6}$$

(Note that $\tau_0^\vee$ and τ occur in $\pi_0^\vee|K_{0,\infty}$ and $\pi|K_\infty$ with multiplicity one since $\mathsf{G}_0(\mathbb{R})$ and $\mathsf{G}(\mathbb{R})$ are real-rank-one unitary groups and π_0 and π are irreducible.) Now for each $g_{\mathrm{f}} \in \mathsf{G}(\mathbb{A}_{\mathrm{f}})$, putting

$$\begin{aligned} &\Phi_{f,F}(g_{\mathrm{f}}\,; v_0{}^\vee \otimes v)(g_\infty) \\ &\qquad = \int_{\mathsf{G}_0(\mathbb{Q})\backslash\mathsf{G}_0(\mathbb{A})} \psi_0(v_0{}^\vee)(g_0)\cdot\psi(v)(g_0 g_{\mathrm{f}} g_\infty)\,d\dot{g}_0, \\ &v_0{}^\vee \in H^\vee_{\pi_0},\ v \in H_\pi,\ g_\infty \in \mathsf{G}(\mathbb{R}), \end{aligned} \tag{5.3.7}$$

we get an element $\Phi_{f,F}(g_{\mathrm{f}}\,;-)$ of $\mathcal{I}^{\mathrm{mod}}(\pi_0|\pi)$. From the definitions of $\Phi_{f,F}(g_{\mathrm{f}}\,;-)$ and $\mathcal{W}_{f,F}$ we have

$$\begin{aligned} &\Phi_{f,F}(g_{\mathrm{f}}\,; \iota^{\pi_0^\vee}_{\tau_0^\vee}(w_0^\vee)\otimes\iota^\pi_\tau(w))(g_\infty) = \langle \mathcal{W}_{f,F}(g_{\mathrm{f}} g_\infty), w_0^\vee \otimes w\rangle, \\ &\quad w_0^\vee \in W_0^\vee,\ w \in W. \end{aligned} \tag{5.3.8}$$

Since $\mathcal{W}_{f,F}(\mathsf{e}_\infty) \neq 0$ for an element $\mathsf{e}_\infty \in \mathsf{G}(\mathbb{R})$, there exist some $u_0^\vee \in W_0^\vee$ and $u \in W$ such that

$$c_0 = \langle \mathcal{W}_{f,F}(\mathsf{e}_\infty), u_0^\vee \otimes u\rangle \neq 0.$$

Because $\dim_{\mathbb{C}}\mathcal{I}^{\mathrm{mod}}(\pi_0|\pi) \leqslant 1$, we then have $\dim_{\mathbb{C}}\mathcal{I}^{\mathrm{mod}}(\pi_0|\pi) = 1$ and $\Phi_{f,F} := \Phi_{f,F}(\mathtt{I}_{n+1}\,;-)$ provides a basis of the space $\mathcal{I}^{\mathrm{mod}}(\pi_0|\pi)$. Hence for every $g_{\mathrm{f}} \in \mathsf{G}(\mathbb{A}_{\mathrm{f}})$, we can write

$$\Phi_{f,F}(g_{\mathrm{f}}\,;-) = \mathcal{W}^{\mathrm{f}}_{f,F}(g_{\mathrm{f}}) \cdot \Phi_{f,F}$$

with a unique complex number $\mathcal{W}^{\mathrm{f}}_{f,F}(g_{\mathrm{f}})$; then $\mathcal{W}^{\mathrm{f}}_{f,F}(\mathtt{I}_{n+1}) = 1$ is obvious. We have the equation

$$\mathcal{W}^{\mathrm{f}}_{f,F}(g_{\mathrm{f}}) = c_0^{-1}\langle \mathcal{W}_{f,F}(\mathsf{e}_\infty g_{\mathrm{f}}), u_0^\vee \otimes u\rangle, \quad g_{\mathrm{f}} \in \mathsf{G}(\mathbb{A}_{\mathrm{f}}),$$

from which the smoothness of $\mathcal{W}^{\mathrm{f}}_{f,F}$ follows. Now putting

$$\begin{aligned}&\langle \mathcal{W}^{\infty}_{f,F}(g_\infty), w_0^\vee \otimes w\rangle = \Phi_{f,F}(\iota^{\pi_0^\vee}_{\tau_0^\vee}(w_0^\vee) \otimes \iota^{\pi}_{\tau}(w))(g_\infty),\\ &\quad w_0^\vee \in W_0^\vee,\ w \in W,\ g_\infty \in \mathsf{G}(\mathbb{R}),\end{aligned}$$

we get $\mathcal{W}^{\infty}_{f,F} : \mathsf{G}(\mathbb{R}) \to W_0 \otimes W^\vee$ with the desired property. The uniquness of $\mathcal{W}^{\infty}_{f,F}$ and $\mathcal{W}^{\mathrm{f}}_{f,F}$ is clear. □

6. Zeta Integrals and Basic Identity

In the first place, we introduce the Rankin-Selberg integrals for a pair of vector valued cusp forms f on $\mathsf{G}_0(\mathbb{A})$ and F on $\mathsf{G}(\mathbb{A})$, that is considerd by Murase-Sugano when f and F are scalar valued holomorphic automorphic forms. The main purpose of this section is to recall the basic identity that relates the Rankin-Selberg integral to an integral transform of the global Shintani function associated with f and F (Proposition 6.1.1). In 6.2, we define the local zeta integrals for real Shintani functions.

6.1. Rankin-Selberg integals and basic identity

Let (τ_0, W_0) and (τ, W) be irreducible unitary representations of $K_{0,\infty}$ and K_∞ respectively. We assume that τ_0 occurs in $\tau \circ (\iota_0|K_{0,\infty})$ and take a $K_{0,\infty}$-inclusion $i^{\tau_0}_{\tau} : W_0 \to W$. As in Lemma 4.1.1, we form τ_1, a representation of $K_{1,\infty}$. Let $\omega : \mathbb{A}_E^*/E^* \to \mathbb{C}^*$ be an idele class character satisfying the conditions (4.1.4) and (4.1.5).

Let $dg = d_{\mathsf{G}(\mathbb{A})}(g)$ be a Haar mesure of $\mathsf{G}(\mathbb{A})$ and $d\dot{g}$ the corresponding quotient measure on $\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})$. For a W_0-valued cusp form $f \in \mathfrak{S}_{\tau_0^\vee}(\mathsf{G}_0\,; K_{0,\mathrm{f}})$, we have introduced the W-valued Eisenstein series $\mathbf{E}(\mathsf{P}_1\,; f\otimes\omega\,; s\,; g_1)$, $g_1 \in$

$\mathsf{G}_1(\mathbb{A})$, $s \in \mathbb{C}$. Now we take a $W^\vee$-valued cusp form $F \in \mathfrak{S}_\tau(\mathsf{G}\ ; K_{\mathrm{f}})$ and consider the following zeta integral after Murase-Sugano.

$$(6.1.1)\quad Z_{f\otimes\omega,F}(s) = \int_{\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})} \langle \mathbf{E}(\mathsf{P}_1\ ; f\otimes\omega\ ; s-\frac{1}{2}\ ; g), F(g)\rangle d\dot{g}, \quad s \in \mathbb{C}.$$

It turns out that the integral converges absolutely and the resulting function $Z_{f\otimes\omega,F}(s)$ is meromorphic on $\mathbb{C}$.

The identity in the next proposition is the so called *basic identity*, which has been established by [5] in the present context. For $g_1 \in \mathsf{G}_1(\mathbb{A})$, we write

$$(6.1.2)\qquad \begin{aligned} g_1 = {} & \mathsf{m}_1(\beta(g_1)\ ; t(g_1))n_1(g_1)k_{1,\infty}(g_1)k_{1,\mathrm{f}}(g_1), \\ & \beta(g_1) \in \mathsf{G}_0(\mathbb{A}),\ t(g_1) \in \mathbb{A}_E^*,\ n_1(g_1) \in \mathsf{N}_1(\mathbb{A}), \\ & k_{1,\infty}(g_1) \in K_{1,\infty},\ k_{1,\mathrm{f}}(g_1) \in K_{1,\mathrm{f}}. \end{aligned}$$

We remark that such a decomposition of g_1 is not unique.

Proposition 6.1.1 (Murase-Sugano). *Let f, ω and F be as above and $\mathcal{W}_{f,F} : \mathsf{G}(\mathbb{A}) \to W_0 \otimes W^\vee$ the global Shintani function associated with f and F. For $s \in \mathbb{C}$ with* $\mathrm{Re}(s) > (n+1)/2$, *we have the identity*

$$(6.1.3)\quad \begin{aligned} Z_{f\otimes\omega,F}(s) = \int_{\mathsf{G}_0(\mathbb{A})\backslash\mathsf{G}(\mathbb{A})} & \epsilon^{\tau}_{\tau_0} \circ (\mathtt{I}_{W_0} \otimes \tau_1^\vee(k_{1,\infty}(g))) \circ \mathcal{W}_{f,F}(\beta(g)^{-1}g) \\ & \times \omega(t(g))|t(g)|_E^{s+n/2} d_{\mathsf{G}_0(\mathbb{A})\backslash\mathsf{G}(\mathbb{A})}(\dot{g}), \end{aligned}$$

where $\epsilon^{\tau_0}_{\tau} : W_0 \otimes_{\mathbb{C}} W^\vee \to \mathbb{C}$ is the map defined by

$$(6.1.4)\qquad \epsilon^{\tau}_{\tau_0}(w_0 \otimes w^\vee) = \langle i^{\tau_0}_{\tau}(w_0), w^\vee\rangle, \quad w_0 \in W_0,\ w^\vee \in W^\vee$$

and $d_{\mathsf{G}_0(\mathbb{A})\backslash\mathsf{G}(\mathbb{A})}(\dot{g})$ denotes the quotient measure of $d_{\mathsf{G}(\mathbb{A})}(g)$ by $d_{\mathsf{G}_0(\mathbb{A})}(g_0)$.

Proof. We reproduce the proof here for completeness following [2], [3], [4] and [5]. We only consider the case when G is $\mathbb{Q}$-isotropic; otherwise the proof is easier. First substituting the expression (4.1.9) to (6.1.1) and then dividing the range of summation into $\mathsf{G}(\mathbb{Q})$-orbits, we get

$$Z_{f\otimes\omega,F}(s) = Z_{\mathtt{I}_{n+2}}(s) + Z_{\mathtt{x}_0}(s)$$

with

$$Z_{\mathtt{y}}(s)=\int_{\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})}\sum_{\gamma_1\in\mathfrak{X}_{\mathtt{y}}}\left\langle\Psi\left(f\otimes\omega\,;s+\frac{n}{2}\,;\gamma_1 g\right),F(g)\right\rangle d\dot{g},$$

where $\mathfrak{X}_{\mathtt{y}}=\mathsf{P}_1(\mathbb{Q})\backslash\big(\mathsf{P}_1(\mathbb{Q})\mathtt{y}\mathsf{G}(\mathbb{Q})\big)$ with $\mathtt{y}=\mathtt{I}_{n+1}$ or $\mathtt{x}_0$ (Lemma 3.3.1). Noting the bijection $\big(\mathtt{y}^{-1}\mathsf{P}_1(\mathbb{Q})\mathtt{y}\cap\mathsf{G}(\mathbb{Q})\big)\backslash\mathsf{G}(\mathbb{Q})\cong\mathfrak{X}_{\mathtt{y}}$, we can rewrite the integral $Z_{\mathtt{y}}(s)$ as

$$\begin{aligned}Z_{\mathtt{y}}(s)&=\int_{\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})}\\&\qquad\times\sum_{\gamma\in(\mathtt{y}^{-1}\mathsf{P}_1(\mathbb{Q})\mathtt{y}\cap\mathsf{G}(\mathbb{Q}))\backslash\mathsf{G}(\mathbb{Q})}\left\langle\Psi\left(f\otimes\omega\,;s+\frac{n}{2}\,;\mathtt{y}\gamma g\right),F(\gamma g)\right\rangle d\dot{g}\\&=\int_{(\mathtt{y}^{-1}\mathsf{P}_1(\mathbb{Q})\mathtt{y}\cap\mathsf{G}(\mathbb{Q}))\backslash\mathsf{G}(\mathbb{A})}\left\langle\Psi\left(f\otimes\omega\,;s+\frac{n}{2}\,;\mathtt{y}g\right),F(g)\right\rangle d\dot{g}.\end{aligned}$$

Now we examine two integrals $Z_{\mathtt{y}}(s)$ with $\mathtt{y}=\mathtt{I}_{n+2},\ \mathtt{x}_0$ separately. We first consider the case $\mathtt{y}=\mathtt{x}_0$. Let P and N be as in Lemma 3.3.1. Along the decomposition $\mathsf{G}(\mathbb{A})=\mathsf{P}(\mathbb{A})K_{\mathbb{A}}=\mathsf{M}(\mathbb{A})\mathsf{N}(\mathbb{A})K_{\mathbb{A}}$ with $K_{\mathbb{A}}=K_\infty K_{\mathrm{f}}$ and $\mathsf{P}=\mathsf{MN}$ a Levi decomposition, we can write $d_{\mathsf{G}(\mathbb{A})}(g)=dm\cdot dn\cdot dk$ with Haar measures dm, dn and dk of $\mathsf{M}(\mathbb{A})$, $\mathsf{N}(\mathbb{A})$ and $K_{\mathbb{A}}$ respectively. We may assume that $\mathrm{vol}(K_{\mathbb{A}})=1$. For $m\in\mathsf{M}(\mathbb{A})$, $n\in\mathsf{N}(\mathbb{A})$ and $k\in K_{\mathbb{A}}$, noting $\mathtt{x}_0n\mathtt{x}_0^{-1}\in\mathsf{N}_1(\mathbb{A})$ and $\mathtt{x}_0m\mathtt{x}_0^{-1}\in\mathsf{P}_1(\mathbb{A})$ (Lemma 3.3.1), we have

$$\Psi\left(f\otimes\omega\,;s+\frac{n}{2}\,;\mathtt{x}_0mnk\right)=\tau(k_\infty)^{-1}\Psi\left(f\otimes\omega\,;s+\frac{n}{2}\,;\mathtt{x}_0m\right).$$

Using this, we have

$$\begin{aligned}Z_{\mathtt{x}_0}(s)&=\int_{\mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{A})}\int_{\mathsf{N}(\mathbb{Q})\backslash\mathsf{N}(\mathbb{A})}\int_{K_{\mathbb{A}}}\\&\qquad\times\left\langle\Psi\left(f\otimes\omega\,;s+\frac{n}{2}\,;\mathtt{x}_0mnk\right),F(mnk)\right\rangle d\dot{m}d\dot{n}dk\\&=\int_{\mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{A})}\int_{K_{\mathbb{A}}}\Big\langle\tau(k_\infty)^{-1}\Psi\left(f\otimes\omega\,;s+\frac{n}{2}\,;\mathtt{x}_0m\right),\\&\qquad\qquad\tau^\vee(k_\infty)^{-1}\int_{\mathsf{N}(\mathbb{Q})\backslash\mathsf{N}(\mathbb{A})}F(mn)d\dot{n}\Big\rangle dkd\dot{m}\\&=\int_{\mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{A})}\left\langle\Psi\left(f\otimes\omega\,;s+\frac{n}{2}\,;\mathtt{x}_0m\right),\int_{\mathsf{N}(\mathbb{Q})\backslash\mathsf{N}(\mathbb{A})}F(mn)d\dot{n}\right\rangle d\dot{m}.\end{aligned}$$

By the cuspidality of F, the integral of $F(mn)$ over $n \in \mathsf{N}(\mathbb{Q})\backslash\mathsf{N}(\mathbb{A})$ in the right-hand side of the last equality vanishes. Hence $Z_{\mathtt{x}_0}(s) = 0$ for all $s \in \mathbb{C}$ if $\mathrm{Re}(s) > (n+1)/2$.

Next we consider the case $\mathtt{y} = \mathtt{I}_{n+2}$. Since $\mathsf{P}_1(\mathbb{Q}) \cap \mathsf{G}(\mathbb{Q}) = \mathsf{G}_0(\mathbb{Q})$ (Lemma 3.3.1), we have

$$
\begin{aligned}
(6.1.5)\quad & Z_{\mathtt{I}_{n+2}}(s) \\
&= \int_{\mathsf{G}_0(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})} \left\langle \Psi\left(f\otimes\omega\ ; s+\frac{n}{2}\ ; g\right), F(g)\right\rangle d\dot{g} \\
&= \int_{\mathsf{G}_0(\mathbb{A})\backslash\mathsf{G}(\mathbb{A})} \int_{\mathsf{G}_0(\mathbb{Q})\backslash\mathsf{G}_0(\mathbb{A})} \left\langle \Psi\left(f\otimes\omega\ ; s+\frac{n}{2}\ ; g_0 g\right), F(g_0 g)\right\rangle d\dot{g}_0 d\dot{g} \\
&= \int_{\mathsf{G}_0(\mathbb{A})\backslash\mathsf{G}(\mathbb{A})} \int_{\mathsf{G}_0(\mathbb{Q})\backslash\mathsf{G}_0(\mathbb{A})} \\
&\quad \times \left\langle \Psi\left(f\otimes\omega\ ; s+\frac{n}{2}\ ; g_0\beta(g)^{-1} g\right), F(g_0\beta(g)^{-1} g)\right\rangle d\dot{g}_0 d\dot{g}.
\end{aligned}
$$

The last equality follows by a change of variable $g_0 \to g_0\beta(g)^{-1}$. For $g_0' \in \mathsf{G}_0(\mathbb{A})$, we have

$$
g_0' = \mathsf{m}_1(g_0'\ ; 1)\mathsf{n}_1\big((g_0'^{-1} - \mathtt{I}_n)\mathsf{a}\ ; \mathsf{S}_0(\mathsf{a}, \mathsf{a} - g_0'\mathsf{a})\big)
$$

by a computation. Thus we get

$$
g_0' g = \mathsf{m}_1\big(g_0'\beta(g)\ ; t(g)\big) n_1' k_{1,\infty}(g) k_{1,\mathrm{f}}(g)
$$

with some $n_1' \in \mathsf{N}_1(\mathbb{A})$. Hence we may take

$$
\begin{aligned}
(6.1.6)\qquad & \beta(g_0' g) = g_0'\beta(g), \quad t(g_0' g) = t(g), \\
& k_{1,\infty}(g_0' g) = k_{1,\infty}(g), \quad k_{1,\mathrm{f}}(g_0' g) = k_{1,\mathrm{f}}(g)
\end{aligned}
$$

for $g_0' \in \mathsf{G}_0(\mathbb{A})$ and $g \in \mathsf{G}(\mathbb{A})$. For given $g_0 \in \mathsf{G}_0(\mathbb{A})$ and $g \in \mathsf{G}(\mathbb{A})$, put $g_0' = g_0\beta(g)^{-1}$. Then using (6.1.6), we have

$$
\begin{aligned}
(6.1.7)\qquad & \Psi\left(f\otimes\omega\ ; s+\frac{n}{2}\ ; g_0\beta(g)^{-1} g\right) \\
&= \omega(t(g_0' g))|t(g_0' g)|_E^{s+n/2} \cdot \tau_1(k_{1,\infty}(g_0' g))^{-1} \circ i_\tau^{\tau_0} \circ f(\beta(g_0' g)) \\
&= \omega(t(g))|t(g)|_E^{s+n/2} \cdot \tau_1(k_{1,\infty}(g))^{-1} \circ i_\tau^{\tau_0} \circ f(g_0).
\end{aligned}
$$

Substituting (6.1.7) to the last formula of (6.1.5), we have

$$
\begin{aligned}
&Z_{\mathtt{I}_{n+2}}(s)\\
&= \int_{\mathsf{G}_0(\mathbb{A})\backslash\mathsf{G}(\mathbb{A})}\int_{\mathsf{G}_0(\mathbb{Q})\backslash\mathsf{G}_0(\mathbb{A})}\left\langle \tau_1(k_{1,\infty}(g))^{-1} i^{\tau}_{\tau_0}(f(g_0)), F(g_0\beta(g)^{-1}g)\right\rangle\\
&\quad \times \omega(t(g))|t(g)|_E^{s+n/2} d\dot{g}_0 d\dot{g}\\
&= \int_{\mathsf{G}_0(\mathbb{A})\backslash\mathsf{G}(\mathbb{A})}\int_{\mathsf{G}_0(\mathbb{Q})\backslash\mathsf{G}_0(\mathbb{A})} \epsilon^{\tau}_{\tau_0}\Big(f(g_0)\otimes\tau_1^{\vee}(k_{1,\infty}(g))F(g_0\beta(g)^{-1}g)\Big)\\
&\quad \times \omega(t(g))|t(g)|_E^{s+n/2} d\dot{g}_0 d\dot{g}\\
&= \int_{\mathsf{G}_0(\mathbb{A})\backslash\mathsf{G}(\mathbb{A})} \epsilon^{\tau}_{\tau_0}\circ(\mathtt{I}_{W_0}\otimes\tau_1^{\vee}(k_{1,\infty}(g_1)))\circ\mathcal{W}_{f,F}(\beta(g)^{-1}g)\\
&\quad \times \omega(t(g))|t(g)|_E^{s+n/2} d\dot{g}. \ \square
\end{aligned}
$$

6.2. Local zeta integrals

Retain the situation of 6.1. Let π_0 be an irreducible $(\mathfrak{g}_0, K_{0,\infty})$-module and π an irreducible $(\mathfrak{g}, K_\infty)$-module. Suppose that $\mathcal{W}_{f,F}|\mathsf{G}(\mathbb{R})$ is not identically zero. Then from Proposition 5.3.1, the global Shintani function $\mathcal{W}_{f,F} : \mathsf{G}(\mathbb{A}) \to W_0 \otimes_{\mathbb{C}} W^{\vee}$ decomposes as

$$
\mathcal{W}_{f,F}(g_{\mathrm{f}}g_\infty) = \mathcal{W}^{\infty}_{f,F}(g_\infty)\cdot\mathcal{W}^{\mathrm{f}}_{f,F}(g_{\mathrm{f}}), \quad g_{\mathrm{f}}\in\mathsf{G}(\mathbb{A}_{\mathrm{f}}),\ g_\infty\in\mathsf{G}(\mathbb{R}).
$$

Here $\mathcal{W}^{\infty}_{f,F}\in \mathrm{Sh}_{\tau_0,\tau}(\pi_0,\pi)$ and $\mathcal{W}^{\mathrm{f}}_{f,F} : \mathsf{G}(\mathbb{A}_{\mathrm{f}}) \to \mathbb{C}$ is a smooth function on $\mathsf{G}(\mathbb{A}_{\mathrm{f}})$. Furthermore the pair $(\mathcal{W}^{\infty}_{f,F}, \mathcal{W}^{\mathrm{f}}_{f,F})$ is uniquely determined from f and F if we require $\mathcal{W}^{\mathrm{f}}_{f,F}(\mathtt{I}_{n+1}) = 1$ in addition. Write $\omega = \omega_{\mathrm{f}}\cdot\omega_\infty$ with $\omega_{\mathrm{f}} = \omega|\mathbb{A}^*_{E,f}$ and $\omega_\infty = \omega|\mathbb{C}^*$. Then we have

$$
Z_{f\otimes\omega,F}(s) = Z^{\mathrm{f}}(\mathcal{W}^{\mathrm{f}}_{f,F}\ ;\omega_{\mathrm{f}}\ ;s)Z^{\infty}(\mathcal{W}^{\infty}_{f,F}\ ;\omega_\infty\ ;s)
$$

with

$$
\begin{aligned}
&Z^{\mathrm{f}}(\mathcal{W}^{\mathrm{f}}\ ;\omega_{\mathrm{f}}\ ;s) = \int_{\mathsf{G}_0(\mathbb{A}_{\mathrm{f}})\backslash\mathsf{G}(\mathbb{A}_{\mathrm{f}})} \mathcal{W}^{\mathrm{f}}(\beta(g_{\mathrm{f}})^{-1}g_{\mathrm{f}})\omega_{\mathrm{f}}(t(g_{\mathrm{f}}))|t(g_{\mathrm{f}})|_{\mathbb{A}_{E,\mathrm{f}}}^{s+n/2} d\dot{g}_{\mathrm{f}},\\
&Z^{\infty}(\mathcal{W}^{\infty}\ ;\omega_\infty\ ;s)\\
&\quad = \int_{\mathsf{G}_0(\mathbb{R})\backslash\mathsf{G}(\mathbb{R})} \epsilon^{\tau}_{\tau_0}\circ(\mathtt{I}_{W_0}\otimes\tau_1^{\vee}(k_1(g_\infty))\circ\mathcal{W}^{\infty}(\beta(g_\infty)^{-1}g_\infty)\\
&\qquad \times \omega_\infty(t(g_\infty))|t(g_\infty)|_{\mathbb{C}}^{s+n/2} d\dot{g_\infty}.
\end{aligned}
$$

Thus caluculation of the zeta integral $Z_{f\otimes\omega,F}(s)$ is reduced to those of $Z^{\mathrm{f}}(\mathcal{W}^{\mathrm{f}}\ ;\omega_{\mathrm{f}}\ ;s)$ and $Z^{\infty}(\mathcal{W}^{\infty}\ ;\omega_{\infty}\ ;s)$. Assume f and F are Hecke eigen forms in the sense of [5]. Then the calculation of the zeta integral over finite adeles $Z^{\mathrm{f}}(\mathcal{W}^{\mathrm{f}}\ ;\omega_{\mathrm{f}}\ ;s)$ is completely carried out by Murase-Sugano and one can find the result in [5]. As for $Z^{\infty}(\mathcal{W}^{\infty}\ ;\omega_{\infty}\ ;s)$, they also calculate it under a certain assumption. In the next section we calculate the archimedian local zeta integrals $Z^{\infty}(\mathcal{W}^{\infty}\ ;\omega_{\infty}\ ;s)$ in a rather general situation.

7. Calculation of Archimedian Local Zeta Integrals

The aim of this section is to calculate the local zeta integrals for real Shintani functions introduced in 6.2. We do not impose any condition on $(\mathfrak{g}_0\oplus\mathfrak{g}, K_{0,\infty}\times K_{\infty})$-module for the Shintani functions in question; but the calculation can be done for those functions with a rather special $K_{0,\infty}\times K_{\infty}$-type. The final result is found in Theorem 7.2.1.

In this section, all groups that enter in the discussion are real points of algebraic groups, so we omit the subscript ∞ from notations for such points; for example we write g for the general element of $\mathsf{G}(\mathbb{R})$ in place of g_{∞}. Moreover we put $G_0=\mathsf{G}_0(\mathbb{R})$, $G=\mathsf{G}(\mathbb{R})$, $G_1=\mathsf{G}(\mathbb{R})$, $K_0=K_{0,\infty}$, $K=K_{\infty}$ and $K_1=K_{1,\infty}$.

7.1. A reduction

Let (π_0, H_{π_0}) be an irreducible $(\mathfrak{g}_0, K_0)$-module and (π, H_{π}) an irreducible $(\mathfrak{g}, K)$-module. Let (τ_0, W) and (τ, W) be irreducible unitary representations of K_0 and K respectively such that τ_0 occurs in $\tau\circ(\iota_0|K_0)$. Let (c^+, c^-) be the central character of τ and (c_0^+, c_0^-) that of τ_0. By assumption we have $c_0^-=c^-$. Further we assume that τ_0 and τ occurs in $\pi_0|K_0$ and $\pi|K$ with multiplicity one. Fixing a K_0-embedding $i_{\tau}^{\tau_0}: W_0\to W$, we define a linear map $\epsilon_{\tau_0}^{\tau}: W_0\otimes_{\mathbb{C}} W^{\vee}\to\mathbb{C}$ by the formula (6.1.4). In this setting we consider the *local zeta integrals*, that is defined by

$$Z^{\infty}(\mathcal{W}\ ;\omega\ ;s)=\int_{G_0\backslash G}\epsilon_{\tau_0}^{\tau}\circ(\mathtt{I}_{W_0}\otimes\tau_1^{\vee}(k_1(g)))\circ\mathcal{W}(\beta(g)^{-1}g) \times\omega(t(g))|t(g)|_{\mathbb{C}}^{s+n/2}d\dot{g} \tag{7.1.1}$$

for $\mathcal{W}\in\mathrm{Sh}_{\tau_0,\tau}(\pi_0,\pi)$ and a quasi-character $\omega:\mathbb{C}^*\to\mathbb{C}^*$ such that

$$\omega(t)=t^{-c_0^- - c^+ + c_0^+},\qquad t\in\mathbb{C}^{(1)}, \tag{7.1.2}$$

which means that ω is of the form

$$\omega(y) = \bar{y}^{(b-c_0^+ + c^+ + c_0^-)/2} y^{(b+c_0^+ - c^+ - c_0^-)/2}, \qquad y \in \mathbb{C}^* \tag{7.1.3}$$

with a complex number b.

We define a_r to be the element of G such that

$$\begin{aligned} a_r(\xi_n^+) &= \mathrm{ch}(r)\xi_n^+ + \mathrm{sh}(r)\xi_{n+1}^-, \\ a_r(\xi_{n+1}^-) &= \mathrm{sh}(r)\xi_n^+ + \mathrm{ch}(r)\xi_{n+1}^-, \\ a_r(\xi_i^+) &= \xi_i^+, \quad 1 \leqslant i \leqslant n-1 \end{aligned} \tag{7.1.4}$$

with $\mathrm{sh}(r) = 2^{-1}(r - r^{-1}),\ \mathrm{ch}(r) = 2^{-1}(r + r^{-1})$.

Let Z be the center of G. Note that it is isomorphic to $\mathsf{U}(1)$ and is contained in K. Let dz be the Haar measure of Z with total mass one. Let $d_{ZG_0\backslash G}(\dot{g})$ be the G-invariant measure on $ZG_0\backslash G$ such that

$$\int_{ZG_0\backslash G} \left(\int_Z h(zg)dz \right) d_{ZG_0\backslash G}(\dot{g}) = \int_{G_0\backslash G} h(\dot{g})d(\dot{g})$$

for any left G_0-invariant, positive valued measurable function h.

LEMMA 7.1.1. *There exists a positive constant C_0 such that for any left ZG_0-invariant continuous function $h : G \to \mathbb{R}_+$ the formula*

$$\int_{ZG_0\backslash G} h(g) d_{ZG_0\backslash G}(\dot{g}) = C_0 \int_1^\infty \int_K h(a_r k)\mathrm{sh}(r)(\mathrm{ch}(r))^{2n-1} dk \frac{dr}{r} \tag{7.1.5}$$

holds.

PROOF. This is a consequence of the integral formula found in [10, page 110, Theorem 2.5]. □

LEMMA 7.1.2. *Let $a_r,\ r > 0$ be as in* (7.1.4). *We have*

$$\iota(a_r) = \mathsf{n}(\mathsf{y}\ ; z)^{-1} \mathsf{m}_1\left(\mathtt{I}_n\ ; \frac{1}{\mathrm{ch}(r)} \right) \mathsf{k}_1 \left[\mathtt{I}_n\ ; \begin{pmatrix} \frac{1}{\mathrm{ch}(r)} & -\frac{\mathrm{sh}(r)}{\mathrm{ch}(r)} \\ \frac{\mathrm{sh}(r)}{\mathrm{ch}(r)} & \frac{1}{\mathrm{ch}(r)} \end{pmatrix} \right] \tag{7.1.6}$$

with

$$\mathsf{y} = -(2d)^{1/2}\mathrm{th}(r)\mathsf{v}_n^- + \left(\frac{1}{\mathrm{ch}(r)} - 1\right)\mathsf{a},$$
$$z = -\mathrm{th}^2(r)a + \frac{1}{\mathrm{ch}(r)}\left(1 - \frac{1}{\mathrm{ch}(r)}\right)\mathsf{S}_0(\mathsf{a},\mathsf{a}) + (2d)^{1/2}\frac{\mathrm{th}(r)}{\mathrm{ch}(r)}\mathsf{S}_0(\mathsf{a},\mathsf{v}_n^-) - (2d)^{1/2}\mathrm{th}(r)\mathsf{S}_0(\mathsf{v}_n^-,\mathsf{a}).$$

Proof. A direct computation. □

Proposition 7.1.1. *For any $\mathcal{W} \in \mathrm{Sh}_{\tau_0,\tau}(\pi_0,\pi)$, we have*

$$Z^\infty(\mathcal{W}\ ;\omega\ ;s) = C_0\int_1^\infty \epsilon_{\tau_0}^{\tau}(\mathcal{W}(a_r))\omega(\mathrm{ch}(r))^{-1}\mathrm{sh}(r)(\mathrm{ch}(r))^{-2s+n-1}\frac{dr}{r}.$$

Proof. For notational simplicity, we put

$$\rho(k_1) = \mathtt{I}_{W_0}\otimes\tau_1^\vee(k_1), \quad k_1 \in K_1.$$

For any $z \in Z$ and any $g \in G$, we have

$$\beta(zg) = \beta(g), \quad t(zg) = t(g), \quad k_1(zg) = k_1(g)z$$

because z belongs to K and $\iota(K) \subset K_1$. We then have

$$\begin{aligned}
&\int_{G_0\backslash G}\rho(k_1(g))\mathcal{W}(\beta(g)^{-1}g)\omega(t(g))|t(g)|_{\mathbb{C}}^{s+n/2}dzd\dot{g}\\
&= \int_{ZG_0\backslash G}\int_Z \rho(k_1(zg))\mathcal{W}(\beta(zg)^{-1}zg)\omega(t(zg))|t(zg)|_{\mathbb{C}}^{s+n/2}dzd\dot{g}\\
&= \int_{ZG_0\backslash G}\int_Z \rho(zk_1(g))\mathcal{W}(\beta(g)^{-1}gz)\omega_\infty(t(g))|t(g)|_{\mathbb{C}}^{s+n/2}dzd\dot{g}\\
&= \int_{ZG_0\backslash G}\rho(k_1(g))\int_Z \rho(z)\mathcal{W}(\beta(g)^{-1}gz)dz\omega_\infty(t(g))|t(g)|_{\mathbb{C}}^{s+n/2}d\dot{g}\\
&= \int_{ZG_0\backslash G}\rho(k_1(g))\mathcal{W}(\beta(g)^{-1}g)\omega(t(g))|t(g)|_{\mathbb{C}}^{s+n/2}d\dot{g}.
\end{aligned}$$

Note that the last equality is a consequence of the equation

$$\mathcal{W}(\beta(g)^{-1}gz) = \rho(z)^{-1}\mathcal{W}(\beta(g)^{-1}g), \quad z \in Z.$$

Next we apply the integration formula (7.1.5). By Lemma 7.1.2, we may assume

$$\beta(a_r k) = \mathtt{I}_n, \quad t(a_r k) = \frac{1}{\mathrm{ch}(r)},$$
$$k_1(a_r k) = \mathsf{k}_1\left[\mathtt{I}_n\ ;\begin{pmatrix} \frac{1}{\mathrm{ch}(r)} & -\frac{\mathrm{sh}(r)}{\mathrm{ch}(r)} \\ \frac{\mathrm{sh}(r)}{\mathrm{ch}(r)} & \frac{1}{\mathrm{ch}(r)} \end{pmatrix}\right]k$$

for $k \in K$ and $r > 0$. Thus we have

$$\begin{aligned}
&Z^{\infty}(\mathcal{W}\ ;\omega\ ;s) \\
&= C_0 \int_1^{\infty}\int_K \epsilon^{\tau}_{\tau_0}\left\{\rho\left(\mathsf{k}_1\left[\mathtt{I}_n\ ;\begin{pmatrix} \frac{1}{\mathrm{ch}(r)} & -\frac{\mathrm{sh}(r)}{\mathrm{ch}(r)} \\ \frac{\mathrm{sh}(r)}{\mathrm{ch}(r)} & \frac{1}{\mathrm{ch}(r)} \end{pmatrix}\right]k\right)\mathcal{W}(a_r k)\right\} \\
&\qquad \times \omega(\mathrm{ch}(r))^{-1}\left|\frac{1}{\mathrm{ch}(r)}\right|_{\mathbb{C}}^{s+n/2}\mathrm{sh}(r)(\mathrm{ch}(r))^{2n-1}\frac{dr}{r}dk \\
&= C_0 \int_1^{\infty} \epsilon^{\tau}_{\tau_0}\circ\left\{\int_K \mathtt{I}_{W_0}\otimes\tau_1^{\vee}\left(\mathsf{k}_1\left[\mathtt{I}_n\ ;\begin{pmatrix} \frac{1}{\mathrm{ch}(r)} & -\frac{\mathrm{sh}(r)}{\mathrm{ch}(r)} \\ \frac{\mathrm{sh}(r)}{\mathrm{ch}(r)} & \frac{1}{\mathrm{ch}(r)} \end{pmatrix}\right]k\right)\tau^{\vee}(k)^{-1}dk\right\} \\
&\qquad \times \mathcal{W}(a_r)\omega(\mathrm{ch}(r))^{-1}\left|\frac{1}{\mathrm{ch}(r)}\right|_{\mathbb{C}}^{s+n/2}\mathrm{sh}(r)(\mathrm{ch}(r))^{2n-1}\frac{dr}{r}.
\end{aligned}$$

The last equality follows from

$$\mathcal{W}(a_r k) = (\mathtt{I}_{W_0}\otimes\tau^{\vee}(k)^{-1})\mathcal{W}(a_r), \quad k \in K.$$

To conclude the proof, we have only to give the following remark. By Lemma 4.1.1, we have

$$\tau_1^{\vee}\left(\mathsf{k}_1\left[\mathtt{I}_n\ ;\begin{pmatrix} \frac{1}{\mathrm{ch}(r)} & -\frac{\mathrm{sh}(r)}{\mathrm{ch}(r)} \\ \frac{\mathrm{sh}(r)}{\mathrm{ch}(r)} & \frac{1}{\mathrm{ch}(r)} \end{pmatrix}\right]k\right) = \tau^{\vee}(k)\det\begin{pmatrix} \frac{1}{\mathrm{ch}(r)} & -\frac{\mathrm{sh}(r)}{\mathrm{ch}(r)} \\ \frac{\mathrm{sh}(r)}{\mathrm{ch}(r)} & \frac{1}{\mathrm{ch}(r)} \end{pmatrix}^{-c_0^-} = \tau^{\vee}(k)$$

for any $k \in K$. Hence, we have

$$\int_K \mathtt{I}_{W_0}\otimes\tau_1^{\vee}\left(\mathsf{k}_1\left[\mathtt{I}_n\ ;\begin{pmatrix} \frac{1}{\mathrm{ch}(r)} & -\frac{\mathrm{sh}(r)}{\mathrm{ch}(r)} \\ \frac{\mathrm{sh}(r)}{\mathrm{ch}(r)} & \frac{1}{\mathrm{ch}(r)} \end{pmatrix}\right]k\right)\tau^{\vee}(k)^{-1}dk = \mathtt{I}_{W_0}\otimes\mathtt{I}_{W^{\vee}}. \ \square$$

7.2. The main theorem

In this subsection and the next, we freely use the results in [7]. For unexpected notations, see also [7].

In the sequel we identify G_0 and G with $\mathsf{U}(n-1,1)$ and $\mathsf{U}(n,1)$ respectively by the isomorphisms (3.5.2) and (3.5.3).(Note that $\mathsf{U}(n,1)$ is written by G_n in [7].) Then K_0 and K correspond to K_{n-1} and K_n in the notation of [7] respectively.

Let π_0 be an irreducible $(\mathfrak{g}_0, K_0)$-module with central character z_0, π an irreducible $(\mathfrak{g}, K)$-module with central character z. Let $\tau_0 = \tau_{\tilde{\mu}}^{K_{n-1}}$ and $\tau = \tau_{\tilde{\lambda}}^{K_n}$ with $\tilde{\mu} \in \mathcal{L}_{n-1}^{+}(\pi_0)$ and $\tilde{\lambda} \in \mathcal{L}_{n}^{+}(\pi)$ ([7, Definition 3.2.1]) satisfying the following.

(i) Let $(\mathbf{l}, h, \nu)$ with $\mathbf{l} = (l_j)_{1 \leqslant j \leqslant n} \in \Lambda_n^{+}(\pi)$ be the triple for π defined in [7, 8.2]. Then $\tilde{\lambda} = [\mathbf{l}\ ; z - |\mathbf{l}|]$.

(ii) Let $m_i^{\pm}(\pi)$ with $i \in \{1, \ldots, n-1\}$ be the integers (or $\pm\infty$) such that

$$\Lambda_{n-1}^{+}(\pi_0) = \{\mathbf{p} = (p_j)_{1 \leqslant j \leqslant n-1} \in \Lambda_{n-1}^{+} |\ m_j^{-}(\pi_0) \leqslant p_j \leqslant m_j^{+}(\pi_0), \\ 1 \leqslant j \leqslant n-1\}.$$

(see [7, 8.1]). Then $\tilde{\mu} = [\mathbf{m}\ ; z_0 - |\mathbf{m}|]$ with $\mathbf{m} = (m_j)_{1 \leqslant j \leqslant n-1}$ such that

(a) $m_{j-1} = m_{j-1}^{-}(\pi_0) \geqslant l_j$ for $j \in \{2, \ldots, h\}$ with $\mathbf{l}^{-j} \in \Lambda_n^{+}$;

(b) $m_j = m_j^{+}(\pi_0) \leqslant l_j$ for $j \in \{h+1, \ldots, n-1\}$ with $\mathbf{l}^{+j} \in \Lambda_n^{+}$;

(c) If $0 < h < n$, then

$$\sup(l_{h+1}, m_h^{-}(\eta)) \leqslant m_h \leqslant \inf(l_h, m_h^{+}(\eta));$$

(iii) $z - z_0 = |\mathbf{l}| - |\mathbf{m}|$.

It turns out that $\mathbf{m}$ satisfying the conditions above is unique if exists. We assume the existance of such an $\mathbf{m}$. Then τ_0 occurs in $\tau | K_0$.

THEOREM 7.2.1. *Let ω be a quasi-character of $\mathbb{C}^*$ satisfying the condition* (7.1.2) *with $c^{+} = |\mathbf{l}|$, $c_0^{+} = |\mathbf{m}|$ and $c_0^{-} = z - |\mathbf{l}| = z_0 - |\mathbf{m}|$. Let b be the complex number such that $\omega(y) = y^b$ for $y > 0$. For $s \in \mathbb{C}$ with* $2\mathrm{Re}(s) >$

$\sup(-\tilde{\nu},\tilde{\nu})+\theta_{\mathbf{m}}-b$, the integral $Z^{\infty}(\mathcal{W}\ ;\omega\ ;s)$ with $\mathcal{W}\in\mathrm{Sh}_{\tau_0,\tau}(\pi_0,\pi)$ converges absolutely and given by

$$Z^{\infty}(\mathcal{W}\ ;\omega\ ;s) = C_0\frac{\deg(\tau_0)\gamma_0(\mathcal{W})}{2} \times \frac{\Gamma\left(s+\frac{b-\theta_{\mathbf{m}}+\tilde{\nu}}{2}\right)\Gamma\left(s+\frac{b-\theta_{\mathbf{m}}-\tilde{\nu}}{2}\right)}{\Gamma\left(s+\frac{b-\theta_{\mathbf{m}}-\nu_0+1}{2}\right)\Gamma\left(s+\frac{b-\theta_{\mathbf{m}}+\nu_0+1}{2}\right)} \tag{7.2.1}$$

with C_0 the constant which enters in the integration formula (7.1.5), $\tilde{\nu}$, ν_0 and $\theta_{\mathbf{m}}$ the complex numbers respectively defined by

$$\begin{aligned}\tilde{\nu} &= \nu+\mu_{\mathbf{m}}-l_{h^+},\\ \nu_0^2 &= 2\Omega_{G_0}(\pi_0)+(n-1)^2-(|\mathbf{m}|-\mu_{\mathbf{m}}-z_0)^2\\ &\quad+2\Big(-\sum_{i=1}^{n-1}m_i^2-\sum_{i=1}^{n-1}(n-2i)m_i+\mu_{\mathbf{m}}^2\\ &\quad+(n-2h^+)\mu_{\mathbf{m}}+|\mathbf{l}|-2\sum_{h^+<k\leqslant n}l_k+\theta_{\mathbf{m}}\Big),\end{aligned} \tag{7.2.2}$$

$$\theta_{\mathbf{m}} = \sum_{h^+\leqslant k\leqslant n-1}l_{k+1}-\sum_{1\leqslant k\leqslant h^-}l_k-\sum_{h^+\leqslant i\leqslant n-1}m_i+\sum_{1\leqslant i\leqslant h^-}m_i, \tag{7.2.3}$$

and

$$\gamma_0(\mathcal{W}) = \langle\mathcal{W}(\mathtt{1}_{n+1}),w_{\tau_0}^{\vee}\otimes i_{\tau}^{\tau_0}(w_{\tau_0})\rangle, \tag{7.2.4}$$

where w_{τ_0} is a highest weight vector for τ_0 and $w_{\tau_0}^{\vee}$ that for $\tau_0^{\vee}$ such that $\langle w_{\tau_0},w_{\tau_0}^{\vee}\rangle=1$.

The rest of this subsection is devoted to giving a proof of this theorem. Let $\{\iota_{\mathbf{p}}^{\pi_0}|\ \mathbf{p}\in\Lambda_{n-1}^{+}(\pi_0)\}$ be the standard system for π_0 ([7, 4.1]). For $\mathbf{p}=(p_j)_{1\leqslant j\leqslant n-1}\in\Lambda_{n-1}^{+}$, let $\check{\mathbf{p}}=(\check{p}_j)_{1\leqslant j\leqslant n-1}$ be the dominant weight defined by $\check{p}_j=-p_{n-j}$. Then it is the highest weight of $W(\mathbf{p})^{\vee}$. Since $\Lambda_{n-1}^{+}(\pi_0^{\vee})=\{\check{\mathbf{p}}|\mathbf{p}\in\Lambda_{n-1}^{+}(\pi_0)\}$, we can put the standard system of $\pi_0^{\vee}$ as $\{\iota_{\check{\mathbf{p}}}^{\pi_0^{\vee}}\mid\mathbf{p}\in\Lambda_{n-1}^{+}(\pi_0)\}$ with $\iota_{\check{\mathbf{p}}}^{\pi_0^{\vee}}:W(\mathbf{p})^{\vee}\to H_{\pi_0}^{\vee}$; we assume that it is taken so that

$$\langle \iota_{\mathbf{p}}^{\pi_0}(w_0), \iota_{\check{\mathbf{p}}}^{\pi_0^\vee}(w_0^\vee)\rangle = \langle w_0, w_0^\vee\rangle, \quad (w_0, w_0^\vee) \in W(\mathbf{p}) \times W(\mathbf{p})^\vee$$

holds. Let $\{\iota_{\mathbf{q}}^{\pi} |\ \mathbf{q} \in \Lambda_n^+(\pi)\}$ be the standard system for π.

Let $\Phi \in \mathcal{I}^{\mathrm{mod}}(\pi_0|\pi)$. As in the proof of Theorem 5.1.1, we extend the representation π_0^∞ to η^∞ of $H = ZG_0$ and let $\tilde{\Phi} : \pi^\infty \to C^\infty\mathrm{Ind}_H^G(\eta^\infty)$ be the G-intertwining operator that corresponds to Φ by the isomorphism in [7, Proposition 2.4.1]. Let $\tilde{\Phi}_{\mathbf{l}}$ be the function defined by [7, (6.3.1)] and $\{f_{\mathbf{l}}(\mathbf{n}\ ; r)|\ \mathbf{n} \in \Lambda_{n-1}^+(\eta|\lambda)\}$ the corresponding standard coefficients ([7, Definition 7.1.1 (2)]).

Lemma 7.2.1. *For $\Phi \in \mathcal{I}^{\mathrm{mod}}(\pi_0|\pi)$, let us define a function $\Phi_{\mathbf{m},\mathbf{l}} : G \to W(\mathbf{m}) \otimes W(\mathbf{l})^\vee$ by the formula*

$$\langle \Phi_{\mathbf{m},\mathbf{l}}(g), w_0^\vee \otimes w\rangle = \Phi(\iota_{\check{\mathbf{m}}}^{\pi_0^\vee}(w_0^\vee) \otimes \iota_{\mathbf{l}}^{\pi}(w))(g), \quad w_0^\vee \in W(\mathbf{m})^\vee,\ w \in W(\mathbf{l}).$$

Then $\mathrm{Sh}_{\tau_0,\tau}(\pi_0,\pi) = \{\Phi_{\mathbf{m},\mathbf{l}}|\ \Phi \in \mathcal{I}^{\mathrm{mod}}(\pi_0|\pi)\}$. *We have the formula*

$$\begin{aligned}\langle \Phi_{\mathbf{m},\mathbf{l}}(a_r), w_0^\vee \otimes w\rangle &= f_{\mathbf{l}}(\mathbf{m}\ ; r) \cdot \langle \iota_{\mathbf{m}}^{\pi_0} \circ \mathsf{p}_{\mathbf{m}}^{\mathbf{l}}(w), \iota_{\check{\mathbf{m}}}^{\pi_0^\vee}(w_0^\vee)\rangle,\\ &w_0^\vee \in W(\mathbf{m})^\vee,\ w \in W(\mathbf{l})\end{aligned}$$

for $r > 0$.

Proof. The first part is obvious. The second part follows from the definition of the standard coefficients ([7, (7.1.2)]). (For the definition of $\mathsf{p}_{\mathbf{m}}^{\mathbf{l}}$ see [7, Lemma 3.1.1].) $\square$

Lemma 7.2.2. *Let $\epsilon_{\tau_0}^{\tau} : W(\mathbf{m}) \otimes W(\mathbf{l})^\vee \to \mathbb{C}$ be the map defined in Proposition 6.1.1 determined by the K_0-inclusion $i_\tau^{\tau_0} : W(\mathbf{m}) \to W(\mathbf{l})$ such that $\mathsf{p}_{\mathbf{m}}^{\mathbf{l}} \circ i_\tau^{\tau_0} = 1_{W(\mathbf{m})}$. We then have*

$$\epsilon_{\tau_0}^{\tau}(\Phi_{\mathbf{l},\mathbf{m}}(a_r)) = \deg(\tau_0) f_{\mathbf{l}}(\mathbf{m}\ ; r) \tag{7.2.5}$$

for $r > 0$. Here $\deg(\tau_0) = \dim_{\mathbb{C}} W(\mathbf{m})$.

Proof. Let $\{w_{0,i}\}_{i\in I_0}$ be a basis of $W(\mathbf{m})$ and $\{w_{0,i}^\vee\}_{i\in I_0}$ its dual bases of $W(\mathbf{m})^\vee$. Let $\{w_j\}_{j\in I}$ with $I_0 \subset I$ be a basis of $W(\mathbf{l}_0)$ such that $w_i = i_\tau^{\tau_0}(w_{0,i})$ for $i \in I_0$, and $\{w_j^\vee\}_{j\in I}$ its dual basis.

We have

$$\Phi_{\mathbf{m},\mathbf{l}}(a_r)=\sum_{i\in I_0}\sum_{j\in I}\langle\Phi_{\mathbf{m},\mathbf{l}}(a_r),w_{0,i}^{\vee}\otimes w_j\rangle w_{0,i}\otimes w_j^{\vee}.$$

Since $\epsilon_{\tau_0}^{\tau}(w_{0,i}\otimes w_j^{\vee})=\delta_{ij},\ i\in I_0, j\in I$ by definition, we have

$$\begin{aligned}\epsilon_{\tau_0}^{\tau}\Phi_{\mathbf{m},\mathbf{l}}(a_r)&=\sum_{i\in I_0}\sum_{j\in I}\langle\Phi_{\mathbf{m},\mathbf{l}}(a_r),w_{0,i}^{\vee}\otimes w_j\rangle\epsilon_{\tau_0}^{\tau}(w_{0,i}\otimes w_j^{\vee})\\&=\sum_{i\in I_0}\langle\Phi_{\mathbf{m},\mathbf{l}}(a_r),w_{0,i}^{\vee}\otimes w_i\rangle\\&=f_{\mathbf{l}}(\mathbf{m}\ ;r)\sum_{i\in I_0}\langle\iota_{\mathbf{m}}^{\pi_0}\circ\mathsf{p}_{\mathbf{m}}^{\mathbf{l}}(w_i),\iota_{\check{\mathbf{m}}}^{\pi_0^{\vee}}(w_{0,i}^{\vee})\rangle\\&=f_{\mathbf{l}}(\mathbf{m}\ ;r)\sum_{i\in I_0}\langle\iota_{\mathbf{m}}^{\pi_0}(w_{0,i}),\iota_{\check{\mathbf{m}}}^{\pi_0^{\vee}}(w_{0,i}^{\vee})\rangle\\&=\deg(\tau_0)f_{\mathbf{l}}(\mathbf{m}\ ;r).\end{aligned}$$

The third equality follows from Lemma 6.2.1, and the last equality is a consequence of the equation

$$\langle\iota_{\mathbf{m}}^{\pi_0}(w_{0,i}),\iota_{\check{\mathbf{m}}}^{\pi_0^{\vee}}(w_{0,i}^{\vee})\rangle=\langle w_{0,i},w_{0,i}^{\vee}\rangle=1,\quad i\in I_0.\ \square$$

From the assumptions (i), (ii) and (iii), $\mathbf{m}\in\partial^{(h)}\Lambda_{n-1}^{+}(\eta|\pi)$. Hence by [7, Theorem 8.2.1], there exists a constant γ_0 such that

$$f_{\mathbf{l}}(\mathbf{m}\ ;r)=\gamma_0(\mathrm{ch}(r))^{\alpha_{\mathbf{m}}}{}_2F_1(X^+,X^-\ ;1\ ;\mathrm{th}^2(r)),\quad r>0. \tag{7.3.6}$$

Here we put

$$\begin{aligned}X^{\pm}&=\frac{\pm\nu_0-\mu_{\mathbf{m}}+l_{h^+}-\nu+1}{2},\\\alpha_{\mathbf{m}}&=\mu_{\mathbf{m}}-l_{h^+}-n+\nu+\theta_{\mathbf{m}}.\end{aligned}$$

(Note $\beta_{\mathbf{m}}=||\mathbf{l}|-|\mathbf{m}|-z+z_0|=0$ by the condition (iii).)

Putting $r=1$ in the equation (7.3.6), we obtain $\gamma_0=f_{\mathbf{l}}(\mathbf{m}\ ;1)$. From Proposition 7.1.1 and Lemma 7.1.1, we have

$$\begin{aligned}&Z^{\infty}(\Phi_{\mathbf{m},\mathbf{l}}\ ;\omega\ ;s)\\&=C_0\gamma_0\deg(\tau_0)\int_1^{\infty}{}_2F_1(X^+,X^-\ ;1\ ;\mathrm{th}^2(r))(\mathrm{ch}(r))^{-2s+n-1+\alpha_{\mathbf{m}}-b}\frac{dr}{r}.\end{aligned}$$

Now, making change of variables from r to $x = \mathrm{th}^2(r)$, we have

$$\mathrm{ch}(r) = (1-x)^{-1/2}, \quad \mathrm{sh}(r) = x^{1/2}(1-x)^{-1/2},$$
$$\frac{dr}{r} = \frac{1}{2}x^{-1/2}(1-x)^{-1}dx,$$

hence

$$Z^\infty(\Phi_{\mathbf{m},\mathbf{l}}\ ;\omega\ ;s) = C_0\frac{\gamma_0 \deg(\tau_0)}{2}\int_0^1 {}_2F_1(X^+, X^-\ ;1\ ;x)(1-x)^{\sigma-1}dx$$

with

$$\sigma = s - \frac{n+\alpha_{\mathbf{m}}-b}{2}.$$

We need a lemma.

Lemma 7.2.3. *For σ, a, $b \in \mathbb{C}$ such that $\mathrm{Re}(\sigma) > \sup(0, \mathrm{Re}(a+b)-1)$, the formula*

$$\int_0^1 (1-x)^{\sigma-1}{}_2F_1(a,b\ ;1\ ;x)dx = \frac{\Gamma(\sigma)\Gamma(1+\sigma-a-b)}{\Gamma(1+\sigma-a)\Gamma(1+\sigma-b)}$$

holds.

Proof. See [11, page 399, formula (4)]. □

Applying the lemma above, we have the conclusion in Theorem 7.2.1.

7.3. A special case

Recall the setting of [7, 9.3]. Let π_0 be a discrete series representation of $G_0 = \mathsf{U}(n-1,1)$ with Harish-Chandra parameter $\mu = [(\mu_i)_{1\leqslant i\leqslant n-1}\ ;\mu_n] \in \Xi^{n-1}_{(h)}$, and π a discrete series representation of $G = \mathsf{U}(n,1)$ with Harish-Chandra parameter $\lambda = [(\lambda_j)_{1\leqslant j\leqslant n}\ ;\lambda_{n+1}] \in \Xi^n_{(k)}$ such that $1 < k = h+1 < n$. Then the conditions (i), (ii) and (iii) in 7.2 is equivalent to the following.

(i) $[(l_j)_{1\leqslant j\leqslant n}\ ;l_0]$ is the Blattner parameter of π;

(ii) $[(m_i)_{1\leqslant i\leqslant n-1}\ ;m_0]$ is the Blattner parameter of π_0 and

$$\begin{aligned}\lambda_1 > \mu_1 > \lambda_2 > \cdots > \lambda_h > \mu_h > \lambda_{h+1} > \lambda_{n+1}\\ > \mu_n > \mu_{h+1} > \lambda_{h+2} > \cdots > \mu_{n-1} > \lambda_n\end{aligned}$$

holds;

(iii) $l_0 = m_0$.

The numerical data $(\tilde{\nu}, \nu_0, \theta_{\mathbf{m}})$ which involves in the formula (7.2.1) is given as

$$\begin{aligned}
\tilde{\nu} &= m_{h+1} - l_0 + n - 2(h+1), \\
\nu_0 &= m_{h+1} - l_0 + n - 1 - 2h, \\
\theta_{\mathbf{m}} &= \sum_{h+1<i\leqslant n} l_i - \sum_{1\leqslant i\leqslant h+1} l_i + \sum_{1\leqslant j<h+1} m_j - \sum_{h+1<j\leqslant n-1} m_j.
\end{aligned}$$

Remark 7.3.1. The condition (i), (ii) and (iii) above implies $\dim_{\mathbb{C}} \mathcal{I}^{\mathrm{mod}}(\pi_0|\pi) = 1$ ([8]).

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