

On the Navier-Stokes equations in a rotating
frame and the functional-differential equations of
advanced type - a Fourier analysis approach

(フーリエ解析的手法による回転場内の流体方程式と進み
型関数微分方程式の考察)

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Preface

In this thesis we discuss two different topics – “Global solvability of the Navier-Stokes equations in a rotating frame with spatially almost periodic data” (Part I) and “The functional-differential equation of advanced type” (Part II). In Part I, we consider global solvability of the Navier-Stokes equations in a rotating frame with spatially almost periodic data. Global solvability is proven by means of techniques of fast singular oscillating limits and bootstrapping from a global-in-time unique solution to the extended 2D-Navier-Stokes equations. In Part II, we construct solutions to the equation $f'(x) = af(\lambda x)$, $x \in \mathbb{R}$, $f(0) = 0$ for constants $\lambda > 1$ and $a \neq 0$. By our method, numerical computations can be made effectively. Although these topics look completely unrelated, there is some common flavor. It is a substantial application of Fourier analysis to the theory of differential equations. For this reason we put these two topics into one thesis.

Part I

Global solvability of the Navier-Stokes equations in a rotating frame with spatially almost periodic data

Chapter 1

Global solvability of the Navier-Stokes equations in a rotating frame with spatially almost periodic data

abstract

We consider global solvability of the Navier-Stokes equations in a rotating frame with spatially almost periodic data. The Coriolis force appears in almost all of the models of oceanography and meteorology dealing with large-scale phenomena. To consider global solvability, we use FM_0 space, i.e. Fourier preimage of the space of all finite Radon measures with no point mass at the origin proposed by Giga, Inui, Mahalov and Matsui in 2005. Global solvability is proven by means of techniques of fast singular oscillating limits and bootstrapping from a global-in-time unique solution to the extended 2D-Navier-Stokes equations.

1.1 Introduction

One of the most important unresolved questions concerning the Navier-Stokes equations is the global regularity and uniqueness of the solutions to the initial

value problem. This question was posed in 1934 by Leray [30, 31] and is still left open for three dimensional flow. However if we pose some conditions on initial velocity, the smooth solution exists globally-in-time. More precisely, Kato [25], and Giga and Miyakawa [20] showed that if the initial velocity is small enough in L^n norm, then the unique smooth solution exists globally-in-time. This smallness condition is generalized by many authors (see [9, 21, 27, 35, 40]). In particular, by Planchon [35] and Cannone [9], this smallness of initial data condition was interpreted by means of an oscillation property in Besov spaces. For example, using a Besov norm instead of a Lebesgue norm is that the condition of being small enough in a Besov space is satisfied by highly oscillating data even if Lebesgue norm is large enough.

When an initial vector is close enough to a two-dimensional vector field, the unique smooth solution exists globally-in-time (see [12, 24]).

Babin, Mahalov and Nicolaenko [4] considered global solvability of the Navier-Stokes equations in a rotating frame with periodic initial data (see also [3, 5, 6, 7, 32]). They proved existence on infinite intervals of regular solutions to the 3D-Navier-Stokes equations with the Coriolis force. Chemin, Desjardins, Gallagher and Grenier [11] derived dispersion estimates on a linearized version of the 3D-Navier-Stokes equations with the Coriolis force. To construct such estimate, they handled eigenvalues and eigenfunctions of the Coriolis operator. Using the dispersive effect, they showed that there exists a global-in-time unique solution to the 3D-Navier-Stokes equations with a large Coriolis force with no smallness assumption on the initial data provided that the initial data decays at space infinity. Although these two results resemble each other, the mechanism is quite different. For periodic initial data there expects no dispersive effect for regularization of the flow, although the flow looks like two dimensional one for a large Coriolis force.

Problems concerning large-scale atmospheric and oceanic flows are known to be dominated by rotational effects. The Coriolis force appears in almost all of the models of oceanography and meteorology dealing with large-scale phenomena. For example, oceanic circulation featuring Typhoon, Hurricane and

Cyclone are caused by the large rotation. There is no doubt that other physical effects are of similar significance like salinity, natural boundary conditions and so on. However the first step in the study of more complex model is to understand the behavior of rotating fluids. This problem attracted many physicists and mathematicians. See [34] for references.

Let us mention almost periodic functions. Giga, Mahalov and Nicolaenko [19] proved existence of a local-in-time unique classical solution of the Navier-Stokes equations (with or without the Coriolis force) when the initial velocity is spatially almost periodic. They showed that the solutions is always spatially almost periodic any time provided that the solution exists. This fact follows from continuous dependence of the solution with respect to initial data in uniform topology. Giga, Inui, Mahalov and Matsui [15] established unique local existence for the Cauchy problem of the Navier-Stokes equations with the Coriolis force when initial data is in FM_0 , Fourier preimage of the space of all finite Radon measures with no point mass at the origin. Some almost periodic functions are in FM_0 . They also showed that the length of existence time-interval of mild solution is independent of the rotation speed. Giga, Jo, Mahalov and the author [18] considered properties of the solution to the Navier-Stokes equations with the Coriolis force in FM_0 . They showed that when the initial data is almost periodic, the complex amplitude is analytic in time. In particular, a new mode cannot be created at any positive time.

In this paper we discuss existence on long time intervals of regular solutions to the 3D-Navier-Stokes equations in a rotating frame with spatially almost periodic data. (It is equivalent to 3D-Navier-Stokes equations for fully three dimensional initial data characterized by uniformly large vorticity. See [7, 23, 33] for example.) Since the initial data does not decay at space infinity, we are unable to use dispersion estimate by [11].

The Cauchy problem for the 3D-Navier-Stokes equations with the Coriolis force (NSC) are described as follows:

$$\begin{cases} \partial_t v^\Omega + (v^\Omega, \nabla)v^\Omega + \Omega e_3 \times v^\Omega - \Delta v^\Omega = -\nabla p^\Omega, \\ \nabla \cdot v^\Omega = 0, v^\Omega|_{t=0} = v_0, \end{cases} \quad (1.1.1)$$

where $v^\Omega = v^\Omega(x, t) = (v^{\Omega,1}(x, t), v^{\Omega,2}(x, t), v^{\Omega,3}(x, t))$ is the unknown velocity vector field and $p^\Omega = p^\Omega(x, t)$ is the unknown scalar pressure at the point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ in space and time $t > 0$ while $v_0 = v_0(x)$ is the given initial velocity field. Here $\Omega \in \mathbb{R}$ is the Coriolis parameter, which is twice the angular velocity of the rotation around the vertical unit vector $e_3 = (0, 0, 1)$, the kinematic viscosity coefficient is normalized by one. By \times we denote the exterior product, and hence, the Coriolis term is represented by $e_3 \times u = Ju$ with the corresponding skew-symmetric 3×3 matrix J .

We shall give the main ideas of the proof. First, we analyze the nonlinear term of NSC. We introduce operators

$$\begin{cases} F_{(0,0,0)} : \text{operator for pure two dimensional interactions,} \\ F_{(1,0,1)} : \text{skew-symmetric-catalytic operator,} \\ F_{(1,1,0)} : \text{non-skew-symmetric-catalytic operator,} \\ F_{(1,1,1)} : \text{operator for strict three dimensional interactions,} \\ F_c^{\Omega,t} : \text{non-resonant operator,} \end{cases}$$

and write NSC in the form

$$\partial_t u = \Delta u + \sum_{\mu \in D} F_\mu(u, u) + F_c^{\Omega,t}(u, u),$$

where $D = \{(0, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$. If the term $F_c^{\Omega,t}$ is vanishing, then the equations are similar to the 2D-Navier-Stokes equations. We call such a system an extended 2D-Navier-Stokes equation (E2DNS). In fact, the solution to E2DNS is independent of the Coriolis force. The key is to prove global existence of a unique smooth solution to E2DNS. Babin, Mahalov and Nicolaenko [4] used energy inequality of E2DNS to show global unique existence of a solution. However, a straightforward application of energy inequality is impossible if the initial data is almost periodic function. What is worse, there is no good Hilbert space for almost periodic functions, so we cannot use eigenvalues and eigenfunctions of the Coriolis operator as Chemin, Desjardins, Gallagher and Grenier [11] did. To overcome these difficulties, we use FM_0 spaces (Fourier preimage of the space of all finite Radon measures

with no point mass at the origin) proposed by Giga, Inui, Mahalov and Matsui (see [15]). We instead employ mild solutions of E2DNS in FM_0 so that this equation turns into a linear one if we choose an appropriate frequency set. Once the equation becomes linear, it is easy to show that the solution to E2DNS exists globally-in-time.

Babin, Mahalov and Nicolaenko [4] handled periodic L^2 Sobolev spaces, and Chemin, Desjardins, Gallagher and Grenier [11] handled L^2 Sobolev spaces in \mathbb{R}^3 . Thus our result is not included in such results since we use almost periodic functions. Moreover we introduce useful decomposition to clarify the analysis of the nonlinear term of NS, which have never been used before.

This paper is organized as follows. In section 1.2 we define function spaces suitable for almost periodic functions and define Hölder spaces. We also define several important operators (Riesz transforms, the Helmholtz projection and Poincaré-Sobolev group). We recall the results in the Navier-Stokes equations with the Coriolis force on local-in-time unique solvability and give uniqueness result in Section 1.3. In section 1.4 we state long time solvability of the Navier-Stokes equations with the Coriolis force. This is our main result. In Section 1.5 we state three key lemmas, extract the extended 2D-Navier-Stokes equations from the Navier-Stokes equations with the Coriolis force, global solvability of the extended 2D-Navier-Stokes equations and fast singular oscillating limits.

1.2 Function spaces, Riesz transforms, the Helmholtz projection and Poincaré-Sobolev group

In this section we shall give definition of function spaces suitable for almost periodic functions included in $BUC(\mathbb{R}^3)$ (Bounded uniformly continuous functions) . Note that almost periodic functions in the sense of Bohr belonging to BUC are already studied. See [8, 10] for example. To define such function

spaces, we need the definition of frequency sets Λ and $\Lambda(\gamma)$. These sets are different from one defined in [16, Definition 1.1.]

Definition 1.2.1. (Countable sum closed frequency set in \mathbb{R}^3 .) We say that $\Lambda \subset \mathbb{R}^3$ is countable sum closed frequency set in \mathbb{R}^3 if Λ is countable set in \mathbb{R}^3 and it satisfies the following equality:

$$\Lambda = \{a + b : a, b \in \Lambda\}.$$

Remark 1.2.2. \mathbb{Z}^3 , $\{e_1 m_1 + \sqrt{2}e_2 m_2 + e_2 m_3 + e_3 m_4 : m_1, \dots, m_4 \in \mathbb{Z}\}$ and $\{e_1 m_1 + (e_1 + e_2 \sqrt{2})m_2 + (e_2 + e_3 \sqrt{3})m_3 : m_1, m_2, m_3 \in \mathbb{Z}\}$ are countable sum closed frequency sets, where $\{e_j\}_{j=1}^3$ is a standard orthogonal base in \mathbb{R}^3 .

Definition 1.2.3. (Countable sum closed frequency set in \mathbb{R}^3 (depending on γ .) Let

$$\Lambda(\gamma) := \{(n_1, n_2, n_3) \in \mathbb{R}^3 : (n_1, n_2, n_3/\gamma) \in \Lambda\}$$

for $\gamma \in \mathbb{R} \setminus \{0\}$.

Remark 1.2.4. Let $\gamma \in \mathbb{R} \setminus \{0\}$. $\Lambda(\gamma)$ is a countable sum closed frequency set in \mathbb{R}^3 if and only if Λ is also countable sum closed frequency set in \mathbb{R}^3 .

First, we define scalar valued function spaces $X^{s,\Lambda(\gamma)}$, $X_0^{s,\Lambda(\gamma)}$ and $\dot{X}^{s,\Lambda(\gamma)}$.

Definition 1.2.5. (3D-scalar valued function spaces.) For $s \geq 0$, let

$$X^{s,\Lambda(\gamma)} := \{g \in BUC(\mathbb{R}^3) : g(x) = \sum_{n \in \Lambda(\gamma)} a_n e^{in \cdot x}, \|g\|_s < \infty\},$$

where

$$\|g\|_s := \sum_{n \in \Lambda(\gamma)} (1 + |n|^2)^{s/2} |a_n|.$$

The infinite sum is understood in the sense of absolute uniform convergence.

The following function spaces are useful to treat Poincaré-Sobolev group which is defined later. Let us define $X_0^{s,\Lambda(\gamma)}$ as follows:

$$X_0^{s,\Lambda(\gamma)} := \{g \in X^{s,\Lambda(\gamma)} : a_0 = 0\}.$$

Remark 1.2.6. $X_0^{s,\Lambda(\gamma)}$ is a closed subspace of $X^{s,\Lambda(\gamma)}$ with the norm $\|\cdot\|_s$.

Definition 1.2.7. (Homogeneous type spaces) Let us define homogeneous type spaces of order s as follows:

$$\dot{X}^{s,\Lambda(\gamma)} := \{g : (-\Delta)^{s/2}g \in X^{0,\Lambda(\gamma)}, \|(-\Delta)^{s/2}g\|_0 < \infty\} \quad \text{for } s \in \mathbb{R}.$$

Second, we define three-dimensional vector valued function spaces $\mathcal{X}^{s,\Lambda(\gamma)}$ and $\mathcal{X}_\sigma^{s,\Lambda(\gamma)}$.

Definition 1.2.8. (3D-vector valued function spaces.) Let

$$\begin{aligned} \mathcal{X}^{s,\Lambda(\gamma)} &:= \{v = (v^1, v^2, v^3) \in (X^{s,\Lambda(\gamma)}(\mathbb{R}^3))^3 : \\ &\|v\|_s := \|v^1\|_s + \|v^2\|_s + \|v^3\|_s < \infty\}. \end{aligned}$$

Let us define three-dimensional vector valued divergence free function spaces as follows:

$$\begin{aligned} \mathcal{X}_\sigma^{s,\Lambda(\gamma)} &:= \{v = (v^1, v^2, v^3) \in \mathcal{X}^{s,\Lambda(\gamma)} : \\ &n^1 a_n^1 + n^2 a_n^2 + n^3 a_n^3 = 0 \quad \text{for } n = (n^1, n^2, n^3) \in \Lambda(\gamma)\}. \end{aligned}$$

We define $\mathcal{X}_0^{s,\Lambda(\gamma)}$, $\mathcal{X}_{0,\sigma}^{s,\Lambda(\gamma)}$ and $\dot{\mathcal{X}}^{s,\Lambda(\gamma)}$ in the same way since the definitions are similar to $X_0^{s,\Lambda(\gamma)}$, $X_{0,\sigma}^{s,\Lambda(\gamma)}$ and $\dot{X}^{s,\Lambda(\gamma)}$. Clearly, $\mathcal{X}^{s,\Lambda(\gamma)} = \mathcal{X}_0^{s,\Lambda(\gamma)} \oplus \mathbb{C}^3$ (topological direct sum).

Remark 1.2.9. $X^{s,\Lambda(\gamma)}$, $X_0^{s,\Lambda(\gamma)}$, $\mathcal{X}^{s,\Lambda(\gamma)}$, $\mathcal{X}_\sigma^{s,\Lambda(\gamma)}$, $\mathcal{X}_0^{s,\Lambda(\gamma)}$, $\mathcal{X}_{0,\sigma}^{s,\Lambda(\gamma)}$ are Banach spaces.

Let us consider the function space $X_0^{0,\Lambda(\gamma)}$ more precisely. It is easy to see that this function space is a closed subspace of FM_0 (the Fourier preimage of the space of all finite Radon measures with no point mass at the origin) which is introduced in [15]. The space FM_0 is strictly smaller than $\dot{B}_{\infty,1}^0$ as is proved in [15, Appendix A]. Thus the space $X_0^{0,\Lambda(\gamma)}$ is strictly smaller than BUC .

Third, we define two-dimensional vector valued function spaces. To treat the two-dimensional Navier-Stokes equations, it is convenient to set the following operators \mathcal{Q}_0 , \mathcal{Q}_1 , \mathcal{Q}_0^h , \mathcal{Q}_0^3 and function spaces $\mathcal{Q}_0^h \mathcal{X}_0^{s,\Lambda}$, $\mathcal{Q}_0^h \mathcal{X}_{0,\sigma}^{s,\Lambda}$.

Definition 1.2.10. (Splitting vertically oscillating and non-oscillating parts.)

For $u = (u^1, u^2, u^3) \in \mathcal{X}^{s, \Lambda(\gamma)}$,

$$u^j(x) = \sum_{n \in \Lambda(\gamma)} c_n^j e^{in \cdot x} \quad (j = 1, 2, 3),$$

let $\mathcal{Q}_\ell u := (\mathcal{Q}_\ell u^1, \mathcal{Q}_\ell u^2, \mathcal{Q}_\ell u^3)$ ($\ell = 0, 1$) with

$$\mathcal{Q}_0 u^j(x_1, x_2) := \sum_{n \in \Lambda(\gamma), n_3=0} c_n^j e^{in \cdot x}, \quad \mathcal{Q}_1 u^j(x) := \sum_{n \in \Lambda(\gamma), n_3 \neq 0} c_n^j e^{in \cdot x},$$

for $j = 1, 2, 3$.

Remark 1.2.11. A direct calculation yields

$$\mathcal{Q}_0 u^j(x_1, x_2) = \lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r u^j(x_1, x_2, x_3) dx_3$$

and

$$\mathcal{Q}_1 u^j(x) = u^j(x) - \mathcal{Q}_0 u^j(x_1, x_2).$$

See [10] for example.

Definition 1.2.12. (Splitting 2D-two vector and 2D-one vector parts.)

Let $\mathcal{Q}_0^h w := (\mathcal{Q}_0 w^1, \mathcal{Q}_0 w^2, 0)$ and $\mathcal{Q}_0^3 w := (\mathcal{Q}_0 w^3, 0, 0)$.

Remark 1.2.13. It is easy to see that $u = (\mathcal{Q}_0 + \mathcal{Q}_1)u = (\mathcal{Q}_0^h + \mathcal{Q}_0^3 + \mathcal{Q}_1)u$ and that $\|w\|_s = \|\mathcal{Q}_0 w\|_s + \|\mathcal{Q}_1 w\|_s = \|\mathcal{Q}_0^h w\|_s + \|\mathcal{Q}_0^3 w\|_s + \|\mathcal{Q}_1 w\|_s$.

Now we define two-dimensional vector valued function spaces $\mathcal{Q}_0^h \mathcal{X}_0^{s, \Lambda}$, $\mathcal{Q}_0^h \mathcal{X}_{0, \sigma}^{s, \Lambda}$ as follows.

Definition 1.2.14. (2D-vector valued function spaces.) For $s \geq 0$, let

$$\begin{aligned} \mathcal{Q}_0^h \mathcal{X}_0^{s, \Lambda} &:= \{v(x) = (v^1, v^2) \in (BUC(\mathbb{R}^2))^2 : \\ &v^j = \sum_{n \in \Lambda \setminus \{0\}, n_3=0} a_n^j e^{in \cdot x}, \quad \text{for } j = 1, 2, \|v\|_s := \|v^1\|_s + \|v^2\|_s < \infty\}, \end{aligned}$$

where

$$\|v^j\|_s := \sum_{n \in \Lambda \setminus \{0\}, n_3=0} (1 + |n|^2)^{s/2} |a_n^j|.$$

Let

$$\begin{aligned} \mathcal{Q}_0^h \mathcal{X}_{0,\sigma}^{s,\Lambda} := \{v(x) = (v^1, v^2) \in \mathcal{Q}_0^h \mathcal{X}_0^{s,\Lambda} : \\ n_1 a_n^1 + n_2 a_n^2 = 0 \text{ for } n \in \Lambda \text{ with } n_3 = 0\}. \end{aligned}$$

Forth, we define the inhomogeneous Hölder spaces $C^\alpha(\mathbb{R}^d)$ with order α .

Definition 1.2.15. (Inhomogeneous Hölder spaces.) For $k \in \mathbb{Z} \setminus \{0\}$, $\alpha \in \mathbb{R}_+ \setminus \mathbb{Z}_+$ with $\alpha < k$, we define

$$C^\alpha(\mathbb{R}^d) := \{f \in BUC(\mathbb{R}^d) : \|f\|_{L^\infty} + \|f\|_{C^\alpha} < \infty\},$$

$$\|f\|_{C^\alpha} := \sup_{x, h \in \mathbb{R}^d} \frac{1}{|h|^\alpha} |G_h^k(f)(x)|,$$

where

$$G_h^k(f)(x) := \sum_{j=0}^k \binom{k}{j} (-1)^j f(x + jh).$$

Note that the norm of C^α is independent of k .

Remark 1.2.16. By [29, Proposition 0.2.2], we see that

$$\|u\|_{C^\alpha} \approx \sum_{|m| \leq k-1} \|\nabla^m u\|_{L^\infty} + \|u\|_{C^{\alpha-k+1}} \text{ for } k = [\alpha] + 1.$$

Moreover $\|u\|_{C^\alpha} \approx \|u\|_{B_{\infty,\infty}^\alpha}$, where B is the Besov scale.

At the end of this section, we define discrete Riesz transforms, discrete Helmholtz projection and discrete Poincaré-Sobolev group.

Definition 1.2.17. (Riesz transforms and the Helmholtz projection.) For

$$v(x) = \sum_{n \in \Lambda(\gamma) \setminus \{0\}} a_n e^{in \cdot x} \in \mathcal{X}_0^{s,\Lambda(\gamma)}, \quad a_n = (a_n^1, a_n^2, a_n^3)$$

and $1 \leq j, k, \ell \leq 3$, let us define \mathbf{R}^{jk} (Riesz transforms to the almost periodic case) and \mathbf{P} (The Helmholtz projection to the almost periodic case) as follows:

$$\begin{cases} \mathbf{R}^{jk}v^\ell := -\sum_{n \in \Lambda(\gamma) \setminus \{0\}} (n^j n^k / |n|^2) a_n^\ell e^{in \cdot x}, \\ \mathbf{P}v^k := v^k + \sum_{j=1}^3 \mathbf{R}^{jk}v^j, \\ \mathbf{P}v := (\mathbf{P}v^1, \mathbf{P}v^2, \mathbf{P}v^3), \end{cases}$$

Remark 1.2.18. If $v \in \mathcal{X}_0^{s, \Lambda(\gamma)}$, then $\mathbf{P}v \in \mathcal{X}_{0, \sigma}^{s, \Lambda(\gamma)}$.

Now we define discrete Poincaré-Sobolev group.

Definition 1.2.19. (Discrete Poincaré-Sobolev group.) Let $e^{-t\Omega\mathbf{S}}$ be a group (called Poincaré-Sobolev group) generated by the bounded operator

$$-\Omega\mathbf{S} := -\Omega\mathbf{P}\mathbf{J}\mathbf{P} : \mathcal{X}_0^{0, \Lambda(\gamma)} \rightarrow \mathcal{X}_0^{0, \Lambda(\gamma)},$$

where \mathbf{J} is a skew-symmetric 3×3 matrix given by $\mathbf{J}u := e_3 \times u$.

Remark 1.2.20. The function $v_L := e^{-t\Omega\mathbf{S}}v_0$ is a solution to the following linearized equation:

$$\begin{cases} \partial_t v_L + \mathbf{P}\mathbf{J}\mathbf{P}v_L = 0 \\ \operatorname{div} v_L = 0, \quad v_L|_{t=0} = v_0. \end{cases}$$

Definition 1.2.21. \mathcal{R}_n is the discrete vector Riesz operator defined as

$$\begin{cases} \mathcal{R}_n a_n = \frac{n}{|n|} \times a_n \quad \text{for } n \in \Lambda(\gamma) \setminus \{0\}, \\ \mathcal{R}_0 a_0 := 0. \end{cases}$$

Proposition 1.2.22. ([14]). For $v(x) = \sum_{n \in \Lambda(\gamma) \setminus \{0\}} a_n e^{in \cdot x} \in \mathcal{X}_0^{s, \Lambda(\gamma)}$, we can express Poincaré-Sobolev group as follows:

$$e^{-t\Omega\mathbf{S}}v(x) := \sum_{n \in \Lambda(\gamma) \setminus \{0\}} e_n^{-t\Omega\mathbf{S}} a_n e^{in \cdot x},$$

where $e_n^{-t\Omega\mathbf{S}}$ is the discrete Poincaré-Sobolev group expressed as

$$e_n^{-t\Omega\mathbf{S}} a_n := \cos\left(\frac{n_3 \Omega t}{|n|}\right) a_n + \sin\left(\frac{n_3 \Omega t}{|n|}\right) \mathcal{R}_n a_n.$$

Remark 1.2.23. For $v \in \mathcal{X}_0^{s, \Lambda(\gamma)}$, the following estimate is easily obtained:

$$(1/3)\|v\|_s \leq \|e^{-t\Omega\mathbf{S}}v\|_s \leq 3\|v\|_s \quad \text{for all } t, \Omega \in \mathbb{R}.$$

1.3 A local-in-time unique solution to the Navier-Stokes equations with the Coriolis force

In this section we show existence of a local-in-time unique solution to the Navier-Stokes equations with the Coriolis force and its properties. The existence of a local-in-time unique solution was already shown by [15, 18]. Thus we only give a sketch of the proof. To show these results, we have to define a mild solution and a weak solution. We also show uniqueness result in almost periodic case.

Applying the projection \mathbf{P} onto solenoidal subspace to the equations (1.1.1) to annihilate the gradient terms ∇p^Ω , we deduce the integral equation corresponding to (1.1.1):

$$v^\Omega(t) = e^{t(\Delta - \Omega \mathbf{S})} v_0 + \int_0^t e^{(t-s)(\Delta - \Omega \mathbf{S})} \mathbf{P} \nabla \cdot (v^\Omega(s) \otimes v^\Omega(s)) ds \quad (1.3.1)$$

by Duhamel's principle.

Definition 1.3.1. (A mild solution.) A solution $v^\Omega \in C([0, T], \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)})$ of (1.3.1) is called a mild solution.

Definition 1.3.2. (A weak solution.) We call $(v^\Omega, p^\Omega) \in L^\infty(0, T; \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)}) \times \mathcal{S}'(\mathbb{R}^3 \times (0, T))$ a weak solution to the equations (1.1.1) on $(0, T) \times \mathbb{R}^3$ with a initial data v_0 if (v^Ω, p^Ω) satisfies $\nabla \cdot v^\Omega = 0$ in \mathcal{S}' for almost every $t \in (0, T)$ and

$$\int_0^T \left\{ \langle v^\Omega(s), \partial_s \Phi(s) \rangle + \langle v^\Omega(s), \Delta \Phi(s) \rangle + \langle v^\Omega \otimes v^\Omega, \nabla \Phi(s) \rangle - \Omega \langle v^\Omega(s), e_3 \times \Phi(s) \rangle + \langle p^\Omega(s), \nabla \cdot \Phi(s) \rangle \right\} ds = -\langle v_0, \Phi(0) \rangle \quad (1.3.2)$$

for any $\Phi \in C^1([0, T] \times \mathbb{R}^3)^3$ such that $\Phi(s, \cdot) \in (\mathcal{S}(\mathbb{R}^3))^3$ for all $s \in [0, T]$ and $\Phi(T, \cdot) \equiv 0$. Here $\langle v^\Omega, \Phi \rangle$ is the canonical pairing of $v^\Omega \in \mathcal{S}'$ and $\Phi \in \mathcal{S}$, and $\langle v^\Omega \otimes v^\Omega, \nabla \Phi \rangle := \sum_{j,k=1}^3 \langle v^{\Omega,j} v^{\Omega,k}, \partial_j \Phi^k \rangle$.

Theorem 1.3.3. Assume that $v_0 = \sum_{n \in \Lambda(\gamma) \setminus \{0\}} a_n e^{in \cdot x} \in \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)}$. Then there is a local-in-time unique mild solution v^Ω satisfying

$$v^\Omega \in C([0, T_{v_0}], \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)}), \quad T_{v_0} \geq \frac{C}{\|v_0\|_0}, \quad \sup_{0 < t < T_{v_0}} \|v^\Omega\|_0 \leq 10 \|v_0\|_0,$$

where C is a positive constant independent of Ω .

Moreover $v^\Omega = v^\Omega(x, t)$ is expressed as

$$v^\Omega(x, t) = \sum_{n \in \Lambda(\gamma) \setminus \{0\}} a_n^\Omega(t) e^{in \cdot x}, \quad a_n^\Omega(t) \rightarrow a_n \quad (t \rightarrow 0).$$

Assume that p^Ω is a distribution in $\mathcal{S}'(\mathbb{R}^3 \times (0, T_{v_0}))$ satisfying:

$$\begin{aligned} \int_0^T \langle p^\Omega(s), \nabla \cdot \Phi(s) \rangle ds := \\ - \int_0^T \langle v^\Omega \otimes v^\Omega, \nabla \Phi(s) \rangle ds + \Omega \int_0^T \langle v^\Omega(s), e_3 \times \Phi(s) \rangle ds. \end{aligned}$$

Then the mild solution v^Ω and the distribution p^Ω satisfy (1.3.2). Moreover the functions $\partial_t v^\Omega$, Δv^Ω , $e_3 \times v^\Omega$, $(v^\Omega, \nabla) v^\Omega$ and ∇p^Ω are in $BUC(\mathbb{R}^3)^3$, and therefore they satisfy (1.1.1) in classical sense.

Next we state a uniqueness result. Let $v_0 \in \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)}$. If (v, p) exists and in $L^\infty(0, T : \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)}) \times L^\infty(0, T : X^{0,\Lambda(\gamma)} + \dot{X}^{1,\Lambda(\gamma)})$, then the weak solution $(v^\Omega, \nabla p^\Omega)$ to the equation (1.3.2) is uniquely determined and the gradient of pressure is expressed as

$$\partial_t p^\Omega = \sum_{j,k=1}^3 \mathbf{R}^{\ell,k} \partial_j v^{\Omega,j} v^{\Omega,k} + \Omega (\mathbf{R}^{\ell,2} v^{\Omega,1} - \mathbf{R}^{\ell,1} v^{\Omega,2})(t) \quad (1.3.3)$$

in $\mathcal{S}'(\mathbb{R}^3)$ (for a.e. t).

Remark 1.3.4. It is easy to see that

$$\begin{aligned} X^{0,\Lambda(\gamma)} + \dot{X}^{1,\Lambda(\gamma)} \approx \left\{ \sum_{n \in \Lambda(\gamma)} b_n e^{in \cdot x} \in \mathcal{S}'(\mathbb{R}^3) : \right. \\ \left. \sum_{n \in \Lambda(\gamma), |n| < 1} |n| |b_n| < \infty \quad \text{and} \quad \sum_{n \in \Lambda(\gamma), |n| \geq 1} |b_n| < \infty \right\}. \end{aligned}$$

Remark 1.3.5. Since $\mathcal{X}^{0,\Lambda(\gamma)} \subset (BUC(\mathbb{R}^3))^3$, any mild solution v^Ω is in $(C^\alpha((0, T_{v_0}) \times \mathbb{R}^3))^3$ for every $\alpha > 0$ (see [28, Proposition 15.1] for example). More precisely, we have

$$\|v^\Omega\|_{(C^\alpha(\mathbb{R}^3))^3}(t) \leq F_\alpha(\|v^\Omega\|_{(L^\infty)^3}(t)) \quad t > 0. \quad (1.3.4)$$

for some continuous and increasing function F_α satisfying $F_\alpha(0) = 0$.

If we impose conditions on the initial values, we can choose the pressure p from suitable function spaces. Moreover, if we restrict the frequency set Λ to \mathbb{Z}^3 , the following remark is always satisfied without additional condition.

Remark 1.3.6. If $v_0 \in \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)} \cap \dot{\mathcal{X}}^{-1,\Lambda(\gamma)}$, then $p^\Omega \in L^\infty(0, T_{v_0} : X^{0,\Lambda(\gamma)})$ and $p \in C^\alpha((0, T_{v_0}) \times \mathbb{R}^3)$ for any $\alpha > 0$. Moreover if $\Lambda = \mathbb{Z}^3$, then $\dot{\mathcal{X}}^{-1,\Lambda(\gamma)} \subset \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)}$.

Proof of Theorem 1.3.3. To prove Theorem 1.3.3 (existence of a mild solution), the following proposition is used. Its proof can be obtained by direct calculation. Note that the case of FM_0 space is already obtained (see [15, 18]). The proof is similar and therefore omitted.

Proposition 1.3.7. If $v \in \mathcal{X}_0^{0,\Lambda(\gamma)}$, then $e^{t(\Delta-\Omega\mathbf{S})}v \in \mathcal{X}_0^{0,\Lambda(\gamma)}$ for $t \geq 0$. More precisely, $\|e^{t(\Delta-\Omega\mathbf{S})}v\|_0 \leq \|v\|_0$. Moreover, $\|e^{t(\Delta-\Omega\mathbf{S})}v - v\|_0 \rightarrow 0$ ($t \rightarrow 0$). If $v_1, v_2 \in \mathcal{X}_0^{0,\Lambda(\gamma)}$, then $e^{t(\Delta-\Omega\mathbf{S})}\mathbf{P}\nabla \cdot (v_1 \otimes v_2) \in \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)}$ for $t > 0$, and

$$\|e^{t(\Delta-\Omega\mathbf{S})}\mathbf{P}\nabla \cdot (v_1 \otimes v_2)\|_0 \leq \frac{C}{t^{1/2}}\|v_1\|_0\|v_2\|_0.$$

By using the above proposition, we can find a local-in-time unique mild solution to the equation (1.3.1) in $C([0, T_{v_0}] : \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)})$. For the detail, see [15]. We can easily confirm that the mild solution v^Ω and the distribution p^Ω satisfy (1.3.2) (for example, see [13, 38]). By Remark 1.3.5, we can ensure that the functions $\partial_t v^\Omega$, Δv^Ω , $e_3 \times v^\Omega$, $(v^\Omega, \nabla)v^\Omega$ and ∇p^Ω are in $BUC(\mathbb{R}^3)^3$.

Next we show that the weak solution to the equation (1.3.2) is unique in certain function spaces. The proof is quite similar to [26, Proof of Theorem 1]. However, for the convenience of the reader, we give the detail. According

to [26], Kato defined some approximated Riesz transforms. We do not need to consider such approximation since treating Riesz transforms in $\mathcal{X}_0^{s,\Lambda(\gamma)}$ is easy.

We first take a test function Φ whose j th component is $\mathbf{R}^{\ell,j}\tilde{\varphi}$, where $\tilde{\varphi} \in C^1([0, T] \times \mathbb{R}^3)$ is such that $\tilde{\varphi}(s, \cdot) \in \mathcal{S}(\mathbb{R}^3)$ for $0 \leq s \leq T$ and $\tilde{\varphi}(T, \cdot) \equiv 0$. We have that

$$\begin{aligned} \int_0^T \langle \partial_\ell p^\Omega(s), \tilde{\varphi}(s) \rangle ds = \\ \int_0^T \left\langle \sum_{i,j=1}^3 \mathbf{R}^{\ell,j} \partial_i v^{\Omega,i} v^{\Omega,j} + \Omega \langle (\mathbf{R}^{\ell,2} v^{\Omega,1} - \mathbf{R}^{\ell,1} v^{\Omega,2})(s), \tilde{\varphi}(s) \rangle \right\rangle ds. \end{aligned}$$

Thus the expression for the pressure (1.3.3) holds and the functions v^Ω and v_0 satisfy the following equation:

$$\begin{aligned} \int_0^T \left\{ \langle v^\Omega(s), \partial_s \Phi(s) \rangle + \langle v^\Omega(s), \Delta \Phi(s) \rangle + \langle \mathbf{P} \nabla \cdot (v^\Omega \otimes v^\Omega)(s), \Phi(s) \rangle \right. \\ \left. - \Omega \langle v^\Omega(s), e_3 \times \Phi(s) \rangle + \Omega \sum_{\ell=1}^2 \langle (\mathbf{R}^{\ell,2} v^{\Omega,1} - \mathbf{R}^{\ell,1} v^{\Omega,2})(s), \Phi^\ell(s) \rangle \right\} ds \\ = -\langle v_0, \Phi(0) \rangle. \quad (1.3.5) \end{aligned}$$

Second, we take a test function of the following form:

$$\Phi(s, x) = \begin{cases} \eta(s) (e^{(t-s+\delta)(\Delta+\Omega\mathbf{S})} \varphi)(x), & 0 < s < t + \delta, \\ 0, & t + \delta \leq s < T \end{cases}$$

and

$$\eta(s) := \int_s^\infty \rho_\epsilon(s' - t) ds',$$

where $\varphi \in (\mathcal{S}(\mathbb{R}^3))^3$ and $\rho \in C(\mathbb{R})$ with $\rho \geq 0$, $\text{supp } \rho \subset (-1, 1)$, $\int \rho = 1$ and $\rho_\epsilon(s) = \epsilon^{-1} \rho(s/\epsilon)$.

By using such test functions, the first term of the left hand side of (1.3.5) is estimated as follows:

$$\begin{aligned} \int_0^T \langle v^\Omega(s), \partial_s \Phi(s) \rangle ds = \\ - \int_0^T \langle v^\Omega(s), \Delta \Phi(s) + \Omega \mathbf{P} \mathbf{J} \mathbf{P} \Phi(s) \rangle + \langle v^\Omega(s), e^{(t-s+\delta)(\Delta+\Omega\mathbf{S})} \varphi \rangle \rho_\epsilon(s-t) ds. \end{aligned}$$

Since v^Ω is in $L^\infty(0, T : \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)})$ by assumption, the function $U(s) := \langle v^\Omega(s), e^{(t-s+\delta)(\Delta+\Omega\mathbf{S})}\varphi \rangle$ is in $L^\infty([0, T])$. Thus we have

$$\int_0^T U(s)\rho(s-t)ds \rightarrow \langle v^\Omega(t), e^{\delta(\Delta+\Omega\mathbf{S})}\varphi \rangle \quad (\epsilon \rightarrow 0) \quad \text{a. e. } t \in [0, T]$$

by the Lebesgue differentiation theorem (see [39]). Moreover we have

$$\begin{aligned} \int_0^T \langle v^\Omega(s), \mathbf{PJP}\Phi(s) \rangle ds &= \int_0^T \langle \mathbf{P}Jv^\Omega(s), \Phi(s) \rangle ds = \\ &= \langle v^\Omega(s), e_3 \times \Phi(s) \rangle + \sum_{\ell=1}^2 \langle (\mathbf{R}^{\ell,2}v^{\Omega,1} - \mathbf{R}^{\ell,1}v^{\Omega,2})(s), \Phi^\ell(s) \rangle ds. \end{aligned}$$

Combining the above estimates, we have

$$\langle v^\Omega(t) - e^{t(\Delta-\Omega\mathbf{S})}v_0 + \int_0^t e^{(t-s)(\Delta-\Omega\mathbf{S})}\mathbf{P}\nabla \cdot (v^\Omega \otimes v^\Omega)(s)ds, \varphi \rangle = 0 \quad \text{a.e. } t.$$

Since $\varphi \in (\mathcal{S}(\mathbb{R}^3))^3$ can be taken arbitrarily, we have

$$v^\Omega(t) - e^{t(\Delta-\Omega\mathbf{S})}v_0 + \int_0^t e^{(t-s)(\Delta-\Omega\mathbf{S})}\mathbf{P}\nabla \cdot (v^\Omega \otimes v^\Omega)(s)ds = 0 \quad (1.3.6)$$

for a.e. $(t, x) \in [0, T] \times \mathbb{R}^3$. We notice that v is identified with the $\mathcal{X}^{0,\Lambda(\gamma)}$ -valued continuous function on $(0, T)$ since $v^\Omega \in L^\infty(0, T : \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)})$. Finally, we shall confirm that the solution to (1.3.6) is unique. Assume that v_1 and v_2 are solutions to the integral equation (1.3.6) in $L^\infty(0, T, \mathcal{X}^{0,\Lambda(\gamma)})$ for the same initial data and Coriolis parameter Ω . Then we have

$$\begin{aligned} \|v_1 - v_2\|_0(t) &\leq \int_0^t \frac{C}{(s-t)^{1/2}} (\|v_1(s)\|_0 + \|v_2(s)\|_0) \|v_1 - v_2\|_0(s) ds \\ &\leq Ct^{1/2} \sup_{0 < s \leq t} (\|v_1(s)\|_0 + \|v_2(s)\|_0) \sup_{0 < s \leq t} \|v_1 - v_2\|_0(s), \end{aligned}$$

which means that $v_1 = v_2$. ■

1.4 Global solvability of the 3D Navier-Stokes equations in a rotating frame with spatially almost periodic data

In this section we state a result of global solvability of the 3D-Navier-Stokes equations in a rotating frame with spatially almost periodic data. This is our main result.

Theorem 1.4.1. Let Λ be a countable sum closed frequency set. There exists a set $\Gamma \subset \mathbb{R} \setminus \{0\}$ (depending on Λ) whose complement set is countable.

Let us impose the following two assumptions.

- (1) Take $\gamma \in \Gamma$.
- (2) Take $v_0 \in \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)}$ such that the initial value problem for the 2D Navier-Stokes equations admits a global-in-time unique solution in $C([0, \infty) \mathcal{Q}_0^h \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)})$ with a initial data $\mathcal{Q}_0^h v_0 \in \mathcal{Q}_0^h \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)}$.

Then for any $T > 0$ there exists Ω_0 depending only on v_0 and T such that if $|\Omega| > \Omega_0$, then there exists a mild solution $v^\Omega \in C([0, T] : \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)})$ of equation (1.1.1) with a initial data $v_0 \in \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)}$.

Remark 1.4.2. If $\mathcal{Q}_0^h v_0$ is a periodic function, there exists a global-in-time unique solution to the 2D Navier-Stokes equations in $C([0, \infty) \mathcal{Q}_0^h \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)})$. See Appendix.

Remark 1.4.3. If $\|(-\Delta)^{-\frac{1}{2}} \mathcal{Q}_0^h v_0\|_0$ is small enough, there exists a global-in-time unique solution to the 2D Navier-Stokes equations in $C([0, \infty) \mathcal{Q}_0^h \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)})$. See [16, 17].

Proof. We use boot a strapping argument. This technique is based on [4, Theorem 8.2]. By key lemmas to be described in the next section, there is $w \in C([0, \infty) : \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)})$ satisfying the following property (we call fast singular oscillating limits):

For all $\epsilon > 0$, there is $\Omega_0 > 0$ (depending only on $\|v_0\|_0$),

$$\|v^\Omega(t) - e^{t\Omega\mathbf{S}} w(t)\|_0 < \epsilon \quad \text{for } t \in [0, T_{v_0}], \quad \Omega > \Omega_0 \quad \text{and} \quad w|_{t=0} = v_0,$$

where v^Ω is a local-in-time unique solution to the equation (1.1.1) described in the previous section, and $w|_{t=0} = v_0$. Now let us take $T > 0$ and $\rho > 0$ arbitrarily. Let $M = M(\|v_0\|, T)$ be such that $\sup_{0 < t < T} \|w\| = M/3$. Since $\|v_0\| \leq \sup_{0 < t < T} \|w\| = M/3 \leq M + \rho$, there is T_v such that $\sup_{0 < t < T_v} \|v\| \leq 10(M + \rho)$ by the local existence result. Note that we can improve this bound by fast singular oscillating limits.

Now we estimate $\|v^\Omega(T_{v_0})\|_0$. There is $\Omega'_0 > 0$,

$$\|v^\Omega(t) - e^{t\Omega\mathbf{S}}w(t)\|_0 < \rho \quad \text{for } 0 < t < T_{v_0}, \quad \Omega > \Omega'_0.$$

Since

$$\|v^\Omega(T_{v_0})\|_0 \leq \|v^\Omega(T_{v_0}) - e^{T_{v_0}\Omega\mathbf{S}}w(T_{v_0})\|_0 + 3\|w(T_{v_0})\|_0 \leq M + \rho,$$

then we can extend the existence time as follows:

$$\|v^\Omega(2T_{v_0})\|_0 \leq 2(M + \rho) \quad \text{for } 0 < t < 2T_{v_0}.$$

Repeating this argument, we can say

$$\sup_{0 < t < T} \|v^\Omega(t)\|_0 \leq 2(M + \rho) \quad \text{for sufficiently large } \Omega.$$

This is the desired estimate. ■

1.5 Key lemmas

In this section we give key lemmas which are needed to prove the main result. More precisely, we prove existence of the key function w described in the proof of the main theorem. To show existence of the key function w , we have to prove the following three key lemmas, extract the extended 2D-Navier-Stokes equations(E2DNS) from the Navier-Stokes equations with the Coriolis force, global solvability of the E2DNS and fast singular oscillating limits.

1.5.1 Extract the extended 2D-Navier-Stokes equations from the Navier-Stokes equations with the Coriolis force

To state the lemma, we need operators $\{F_\mu\}_{\mu \in \{0,1\}^3}$ and $F_c^{\Omega,t}$ satisfying appropriate conditions. The key lemma is as follows.

Lemma 1.5.1. Let $D := \{(0, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$. There exist bilinear operators (see also Definition 1.5.13)

$$F_\mu : \mathcal{Q}_{\mu_2} \mathcal{X}^{s,\Lambda(\gamma)} \times \mathcal{Q}_{\mu_3} \mathcal{X}^{s+1,\Lambda(\gamma)} \rightarrow \mathcal{Q}_{\mu_1} \mathcal{X}_{0,\sigma}^{s,\Lambda(\gamma)} \quad \text{for } \mu \in D$$

and

$$F_c^{\Omega,t} : \mathcal{X}^{s,\Lambda(\gamma)} \times \mathcal{X}^{s+1,\Lambda(\gamma)} \rightarrow \mathcal{X}_{0,\sigma}^{s,\Lambda(\gamma)},$$

$$\begin{cases} F_{(0,0,0)} : \text{operator for pure two dimensional interactions,} \\ F_{(1,0,1)} : \text{skew-symmetric-catalytic operator,} \\ F_{(1,1,0)} : \text{non-skew-symmetric-catalytic operator,} \\ F_{(1,1,1)} : \text{operator for strict three dimensional interactions,} \\ F_c^{\Omega,t} : \text{non-resonant operator,} \end{cases}$$

satisfying

$$\|e^{t\Delta} \mathcal{Q}_{\mu_1} F_\mu(\mathcal{Q}_{\mu_2} u_1, \mathcal{Q}_{\mu_3} u_2)\|_s \leq \frac{C}{t^{1/2}} (\|\mathcal{Q}_{\mu_2} u_1\|_0 \|\mathcal{Q}_{\mu_3} u_2\|_s + \|\mathcal{Q}_{\mu_2} u_1\|_s \|\mathcal{Q}_{\mu_3} u_2\|_0),$$

for $\mu \in D$,

$$\|e^{t\Delta} F_c^{\Omega,t}(u_1, u_2)\|_s \leq \frac{C}{t^{1/2}} (\|u_1\|_0 \|u_2\|_s + \|u_1\|_s \|u_2\|_0),$$

such that the function $u := e^{t\Omega \mathbf{S}} v \in C([0, T_{v_0}] : \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)})$ satisfies the following equation:

$$\begin{cases} \partial_t u = \Delta u + \sum_{\mu \in D} \mathcal{Q}_{\mu_1} F_\mu(\mathcal{Q}_{\mu_2} u, \mathcal{Q}_{\mu_3} u) + F_c^{\Omega,t}(u, u), \\ \operatorname{div} u = 0, \quad u(0) = v_0 = \sum_{n \in \Lambda(\gamma) \setminus \{0\}} a_n e^{in \cdot x} \in \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)} \end{cases} \quad (1.5.1)$$

if and only if v is a solution to the equation (1.1.1).

Remark 1.5.2. The operator $\mathcal{Q}_0 F_{(0,0,0)}(\mathcal{Q}_0 \cdot, \mathcal{Q}_0 \cdot)$ can be expressed as follows:

$$\begin{aligned} \mathcal{Q}_0 F_{(0,0,0)}(\mathcal{Q}_0 u, \mathcal{Q}_0 u) &= \mathcal{Q}_0 \mathbf{P}(\mathcal{Q}_0 u, \nabla) \mathcal{Q}_0 u \\ &= \mathcal{Q}_0^h \mathbf{P}(\mathcal{Q}_0^h u, \nabla) \mathcal{Q}_0^h u + (\mathcal{Q}_0^h u, \nabla) \mathcal{Q}_0^3 u. \end{aligned}$$

The operators $F_{(1,1,0)}$, $F_{(1,0,1)}$ and $F_{(1,1,1)}$ can be also expressed by using explicit formula (see Remark 1.5.12 and Remark 1.5.14).

Remark 1.5.3. The key function w is the solution to the following equation (see the next section):

$$\begin{cases} \partial_t w = \Delta w + \sum_{\mu \in D \setminus (1,1,1)} \mathcal{Q}_{\mu_1} F_{\mu}(\mathcal{Q}_{\mu_2} w, \mathcal{Q}_{\mu_3} w), \\ \operatorname{div} w = 0, \quad u(0) = v_0 \in \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)}. \end{cases} \quad (1.5.2)$$

Remark 1.5.4. $F_{(1,0,1)}$ is skew symmetric. However $F_{(1,1,0)}$ is not skew symmetric (see Appendix). Thus we cannot construct energy inequality of (1.5.2).

Proof of Lemma 1.5.1. For the sake of simplicity, we set $v := v^\Omega$ in the proof. By an easy calculation, we can say that v is a solution to the equation (1.1.1) with a initial data $v_0 \in \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)}$ if and only if the Fourier sum $\sum_{n \in \Lambda(\gamma) \setminus \{0\}} c_n(t) e^{in \cdot x} := u(x, t) = e^{t\Omega \mathbf{S}} v(x, s)|_{s=t}$ satisfies the following infinitely dimensional ordinary differential equations:

$$\begin{cases} \partial_t c_n(t) = -|n|^2 c_n(t) + B_n^\Omega(c(t), c(t)), \\ n_1 c_n^1(t) + n_2 c_n^2(t) + n_3 c_n^3(t) = 0 \quad \text{for } n \in \Lambda(\gamma) \setminus \{0\}, \\ c_n(0) = c_n = a_n, \end{cases}$$

where

$$B_n^\Omega(c_1, c_2) := e_n^{-t\Omega \mathbf{S}} \mathbf{P}_n \sum_{n=k+m} (e_k^{t\Omega \mathbf{S}} c_{1,k}, im) e_m^{t\Omega \mathbf{S}} c_{2,m}.$$

Let us now consider the nonlinear part B_n^Ω more precisely. To analyze the nonlinear part, we need several definitions.

Definition 1.5.5. For $k, m, n \in \Lambda(\gamma)$ and $\sigma, \delta \in \{0, 1\}^3$, let

$$\begin{cases} \theta_k^0(\Omega t) := \cos\left(\frac{k_3}{|k|}\Omega t\right), \theta_k^1(\Omega t) := \sin\left(\frac{k_3}{|k|}\Omega t\right), \\ \alpha_{\delta, \sigma} = \prod_{j=1}^3 \frac{(-1)^{\delta_j \sigma_j (-i)^{\delta_j}}}{8}, \\ \omega_{nkm}^\sigma := (-1)^{\sigma_1 \frac{n_3}{|n|}} + (-1)^{\sigma_2 \frac{k_3}{|k|}} + (-1)^{\sigma_3 \frac{m_3}{|m|}}, \\ b_{nkm}^\delta(c_k, c_m) := \mathcal{R}_n^{\delta_1} \mathbf{P}_n(\mathcal{R}_k^{\delta_2} c_k(t), im) \mathcal{R}_m^{\delta_3} c_m(t), \\ \mathcal{R}_n^0 c_n = c_n, \quad \mathcal{R}_n^1 c_n = \mathcal{R}_n c_n = \frac{n}{|n|} \times c_n. \end{cases}$$

Remark 1.5.6. Since $\{c_m(t)\}_{m \in \Lambda(\gamma)}$ is divergence free, $\mathcal{R}_n^{\delta_1}$ and \mathbf{P}_n are commutative, we have

$$b_{nkm}^\delta(c_k, c_m) = \mathbf{P}_n \mathcal{R}_n^{\delta_1}(\mathcal{R}_k^{\delta_2} c_k(t), im) \mathcal{R}_m^{\delta_3} c_m(t).$$

Since $|\mathbf{P}_n|, |\mathcal{R}_n^\delta| \leq 1$ for $n \in \Lambda(\gamma) \setminus \{0\}$, and $\mathcal{R}_0 c_0 = 0$, we have

$$|b_{nkm}^\delta(c_k, c_m)| \leq |n| |c_k| |c_m| \quad \text{for } \delta \in \{0, 1\}^3, \quad n, k, m \in \Lambda(\gamma).$$

Direct calculations yield

$$\begin{aligned} B_n^\Omega(c(t), c(t)) &= \sum_{n=k+m} \sum_{\delta \in \{0,1\}^3} \theta_n^{\delta_1} \theta_k^{\delta_2} \theta_m^{\delta_3} b_{nkm}^\delta(c_k(t), c_m(t)) \\ &= \sum_{n=k+m} \sum_{\delta, \sigma \in \{0,1\}^3} \alpha_{\delta, \sigma} e^{i\Omega t \omega_{nkm}^\sigma} b_{nkm}^\delta(c_k(t), c_m(t)). \end{aligned}$$

We now define the resonant frequency set \mathcal{K}^σ (the non-resonant frequency set is its complementary set) and $\Lambda(\gamma)_\mu$ which is applied to extract the extended 2D-Navier-Stokes equations.

Definition 1.5.7. (Resonant frequency set.) For $\sigma \in \{0, 1\}^3$, let

$$\mathcal{K}^\sigma := \{(n, k, m) \in (\Lambda(\gamma))^3 : \omega_{nkm}^\sigma = 0\}.$$

Definition 1.5.8. (Splitting vertically oscillating and non-oscillating parts.)

For $\mu \in \{0, 1\}^3$, let

$$\begin{aligned} \Lambda(\gamma)_\mu &:= \{(n, k, m) \in (\Lambda(\gamma))^3 : (n_1, n_2, n_3/\gamma) \in \Lambda_{\mu_1}, \\ &\quad (k_1, k_2, k_3/\gamma) \in \Lambda_{\mu_2}, (m_1, m_2, m_3/\gamma) \in \Lambda_{\mu_3}\}, \end{aligned}$$

where $\Lambda_0 := \{0\}$ and $\Lambda_1 := \Lambda \setminus \{0\}$.

Remark 1.5.9. It is easy to see that

$$(\Lambda(\gamma))^3 = \bigcup_{\sigma, \mu \in \{0,1\}^3} \mathcal{K}^\sigma \cup (\mathcal{K}^\sigma)^c = \bigcup_{\sigma \in \{0,1\}^3} (\mathcal{K}^\sigma)^c \cup \bigcup_{\sigma, \mu \in \{0,1\}^3} (\mathcal{K}^\sigma \cap \Lambda(\gamma)_\mu)$$

Moreover, the right hand side of each frequency set is disjoint.

Now we define operators $\{B_\mu\}_{\mu \in \{0,1\}^3}$ and $B_{n,\Omega,t}^c$.

Definition 1.5.10. (B_n^μ and $B_{n,\Omega,t}^c$.) For $\mu \in \{0,1\}^3$, let

$$B_n^\mu(c_1, c_2) := \sum_{\delta, \sigma \in \{0,1\}^3} \sum_{n=k+m, (n,k,m) \in \mathcal{K}^\sigma \cap \Lambda(\gamma)_\mu} \alpha_{\delta, \sigma} b_{nkm}^\delta(c_{1,k}, c_{2,m}),$$

$$B_{n,\Omega,t}^c(c_1, c_2) := \sum_{\delta, \sigma \in \{0,1\}^3} \sum_{n=k+m, (n,k,m) \in (\mathcal{K}^\sigma)^c} \alpha_{\delta, \sigma} e^{i\Omega t \omega_{nkm}^\sigma} b_{nkm}^\delta(c_{1,k}, c_{2,m}).$$

Remark 1.5.11. By Remark 1.5.6, we easily have

$$\begin{aligned} |B_n^\mu(c_1, c_2)| &\leq \sum_{n=k+m, (n,k,m) \in \mathcal{K}^\sigma \cap \Lambda(\gamma)_\mu} |n| |c_{1,k}| |c_{2,m}| \\ &\leq \sum_{n=k+m, (n,k,m) \in \Lambda(\gamma)_\mu} |n| |c_{1,k}| |c_{2,m}| \end{aligned}$$

for $\mu \in D$ and $\sigma \in \{0,1\}^3$,

$$|B_{n,\Omega,t}^c(c_1, c_2)| \leq C \sum_{n=k+m, (n,k,m) \in (\mathcal{K}^\sigma)^c} |n| |c_{1,k}| |c_{2,m}| \leq C \sum_{n=k+m} |n| |c_{1,k}| |c_{2,m}|.$$

Remark 1.5.12. The following explicit formula is important in the next section:

$$B_n^{(1,1,1)}(c_1, c_2) := \sum_{\delta, \sigma \in \{0,1\}^3} \sum_{n=k+m, \omega_{nkm}^\sigma = 0, n_3, k_3, m_3 \neq 0} \alpha_{\delta, \sigma} b_{nkm}^\delta(c_{1,k}, c_{2,m}).$$

By the definition of B^μ , $B_{n,\Omega,t}^c$, and a direct calculation yields

$$\begin{aligned} B_n^\Omega(c(t), c(t)) &= \sum_{n=k+m} \sum_{\delta, \sigma \in \{0,1\}^3} \alpha_{\delta, \sigma} e^{i\Omega t \omega_{nkm}^\sigma} b_{nkm}^\delta(c_k(t), c_m(t)) \\ &= \left(\sum_{\mu \in \{0,1\}^3} B_n^\mu + B_{n,\Omega,t}^c \right)(c(t), c(t)). \quad (1.5.3) \end{aligned}$$

Now we define the resonant operators F_μ (these are independent of Ω), and the non-resonant operator $F_c^{\Omega,t}$ (these are depending on Ω).

Definition 1.5.13. (F_μ and $F_c^{\Omega,t}$.) For $u_1(x) := \sum_{n \in \Lambda(\gamma) \setminus \{0\}} c_{1,n} e^{in \cdot x}$ and $u_2(x) := \sum_{n \in \Lambda(\gamma) \setminus \{0\}} c_{2,n} e^{in \cdot x}$, let

$$F_\mu(u_1, u_2) = \mathcal{Q}_{\mu_1} F_\mu(\mathcal{Q}_{\mu_2} u_1, \mathcal{Q}_{\mu_3} u_2) := \sum_{n \in \Lambda(\gamma)} B_n^\mu(c_1, c_2) e^{in \cdot x},$$

$$F_c^{\Omega,t}(u_1, u_2) := \sum_{n \in \Lambda(\gamma)} B_{n,\Omega,t}^c(c_1, c_2) e^{in \cdot x}.$$

By (1.5.3), we can say that v is the solution to (1.1.1) with a initial data $v_0 \in \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)}$ if and only if $u(x, t) \in C([0, T_{v_0}] : \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)})$, satisfies the following nonlinear equations:

$$\begin{cases} \partial_t u = \Delta u + \sum_{\mu \in \{0,1\}^3} \mathcal{Q}_{\mu_1} F_\mu(\mathcal{Q}_{\mu_2} u, \mathcal{Q}_{\mu_3} u) + F_c^{\Omega,t}(u, u), \\ \operatorname{div} u = 0, \quad u(0) = v_0. \end{cases}$$

By Remark 1.5.11, we have

$$\|e^{t\Delta} \mathcal{Q}_{\mu_1} F_\mu(\mathcal{Q}_{\mu_2} u_1, \mathcal{Q}_{\mu_3} u_2)\|_s \leq \frac{C}{t^{1/2}} (\|\mathcal{Q}_{\mu_2} u_1\|_0 \|\mathcal{Q}_{\mu_3} u_2\|_s + \|\mathcal{Q}_{\mu_2} u_1\|_s \|\mathcal{Q}_{\mu_3} u_2\|_0) \quad (1.5.4)$$

for $\mu \in \{0, 1\}^3$ and

$$\|e^{t\Delta} F_c^{\Omega,t}(u_1, u_2)\|_s \leq \frac{C}{t^{1/2}} (\|u_1\|_0 \|u_2\|_s + \|u_1\|_s \|u_2\|_0). \quad (1.5.5)$$

Recall that $D := \{(0, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$. To complete the proof of the key lemma, it suffices to prove the following equality:

$$\sum_{\mu \in \{0,1\}^3} \mathcal{Q}_{\mu_1} F_\mu(\mathcal{Q}_{\mu_2} u, \mathcal{Q}_{\mu_3} u) = \sum_{\mu \in D} \mathcal{Q}_{\mu_1} F_\mu(\mathcal{Q}_{\mu_2} u, \mathcal{Q}_{\mu_3} u).$$

It means that it suffices to show the following four equalities:

$$B_n^{(0,1,1)}(c, c) = 0, \quad B_n^{(1,0,0)}(c, c) = 0, \quad B_n^{(0,1,0)}(c, c) = 0, \quad B_n^{(0,0,1)}(c, c) = 0.$$

Indeed, for $c := \{c_\lambda\}_{\lambda \in \Lambda(\gamma) \setminus \{0\}}$, the following three equalities

$$B_n^{(1,0,0)}(c, c) = 0, \quad B_n^{(0,1,0)}(c, c) = 0 \quad \text{and} \quad B_n^{(0,0,1)}(c, c) = 0$$

are clear. For example, if $k_3 = m_3 = 0$, n_3 must be 0 since $n = k + m$. It means that $B_n^{(1,0,0)}(c, c) = B_n^{(0,0,0)}(c, c)$ and it is a contradiction. Thus our task is to show $B_n^{(0,1,1)}(c, c) = 0$. Since

$$\theta_n^1 \theta_k^{\delta_2} \theta_m^{\delta_3} = 0 \quad \text{for } (n, k, m) \in \Lambda_{(0,1,1)} \quad \text{and } \delta_2, \delta_3 \in \{0, 1\},$$

$\theta_n^0 \theta_k^1 \theta_m^0$ and $\theta_n^0 \theta_k^0 \theta_m^1$ do not have any resonant term for $(n, k, m) \in \Lambda_{(0,1,1)}$, we see that

$$\begin{aligned} \sum_{\sigma \in \{0,1\}^3} \sum_{n=k+m, (n,k,m) \in \mathcal{K}^\sigma \cap \Lambda_{(0,1,1)}} \theta_n^{\delta_1} \theta_k^{\delta_2} \theta_m^{\delta_3} b_{nkm}^\delta(c_k(t), c_m(t)) = \\ \sum_{\sigma \in \{0,1\}^3} \sum_{n=k+m, (n,k,m) \in \mathcal{K}^\sigma \cap \Lambda_{(0,1,1)}} \alpha_{\delta, \sigma} b_{nkm}^\delta(c_k(t), c_m(t)) = 0 \end{aligned}$$

for $\delta \in \{(1, 0, 0), (1, 1, 0), (1, 0, 1), (1, 1, 1), (0, 1, 0), (0, 0, 1)\}$. (Note that terms $\theta_n^0 \theta_k^0 \theta_m^0$ and $\theta_n^0 \theta_k^1 \theta_m^1$ have a resonant term and these coefficients are $1/2$ and $-1/2$ respectively.) A direct calculation yields $\mathcal{K}^\sigma \cap \Lambda_{(0,1,1)} = \emptyset$ if $n_3 = k_3 + m_3$, $\sigma_1 \in \{0, 1\}$ and $(\sigma_2, \sigma_3) \in \{(0, 1), (1, 0)\}$. Thus we have

$$\begin{aligned} B_n^{(0,1,1)}(c_1, c_2) &= \sum_{\sigma_1 \in \{0,1\}, (\sigma_2, \sigma_3) \in \{(1,1), (0,0)\}} \sum_{n=k+m, (n,k,m) \in \mathcal{K}^\sigma \cap \Lambda_{(0,1,1)}} \\ &\quad \alpha_{\delta, \sigma} b_{nkm}^{(0,0,0)}(c_{1,k}(t), c_{2,m}(t)) + \alpha_{\delta, \sigma} b_{nkm}^{(0,1,1)}(c_{1,k}(t), c_{2,m}(t)) \\ &= \frac{1}{2} \sum_{n=k+m, |k|=|m| \neq 0, k_3 = -m_3 \neq 0} \left(b_{nkm}^{(0,0,0)}(c_{1,k}, c_{2,m}) - b_{nkm}^{(0,1,1)}(c_{1,k}, c_{2,m}) \right). \end{aligned}$$

Remark 1.5.14. By the similar calculation, we also have

$$\begin{aligned} B_n^{(1,0,1)}(c_1, c_2) &= \frac{1}{2} \sum_{n=k+m, |n|=|m| \neq 0, n_3 = m_3 \neq 0} \\ &\quad \left(b_{nkm}^{(0,0,0)}(c_{1,k}, c_{2,m}) + b_{nkm}^{(1,0,1)}(c_{1,k}, c_{2,m}) \right) \end{aligned}$$

and

$$\begin{aligned} B_n^{(1,1,0)}(c_1, c_2) &= \frac{1}{2} \sum_{n=k+m, |n|=|k| \neq 0, n_3 = k_3 \neq 0} \\ &\quad \left(b_{nkm}^{(0,0,0)}(c_{1,k}, c_{2,m}) + b_{nkm}^{(1,1,0)}(c_{1,k}, c_{2,m}) \right). \end{aligned}$$

Now we show $B_n^{(0,1,1)} = 0$. Direct calculations yield

$$\begin{aligned} -b_{nkm}^{(0,1,1)}(c_k, c_m) &= -\mathbf{P}_n(\mathcal{R}_k c_k, im)\mathcal{R}_m c_m = \\ &+ \mathbf{P}_n(\mathcal{R}_k c_k \times (im \times \mathcal{R}_m c_m)) - \mathbf{P}_n(im(\mathcal{R}_k c_k, \mathcal{R}_m c_m)) = \\ &+ \mathbf{P}_n(c_m \times (ik \times c_k)) - \mathbf{P}_n(im(\mathcal{R}_k c_k, \mathcal{R}_m c_m)). \end{aligned}$$

On the other hand,

$$b_{nkm}^{(0,0,0)}(c_k, c_m) = -\mathbf{P}_n(c_k \times (im \times c_m)) + \mathbf{P}_n(im(c_k, c_m)).$$

Since k and m are symmetric, we have

$$\begin{aligned} \frac{1}{2} \sum_{n=k+m, |k|=|m| \neq 0} (\mathbf{P}_n(c_m \times (ik \times c_k)) - \mathbf{P}_n(c_k \times (im \times c_m))) &= 0, \\ \sum_{n=k+m, |k|=|m| \neq 0} \mathbf{P}_n(im(\mathcal{R}_k c_k, \mathcal{R}_m c_m)) &= \\ \sum_{n=k+m, |k|=|m| \neq 0} \mathbf{P}_n(i(n-k)(\mathcal{R}_k c_k, \mathcal{R}_m c_m)) &= \\ - \sum_{n=k+m, |k|=|m| \neq 0} \mathbf{P}_n(ik(\mathcal{R}_k c_k, \mathcal{R}_m c_m)) &= \\ - \sum_{n=k+m, |k|=|m| \neq 0} \mathbf{P}_n(im(\mathcal{R}_k c_k, \mathcal{R}_m c_m)) &= 0 \quad \text{and also} \\ \sum_{n=k+m, |k|=|m| \neq 0} \mathbf{P}_n(im(c_k, c_m)) &= 0. \end{aligned}$$

Combining these estimates, we have the desired result.

■

1.5.2 Global solvability of extended 2D-Navier-Stokes equations

In this subsection, we show global-in-time existence of the key function w which is the solution to the following extended 2D-Navier-Stokes equations:

$$\begin{cases} \partial_t w = \Delta w + \sum_{\mu \in D} \mathcal{Q}_{\mu_1} F_{\mu}(\mathcal{Q}_{\mu_2} w, \mathcal{Q}_{\mu_3} w), \\ \nabla \cdot w = 0, \quad w|_{t=0} = v_0 \in \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)}, \\ D = \{(0,0,0), (1,0,1), (1,1,0), (1,1,1)\}. \end{cases} \quad (1.5.6)$$

Unfortunately, we cannot construct the energy inequality of (1.5.6). To overcome this difficulty, we choose appropriate $\gamma \in \mathbb{R}$ and eliminate the term $F_{(1,1,1)}$. Then we can divide (1.5.6) into three good equations, the 2D-Navier-Stokes equation and two linear equations.

Definition 1.5.15. (A set of γ .) Assume that $\tilde{n}_3 = n_3/\gamma$, $\tilde{k}_3 = k_3/\gamma$, $\tilde{m}_3 = m_3/\gamma$, $\tilde{n} = (n_1, n_2, \tilde{n}_3)$, $\tilde{k} = (k_1, k_2, \tilde{k}_3)$ and $\tilde{m} = (m_1, m_2, \tilde{m}_3)$. (Note that $\tilde{n}, \tilde{k}, \tilde{m} \in \Lambda \setminus \{0\}$ and $n, k, m \in \Lambda(\gamma)$.) Let us define

$$P_{\tilde{n}\tilde{k}\tilde{m}}(\gamma) := |n|^8 |k|^8 |m|^8 \prod_{\sigma \in \{0,1\}^3} \omega_{nkm}^\sigma$$

and

$$\Gamma := \{\gamma \in \mathbb{R} \setminus \{0\} : P_{\tilde{n}\tilde{k}\tilde{m}}(\gamma) \neq 0 \text{ for all } \tilde{n}, \tilde{k}, \tilde{m} \in \Lambda\}.$$

Remark 1.5.16. By an easy calculation, $P_{nkm}(\gamma)$ is a polynomial of order 8 for fixed $\tilde{n}, \tilde{k}, \tilde{m} \in \Lambda \setminus \{0\}$. Thus the complement set of Γ is countable (see [4, Lemma 3.2.]).

Remark 1.5.17. If $\gamma \in \Gamma$, then $\cup_{\sigma \in \{0,1\}^3} \mathcal{K}^\sigma = \emptyset$ for $(n, k, m) \in \Lambda(\gamma)_{(1,1,1)}$. Thus $F_{(1,1,1)}(u, u) = 0$ by Remark 1.5.12.

The key lemma is as follows.

Lemma 1.5.18. Let us impose the following two assumptions.

- (1) Take $\gamma \in \Gamma$.
- (2) Take $v_0 \in \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)}$ such that the initial value problem for the 2D Navier-Stokes equations admits a global-in-time unique solution in $C([0, \infty) \mathcal{Q}_0^h \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)})$ with a initial data $\mathcal{Q}_0^h v_0 \in \mathcal{Q}_0^h \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)}$.

Then there exist a global-in-time unique solution w to the equation (1.5.6) such that

$$w \in C([0, \infty) : \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)}).$$

Proof. Let $\gamma \in \Gamma$. By Remark 1.5.2 and Remark 1.5.17, the equation (1.5.6)

can be divided into three parts as follows:

$$\begin{cases} \partial_t \mathcal{Q}_0^h w = \Delta \mathcal{Q}_0^h w + \mathcal{Q}_0^h \mathbf{P}(\mathcal{Q}_0^h w, \nabla) \mathcal{Q}_0^h w, \\ \nabla \cdot \mathcal{Q}_0^h w = 0, \quad \mathcal{Q}_0^h w|_{t=0} = \mathcal{Q}_0^h v_0 \in \mathcal{X}_0^h \mathcal{X}_{0,\sigma}^{0,\Lambda}, \end{cases} \quad (1.5.7)$$

$$\begin{cases} \partial_t \mathcal{Q}_0^3 w = \Delta \mathcal{Q}_0^3 w + (\mathcal{Q}_0^h w, \nabla) \mathcal{Q}_0^3 w, \\ \mathcal{Q}_0^3 w|_{t=0} = \mathcal{Q}_0^3 v_0 \in X_0^{0,\Lambda(\gamma)}, \end{cases} \quad (1.5.8)$$

and

$$\begin{cases} \partial_t \mathcal{Q}_1 w = \Delta \mathcal{Q}_1 w + \mathcal{Q}_1 F_{(1,0,1)}(\mathcal{Q}_0 w, \mathcal{Q}_1 w) + \mathcal{Q}_1 F_{(1,1,0)}(\mathcal{Q}_1 w, \mathcal{Q}_0 w), \\ \nabla \cdot \mathcal{Q}_1 w = 0, \quad \mathcal{Q}_1 w|_{t=0} = \mathcal{Q}_1 v_0 \in \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)}. \end{cases} \quad (1.5.9)$$

By the assumption, there exists a global-in-time unique solution $\mathcal{Q}_0^h w$ to the equation (1.5.7). Since the equation (1.5.8) is linear and by (1.5.4), there exists a global-in-time unique solution $\mathcal{Q}_0^3 w$ to the equation (1.5.8),

$$\mathcal{Q}_0^3 w \in C([0, \infty) : X_0^{0,\Lambda(\gamma)}).$$

The equation (1.5.9) is linear and by (1.5.4), we can also obtain a global-in-time unique solution $\mathcal{Q}_1 w$ to the equation (1.5.9),

$$\mathcal{Q}_1 w \in C([0, \infty) : \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)}).$$

Combining these estimates, we obtain global-in-time existence of the key function w . ■

1.5.3 Fast singular oscillating limits

In this subsection, we show that the solution u^Ω to the equation (2.2.2) is close enough (in some topology) to the solution w to the equation (1.5.6) if the Coriolis parameter Ω is large enough.

The key lemma is as follows.

Lemma 1.5.19. Let $v^\Omega \in \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)}$ be a local-in-time unique solution to the equation (1.1.1), and $w \in \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)}$ be a global-in-time unique solution to the equation (1.5.6). For all $\epsilon > 0$, there is Ω_0 s.t.

$$\|v^\Omega - e^{t\Omega \mathbf{S}} w\|_0(t) < \epsilon \quad \text{for } 0 < t \leq T_{v_0}, \quad \Omega > \Omega_0,$$

where T_{v_0} is depending only on $\|v_0\|_0$, independent of Ω .

Proof. Let $u := e^{-t\Omega\mathbf{S}}v$. The idea of the proof is based on [4, Theorem 4.2]. Let us define a frequency set $\mathcal{P}_\eta\Lambda(\gamma)$ consist of finite element. For $\eta = 1, 2, \dots$, let

$$\mathcal{P}_\eta\Lambda(\gamma) := \{(n_1, n_2, n_3) \in \mathbb{R}^3 : n_1, n_2 \in \mathcal{P}_\eta\Lambda, n_3/\gamma \in \mathcal{P}_\eta\Lambda\},$$

where $\mathcal{P}_\eta\Lambda := \{n_1, \dots, n_\eta \in \Lambda : n_k \neq n_\ell (k \neq \ell), |n_j| \leq \eta (j = 1, \dots, \eta)\}$.

Let us define almost periodic functions consist of the frequency set $\mathcal{P}_\eta\Lambda(\gamma)$. For $u(x) = \sum_{n \in \Lambda(\gamma)} c_n e^{in \cdot x}$, let

$$\mathcal{P}_\eta u(x) := u_\eta(x) := \sum_{n \in \mathcal{P}_\eta\Lambda(\gamma)} c_n e^{in \cdot x},$$

Remark 1.5.20. Let $s_1 \geq s_2 \geq 0$, $s_1 - s_2 = s$. For $u \in \mathcal{X}^{s_2, \Lambda(\gamma)}$, we easily have $\|u_\eta - u\|_{s_2} \rightarrow 0$ ($\eta \rightarrow \infty$) and $\|u_\eta\|_{s_1} \leq (1 + \eta^2)^{s/2} \|u_\eta\|_{s_2} \leq (1 + \eta^2)^{s/2} \|u\|_{s_2}$.

To prove Lemma 1.5.19, we need to define the following operators $\tilde{B}_{n, \Omega, t}^c$ and $\tilde{F}_c^{\Omega, t}$ derived from $B_{n, \Omega, t}^c$ and $F_c^{\Omega, t}$ respectively.

Definition 1.5.21. Let

$$\tilde{B}_{n, \Omega, t}^c := \sum_{\delta, \sigma \in \{0, 1\}^3} \sum_{n=k+m, (n, k, m) \in (\mathcal{K}^\sigma)^c} \frac{\alpha_{\delta, \sigma}}{i\Omega\omega_{nkm}^\sigma} e^{i\Omega t \omega_{nkm}^\sigma} b_{nkm}^\delta$$

and

$$\tilde{F}_c^{\Omega, t} := \sum_{n \in \Lambda(\gamma)} \tilde{B}_{n, \Omega, t}^c e^{in \cdot x}.$$

Remark 1.5.22. We see that

$$\begin{aligned} \partial_t \left(\mathcal{P}_\eta \tilde{F}_c^{\Omega, t}(u_\eta(t), u_\eta(t)) \right) &= \mathcal{P}_\eta F_c^{\Omega, t}(u_\eta(t), u_\eta(t)) \\ &\quad + \mathcal{P}_\eta \tilde{F}_c^{\Omega, t}(\partial_t u_\eta(t), u_\eta(t)) + \mathcal{P}_\eta \tilde{F}_c^{\Omega, t}(u_\eta(t), \partial_t u_\eta(t)), \end{aligned}$$

$$\|\mathcal{P}_\eta \tilde{F}_c^{\Omega, t}(u_\eta, u_\eta)\|_0 \leq \frac{\beta_1(\eta)}{\Omega} (1 + \eta^2)^{1/2} \|u_\eta\|_0 \|u_\eta\|_0$$

and

$$\|e^{t\Delta} \mathcal{P}_\eta \tilde{F}_c^{\Omega, t}(u_\eta, u_\eta)\|_0 \leq \frac{\beta_1(\eta)}{t^{1/2}\Omega} \|u_\eta\|_0^2$$

for $0 < t < T_{v_0}$, where $\beta_1(\eta) := \max\{|\omega_{nkm}^\sigma|^{-1} : n, k, m \in \mathcal{P}_\eta\Lambda(\gamma)\}$.

Now we define the function y^Ω as follows:

$$y^\Omega := (u^\Omega - w) - \tilde{F}_c^{\Omega,t}(u_\eta^\Omega, u_\eta^\Omega), \quad y(0) = \tilde{F}_c^{\Omega,t}(u_\eta(0), u_\eta(0)).$$

By using Remark 1.5.22 and Lemma 1.5.23 which is given later, we have

$$\begin{aligned} \|u^\Omega - w\|_0 &\leq \|\mathcal{P}_\eta(u^\Omega - w)\|_0 + \|(I - \mathcal{P}_\eta)(u^\Omega - w)\|_0 \\ &\leq \|\mathcal{P}_\eta \tilde{F}_c^{\Omega,t}(u_\eta, u_\eta)\|_0 + \|y_\eta\|_0 + \|(I - \mathcal{P}_\eta)(u^\Omega - w)\|_0 \\ &\leq \frac{4\beta_1(\eta)}{\Omega} (1 + \beta_1(\eta)\eta^4)^{1/2} \|v_0\|_0^2 \\ &\quad + C \left(\frac{\beta_2(\eta, \|v_0\|_0)}{\Omega} + \|v_0\|_0 \|(I - \mathcal{P}_\eta)v_0\|_0 \right) \\ &\quad + 20\|(I - \mathcal{P}_\eta)v_0\|_0, \end{aligned}$$

where I is the identity operator and

$$\beta_2(\eta, x) := \beta_1(\eta)(1 + \eta^2)^{3/2}(1 + x^3).$$

First we take η sufficiently large (depending only on $\|v_0\|$, independent of Ω), and then we take Ω sufficiently large (depending on η and $\|v_0\|$), we have the desired result. ■

Lemma 1.5.23. Let $y^\Omega := (u^\Omega - w) - \tilde{F}_c^{\Omega,t}(u_\eta^\Omega, u_\eta^\Omega)$ and $\beta_2(\eta, x) := \beta_1(\eta)(1 + \eta^2)^{3/2}(1 + x^3)$. The following inequality holds:

$$\|y_\eta^\Omega\|_0 \leq C \left(\frac{\beta_2(\eta, \|v_0\|_0)}{\Omega} + \|v_0\|_0 \|(I - \mathcal{P}_\eta)v_0\|_0 \right)$$

for $0 < t < T_{v_0}$, where T_{v_0} is depending only on $\|v_0\|_0$.

Proof. Let $F := \sum_{\mu \in D} F_\mu$. Recall that the function u^Ω and w is the solution to the following equations:

$$\begin{cases} \partial_t u^\Omega = \Delta u^\Omega + F(u^\Omega, u^\Omega) + F_c^{\Omega,t}(u^\Omega, u^\Omega), \\ \nabla \cdot u^\Omega = 0, \quad u^\Omega|_{t=0} = v_0 \in \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)}, \end{cases}$$

and

$$\begin{cases} \partial_t w = \Delta w + F(w, w), \\ \nabla \cdot w = 0, \quad w|_{t=0} = v_0 \in \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)}. \end{cases}$$

A direct calculation yields that the function y^Ω satisfies the following equation:

$$\partial_t y_\eta^\Omega - \Delta y_\eta^\Omega - L(u_\eta^\Omega, w_\eta) y_\eta^\Omega = \sum_{j=1}^6 E_j,$$

where

$$\begin{cases} L(u_\eta^\Omega, w_\eta) y_\eta^\Omega := \mathcal{P}_\eta (F(u_\eta^\Omega, y_\eta^\Omega) + F(y_\eta^\Omega, w_\eta)), \\ E_1 := \mathcal{P}_\eta (F(u^\Omega, u^\Omega - u_\eta^\Omega) + F(u^\Omega - u_\eta^\Omega, u_\eta^\Omega)), \\ E_2 := -\mathcal{P}_\eta (F(w, w - w_\eta) + F(w - w_\eta, w_\eta)), \\ E_3 := \mathcal{P}_\eta (F_c^{\Omega, t}(u^\Omega, u^\Omega - u_\eta^\Omega) + F_c^{\Omega, t}(u^\Omega - u_\eta^\Omega, u_\eta^\Omega)), \\ E_4 := \mathcal{P}_\eta (\tilde{F}_c^{\Omega, t}(\partial_t u_\eta^\Omega, u_\eta^\Omega) + \tilde{F}_c^{\Omega, t}(u_\eta^\Omega, \partial_t u_\eta^\Omega)), \\ E_5 := \Delta \mathcal{P}_\eta \tilde{F}_c^{\Omega, t}(u_\eta^\Omega, u_\eta^\Omega), \\ E_6 := L(u_\eta^\Omega, w_\eta) \mathcal{P}_\eta \tilde{F}_c^{\Omega, t}(u_\eta^\Omega, u_\eta^\Omega). \end{cases}$$

Note that by Remark 1.5.22, we see that

$$-\partial_t (\mathcal{P}_\eta \tilde{F}_c^{\Omega, t}(u_\eta(t), u_\eta(t))) + \mathcal{P}_\eta F_c^{\Omega, t}(u_\eta(t), u_\eta(t)) = E_4$$

and E_3 includes $\mathcal{P}_\eta F_c^{\Omega, t}(u_\eta(t), u_\eta(t))$. Moreover, these operators satisfy the following estimates:

$$\begin{cases} \|e^{t\Delta} L(u_\eta^\Omega, w_\eta) y_\eta^\Omega\|_0 \leq (C/t^{1/2})(\|u^\Omega\|_0 + \|w\|_0) \|y_\eta\|_0, \\ \|e^{t\Delta} E_1\|_0 \leq (C/t^{1/2}) \|u^\Omega\|_0 \|u^\Omega - u_\eta^\Omega\|_0, \\ \|e^{t\Delta} E_2\|_0 \leq (C/t^{1/2}) \|w\|_0 \|w_\eta - w\|_0, \\ \|e^{t\Delta} E_3\|_0 \leq (C/t^{1/2}) \|u^\Omega\|_0 \|u^\Omega - u_\eta^\Omega\|_0, \\ \|e^{t\Delta} E_4\|_0 \leq (\beta_1(\eta)/(t^{1/2}\Omega)) (1 + \eta^2)^{3/2} (\|u^\Omega\|_0^2 + \|u^\Omega\|_0^3), \\ \|e^{t\Delta} E_5\|_0 \leq (\beta_1(\eta)/(t^{1/2}\Omega)) (1 + \eta^2)^{3/2} \|u^\Omega\|_0^2, \\ \|e^{t\Delta} E_6\|_0 \leq (C/t^{1/2})(\|u^\Omega\|_0 + \|w\|_0) \frac{\beta_1(\eta)}{\Omega} (1 + \eta^2)^{1/2} \|u^\Omega\|_0^2 \end{cases}$$

by (1.5.4), (1.5.5), Remark 1.5.20, Remark 1.5.22 and the following inequality:

$$\begin{aligned} \|\partial_t u_\eta^\Omega\|_0 &\leq \|\Delta u_\eta\|_0 + \|\mathcal{P}_\eta \nabla \cdot (u^\Omega \otimes u^\Omega)\|_0 \\ &\leq (1 + \eta^2) \|u^\Omega\|_0 + (1 + \eta^2)^{1/2} \|u^\Omega\|_0^2. \end{aligned}$$

By the above estimates and the definition of β_2 , we have

$$\sum_{j=1}^6 \|e^{t\Delta} E_j\|_0 \leq \frac{C}{t^{1/2}} \left(\frac{\beta_2(\eta, \|v_0\|_0)}{\Omega} + 3(\|v_0\|_0 \|(I - \mathcal{P}_\eta)v_0\|_0) \right)$$

for $0 < t < T_{v_0}$. Thus by usual argument, we have the following estimate

$$\sup_{0 < t < T_*} \|y_\eta^\Omega\|_0(t) \leq \frac{\beta_1(\eta)}{\Omega} (1 + \eta^2)^{1/2} \|v_0\|_0^2 + CT_*^{1/2} \sup_{0 < t < T_*} \left\{ 2\|v_0\|_0 \|y_\eta^\Omega\|_0(t) + \frac{\beta_2(\eta, \|v_0\|_0)}{\Omega} + 3\|v_0\|_0 \|(I - \mathcal{P}_\eta)v_0\|_0 \right\},$$

where $T_* = (1/(4C\|v_0\|_0))^2$ (depending only on $\|v_0\|_0$). We set T_{v_0} as minimal value compare T_* with locally existence time described in Theorem 1.3.3. We have

$$\|y_\eta^\Omega\|_0 \leq 2 \left(\frac{\beta_1(\eta)}{\Omega} (1 + \eta^2)^{1/2} \|v_0\|_0^2 + \frac{\beta_2(\eta, \|v_0\|_0)}{\Omega} + 3\|v_0\|_0 \|(I - \mathcal{P}_\eta)v_0\|_0 \right)$$

for $0 < t < T_{v_0}$. This is the desired estimate. ■

1.6 Appendix

1.6.1 A global-in-time unique solution to the 2D-Navier-Stokes equations

In this section, we show global-in-time existence of a unique solution to the following two dimensional Navier-Stokes equations with a periodic initial data:

$$\begin{cases} \partial_t \mathcal{Q}_0^h w = \Delta \mathcal{Q}_0^h w + \mathcal{Q}_0^h \mathbf{P}(\mathcal{Q}_0^h w, \nabla) \mathcal{Q}_0^h w, \\ \nabla \cdot \mathcal{Q}_0^h w = 0, \\ \mathcal{Q}_0^h w|_{t=0} = \mathcal{Q}_0^h v_0(x) \in \mathcal{Q}_0^h \mathcal{X}_{0,\sigma}^{0,\Lambda}, \\ \mathcal{Q}_0^h v_0(x) \text{ is a periodic function.} \end{cases} \quad (1.6.1)$$

To show existence of a global-in-time unique solution, we need a priori estimate described in [37] (see also [22, 36]). According to [37], they established a priori estimate in BUC , bounded uniformly continuous functions. We use its estimate, (1.3.4) and generalized Bernstein's theorem.

Lemma 1.6.1. There is a global-in-time unique solution $(\mathcal{Q}_0^h w, \nabla p)$ to the equation (1.6.1) with $\partial_\ell p = \partial_\ell \sum_{j,k=1}^2 \mathbf{R}^{jk} \mathcal{Q}_0^h w^j \mathcal{Q}_0^h w^k$,

$$\mathcal{Q}_0^h w \in C([0, \infty), \mathcal{Q}_0^h \mathcal{X}_{0,\sigma}^{0,\Lambda}).$$

Proof. Since the initial data is a periodic function, we can apply generalized Bernstein's theorem which is given at the end of this subsection. (For the classical Bernstein's theorem, see [2] for example. However it describes only one-dimensional case. Thus we cannot apply its estimate.) By [37], we have

$$\|\mathcal{Q}_0^h w(t)\|_{L^\infty} \leq C_1 \|\mathcal{Q}_0^h w(\epsilon)\|_{L^\infty} \exp\{C_2 t \|\text{rot } \mathcal{Q}_0^h w(\epsilon)\|_{L^\infty}\}$$

for $t > \epsilon > 0$. Note that the initial data is in $\mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)} \subset (BUC)^3$. By (1.3.4) and by generalized Bernstein's theorem (see Lemma 1.6.2), we have

$$\|\mathcal{Q}_0^h w\|_0 \leq C \|\mathcal{Q}_0^h w\|_{C^\alpha} \leq C F_\alpha (\|\mathcal{Q}_0^h w(t)\|_{L^\infty}), \quad t > 0$$

for $\alpha \in \mathbb{R}_+ \setminus \mathbb{Z}_+$, $\alpha > 1$. By these estimates, we can easily obtain a global-in-time solution. ■

Lemma 1.6.2. (Generalized Bernstein's theorem.) Let $D \subset \mathbb{R}^d$ be a countable set satisfying $\sum_{N=-\infty}^{\infty} \sum_{\kappa \in \{-1,1\}^d} |D_N^\kappa|^{1/2} / 5^{N\alpha} < \infty$, where

$$D_N^\kappa := \{\lambda \in D : 5^N < |\lambda \cdot \kappa| \leq 5^{N+1}, \lambda_1 \kappa_1, \dots, \lambda_d \kappa_d \geq 0\}$$

for $\kappa = (\kappa_1, \dots, \kappa_d) \in \{-1, 1\}^d$. Let $f(x) := \sum_{\lambda \in D \setminus \{0\}} c_\lambda e^{i\lambda \cdot x}$ be a almost periodic function in $C^\alpha(\mathbb{R}^d)$ ($\alpha > d/2$). Then we have

$$\sum_{\lambda \in D \setminus \{0\}} |c_\lambda| \leq C \|f\|_{C^\alpha},$$

where C is a positive constant depending only on α .

Remark. If $D = \mathbb{Z}^d$, the condition $\sum_{N=-\infty}^{\infty} \sum_{\kappa \in \{-1,1\}^d} |D_N^\kappa|^{1/2} / 5^{N\alpha} < \infty$ is automatically satisfied.

Proof. Recall that

$$G_h^k(f)(x) := \sum_{j=0}^k \binom{k}{j} (-1)^j f(x + jh).$$

A direct calculation yields

$$G_h^k(f)(x) = (2i)^k e^{i\lambda \cdot h} \left(\sin \frac{\lambda \cdot h}{2} \right)^k f(x).$$

We now set $h_N := \pi/(6 \cdot 5^N)$.

By the Parseval's equality for the almost periodic case (see [8, 10]), we have

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{1}{|B_r|} \int_{B_r} |G_{h_N \kappa}^k(f)(x)|^2 dx \\ \geq 2^{2k} \sum_{\lambda \in D_N^\kappa} \left(\sin \frac{h_N(\lambda \cdot \kappa)}{2} \right)^{2k} |c_\lambda|^2 \geq 2^{2k} \left(\frac{1}{2} \right)^{2k} \sum_{\lambda \in D_N^\kappa} |c_\lambda|^2, \end{aligned}$$

where $B_r := \{x \in \mathbb{R}^d : |x| < r\}$. Therefore we have

$$\begin{aligned} \sum_{\lambda \in D \setminus \{0\}} |c_\lambda| &= \sum_{N=-\infty}^{\infty} \sum_{\kappa \in \{-1,1\}^d} \sum_{\lambda \in D_N^\kappa} |c_\lambda| \\ &\leq \sum_{N=-\infty}^{\infty} \sum_{\kappa \in \{-1,1\}^d} \left(\sum_{\lambda \in D_N^\kappa} |c_\lambda|^2 \right)^{1/2} |D_N^\kappa|^{1/2} \\ &\leq \sum_{N=-\infty}^{\infty} \sum_{\kappa \in \{-1,1\}^d} \left(\lim_{r \rightarrow \infty} \frac{1}{|B_r|} \int_{B_r} |G_{h_N \kappa}^k(f)(x)|^2 dx \right)^{1/2} |D_N^\kappa|^{1/2} \\ &\leq C \|F\|_{C^\alpha} \sum_{N=-\infty}^{\infty} \sum_{\kappa \in \{-1,1\}^d} |h_N|^\alpha |D_N^\kappa|^{1/2}. \end{aligned}$$

Since $|h_N|^\alpha \leq C/5^{N\alpha}$, we have $\sum_N \sum_{\kappa \in \{-1,1\}^d} |h_N|^\alpha |D_N^\kappa|^{1/2} < \infty$.

Thus we complete the proof. ■

1.6.2 Skew-symmetry of key operators

Let us consider the operators $F_{(1,0,1)}$ and $F_{(1,1,0)}$ more precisely. We point out that $F_{(1,0,1)}$ is skew symmetric. However $F_{(1,1,0)}$ is not skew symmetric.

Lemma 1.6.3. There exists at least one periodic (almost periodic) function $w \in C^\infty$ with $\operatorname{div} w = 0$ satisfying

$$\langle F_{(1,1,0)}(w, w), w^* \rangle \neq 0, \quad (1.6.2)$$

where w^* is the conjugate function of w .

Remark 1.6.4. For all $w \in \mathcal{X}_{0,\sigma}^{0,\Lambda(\gamma)}$, it is easy to see that $\langle F_{(1,0,1)}(w, w), w^* \rangle = 0$. See [4, Theorem 5.3] for example.

Proof. Let us set the function w as follows:

$$w(x) = c_{\lambda_1} e^{i\lambda_1 \cdot x} + c_{\lambda_2} e^{i\lambda_2 \cdot x} + c_{\lambda_3} e^{i\lambda_3 \cdot x} + c_{\lambda_1} e^{-i\lambda_1 \cdot x} + c_{\lambda_2}^* e^{-i\lambda_2 \cdot x} + c_{\lambda_3} e^{-i\lambda_3 \cdot x},$$

where

$$\begin{aligned} \lambda_1 &= (-d_1, -d_2, -d_3), \quad \lambda_2 = (-d_1, -d_2, d_3), \quad \lambda_3 = (2d_1, 2d_2, 0), \\ c_{\lambda_1} &= c_{\lambda_3} = (d_2, -d_1, 0), \quad c_{\lambda_2} = (d_2 + id_2, -d_1 + d_3 + i(-d_1 + d_3), d_2 + id_2) \end{aligned}$$

for $d_1, d_2, d_3 \in \mathbb{R} \setminus \{0\}$. Note that the function w satisfy divergence free condition and $|\lambda_1| = |\lambda_2| \neq |\lambda_3|$. Since $c_n^* = c_{-n}$ for $n = \lambda_1, \lambda_3$, Remark 1.5.14, and $(c_{\lambda_1}, i\lambda_3) = 0$, we have

$$\begin{aligned} \sum_{n \in \Lambda(\gamma)} \sum_{-n=k+m, |n|=|k| \neq 0, n_3=m_3 \neq 0} \langle b_{nkm}^{(0,0,0)}(c_k, c_m), c_n \rangle &= \\ &= (c_{\lambda_2}, i\lambda_3)(c_{\lambda_3}, c_{\lambda_1}) + (c_{\lambda_1}, i\lambda_3)(c_{\lambda_3}, c_{\lambda_2}) \\ &\quad - (c_{\lambda_2}^*, i\lambda_3)(c_{\lambda_3}, c_{\lambda_1}) - (c_{\lambda_1}, i\lambda_3)(c_{\lambda_3}, c_{\lambda_2}^*) \\ &= 2i(\operatorname{Im} c_{\lambda_2}, i\lambda_3)(c_{\lambda_3}, c_{\lambda_1}) = -4d_2d_3(d_1^2 + d_2^2). \end{aligned}$$

Since

$$(\mathcal{R}_{-\lambda_1} c_{\lambda_3}, c_{\lambda_1}) = \frac{1}{|\lambda_1|} (\lambda_1 \times c_{\lambda_1}, c_{\lambda_1}) = -\frac{1}{|\lambda_1|} (c_{\lambda_1}, \lambda_1 \times c_{\lambda_1}) = 0,$$

we also have

$$\begin{aligned} \sum_{n \in \Lambda(\gamma)} \sum_{-n=k+m, |n|=|k|} \langle b_{nkm}^{(1,1,0)}(c_k, c_m), c_n \rangle &= \\ &= (\mathcal{R}_{\lambda_2} c_{\lambda_2}, i\lambda_3)(\mathcal{R}_{-\lambda_1} c_{\lambda_3}, c_{\lambda_1}) + (\mathcal{R}_{\lambda_1} c_{\lambda_1}, i\lambda_3)(\mathcal{R}_{-\lambda_2} c_{\lambda_3}, c_{\lambda_2}) \\ &\quad - (\mathcal{R}_{-\lambda_2} c_{\lambda_2}^*, i\lambda_3)(\mathcal{R}_{\lambda_1} c_{\lambda_3}, c_{\lambda_1}) - (\mathcal{R}_{-\lambda_1} c_{\lambda_1}, i\lambda_3)(\mathcal{R}_{\lambda_2} c_{\lambda_3}, c_{\lambda_2}^*) \\ &= 2i(\mathcal{R}_{\lambda_1} c_{\lambda_1}, i\lambda_3)(\mathcal{R}_{-\lambda_2} c_{\lambda_3}, \operatorname{Im} c_{\lambda_2}) \\ &= 4(d_1^2 d_3 + d_2^2 d_3)(d_2 d_3^2 + (d_1^2 + d_2^2) d_2). \end{aligned}$$

Thus we can say that the function w satisfies (1.6.2) for some d_1, d_2 and d_3 . It means that the operator $F_{(1,1,0)}$ is not skew-symmetric.

■

Bibliography

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Part II

The functional differential equations of advanced type.

Chapter 2

The functional-differential equation of advanced type

abstract

Solutions to the equation

$$\begin{cases} f'(x) = af(\lambda x), & x \in \mathbb{R}, \\ f(0) = 0, \end{cases}$$

are constructed for constants $\lambda > 1$ and $a \neq 0$. The solutions are infinitely differentiable and have some symmetry. By our method, numerical computations can be made effectively.

2.1 Introduction

The purpose of this chapter is to construct non-trivial solutions to the functional-differential equation

$$\begin{cases} f'(x) = af(\lambda x), & x \in \mathbb{R} = (-\infty, +\infty), \\ f(0) = 0, \end{cases} \quad (2.1.1)$$

for constants $\lambda > 1$ and $a \neq 0$. Note that the equation (2.1.1) is of the advanced type only if $x > 0$. Our solutions are infinitely differentiable on \mathbb{R} .

Moreover, if $\lambda \geq 2$, then the solution f is bounded and has an infinite number of intervals I such that $f(x) = 0$ for $x \in I$.

In practical applications equation (2.1.1) arises in the study of electrical transmission lines of electrical railway systems [3, 4]. Frederickson [1, 2] (1971) investigated functional-differential equations of advanced type

$$f'(x) = af(\lambda x) + \lambda f(x), \quad x > 0, \quad (2.1.2)$$

for $\lambda > 1$, and proved several properties of solutions. After his works, Kato and McLeod [5] (1971) and Kato [6] (1972) studied the asymptotic behavior of solutions of (2.1.2) as $x \rightarrow \infty$. Unfortunately, the solutions obtained by Theorem 2.2.3 in this paper do not decay at $x \rightarrow \infty$, therefore they do not satisfy the asymptotic behavior expected in [5] (Theorem 10 (ii)).

Frederickson [1] obtained a global existence for the non-trivial solutions to equations

$$f'(x) = F(f(2x)), \quad x \in \mathbb{R},$$

where F is an odd, continuous function with $F(s) > 0$ for $s > 0$. His proof is based on the Schauder fixed point theorem. He also showed that the absolute value of the solution $|f(x)|$ is periodic for $x \geq 0$. In [2], he also considered the equations

$$f'(z) = af(\lambda z) + bf(z),$$

where $a, b \in \mathbb{C}$ and $\lambda > 1$, and derived a global existence theorem. Furthermore, solutions are given of the form of a Dirichlet series

$$\varphi(z, \beta) = \sum_{n \in \mathbb{Z}} c_n e^{\beta \lambda^n z}, \quad \Re(\beta z) \leq 0,$$

with some parameter $\beta \in \mathbb{C}$. In the case of $b = 0$ and $\beta = i$, the solution is analytic in the upper half plane $\{z; \Im z > 0\}$ and continuous on $\{z; \Im z \geq 0\}$. From his result it follows that our solutions of (2.1.1) cannot be real analytic.

Heard [7] (1973) described solutions of (2.1.1) in the form of integrals explicitly. However, he did not give the original shapes of solutions.

Augustynowicz, Leszczyński and Walter [8] (2003) considered the following equations related to (2.1.1):

$$\begin{cases} f'(x) = (f(\lambda x))^{1/\lambda}, & x \in [0, +\infty) \quad \text{with } \lambda > 1, \\ f(0) = f_0. \end{cases} \quad (2.1.3)$$

In the case $f_0 = 0$, there is at least one non-trivial positive solution. For some $\lambda > 1$ it can be analytic, moreover for the case $\lambda = 2$ the equation (2.1.3) admits infinitely many analytic solutions.

Recently, in [9] the author constructed solutions of (2.1.1) with $\lambda = 2$. In this paper we construct solutions of (2.1.1) by another approach based on the Fourier transform and special sequences of numbers given by (2.2.4) and (2.2.5).

This paper is organized as follows. We state the main theorem (Theorem 2.2.3) in Section 2.2 and illustrate the shape of solutions in Section 2.3. We also clarify the importance of parameter λ . We prove lemmas in Section 2.4 and prove the main theorem in Section 2.5.

2.2 Main results

To overcome the difficulties in constructing the solutions to (2.1.1) we apply to the relationship between the Fourier transform and the dilation. Furthermore, this method allows us to demonstrate the shape of solutions by numerical computation.

Let $\lambda > 1$. If f is a solution of the equation

$$\begin{cases} f'(x) = \lambda^2 f(\lambda x), & x \in \mathbb{R}, \\ f(0) = 0, \end{cases} \quad (2.2.1)$$

then $f(ax/\lambda^2)$ is a solution of the equation (2.1.1). We now consider the equation (2.2.1).

First, we state two lemmas, whose proofs will be given in Section 2.4. Before stating lemmas, we recall the definitions of Fourier transform, its inverse and

the sinc function:

$$\hat{f}(\xi) := \mathcal{F}[f](\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi} dx, \quad \mathcal{F}^{-1}[f](\xi) := \frac{1}{2\pi} \int_{\mathbb{R}} f(x)e^{ix\xi} dx,$$

and

$$\text{sinc } \xi := \begin{cases} \sin(\pi\xi)/(\pi\xi), & \xi \neq 0, \\ 1, & \xi = 0. \end{cases}$$

Lemma 2.2.1. *The product*

$$\prod_{k=1}^{\infty} \text{sinc} \left(\frac{\xi}{2\lambda^k \pi} \right), \quad \xi \in \mathbb{R}$$

converges pointwisely and in $L^1(\mathbb{R})$ for any $\lambda > 1$.

Lemma 2.2.2. *Let $\lambda > 1$, and let*

$$u := \mathcal{F}^{-1}[U], \quad U(\xi) := \exp \left(-\frac{i\xi}{2(\lambda-1)} \right) \prod_{k=1}^{\infty} \text{sinc} \left(\frac{\xi}{2\lambda^k \pi} \right). \quad (2.2.2)$$

Then u has the following properties:

$$\begin{aligned} u &\in C^\infty(\mathbb{R}), \\ u(x) &> 0 \text{ for } x \in \left(0, \frac{1}{\lambda-1} \right), \quad u(x) = 0 \text{ for } x \notin \left(0, \frac{1}{\lambda-1} \right), \\ u(x) &= u(1/(\lambda-1) - x), \\ \int_{\mathbb{R}} u(x) dx &= 1, \end{aligned}$$

and

$$u'(x) = \lambda^2 u(\lambda x) \quad \text{for } x \in \left[0, \min \left(\frac{1}{\lambda}, \frac{1}{\lambda(\lambda-1)} \right) \right]. \quad (2.2.3)$$

Secondly, we define sequences of numbers $\{n_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ as follows:

$$\begin{cases} n_1 = 0, & n_2 = 1, \\ n_{2k-1} = 1, & n_{2k} = 0, \quad \text{if } n_k = 1 \quad (k \geq 2), \\ n_{2k-1} = 0, & n_{2k} = 1, \quad \text{if } n_k = 0 \quad (k \geq 2), \end{cases} \quad (2.2.4)$$

and

$$y_k = \sum_{l=1}^{\infty} C_{k,l} \lambda^{l-1}, \quad k = 1, 2, 3, \dots, \quad (2.2.5)$$

where $C_{k,l} \in \{0, 1\}$ ($l = 1, 2, 3, \dots$) are coefficients of the binary system such that

$$k - 1 = \sum_{l=1}^{\infty} C_{k,l} 2^{l-1}, \quad k = 1, 2, 3, \dots.$$

Then we have the following relations:

$$\begin{cases} (-1)^{n_{2k-1}} = (-1)^{n_k}, \\ (-1) \cdot (-1)^{n_{2k}} = (-1)^{n_k}, \end{cases} \quad k = 1, 2, 3, \dots, \quad (2.2.6)$$

$$\begin{cases} y_{2k-1}/\lambda = y_k, \\ y_{2k}/\lambda = y_k + 1/\lambda, \end{cases} \quad k = 1, 2, 3, \dots, \quad (2.2.7)$$

and

$$y_k \geq \lambda^j \quad \text{if} \quad k - 1 \geq 2^j, \quad j = 0, 1, 2, \dots. \quad (2.2.8)$$

Hence $\lim_{k \rightarrow \infty} y_k = \infty$ since $\lambda > 1$. If $\lambda \geq 2$, then y_k is strictly increasing. For example,

$$\begin{aligned} \{n_k\}_{k=1}^{\infty} &= \{0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, \dots\}, \\ \{y_k\}_{k=1}^{\infty} &= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \dots\} \text{ for } \lambda = 2, \\ \{y_k\}_{k=1}^{\infty} &= \{0, 1, 4, 5, 16, 17, 20, 21, 64, 65, 68, 69, 80, \dots\} \text{ for } \lambda = 4, \\ \{y_k\}_{k=1}^{\infty} &= \left\{ 0, 1, \frac{3}{2}, \frac{5}{2}, \frac{9}{4}, \frac{13}{4}, \frac{15}{4}, \frac{19}{4}, \frac{27}{8}, \frac{35}{8}, \frac{39}{8}, \frac{47}{8}, \frac{45}{8}, \dots \right\} \text{ for } \lambda = 3/2. \end{aligned}$$

Our main result is the following:

Theorem 2.2.3. *Let $\lambda > 1$. Then*

$$f(x) = \sum_{k=1}^{\infty} (-1)^{n_k} u(x - y_k) \quad (2.2.9)$$

satisfies (2.2.1), where u , $\{n_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ are as in (2.2.2), (2.2.4) and (2.2.5), respectively. The solution f is in $C^\infty(\mathbb{R})$, non-decay at $x \rightarrow \infty$ and $f(x) = 0$ for $x \leq 0$. Moreover, if $\lambda \geq 2$, then f is bounded.

Remark 2.2.1. (i) The function f is well-defined, since the support of u is $[0, 1/(\lambda - 1)]$ and $\lim_{k \rightarrow \infty} y_k = \infty$. Indeed, by (2.2.8) we have

$$f(x) = \sum_{k=1}^{2^j} (-1)^{n_k} u(x - y_k) \quad \text{for } x \in [0, \lambda^j], \quad j = 0, 1, 2, \dots \quad (2.2.10)$$

(ii) If $\lambda = 2$, then $\{x > 0 : f(x) = 0\} = \{1, 2, 3, \dots\}$. If $\lambda > 2$, then $\{x > 0 : f(x) = 0\} = \cup_{k=1}^{\infty} [y_k + 1/(\lambda - 1), y_{k+1}]$ and its measure is infinite, since $1/(\lambda - 1) < 1 < y_{k+1} - y_k$, $k = 1, 2, 3, \dots$ (see Fig. 2.2 and Fig. 2.3).

(iii) A constant times f is also a solution.

2.3 Examples

For $\lambda = 4, 2, 3/2$, the graphs of the functions $u(x)$ of Theorem 2.2.3 are in Fig. 2.1. We get the numerical data of $u(x)$ by using the operator T in (2.4.1) or (2.4.2). For $\lambda = 4, 3, 2, 31/16, 15/8, 7/4, 3/2, 5/4$, the graphs of the functions $f(x)$ of Theorem 2.2.3 are in Fig. 2.2 – 2.9, respectively.

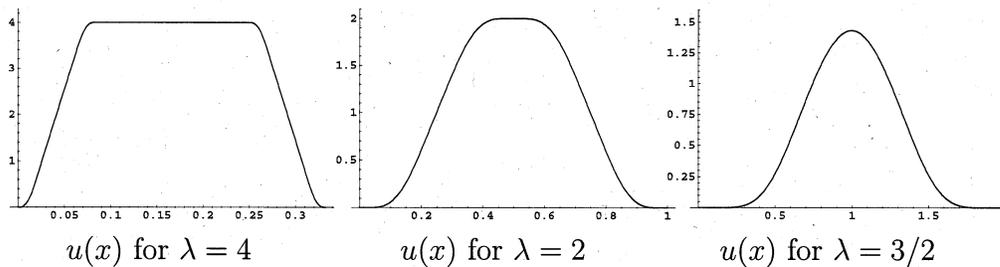


Figure 2.1: $u(x)$

2.4 Proof of Lemmas

Proof of Lemma 2.2.1. Let

$$P_\ell^m(\xi) = \prod_{k=\ell}^m \operatorname{sinc} \left(\frac{\xi}{2\lambda^k \pi} \right).$$

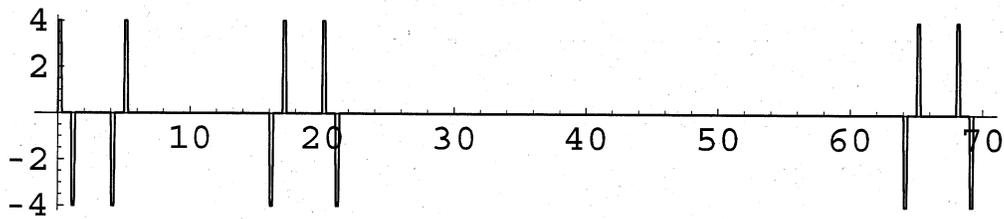


Figure 2.2: $f'(x) = 4^2 f(4x)$

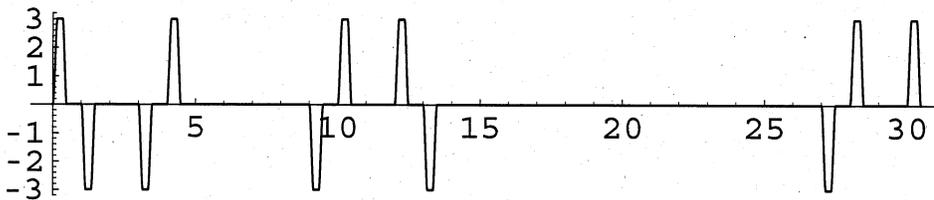


Figure 2.3: $f'(x) = 3^2 f(3x)$

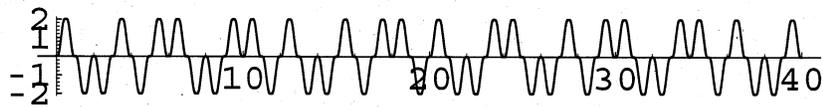


Figure 2.4: $f'(x) = 2^2 f(2x)$

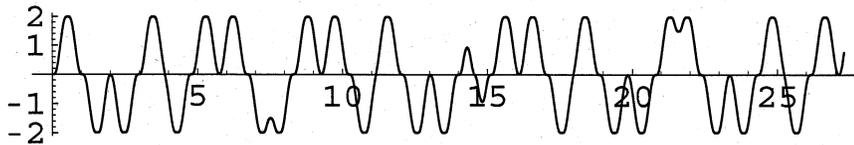


Figure 2.5: $f'(x) = (31/16)^2 f(31x/16)$

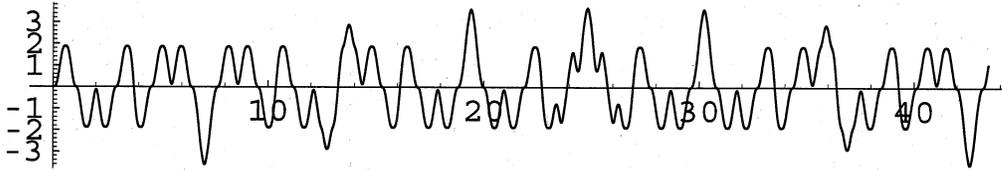


Figure 2.6: $f'(x) = (15/8)^2 f(15x/8)$



Figure 2.7: $f'(x) = (7/4)^2 f(7x/4)$

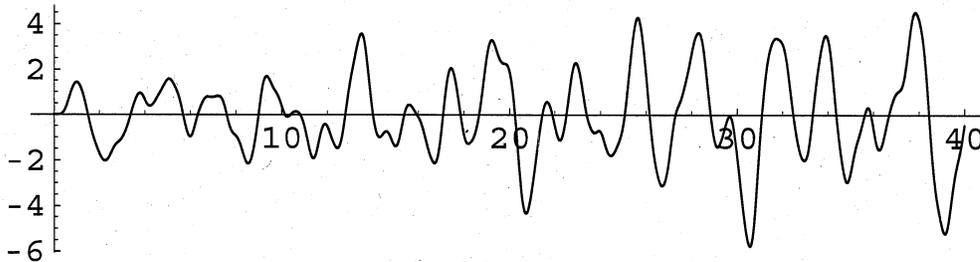


Figure 2.8: $f'(x) = (3/2)^2 f(3x/2)$

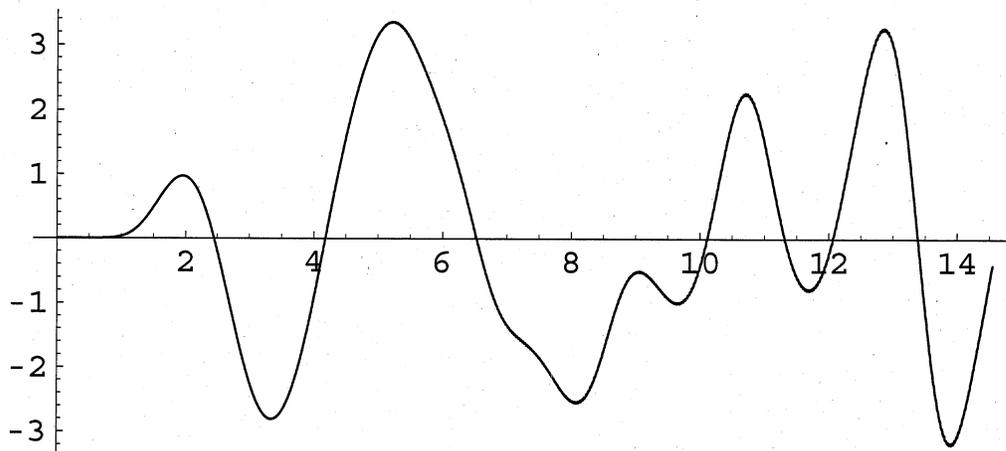


Figure 2.9: $f'(x) = (5/4)^2 f(5x/4)$

We note that there exist constants $\gamma > 0$ and $C > 0$ such that

$$\exp(-C|\xi|) \leq \operatorname{sinc}\left(\frac{\xi}{2\pi}\right) \leq 1 \quad \text{for all } |\xi| \leq \gamma.$$

It follows that, for every ℓ and m ,

$$\exp\left(-C|\xi| \sum_{k=\ell}^{\infty} \frac{1}{\lambda^k}\right) \leq \exp\left(-C|\xi| \sum_{k=\ell}^m \frac{1}{\lambda^k}\right) \leq P_\ell^m(\xi) \leq 1 \quad \text{for all } |\xi| \leq \lambda^\ell \gamma.$$

Since $P_\ell^m(\xi)$ is monotone decreasing with respect to m , $P_\ell^m(\xi)$ converges as $m \rightarrow \infty$ for all $|\xi| \leq \lambda^\ell \gamma$. Hence $P_1^m(\xi)$ converges as $m \rightarrow \infty$ for all $|\xi| \leq \lambda^\ell \gamma$, $\ell = 1, 2, 3, \dots$. Let

$$P_1^\infty(\xi) = \lim_{m \rightarrow \infty} P_1^m(\xi) = \prod_{k=1}^{\infty} \operatorname{sinc}\left(\frac{\xi}{2\lambda^k \pi}\right), \quad \xi \in \mathbb{R}.$$

For $m \geq 2$, we have

$$\begin{aligned} |P_1^m(\xi)| &= \left| \operatorname{sinc}\left(\frac{\xi}{2\lambda\pi}\right) \operatorname{sinc}\left(\frac{\xi}{2\lambda^2\pi}\right) \cdots \operatorname{sinc}\left(\frac{\xi}{2\lambda^m\pi}\right) \right| \\ &\leq \max\left(1, \left|\frac{2\lambda\pi}{\xi} \frac{2\lambda^2\pi}{\xi}\right|\right) \in L^1(\mathbb{R}). \end{aligned}$$

By Lebesgue's convergence theorem, we have $P_1^m \rightarrow P_1^\infty$ in $L^1(\mathbb{R})$.

To prove Lemma 2.2.2, we define an operator $T : L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$ as follows:

$$Tv(x) = \lambda (\chi_{[0,1]} * v)(\lambda x),$$

where $\chi_{[0,1]}$ be the characteristic function of the interval $[0, 1]$. Then T is bounded on $L^1(\mathbb{R})$. Let

$$X = \left\{ v \in L^1(\mathbb{R}) : v(x) \geq 0, \text{supp } v \subset [0, 1/(\lambda - 1)], \right. \\ \left. v(x) = v(1/(\lambda - 1) - x), \int_{\mathbb{R}} v(x) dx = 1 \right\}.$$

Then $Tv \in X$ for $v \in X$ and X is a closed subset of $L^1(\mathbb{R})$. Moreover, for $v \in X$ we have that

$$Tv(x) = \begin{cases} \lambda \int_0^{\lambda x} v(t) dt, & x \in \left[0, \frac{1}{\lambda}\right], \\ \lambda \int_{\lambda x - 1}^{\lambda x} v(t) dt, & x \in \left(\frac{1}{\lambda}, \frac{1}{\lambda(\lambda - 1)}\right], \\ \lambda \int_{\lambda x - 1}^{1/(\lambda - 1)} v(t) dt, & x \in \left(\frac{1}{\lambda(\lambda - 1)}, \frac{1}{\lambda - 1}\right], \\ 0 & \text{otherwise,} \end{cases} \quad \text{if } 1 < \lambda < 2, \quad (2.4.1)$$

and that

$$Tv(x) = \begin{cases} \lambda \int_0^{\lambda x} v(t) dt, & x \in \left[0, \frac{1}{\lambda(\lambda - 1)}\right], \\ \lambda \int_0^{1/(\lambda - 1)} v(t) dt, & x \in \left(\frac{1}{\lambda(\lambda - 1)}, \frac{1}{\lambda}\right], \\ \lambda \int_{\lambda x - 1}^{1/(\lambda - 1)} v(t) dt, & x \in \left(\frac{1}{\lambda}, \frac{1}{\lambda - 1}\right], \\ 0 & \text{otherwise,} \end{cases} \quad \text{if } \lambda \geq 2. \quad (2.4.2)$$

Therefore, if $u \in X$ and $u = Tu$, then u satisfies (2.2.3).

Proof of Lemma 2.2.2. We prove that $u \in X \cap C^\infty(\mathbb{R})$ and $u = Tu$. Let $w = \chi_{[0,1]}$. Then $Tv(x) = \lambda(w * v)(\lambda x)$. Hence

$$\mathcal{F}[Tv](\xi) = \hat{w}(\xi/\lambda)\hat{v}(\xi/\lambda), \quad \hat{w}(\xi) = e^{-i\xi/2} \text{sinc}(\xi/(2\pi)).$$

From the equation

$$\mathcal{F}[T^{k+1}v](\xi) = \hat{w}(\xi/\lambda)\mathcal{F}[T^k v](\xi/\lambda),$$

it follows that

$$\begin{aligned} \mathcal{F}[T^m v](\xi) &= \left(\prod_{k=1}^m \hat{w}(\xi/\lambda^k) \right) \hat{v}(\xi/\lambda^m) \\ &= \left(\prod_{k=1}^m \exp\left(-i\xi/(2\lambda^k)\right) \operatorname{sinc}(\xi/(2\lambda^k\pi)) \right) \hat{v}(\xi/\lambda^m) \\ &= \exp\left(-\frac{i\xi}{2} \sum_{k=1}^m \frac{1}{\lambda^k}\right) \left(\prod_{k=1}^m \operatorname{sinc}(\xi/(2\lambda^k\pi)) \right) \hat{v}(\xi/\lambda^m). \end{aligned}$$

From $\|\hat{v}\|_{L^\infty} \leq \|v\|_{L^1} = \hat{v}(0) = 1$ and the continuity of \hat{v} , it follows that

$$\begin{aligned} &\left| \exp\left(-\frac{i\xi}{2} \sum_{k=1}^m \frac{1}{\lambda^k}\right) \hat{v}(\xi/\lambda^m) \right| \leq 1, \\ \exp\left(-\frac{i\xi}{2} \sum_{k=1}^m \frac{1}{\lambda^k}\right) \hat{v}(\xi/\lambda^m) &\rightarrow \exp\left(-\frac{i\xi}{2(\lambda-1)}\right) \quad \text{as } m \rightarrow \infty. \end{aligned}$$

From Lemma 2.2.1 we have that, for all $v \in X$,

$$\mathcal{F}[T^m v](\xi) \rightarrow \exp\left(-\frac{i\xi}{2(\lambda-1)}\right) P_1^\infty(\xi) = \hat{u}(\xi) \quad \text{as } m \rightarrow \infty, \quad (2.4.3)$$

pointwise and in $L^1(\mathbb{R})$. Therefore, $T^m v \rightarrow u$ uniformly, and so in $L^1(\mathbb{R})$. Since X is closed in $L^1(\mathbb{R})$ and T is bounded on $L^1(\mathbb{R})$, u is in X and $u = Tu$. Moreover, we have that

$$|\xi^n \hat{u}(\xi)| = |\xi^n P_1^\infty(\xi)| \leq |\xi^n P_1^{n+2}(\xi)| \in L^1(\mathbb{R}), \quad \text{for every } n = 0, 1, 2, \dots$$

This establishes $u \in C^\infty(\mathbb{R})$. If we assume that $u(x_0) = 0$ for some $x_0 \in \left(0, \frac{1}{2(\lambda-1)}\right]$, then, using $u = Tu$, $u \geq 0$ and (2.4.1) or (2.4.2), we have $u \equiv 0$ in contradiction to $\int_{\mathbb{R}} u(x) dx = 1$. Therefore we have $u(x) > 0$ for $x \in (0, 1/(\lambda-1))$.

2.5 Proof of the main result

Let u be as in (2.2.2). It is clear that $u'(x) = \lambda^2 u(\lambda x)$ for $x \in (-\infty, 0]$. Since u satisfies (2.2.3), it can be extended uniquely to the right (increasing value of x) by using (2.2.1). This will be the solution f in Theorem 2.2.3. Actually, we can prove that

$$u(x) = f(x) \quad \text{for } x \in \left[0, \min\left(\frac{1}{\lambda}, \frac{1}{\lambda(\lambda-1)}\right)\right], \quad (2.5.1)$$

and that

$$f'(x) = \lambda^2 f(\lambda x) \quad \text{for } x \in \mathbb{R}. \quad (2.5.2)$$

The equation (2.5.1) follows from (2.2.10) with $j = 0$. Let

$$f_j(x) = \sum_{k=1}^{2^j} (-1)^{n_k} u(x - y_k), \quad j = 0, 1, 2, \dots$$

Then we have by (2.2.10) that

$$f(x) = f_j(x) \quad \text{for } x \in [0, \lambda^j], \quad j = 0, 1, 2, \dots$$

To prove (2.5.2) we show that

$$f_j'(x) = \lambda^2 f_{j+1}(\lambda x) \quad \text{for } x \in \mathbb{R}, \quad j = 1, 2, 3, \dots \quad (2.5.3)$$

Then we have that

$$f'(x) = f_j'(x) = \lambda^2 f_{j+1}(\lambda x) = \lambda^2 f(\lambda x) \quad \text{for } x \in [0, \lambda^j], \quad j = 1, 2, 3, \dots$$

Let

$$P_\ell^\infty(\xi) = \prod_{k=\ell}^{\infty} \operatorname{sinc}\left(\frac{\xi}{2\lambda^k\pi}\right), \quad \ell = 1, 2.$$

Then

$$P_1^\infty(\xi/\lambda) = P_2^\infty(\xi).$$

From the equations

$$\begin{aligned} \mathcal{F}[u(\cdot - y_k)](\xi) &= \hat{u}(\xi) \exp(-iy_k\xi) \\ &= \exp\left(-\frac{i\xi}{2(\lambda-1)}\right) P_1^\infty(\xi) \exp(-iy_k\xi), \end{aligned}$$

and

$$i\xi \operatorname{sinc}\left(\frac{\xi}{2\lambda\pi}\right) = \lambda \left(\exp\left(\frac{i\xi}{2\lambda}\right) - \exp\left(-\frac{i\xi}{2\lambda}\right) \right),$$

it follows that

$$\begin{aligned} \mathcal{F}[f_j'](\xi) &= i\xi \mathcal{F}[f_j](\xi) \\ &= i\xi \exp\left(-\frac{i\xi}{2(\lambda-1)}\right) P_1^\infty(\xi) \sum_{k=1}^{2^j} (-1)^{n_k} \exp(-iy_k\xi) \\ &= \lambda \exp\left(-\frac{i\xi}{2(\lambda-1)}\right) \\ &\quad \times \left(\exp\left(\frac{i\xi}{2\lambda}\right) - \exp\left(-\frac{i\xi}{2\lambda}\right) \right) P_2^\infty(\xi) \sum_{k=1}^{2^j} (-1)^{n_k} \exp(-iy_k\xi) \\ &= \lambda \exp\left(-\frac{i\xi}{2\lambda(\lambda-1)}\right) \\ &\quad \times \left(1 - \exp\left(-\frac{i\xi}{\lambda}\right) \right) P_2^\infty(\xi) \sum_{k=1}^{2^j} (-1)^{n_k} \exp(-iy_k\xi) \\ &= \lambda \exp\left(-\frac{i\xi}{2\lambda(\lambda-1)}\right) P_2^\infty(\xi) \\ &\quad \times \left(\sum_{k=1}^{2^j} (-1)^{n_k} \exp(-iy_k\xi) - \sum_{k=1}^{2^j} (-1)^{n_k} \exp\left(-iy_k\xi - \frac{i\xi}{\lambda}\right) \right). \end{aligned}$$

By the relations (2.2.6) and (2.2.7) we have

$$\begin{aligned} \mathcal{F}[f_j'](\xi) &= \lambda \exp\left(-\frac{i\xi}{2\lambda(\lambda-1)}\right) P_2^\infty(\xi) \\ &\quad \times \left(\sum_{k=1}^{2^j} (-1)^{n_{2k-1}} \exp\left(-\frac{iy_{2k-1}\xi}{\lambda}\right) + \sum_{k=1}^{2^j} (-1)^{n_{2k}} \exp\left(-\frac{iy_{2k}\xi}{\lambda}\right) \right) \\ &= \lambda \exp\left(-\frac{i(\xi/\lambda)}{2(\lambda-1)}\right) P_1^\infty(\xi/\lambda) \sum_{k=1}^{2^{j+1}} (-1)^{n_k} \exp(-iy_k\xi/\lambda) \\ &= \lambda \mathcal{F}[f_{j+1}](\xi/\lambda) \\ &= \lambda^2 \mathcal{F}[f_{j+1}(\lambda \cdot)](\xi). \end{aligned}$$

This establishes (2.5.3).

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On the Navier-Stokes
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functional-differential
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a Fourier analysis approach

(フーリエ解析的手法による
回転場内の流体方程式と進み
型関数微分方程式の考察)

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