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A Limit Theorem for Weyl Transformation in Infinite-Dimensional Torus and Central Limit Theorem for Correlated Multiple Wiener Integrals

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Abstract. We show that under many of the probabilities on $\mathbb{T}^\infty$, infinite-dimensional torus, a random system $(1/\sqrt{N}\sum_{i=1}^N f(x_i+p\alpha_i))$ converges to a centered Gaussian system whose covariance is determined only by the distribution of $(\alpha_i)_{i=1}^\infty$ over $\mathbb{T}$. Moreover we show the convergence of a system of symmetric statistics to that of correlated multiple Wiener integrals defined by the Gaussian system.

Also we study the central limit theorem for a sequence of the correlated multiple Wiener integrals.

0. Introduction and Notation

We study a limit theorem for a random system of Weyl transformation in infinite-dimensional torus $\mathbb{T}^\infty$.

In the previous paper [16], taking infinite-dimensional Lebesgue probability $\mathbf{P}^\infty$ as underlying probability on $\mathbb{T}^\infty$, we showed that on $(\mathbb{T}^\infty, \mathbf{P}^\infty)$

$$\left(\frac{1}{\sqrt{N}}\sum_{i=1}^N f(x_i+p\alpha_i)\right)_{p\in\mathbb{Z}, f\in\mathcal{CL}_2} \tag{0.1}$$
$$\overset{\text{f.d.}}{\Longrightarrow} \quad \left(I^{(p)}(f)\right)_{p\in\mathbb{Z}, f\in\mathcal{CL}_2} \quad \text{as } N\to\infty,$$

if and only if $\alpha=(\alpha_1,\alpha_2,\ldots)\in\mathbb{T}^\infty_{dx}$, i.e., α is uniformly distributed over $\mathbb{T}$. (We expressed this phenomenon by saying that the disappearance of dependency happens.) Here $\mathcal{CL}_2$ is the totality of square integrable real functions on $\mathbb{T}$ with $\int_{\mathbb{T}} f(x)dx=0$, " $\overset{\text{f.d.}}{\Longrightarrow}$ " means the convergence in finite dimensional distribution, $I^{(p)}(f)$ is the Wiener integral of f with respect to $B^{(p)}$ and $\{B^{(p)}=(B^{(p)}(t))_{0\le t\le 1}\}_{p\in\mathbb{Z}}$ is a sequence of independent 1-dimensional Brownian motions starting at 0. Moreover this was generalized

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to the convergence of a system of symmetric statistics to that of multiple Wiener integrals, i.e.,

$$\Big((\tfrac{1}{\sqrt{N}})^n \sum_{1\le i_1<\cdots<i_n\le N} h(x_{i_1}+p\alpha_{i_1},\ldots,x_{i_n}+p\alpha_{i_n})\Big)_{p\in\mathbb{Z},h\in\mathcal{CSL}_2^n}$$

$$\text{(0.2)}\qquad \overset{\text{f.d.}}{\Longrightarrow} \Big(\tfrac{1}{n!} I_n^{(p)}(h)\Big)_{p\in\mathbb{Z},h\in\mathcal{CSL}_2^n} \quad \text{as } N\to\infty.$$

Here $I_n^{(p)}(h)$ is the n-ple Wiener integral of $h\in\mathcal{CSL}_2^n$ (for which see (0.7) below) with respect to $B^{(p)}$.

In the present paper we take, as underlying probability, product probability and generally absolutely continuous one relative to it. When a product probability $\boldsymbol{\mu}$ is mostly dominated by $\mathbf{P}^\infty$, i.e., the singular part of $\boldsymbol{\mu}$ with respect to $\mathbf{P}^\infty$ is small in a certain sense, we show that on any absolutely continuous probability space relative to $\boldsymbol{\mu}$,

$$\Big(\tfrac{1}{\sqrt{N}}\sum_{i=1}^N f(x_i+p\alpha_i)\Big)_{p\in\mathbb{Z},f\in C(\mathbb{T})\cap\mathcal{CL}_2}$$

$$\text{(0.3)}\qquad \overset{\text{f.d.}}{\Longrightarrow} \Big(I^{(p)}(f)\Big)_{p\in\mathbb{Z},f\in C(\mathbb{T})\cap\mathcal{CL}_2} \quad \text{as } N\to\infty,$$

if and only if $\alpha\in\mathbb{T}^\infty_{dx}$, and that for general $\alpha\in\mathbb{T}^\infty$, chosen a subsequence $\{N_m\}_{m=1}^\infty$ of $\{1,2,3,\ldots\}$ and a probability μ on $\mathbb{T}$ so that $\alpha\in\mathbb{T}^\infty_{\mu,\{N_m\}}$ (for which see (0.8) below), the convergence (0.3) holds by replacing $\{N\}$ and $(I^{(p)}(f))$ by $\{N_m\}$ and $(I^{(p)}(f;\mu))$, respectively, where $(I^{(p)}(f;\mu))$ is a centered, correlated Gaussian system (cf. (1.1)). For details, see Theorem 1.2. For this reason one may say that under many of the probabilities on $\mathbb{T}^\infty$ a random system $(\frac{1}{\sqrt{N}}\sum_{i=1}^N f(x_i+p\alpha_i))_{p\in\mathbb{Z},f\in C(\mathbb{T})\cap\mathcal{CL}_2}$ converges to a centered Gaussian system whose covariance is determined only by the distribution of $(\alpha_1,\alpha_2,\ldots)$ over $\mathbb{T}$.

We further improve the convergence (0.2) in the following way: In case the $\boldsymbol{\mu}$ above satisfies some additional conditions and $\alpha\in\mathbb{T}^\infty_{\mu,\{N_m\}}$, on any absolutely continuous probability space relative to $\boldsymbol{\mu}$

$$\Big((\tfrac{1}{\sqrt{N_m}})^n \sum_{1\le i_1<\cdots<i_n\le N_m} h(x_{i_1}+p\alpha_{i_1},\ldots,x_{i_n}+p\alpha_{i_n})\Big)_{p\in\mathbb{Z},h\in\mathcal{CSL}_2^n}$$

$$\text{(0.4)}\qquad \overset{\text{f.d.}}{\Longrightarrow} \Big(\tfrac{1}{n!} I_n^{(p)}(h;\mu)\Big)_{p\in\mathbb{Z},h\in\mathcal{CSL}_2^n} \quad \text{as } m\to\infty,$$

where $I_n^{(p)}(\cdot;\mu)$ is the n-ple multiple Wiener integral with respect to $I^{(p)}(\cdot;\mu)$. See Theorem 2.1.

Next we study the central limit theorem (abbr. CLT) for a sequence $\{I_n^{(p)}(\cdot;\mu)\}_{p=1}^{\infty}$. For general $\mu(dx)$, the sequence is correlated in p, but when $\mu(dx)$ is the Lebesgue measure dx, it is independent. So it is expected that if $\mu(dx)$ is not very far from dx in some sense, then the dependence of $\{I_n^{(p)}(\cdot;\mu)\}_{p=1}^{\infty}$ will be not very large, so that the CLT for $\{I_n^{(p)}(\cdot;\mu)\}_{p=1}^{\infty}$ must hold. Indeed, when $\mu(dx)$ is absolutely continuous relative to dx, and its density is continuous and positive, we show that for any non-zero $h \in \mathcal{CSL}_2^n$, $\frac{1}{\sqrt{P}}\sum_{p=1}^{P} I_n^{(p)}(h;\mu)$ converges to a nondegenerate Gaussian as $P \to \infty$. See Theorem 3.3.

In Section 4, combining this with the convergence (0.4), we present a justification of the claim of Sobol' et al, by which our study in [16] was inspired. In the last section, related topics on the disappearance of dependency are introduced.

Let us here explain the notation used in this paper.

Let $\mathbb{T} \cong [0,1)$ be the 1-dimensional torus and $\mathbf{P}(dx) = dx$ the Lebesgue measure on it. For $m = 1,2,\ldots,\infty$ let $\mathbb{T}^m$ be the m-dimensional torus, i.e., $\mathbb{T}^m = \underbrace{\mathbb{T}\times\cdots\times\mathbb{T}}_{m}$. Let $\mathfrak{F}$ be a σ-algebra on $\mathbb{T}^\infty$ generated by cylindrical sets and $\mathbf{P}^\infty$ the Lebesgue measure on $(\mathbb{T}^\infty, \mathfrak{F})$, i.e., $\mathbf{P}^\infty(dx_1dx_2\cdots) = \prod_{i=1}^{\infty} dx_i$. The addition and scalar multiplication on $\mathbb{T}^m$ are always considered coordinatewise in the sense of modulo 1.

Let

$$(0.5)\qquad \mathcal{L}_2^n = L_2(\mathbb{T}^n; dx_1\cdots dx_n),$$

$$(0.6)\qquad \mathcal{SL}_2^n = \left\{ h \in \mathcal{L}_2^n;\ \begin{array}{r} h \text{ is symmetric in each two variables, i.e.,} \\ h(\ldots,\overset{i}{x},\ldots,\overset{j}{y},\ldots) = h(\ldots,\overset{i}{y},\ldots,\overset{j}{x},\ldots) \end{array} \right\},$$

$$(0.7)\qquad \mathcal{CSL}_2^n = \left\{ h \in \mathcal{SL}_2^n;\ \begin{array}{r} \int_{\mathbb{T}} h(x_1,\ldots,x_{n-1},y)dy = 0 \\ \text{a.a. } (x_1,\ldots,x_{n-1}) \in \mathbb{T}^{n-1} \end{array} \right\}.$$

When $n = 1$, $\mathcal{L}_2^1 = \mathcal{SL}_2^1$ and $\mathcal{CSL}_2^1$ are simply written as $\mathcal{L}_2$ and $\mathcal{CL}_2$, respectively. The norm and inner product on $\mathcal{L}_2^n$ are denoted by $\|\cdot\|_{\mathcal{L}_2^n}$ and $(\cdot,*)_{\mathcal{L}_2^n}$, respectively. In some cases, they may be written in dropping the subscript $\mathcal{L}_2^n$. For $f_1,\dots,f_n \in \mathcal{L}_2$, $f_1\otimes\cdots\otimes f_n \in \mathcal{L}_2^n$ is defined by

$$f_1\otimes\cdots\otimes f_n(x_1,\dots,x_n) \ := \ f_1(x_1)\times\cdots\times f_n(x_n).$$

If $f_1=\cdots=f_n=f$, this is denoted by $f^{\otimes n}$. Note that $f^{\otimes n}\in\mathcal{CSL}_2^n$ if $f\in\mathcal{CL}_2$.

Functions on $\mathbb{T}$ are identified with 1-periodic functions on $\mathbb{R}$ in an obvious way. By this identification a continuous function on $\mathbb{T}$ is regarded as a continuous 1-periodic function on $\mathbb{R}$. Let $C(\mathbb{T})$ be the space of all such functions with the supremum norm $\|\cdot\|_\infty$.

For $f\in L_1(\mathbb{T};dx)$ and generally a finite measure $\nu(dx)$ on $\mathbb{T}$, $\widehat{f}(n)$ and $\widehat{\nu}(n)$ are the n-th Fourier coefficients of f and ν, respectively, i.e., $\widehat{f}(n)=\int_{\mathbb{T}} f(x)e^{-\sqrt{-1}\,2n\pi x}dx$ and $\widehat{\nu}(n)=\int_{\mathbb{T}} e^{-\sqrt{-1}\,2n\pi x}\nu(dx)$. For a probability measure μ on $\mathbb{T}$ and a subsequence $\{N_m\}_{m=1}^\infty$ of $\{1,2,3,\dots\}$, let

$$(0.8)\qquad \mathbb{T}^\infty_{\mu,\{N_m\}} = \left\{\begin{array}{l} \alpha=(\alpha_1,\alpha_2,\dots)\in\mathbb{T}^\infty; \\ \lim_{m\to\infty}\frac{1}{N_m}\sum_{i=1}^{N_m}\delta_{\alpha_i}(dx)=\mu(dx) \ \text{ weakly} \end{array}\right\}.$$

If $N_m=m$, this is simply written as $\mathbb{T}^\infty_\mu$.

1. Convergence to a Centered Gaussian System

In this section underlying probability space $(\mathbb{T}^\infty,\boldsymbol{\mathfrak{F}},\boldsymbol{\mu})$ is a product probability space, i.e., for a sequence $\{\mathbf{P}_i\}_{i=1}^\infty$ of probability measures on $\mathbb{T}$, $\boldsymbol{\mu}=\prod_{i=1}^\infty \mathbf{P}_i$. Set $\boldsymbol{\mathfrak{F}}_n:=\sigma(X_1,\dots,X_n)$, where X_i is the coordinate function, i.e., $X_i(x)=x_i, x=(x_j)_{j=1}^\infty\in\mathbb{T}^\infty$. Note that $\boldsymbol{\mathfrak{F}}=\sigma(\bigcup_n \boldsymbol{\mathfrak{F}}_n)$. For a probability measure μ on $\mathbb{T}$ let $(I^{(p)}(f;\mu))_{p\in\mathbb{Z},f\in\mathcal{CL}_2}$ be a centered Gaussian system on a probability space $(\Omega,\mathcal{F},P)$ such that

$$(1.1)\qquad E\Big[I^{(p)}(f;\mu)I^{(q)}(g;\mu)\Big] \ = \ \sum_{|i|\ge 1}\widehat{f}(i)\,\overline{\widehat{g}(i)}\,\widehat{\mu}(i(q-p)), \qquad {}^\forall f,g\in\mathcal{CL}_2, {}^\forall p,q\in\mathbb{Z}.$$

THEOREM 1.1. *Let $\alpha \in \mathbb{T}^\infty$ and let $\{N_m\}_{m=1}^\infty$ be a subsequence of $\{1,2,3,\ldots\}$. The following* (A) *and* (B) *are equivalent:*

$$\text{(A)}\quad \Big(\tfrac{1}{\sqrt{N_m}}\sum_{i=1}^{N_m} f(x_i+p\alpha_i)\Big)_{p\in\mathbb{Z},f\in C(\mathbb{T})\cap\mathcal{CL}_2} \quad \textit{under } \boldsymbol{\mu}$$
$$\overset{\text{f.d.}}{\Longrightarrow}\ \Big(I^{(p)}(f;\mu)\Big)_{p\in\mathbb{Z},f\in C(\mathbb{T})\cap\mathcal{CL}_2} \quad \textit{as } m\to\infty.$$

$$\text{(B)}\quad \text{(B.1)}\quad \lim_{m\to\infty}\tfrac{1}{N_m}\sum_{i=1}^{N_m}\Big(\int_{\mathbb{T}} f(x+p\alpha_i)\mathbf{P}_i(dx)\Big)^2 \;=\; 0,$$
$$\forall f\in C(\mathbb{T})\cap\mathcal{CL}_2,\ \forall p\in\mathbb{Z}$$
$$\text{(B.2)}\quad \lim_{m\to\infty}\tfrac{1}{\sqrt{N_m}}\sum_{i=1}^{N_m}\int_{\mathbb{T}} f(x+p\alpha_i)\mathbf{P}_i(dx) \;=\; 0,$$
$$\forall f\in C(\mathbb{T})\cap\mathcal{CL}_2,\ \forall p\in\mathbb{Z}$$
$$\text{(B.3)}\quad \lim_{m\to\infty}\tfrac{1}{N_m}\sum_{i=1}^{N_m}\int_{\mathbb{T}} f(x+p\alpha_i)g(x+q\alpha_i)\mathbf{P}_i(dx)$$
$$=\; E\Big[I^{(p)}(f;\mu)I^{(q)}(g;\mu)\Big],$$
$$\forall f,g\in C(\mathbb{T})\cap\mathcal{CL}_2,\ \forall p,q\in\mathbb{Z}\,;p\neq q.$$

To prove the theorem we present a lemma:

LEMMA 1.1. *Let $\alpha\in\mathbb{T}^\infty$. Let $p_0\in\mathbb{Z}$, $K\in\mathbb{N}$ and $f_1,\ldots,f_L\in C(\mathbb{T})\cap\mathcal{CL}_2$. For $(a_{pk})_{\substack{1\le p\le K\\1\le k\le L}}\in S^{KL-1}$, i.e., $(a_{pk})_{\substack{1\le p\le K\\1\le k\le L}}\in\mathbb{R}^{KL}$ with $\sum_{\substack{1\le p\le K\\1\le k\le L}} a_{pk}^2 = 1$, set*

$$\zeta_{Ni} \;:=\; \tfrac{1}{\sqrt{N}}\sum_{\substack{1\le p\le K\\1\le k\le L}} a_{pk}f_k(x_i+(p+p_0)\alpha_i),\quad 1\le i\le N,$$

$$T_N(t) \;:=\; \prod_{i=1}^{N}(1+\sqrt{-1}\,t\zeta_{Ni}),\quad t\in\mathbb{R}.$$

Let $\boldsymbol{\lambda}$ be a probability measure on $(\mathbb{T}^\infty,\mathfrak{F})$. Then, for $^\forall v\ge 0$ and $^\forall$ sub-

sequence $\{N_j\}_{j=1}^\infty$ *of* $\{1,2,3,\ldots\}$

$$\limsup_{j\to\infty} \mathbf{E}^{\boldsymbol{\lambda}}\left[|e^{\sqrt{-1}\,t\sum_{i=1}^{N_j}\zeta_{N_j i}} - T_{N_j}(t)e^{-\frac{t^2}{2}v}|\right]$$

$$\leq \Big(1+e^{\frac{t^2}{2}K\sum_{k=1}^{L}\|f_k\|_\infty^2}\Big)\lim_{\varepsilon\downarrow 0}\limsup_{j\to\infty}\boldsymbol{\lambda}\Big(|\sum_{i=1}^{N_j}\zeta_{N_j i}^2 - v| > \varepsilon\Big).$$

PROOF. Using an expression

$$e^z = (1+z)e^{\frac{z^2}{2}-r(z)}, \quad \operatorname{Re} z > -1 \tag{1.2}$$

where

$$r(z) = z^3\int_0^1 \frac{s^2}{1+zs}\,ds, \tag{1.3}$$

we write for $0<\varepsilon\leq 1$ and $v\geq 0$

$$\begin{aligned}
& e^{\sqrt{-1}\,t\sum_{i=1}^{N}\zeta_{Ni}} \\
&= e^{\sqrt{-1}\,t\sum_{i=1}^{N}\zeta_{Ni}} 1_{\max_{1\leq i\leq N}|\zeta_{Ni}|>\varepsilon} \\
&+ e^{\sqrt{-1}\,t\sum_{i=1}^{N}\zeta_{Ni}} 1_{\{\max_{1\leq i\leq N}|\zeta_{Ni}|\leq\varepsilon,\,|\sum_{i=1}^{N}\zeta_{Ni}^2 - v|>\varepsilon\}} \\
&- T_N(t)e^{-\frac{t^2}{2}v} 1_{\{\max_{1\leq i\leq N}|\zeta_{Ni}|>\varepsilon \text{ or } |\sum_{i=1}^{N}\zeta_{Ni}^2 - v|>\varepsilon\}} \\
&+ T_N(t)\Big(e^{-\frac{t^2}{2}\sum_{i=1}^{N}\zeta_{Ni}^2 - \sum_{i=1}^{N} r(\sqrt{-1}\,t\zeta_{Ni})} - e^{-\frac{t^2}{2}v}\Big) \\
&\quad \times 1_{\{\max_{1\leq i\leq N}|\zeta_{Ni}|\leq\varepsilon,\,|\sum_{i=1}^{N}\zeta_{Ni}^2 - v|\leq\varepsilon\}} \\
&+ T_N(t)e^{-\frac{t^2}{2}v}.
\end{aligned}$$

Then

$$\begin{aligned}
& |e^{\sqrt{-1}\,t\sum_{i=1}^{N}\zeta_{Ni}} - T_N(t)e^{-\frac{t^2}{2}v}| \\
&\leq 1_{\max_{1\leq i\leq N}|\zeta_{Ni}|>\varepsilon} + 1_{|\sum_{i=1}^{N}\zeta_{Ni}^2 - v|>\varepsilon} \\
&\quad + |T_N(t)| 1_{\{\max_{1\leq i\leq N}|\zeta_{Ni}|>\varepsilon \text{ or } |\sum_{i=1}^{N}\zeta_{Ni}^2 - v|>\varepsilon\}} \\
&\quad + |T_N(t)|\sup\left\{|e^z - e^w|\,;\ \begin{array}{l} |z|,|w|\leq (\frac{t^2}{2}+\frac{|t|^3}{3})(1+v), \\ |z-w|\leq\varepsilon(\frac{t^2}{2}+\frac{|t|^3}{3}(1+v)) \end{array}\right\}.
\end{aligned}$$

Hence, noting that

$$|T_N(t)| \leq e^{\frac{t^2}{2}K\sum_{k=1}^{L}\|f_k\|_\infty^2}, \tag{1.4}$$

$$\max_{1\leq i\leq N}|\zeta_{Ni}| \leq \frac{1}{\sqrt{N}}\sqrt{K\sum_{k=1}^{L}\|f_k\|_\infty^2}, \tag{1.5}$$

we have the conclusion. □

Proof of " (B) $\Rightarrow$ (A) ". We suppose (B). Let $K, L \in \mathbb{N}$ be fixed arbitrarily. Let $p_0 \in \mathbb{Z}$, $f_1, \dots, f_L \in C(\mathbb{T}) \cap \mathcal{CL}_2$ and $(a_{pk})_{\substack{1\leq p\leq K\\1\leq k\leq L}} \in S^{KL-1}$. By the Cramér-Wold device (cf. [1]) it suffices to show that for $^\forall t \in \mathbb{R}$

$$\begin{aligned}
&\lim_{m\to\infty}\mathbf{E}^{\boldsymbol{\mu}}\left[e^{\sqrt{-1}\,t\sum_{p=1}^{K}\sum_{k=1}^{L}a_{pk}\frac{1}{\sqrt{N_m}}\sum_{i=1}^{N_m}f_k(x_i+(p+p_0)\alpha_i)}\right]\\
&\quad= e^{-\frac{t^2}{2}E\left[\left(\sum_{p=1}^{K}\sum_{k=1}^{L}a_{pk}I^{(p+p_0)}(f_k;\mu)\right)^2\right]}.
\end{aligned}$$

Let ζ_{Ni} and $T_N(t)$ be as in Lemma 1.1. From (B.2) note that

$$\lim_{m\to\infty}\frac{1}{N_m}\sum_{i=1}^{N_m}\mathbf{E}_i[f(\cdot+p\alpha_i)] = \mathbf{E}[f], \quad {}^\forall f\in C(\mathbb{T}), {}^\forall p\in\mathbb{Z}.$$

We write

$$\begin{aligned}
\sum_{i=1}^{N}\zeta_{Ni}^2 &= \sum_{\substack{1\leq p\leq K\\1\leq k\leq L}}\sum_{\substack{1\leq q\leq K\\1\leq l\leq L}}a_{pk}a_{ql}\\
&\quad\times\Bigg(\frac{1}{N}\sum_{i=1}^{N}(f_k(x_i+(p+p_0)\alpha_i)f_l(x_i+(q+p_0)\alpha_i)\\
&\qquad\qquad-\mathbf{E}_i[f_k(\cdot+(p+p_0)\alpha_i)f_l(\cdot+(q+p_0)\alpha_i)])\\
&\qquad\qquad+\frac{1}{N}\sum_{i=1}^{N}\mathbf{E}_i[f_k(\cdot+(p+p_0)\alpha_i)f_l(\cdot+(q+p_0)\alpha_i)]\Bigg).
\end{aligned}$$

By the note above and (B.3)

$$\lim_{m\to\infty}\frac{1}{N_m}\sum_{i=1}^{N_m}\mathbf{E}_i[f_k(\cdot+p\alpha_i)f_l(\cdot+q\alpha_i)] = E\Big[I^{(p)}(f_k;\mu)I^{(q)}(f_l;\mu)\Big],$$

and also by the strong law of large numbers (abbr. SLLN)

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}(f_k(x_i+p\alpha_i)f_l(x_i+q\alpha_i) \\ -\mathbf{E}_i[f_k(\cdot+p\alpha_i)f_l(\cdot+q\alpha_i)]) \;=\; 0 \quad \boldsymbol{\mu}\text{-a.s.}$$

Hence substituting these into the expression of $\sum_{i=1}^{N_m}\zeta_{N_m i}^2$, we have

$$\lim_{m\to\infty}\sum_{i=1}^{N_m}\zeta_{N_m i}^2 \;=\; E\Big[\Big(\sum_{\substack{1\le p\le K\\1\le k\le L}}a_{pk}I^{(p+p_0)}(f_k;\mu)\Big)^2\Big] \quad \boldsymbol{\mu}\text{-a.s.} \tag{1.6}$$

Next let us check

$$\lim_{m\to\infty}\mathbf{E}^{\boldsymbol{\mu}}[T_{N_m}(t)\,;A] \;=\; \boldsymbol{\mu}(A), \qquad {}^{\forall}t\in\mathbb{R},\,{}^{\forall}A\in\mathfrak{F}. \tag{1.7}$$

To do so, for $m>n\ge 1$ we divide $\mathbf{E}^{\boldsymbol{\mu}}[T_{N_m}(t)\,;A]$ as

$$\mathbf{E}^{\boldsymbol{\mu}}[T_{N_m}(t)\,;\,A] \;=\; \mathbf{E}^{\boldsymbol{\mu}}\Big[T_{N_m}(t)(\boldsymbol{\mu}(A\,|\,\mathfrak{F}_{N_m})-\boldsymbol{\mu}(A\,|\,\mathfrak{F}_{N_n}))\Big] \\ +\mathbf{E}^{\boldsymbol{\mu}}\Big[T_{N_m}(t)\boldsymbol{\mu}(A\,|\,\mathfrak{F}_{N_n})\Big].$$

By (1.4)

$$|\text{the first term}| \;\le\; e^{\frac{t^2}{2}K\sum_{k=1}^{L}\|f_k\|^2}\,\mathbf{E}^{\boldsymbol{\mu}}\Big[|\boldsymbol{\mu}(A\,|\,\mathfrak{F}_{N_m})-\boldsymbol{\mu}(A\,|\,\mathfrak{F}_{N_n})|\Big].$$

Since $\boldsymbol{\mu}(A\,|\,\mathfrak{F}_N)\to 1_A$ in L_1 as $N\to\infty$,

$$\limsup_{m\to\infty}|\text{the first term}| \;\le\; e^{\frac{t^2}{2}K\sum_{k=1}^{L}\|f_k\|^2}\,\mathbf{E}^{\boldsymbol{\mu}}\Big[|1_A-\boldsymbol{\mu}(A\,|\,\mathfrak{F}_{N_n})|\Big] \\ \longrightarrow\; 0 \qquad \text{as } n\to\infty.$$

By the independence of $\mathfrak{F}_{N_n}$ and $\sigma(\zeta_{N_m i};N_n+1\le i\le N_m)$ and by (1.2),

the second term

$$= \;\mathbf{E}^{\boldsymbol{\mu}}\Big[\prod_{i=1}^{N_n}\Big(1+\frac{\sqrt{-1}\,t}{\sqrt{N_m}}\sum_{\substack{1\le p\le K\\1\le k\le L}}a_{pk}f_k(x_i+(p+p_0)\alpha_i)\Big)\boldsymbol{\mu}(A\,|\,\mathfrak{F}_{N_n})\Big]$$

$$\times\, e^{\sqrt{-1}\,t\sum_{p=1}^{K}\sum_{k=1}^{L}a_{pk}\frac{1}{\sqrt{N_m}}\sum_{i=N_n+1}^{N_m}\mathbf{E}_i[f_k(\cdot+(p+p_0)\alpha_i)]}$$
$$\times\, e^{\frac{t^2}{2}\frac{1}{N_m}\sum_{i=N_n+1}^{N_m}\left(\sum_{p=1}^{K}\sum_{k=1}^{L}a_{pk}\mathbf{E}_i[f_k(\cdot+(p+p_0)\alpha_i)]\right)^2}$$
$$\times\, e^{\sum_{i=N_n+1}^{N_m} r\left(\frac{\sqrt{-1}\,t}{\sqrt{N_m}}\sum_{p=1}^{K}\sum_{k=1}^{L}a_{pk}\mathbf{E}_i[f_k(\cdot+(p+p_0)\alpha_i)]\right)}.$$

Clearly

$$\text{the first product} \quad\longrightarrow\quad \mathbf{E}^{\boldsymbol{\mu}}\left[\boldsymbol{\mu}(A\,|\,\mathfrak{F}_{N_n})\right] \;=\; \boldsymbol{\mu}(A) \quad \text{as } m\to\infty.$$

By (B.2) and (B.1), the second and third products $\to 1$. By the fact $|r(\sqrt{-1}\,x)| \le \frac{|x|^3}{3}$ (cf. (1.3)), the fourth product $\to 1$. Hence, putting the above we have (1.7).

By (1.6) and (1.7), Lemma 1.1 gives us the desired conclusion. □

Proof of " (A) ⇒ (B) ". We suppose (A). The proof is done in three steps.

1° *For* $^\forall f \in C(\mathbb{T})$ *and* $^\forall p \in \mathbb{Z}$

$$\lim_{m\to\infty}\frac{1}{N_m}\sum_{i=1}^{N_m} f(x_i+p\alpha_i) \;=\; \mathbf{E}[f] \quad \boldsymbol{\mu}\text{-}a.s.$$

Proof. Let $p\in\mathbb{Z}$ and $f\in C(\mathbb{T})\cap\mathcal{CL}_2$. By the assumption

$$\frac{1}{\sqrt{N_m}}\sum_{i=1}^{N_m} f(x_i+p\alpha_i) \;\Longrightarrow\; I^{(p)}(f;\mu) \qquad \text{as } m\to\infty,$$

and so

$$\lim_{m\to\infty}\frac{1}{N_m}\sum_{i=1}^{N_m} f(x_i+p\alpha_i) \;=\; 0 \quad \text{in } \boldsymbol{\mu}.$$

But, since $\left\{\frac{1}{N}\sum_{i=1}^{N} f(x_i+p\alpha_i)\right\}_{N=1}^{\infty}$ is uniformly bounded, and hence uniformly integrable, this convergence also holds in $L_1(\boldsymbol{\mu})$. Therefore, by taking expectation

$$\lim_{m\to\infty}\frac{1}{N_m}\sum_{i=1}^{N_m}\mathbf{E}_i[f(\cdot+p\alpha_i)] \;=\; 0.$$

On the other hand, by the SLLN

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\Big(f(x_i+p\alpha_i)-\mathbf{E}_i[f(\cdot+p\alpha_i)]\Big) \;=\; 0 \quad \boldsymbol{\mu}\text{-a.s.}$$

Combining these we have

$$\lim_{m\to\infty}\frac{1}{N_m}\sum_{i=1}^{N_m}f(x_i+p\alpha_i) \;=\; 0 \quad \boldsymbol{\mu}\text{-a.s.}$$

From this the assertion for general $f\in C(\mathbb{T})$ will be clear. □

2° (B.1) *and* (B.2) *hold.*

PROOF. Let $p\in\mathbb{Z}$ and $f\in C(\mathbb{T})\cap\mathcal{CL}_2$. We may suppose that $\mathbf{E}[f^2]=\int_{\mathbb{T}}f(x)^2dx=1$. For simplicity set $\zeta_{Ni}:=\frac{1}{\sqrt{N}}f(x_i+p\alpha_i)$, $1\le i\le N$. By the assumption $\sum_{i=1}^{N_m}\zeta_{N_mi}\Rightarrow I^{(p)}(f;\mu)\sim\mathfrak{N}(0,1)$, and by 1° $\sum_{i=1}^{N_m}\zeta_{N_mi}^2\to 1$ $\boldsymbol{\mu}$-a.s. Therefore we apply Lemma 1.1 to have

$$\lim_{m\to\infty}\mathbf{E}^{\boldsymbol{\mu}}\Big[\prod_{i=1}^{N_m}(1+\sqrt{-1}\,t\zeta_{N_mi})\Big] \;=\; 1, \qquad {}^\forall t\in\mathbb{R}.$$

Since, by (1.2)

$$\begin{aligned} &e^{\sqrt{-1}\,t\frac{1}{\sqrt{N_m}}\sum_{i=1}^{N_m}\mathbf{E}_i[f(\cdot+p\alpha_i)]}\\ &\quad=\;\mathbf{E}^{\boldsymbol{\mu}}\Big[\prod_{i=1}^{N_m}(1+\sqrt{-1}\,t\zeta_{N_mi})\Big]\\ &\qquad\times e^{-\frac{t^2}{2}\frac{1}{N_m}\sum_{i=1}^{N_m}\left(\mathbf{E}_i[f(\cdot+p\alpha_i)]\right)^2}e^{-\sum_{i=1}^{N_m}r\left(\frac{\sqrt{-1}\,t}{\sqrt{N_m}}\mathbf{E}_i[f(\cdot+p\alpha_i)]\right)}, \end{aligned}$$

and hence

$$\begin{aligned} &e^{\frac{t^2}{2}\frac{1}{N_m}\sum_{i=1}^{N_m}\left(\mathbf{E}_i[f(\cdot+p\alpha_i)]\right)^2}\\ &\quad=\;\Big|\mathbf{E}^{\boldsymbol{\mu}}\Big[\prod_{i=1}^{N_m}(1+\sqrt{-1}\,t\zeta_{N_mi})\Big]\Big|\,\Big|e^{-\sum_{i=1}^{N_m}r\left(\frac{\sqrt{-1}\,t}{\sqrt{N_m}}\mathbf{E}_i[f(\cdot+p\alpha_i)]\right)}\Big|, \end{aligned}$$

letting $m \to \infty$ in the second expression yields

$$\lim_{m\to\infty} \tfrac{1}{N_m} \sum_{i=1}^{N_m} \Big(\mathbf{E}_i[f(\cdot + p\alpha_i)]\Big)^2 \quad = \quad 0,$$

which says (B.1), and next putting this into the first expression yields

$$\lim_{m\to\infty} e^{\sqrt{-1}\,t \frac{1}{\sqrt{N_m}} \sum_{i=1}^{N_m} \mathbf{E}_i[f(\cdot + p\alpha_i)]} \quad = \quad 1, \qquad {}^\forall t \in \mathbb{R},$$

i.e.,

$$\lim_{m\to\infty} \tfrac{1}{\sqrt{N_m}} \sum_{i=1}^{N_m} \mathbf{E}_i[f(\cdot + p\alpha_i)] \quad = \quad 0,$$

which says (B.2). □

3° (B.3) *holds.*

Proof. Let $f, g \in C(\mathbb{T}) \cap \mathcal{CL}_2$ and $p \neq q$. We may suppose that $\int_{\mathbb{T}} f(x)^2 dx = \int_{\mathbb{T}} g(x)^2 dx = 1$. Set

$$\zeta_{Ni} \quad := \quad \tfrac{1}{\sqrt{N}} \Big(\tfrac{1}{\sqrt{2}} f(x_i + p\alpha_i) + \tfrac{1}{\sqrt{2}} g(x_i + q\alpha_i)\Big), \quad 1 \le i \le N.$$

By the assumption

$$\sum_{i=1}^{N_m} \zeta_{N_m i} \quad \Longrightarrow \quad \tfrac{1}{\sqrt{2}}\Big(I^{(p)}(f;\mu) + I^{(q)}(g;\mu)\Big) \qquad \text{as } m \to \infty. \tag{1.8}$$

Since we have (B.1) and (B.2), in the same way as in the proof of (1.7)

$$\lim_{m\to\infty} \mathbf{E}^{\boldsymbol{\mu}} \Big[\prod_{i=1}^{N_m} (1 + \sqrt{-1}\, t\zeta_{N_m i})\Big] \quad = \quad 1, \qquad {}^\forall t \in \mathbb{R}.$$

To view the convergence of $\sum_{i=1}^{N_m} \zeta_{N_m i}^2$, we write

$$\begin{aligned}\sum_{i=1}^{N_m} \zeta_{N_m i}^2 &= \frac{1}{2}\frac{1}{N_m}\sum_{i=1}^{N_m} f(x_i+p\alpha_i)^2 + \frac{1}{2}\frac{1}{N_m}\sum_{i=1}^{N_m} g(x_i+q\alpha_i)^2 \\ &\quad + \frac{1}{N_m}\sum_{i=1}^{N_m}\Big(f(x_i+p\alpha_i)g(x_i+q\alpha_i) \\ &\qquad\qquad - \mathbf{E}_i[f(\cdot+p\alpha_i)g(\cdot+q\alpha_i)]\Big) \\ &\quad + \frac{1}{N_m}\sum_{i=1}^{N_m}\mathbf{E}_i[f(\cdot+p\alpha_i)g(\cdot+q\alpha_i)].\end{aligned}$$

By 1°, the first and second terms $\to \frac{1}{2}$ $\boldsymbol{\mu}$-a.s. By the SLLN, the third term $\to 0$ $\boldsymbol{\mu}$-a.s. The fourth term is bounded in m, and so for $^\forall$subsequence $\{m'\}$ we can take a further subsequence $\{m''\}$ of $\{m'\}$ and $c \in \mathbb{R}$ such that

$$\frac{1}{N_{m''}}\sum_{i=1}^{N_{m''}} \mathbf{E}_i[f(\cdot+p\alpha_i)g(\cdot+q\alpha_i)] \longrightarrow c.$$

Then putting these into the expression of $\sum_{i=1}^{N_m} \zeta_{N_m i}^2$ yields that $\sum_{i=1}^{N_{m''}} \zeta_{N_{m''} i}^2 \to 1+c$ $\boldsymbol{\mu}$-a.s.

Now Lemma 1.1 tells us that $\sum_{i=1}^{N_{m''}} \zeta_{N_{m''} i} \Rightarrow \mathfrak{N}(0, 1+c)$. Combined with (1.8) this implies that

$$c = E\Big[I^{(p)}(f;\mu)I^{(q)}(g;\mu)\Big].$$

Therefore, since the convergence above holds with this value for $^\forall$sub-sequence $\{m'\}$, we must have

$$\lim_{m\to\infty} \frac{1}{N_m}\sum_{i=1}^{N_m} \mathbf{E}_i[f(\cdot+p\alpha_i)g(\cdot+q\alpha_i)] = E\Big[I^{(p)}(f;\mu)I^{(q)}(g;\mu)\Big],$$

which says (B.3). $\square$

Claim 1.1. *Let* $\alpha \in \mathbb{T}^\infty$ *and* $\{N_m\}_{m=1}^\infty$ *be as in Theorem* 1.1. *Suppose the conditions* (B.1) *and* (B.2). *The convergence of* (A) *holds under some absolutely continuous probability relative to* $\boldsymbol{\mu}$, *if and only if the condition* (B.3) *holds. In this case the convergence of* (A) *remains valid under any absolutely continuous one.*

Proof. Suppose the convergence of (A) under an absolutely continuous probability $\boldsymbol{\nu}$ relative to $\boldsymbol{\mu}$. Let ζ_{Ni} be as in 3° above. By the assumption

$$(1.9)\quad \sum_{i=1}^{N_m} \zeta_{N_m i} \quad \text{under } \boldsymbol{\nu} \implies \tfrac{1}{\sqrt{2}}\Big(I^{(p)}(f;\mu) + I^{(q)}(g;\mu)\Big) \qquad \text{as } m \to \infty.$$

By virtue of the conditions (B.1) and (B.2)

$$\lim_{m\to\infty} \mathbf{E}^{\boldsymbol{\nu}}\Big[\prod_{i=1}^{N_m}(1+\sqrt{-1}\,t\zeta_{N_m i})\Big] \;=\; 1, \qquad {}^\forall t \in \mathbb{R},$$

which is easily seen from (1.7). In the same way as in 3°, for any subsequence $\{m'\}$ we take a further subsequence $\{m''\}$ such that $\lim_{m''} \sum_{i=1}^{N_{m''}} \zeta_{N_{m''} i}^2 = 1 + c$ in $\boldsymbol{\nu}$, where

$$c \;=\; \lim_{m''} \tfrac{1}{N_{m''}} \sum_{i=1}^{N_{m''}} \mathbf{E}_i[f(\cdot + p\alpha_i)g(\cdot + q\alpha_i)].$$

Then, by Lemma 1.1 and (1.9), $c = E[I^{(p)}(f;\mu)I^{(q)}(g;\mu)]$. This implies the condition (B.3).

Conversely suppose the condition (B.3). This time let ζ_{Ni} be as in Lemma 1.1. It suffices to show that for ${}^\forall Z \in L_1(\mathbb{T}^\infty, \boldsymbol{\mathfrak{F}}, \boldsymbol{\mu})$ and ${}^\forall t \in \mathbb{R}$

$$\lim_{m\to\infty} \mathbf{E}^{\boldsymbol{\mu}}[Ze^{\sqrt{-1}\,t\sum_{i=1}^{N_m}\zeta_{N_m i}}] \;=\; \mathbf{E}^{\boldsymbol{\mu}}[Z]e^{-\frac{t^2}{2}E\left[\left(\sum_{p=1}^K\sum_{k=1}^L a_{pk}I^{(p+p_0)}(f_k;\mu)\right)^2\right]}.$$

Since elements in $L_1(\mathbb{T}^\infty, \boldsymbol{\mathfrak{F}}, \boldsymbol{\mu})$ are approximated by $\boldsymbol{\mathfrak{F}}$-simple functions, we may assume the Z above to be a defining function.

Let $A \in \boldsymbol{\mathfrak{F}}$ and $t \in \mathbb{R}$ be fixed arbitrarily. Then by (1.6) and (1.7), Lemma 1.1 implies that

$$\lim_{m\to\infty} \mathbf{E}^{\boldsymbol{\mu}}\left[e^{\sqrt{-1}\,t\sum_{i=1}^{N_m}\zeta_{N_m i}}; A\right] \;=\; \boldsymbol{\mu}(A)e^{-\frac{t^2}{2}E\left[\left(\sum_{p=1}^K\sum_{k=1}^L a_{pk}I^{(p+p_0)}(f_k;\mu)\right)^2\right]},$$

which is just the desired conclusion. □

CLAIM 1.2. *The condition* (B.1) + (B.3) *is equivalent to that* $(\mathrm{C})_k$ ($k = 1$ *or* 2) + (M)*:*

$$(\mathrm{C})_k \qquad \lim_{m\to\infty} \frac{1}{N_m}\sum_{i=1}^{N_m} |\widehat{\mathbf{P}_i}(n)|^k \;=\; 0, \qquad {}^\forall n \in \mathbb{Z}\setminus\{0\},$$

$$(\mathrm{M}) \qquad \alpha \in \mathbb{T}^\infty_{\mu,\{N_m\}}.$$

PROOF. The implications (B.1) $\Rightarrow$ $(\mathrm{C})_2$ $\Rightarrow$ $(\mathrm{C})_1$ $\Rightarrow$ (B.1) are easily seen. Indeed, the first implication follows from an equality

$$\Big(\int_{\mathbb{T}} \cos 2n\pi(x+p\alpha_i)\mathbf{P}_i(dx)\Big)^2 + \Big(\int_{\mathbb{T}} \sin 2n\pi(x+p\alpha_i)\mathbf{P}_i(dx)\Big)^2$$
$$= \; |\widehat{\mathbf{P}_i}(\mp n)|^2;$$

the second does from an inequality $|a| \le \frac{1}{2}(\varepsilon^2 + \frac{|a|^2}{\varepsilon^2})$, $\varepsilon > 0$; the third does from the Weierstrass approximation theorem.

Next we show the implication $(\mathrm{C})_1$ + (M) $\Rightarrow$ (B.3). Let $f, g \in C(\mathbb{T}) \cap \mathcal{CL}_2$ and let $p, q \in \mathbb{Z}$ be such that $p \ne q$. By virtue of the Weierstrass approximation theorem we may assume the f and g to be real trigonometric polynomials. Letting

$$f(x) \;=\; \sum_{1\le|k|\le N} c_k e^{\sqrt{-1}\,2\pi kx}, \quad g(x) \;=\; \sum_{1\le|l|\le N} d_l e^{\sqrt{-1}\,2\pi lx},$$

where $\overline{c_k} = c_{-k}$ and $\overline{d_l} = d_{-l}$, we write

$$\frac{1}{N_m}\sum_{i=1}^{N_m} \mathbf{E}_i[f(\cdot + p\alpha_i)g(\cdot + q\alpha_i)]$$
$$= \sum_{1\le|k|\le N} c_k\overline{d_k}\frac{1}{N_m}\sum_{i=1}^{N_m} e^{\sqrt{-1}\,2\pi k(p-q)\alpha_i}$$
$$+ \sum_{\substack{1\le|k|,|l|\le N;\\ k\ne l}} c_k\overline{d_l}\frac{1}{N_m}\sum_{i=1}^{N_m} e^{\sqrt{-1}\,2\pi(kp-lq)\alpha_i}\widehat{\mathbf{P}_i}(l-k).$$

By (M), the first term $\to \sum_{1\le|k|\le N} c_k\overline{d_k}\widehat{\mu}(k(q-p)) = E[I^{(p)}(f;\mu)I^{(q)}(g;\mu)]$, and by $(\mathrm{C})_1$, the second term $\to 0$. Hence we have (B.3).

Finally we show the implication (B.3) + $(\mathrm{C})_1 \Rightarrow$ (M). Let $\{m'\}$ be any subsequence. Because of the compactness of $\mathbb{T}$ there exist a further subsequence $\{m''\}$ and a probability measure ν on $\mathbb{T}$ so that $\alpha \in \mathbb{T}^\infty_{\nu,\{N_{m''}\}}$. From what we have shown above and our assumption

$$\begin{aligned}&\sum_{|i|\ge1} \widehat{f}(i)\,\overline{\widehat{g}(i)}\,\widehat{\nu}(i(q-p)) \\ &\quad= \sum_{|i|\ge1} \widehat{f}(i)\,\overline{\widehat{g}(i)}\,\widehat{\mu}(i(q-p)), \quad {}^\forall f,g \in C(\mathbb{T})\cap\mathcal{CL}_2, {}^\forall p,q\in\mathbb{Z}.\end{aligned}$$

This easily implies $\nu=\mu$, so that we have $\alpha\in\mathbb{T}^\infty_{\mu,\{N_m\}}$, which is just (M). □

We present a sufficient condition for $\{\mathbf{P}_i\}_{i=1}^\infty$ to satisfy the conditions (B.1) and (B.2).

Theorem 1.2. *Suppose that a sequence $\{\mathbf{P}_i\}_{i=1}^\infty$ of probability measures on $\mathbb{T}$ satisfies the following: For some $\mathbb{N}_0\subset\mathbb{N}$,*

(i) $\mathbf{P}_i(dx) \ll dx$ $({}^\forall i\in\mathbb{N}_0)$ *and* $\prod_{i\in\mathbb{N}_0}\int_{\mathbb{T}}\sqrt{\frac{d\mathbf{P}_i}{dx}(x)}dx > 0$,

(ii) $\#(\mathbb{N}\setminus\mathbb{N}_0)\cap\{1,\ldots,N\} = O(\sqrt{N})$ *as* $N\to\infty$ *and*

$$\lim_{N\to\infty}\frac{1}{\sqrt{N}}\sum_{i\in(\mathbb{N}\setminus\mathbb{N}_0)\cap\{1,\ldots,N\}}|\widehat{\mathbf{P}_i}(n)| = 0, \quad {}^\forall n\ne0.$$

Then, for ${}^\forall\alpha\in\mathbb{T}^\infty$, (B.1) *and* (B.2) *hold with the whole sequence $\{1,2,3,\ldots\}$. Thus, the convergence of* (A) *holds under some / any absolutely continuous probability relative to $\boldsymbol{\mu}=\prod_{i=1}^\infty\mathbf{P}_i$, if and only if $\alpha\in\mathbb{T}^\infty_{\mu,\{N_m\}}$. In particular the disappearance of dependency happens as $N\to\infty$ in the sense that on some / any absolutely continuous probability space relative to $\boldsymbol{\mu}=\prod_{i=1}^\infty\mathbf{P}_i$*

$$\left(\frac{1}{\sqrt{N}}\sum_{i=1}^N f(x_i+p\alpha_i)\right)_{p\in\mathbb{Z},f\in C(\mathbb{T})\cap\mathcal{CL}_2}$$

$$\overset{\text{f.d.}}{\Longrightarrow} \quad \left(I^{(p)}(f)\right)_{p\in\mathbb{Z}, f\in C(\mathbb{T})\cap\mathcal{C}\mathcal{L}_2} \quad \text{as } N\to\infty,$$

if and only if $\alpha\in\mathbb{T}^\infty_{dx}$. *Here* $\left\{(B^{(p)}(t))_{0\le t\le 1}\right\}_{p\in\mathbb{Z}}$ *is a sequence of independent* 1*-dimensional Brownian motions starting at* 0*, and* $I^{(p)}(f)$ *is the Wiener integral of* $f\in\mathcal{L}_2$ *with respect to* $B^{(p)}$*, i.e.,* $I^{(p)}(f) = \int_0^1 f(s)dB^{(p)}(s)$.

PROOF. It suffices to check $(\mathrm{C})_2$ and (B.2) with $N_m = m$. First of all note that for a sequence $\{a_k\}_{k=1}^\infty$ with $0 < a_k \le 1$ $({}^\forall k)$

$$\prod_{k=1}^\infty a_k > 0 \quad \underset{\text{iff}}{\Longleftrightarrow} \quad \sum_{k=1}^\infty (1-a_k) < \infty.$$

Set $\rho_i(x) := \frac{d\mathbf{P}_i}{dx}(x)$, $i\in\mathbb{N}_0$. By the assumption (i) and the note above

$$\sum_{i\in\mathbb{N}_0}(1-\int_{\mathbb{T}}\sqrt{\rho_i(x)}\,dx) \quad < \quad \infty.$$

Also, by the fact $\int_{\mathbb{T}}\rho_i(x)dx = 1$

$$\begin{aligned}\int_{\mathbb{T}}|1-\sqrt{\rho_i(x)}|^2dx &\;=\; 2(1-\int_{\mathbb{T}}\sqrt{\rho_i(x)}\,dx),\\ \int_{\mathbb{T}}|1-\rho_i(x)|dx &\;\le\; 2\sqrt{\int_{\mathbb{T}}|1-\sqrt{\rho_i(x)}|^2dx}.\end{aligned}$$

Therefore

$$\sum_{i\in\mathbb{N}_0}\int_{\mathbb{T}}|1-\sqrt{\rho_i(x)}|^2dx \quad < \quad \infty, \tag{1.10}$$

$$\sum_{i\in\mathbb{N}_0}(\int_{\mathbb{T}}|1-\rho_i(x)|dx)^2 \quad < \quad \infty. \tag{1.11}$$

Now we show $(\mathrm{C})_2$. Let $n\in\mathbb{Z}\setminus\{0\}$. Then, by (1.11) and the assumption (ii) we have

$$\frac{1}{N}\sum_{i=1}^N|\widehat{\mathbf{P}_i}(n)|^2 \quad = \quad \frac{1}{N}\sum_{i\in\mathbb{N}_0\cap\{1,\dots,N\}}\left|\int_{\mathbb{T}}e^{-\sqrt{-1}\,2n\pi x}\rho_i(x)dx\right|^2$$

$$
\begin{aligned}
& + \tfrac{1}{N} \sum_{i \in (\mathbb{N} \setminus \mathbb{N}_0) \cap \{1,\ldots,N\}} \Big| \int_{\mathbb{T}} e^{-\sqrt{-1}\,2n\pi x} \mathbf{P}_i(dx) \Big|^2 \\
\leq \quad & \tfrac{1}{N} \sum_{i \in \mathbb{N}_0 \cap \{1,\ldots,N\}} \Big(\int_{\mathbb{T}} |1 - \rho_i(x)| dx \Big)^2 \\
& + \tfrac{1}{\sqrt{N}} \tfrac{1}{\sqrt{N}} \# (\mathbb{N} \setminus \mathbb{N}_0) \cap \{1, \ldots, N\} \\
\longrightarrow \quad & 0 \qquad \text{as } N \to \infty,
\end{aligned}
$$

which is just $(\mathrm{C})_2$.

Next we show (B.2). Let $f \in C(\mathbb{T}) \cap \mathcal{CL}_2$ and $p \in \mathbb{Z}$. We divide $\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \mathbf{E}_i[f(\cdot + p\alpha_i)]$ into two terms as

$$
\begin{aligned}
\tfrac{1}{\sqrt{N}} \sum_{i=1}^{N} \mathbf{E}_i[f(\cdot + p\alpha_i)] \quad = \quad & \tfrac{1}{\sqrt{N}} \sum_{i \in \mathbb{N}_0 \cap \{1,\ldots,N\}} \mathbf{E}_i[f(\cdot + p\alpha_i)] \\
& + \tfrac{1}{\sqrt{N}} \sum_{i \in (\mathbb{N} \setminus \mathbb{N}_0) \cap \{1,\ldots,N\}} \mathbf{E}_i[f(\cdot + p\alpha_i)].
\end{aligned}
$$

As for the first term, noting that

$$
\mathbf{E}_i[f(\cdot + p\alpha_i)] \quad = \quad \int_{\mathbb{T}} f(x + p\alpha_i)(\sqrt{\rho_i(x)} + 1)(\sqrt{\rho_i(x)} - 1) dx,
$$

we have by (1.10)

|the first term|

$$
\begin{aligned}
= \quad & \Big| \tfrac{1}{\sqrt{N}} \sum_{i \in \mathbb{N}_0 \cap \{1,\ldots,M\}} \mathbf{E}_i[f(\cdot + p\alpha_i)] \\
& + \tfrac{1}{\sqrt{N}} \sum_{i \in \mathbb{N}_0 \cap \{M+1,\ldots,N\}} \int_{\mathbb{T}} f(x + p\alpha_i)(\sqrt{\rho_i(x)} + 1)(\sqrt{\rho_i(x)} - 1) dx \Big| \\
\leq \quad & \tfrac{M}{\sqrt{N}} \|f\|_\infty + \tfrac{1}{\sqrt{N}} \Big(\sum_{i \in \mathbb{N}_0 \cap \{M+1,\ldots,N\}} \int_{\mathbb{T}} f(x + p\alpha_i)^2 (\sqrt{\rho_i(x)} + 1)^2 dx \Big)^{\frac{1}{2}} \\
& \times \Big(\sum_{i \in \mathbb{N}_0 \cap \{M+1,\ldots,N\}} \int_{\mathbb{T}} |1 - \sqrt{\rho_i(x)}|^2 dx \Big)^{\frac{1}{2}} \\
\underset{\substack{\text{first } N \to \infty \\ \text{second } M \to \infty}}{\longrightarrow} \quad & 0.
\end{aligned}
$$

As for the second term, for $^\forall\varepsilon > 0$ we take a trigonometric polynomial $P_\varepsilon(x) = \sum_{1\le|k|\le n} c_k e^{\sqrt{-1}\,2k\pi x}$ such that $\|f - P_\varepsilon\|_\infty < \varepsilon$. Then we have by the assumption (ii)

$$
\begin{aligned}
&|\text{the second term}|\\
&= \Bigg|\frac{1}{\sqrt{N}} \sum_{i\in(\mathbb{N}\setminus\mathbb{N}_0)\cap\{1,\dots,N\}} \int_{\mathbb{T}} (f - P_\varepsilon)(x + p\alpha_i)\mathbf{P}_i(dx)\\
&\quad + \sum_{1\le|k|\le n} c_k \frac{1}{\sqrt{N}} \sum_{i\in(\mathbb{N}\setminus\mathbb{N}_0)\cap\{1,\dots,N\}} e^{\sqrt{-1}\,2k\pi p\alpha_i} \int_{\mathbb{T}} e^{\sqrt{-1}\,2k\pi x}\mathbf{P}_i(dx)\Bigg|\\
&\le \varepsilon \frac{1}{\sqrt{N}} \#\,(\mathbb{N}\setminus\mathbb{N}_0)\cap\{1,\dots,N\}\\
&\quad + \sum_{1\le|k|\le n} |c_k| \frac{1}{\sqrt{N}} \sum_{i\in(\mathbb{N}\setminus\mathbb{N}_0)\cap\{1,\dots,N\}} |\widehat{\mathbf{P}_i}(-k)|\\
&\xrightarrow[\substack{\text{first } N\to\infty\\ \text{second } \varepsilon\to 0}]{} 0.
\end{aligned}
$$

These convergences clearly imply (B.2). □

2. Convergence to a System of Correlated Multiple Wiener Integrals

In this section we study the convergence of a system of symmetric statistics to that of correlated multiple Wiener integrals. As in the previous section let $(I^{(p)}(f;\mu))_{p\in\mathbb{Z}, f\in\mathcal{CL}_2}$ be a centered Gaussian system with covariance (1.1), and let us denote by $\boldsymbol{\mu}$ a product probability $\prod_{i=1}^{\infty} \mathbf{P}_i$.

We begin with the following definition:

DEFINITION 2.1. For $\kappa \in [0,\infty)$, we define a real Hilbert space $(H_\kappa, (\cdot,*)_{H_\kappa})$ by

$$
\begin{aligned}
H_\kappa &:= \left\{ (h_n)_{n=1}^{\infty};\ \begin{array}{c} h_n \in \mathcal{CSL}_2^n\ (n = 1,2,\dots), \\ \sum_{n=1}^{\infty} \Big(\frac{1}{n!} + \sum_{r=1}^{n} \frac{\kappa^r}{r!} \frac{1}{\sqrt{(n-r)!}}\Big) \|h_n\|^2_{\mathcal{L}_2^n} < \infty \end{array} \right\},\\
(h^{(1)}, h^{(2)})_{H_\kappa} &:= \sum_{n=1}^{\infty} \Big(\frac{1}{n!} + \sum_{r=1}^{n} \frac{\kappa^r}{r!} \frac{1}{\sqrt{(n-r)!}}\Big) (h_n^{(1)}, h_n^{(2)})_{\mathcal{L}_2^n}, \quad h^{(1)}, h^{(2)} \in H_\kappa.
\end{aligned}
$$

Clearly

$$\{h^f := (f^{\otimes n})_{n=1}^{\infty}; f \in \mathcal{CL}_2\} \quad \subset \quad H_\kappa,$$

and moreover the following holds (cf. Lemma 1 of [16]):

$$\text{c.l.s.}\{h^f\,; f \in \mathcal{CL}_2\} \quad = \quad H_\kappa. \tag{2.1}$$

Also it holds that (cf. Lemma 3 of [16])

$$\|h^\phi - h^\psi\|_{H_\kappa} \quad \leq \quad \text{const}(\kappa, \|\phi\| \vee \|\psi\|)\|\phi - \psi\| \tag{2.2}$$

for ${}^\forall\phi, \psi \in \mathcal{CL}_2$, where

$$\text{const}(\kappa, t) \quad = \quad \sqrt{\sum_{n=1}^{\infty}\Big(\tfrac{1}{n!} + \sum_{r=1}^{n}\tfrac{\kappa^r}{r!}\tfrac{1}{\sqrt{(n-r)!}}\Big)n^2 t^{2(n-1)}}.$$

Definition 2.2. We define symmetric statistics $\sigma_n^N(\cdot\,; h)$ $(h \in \mathcal{SL}_2^n)$ and $Y_N(\cdot\,; h)$ $(h = (h_n) \in H_0)$ by

$$\sigma_n^N(y; h) \quad := \quad \begin{cases} \displaystyle\sum_{1\leq i_1<\cdots<i_n\leq N} h(y_{i_1}, \ldots, y_{i_n}) & n \leq N \\ 0 & n > N, \end{cases}$$

$$Y_N(y; h) \quad := \quad \sum_{n=1}^{\infty}\Big(\tfrac{1}{\sqrt{N}}\Big)^n \sigma_n^N(y; h_n).$$

As a CONS of $\mathcal{CL}_2$, we take

$$\{\phi_k\}_{k=1}^{\infty} \quad = \quad \Big\{\sqrt{2}\cos 2\pi n x, \sqrt{2}\sin 2\pi m x\Big\}_{n,m\in\mathbb{N}}.$$

Claim 2.1. *For $\psi \in \mathcal{CL}_2$ and $R \in \mathbb{N}$, set $\psi^{(R)} := \sum_{k=1}^{R}(\psi, \phi_k)\phi_k$. Suppose that a sequence $\{\mathbf{P}_i\}_{i=1}^{\infty}$ satisfies the conditions* (i) *and* (ii) *in Theorem*

1.2. *Then, for* $^\forall\alpha \in \mathbb{T}^\infty_{\mu,\{N_m\}}$ *and* $^\forall\boldsymbol{\nu} \ll \boldsymbol{\mu}$

$$\Big(1 + Y_{N_m}(\cdot + p\alpha\,; h^{\psi^{(R)}})\Big)_{p\in\mathbb{Z},\psi\in\mathcal{CL}_2} \quad \textit{under } \boldsymbol{\nu}$$
$$\overset{\text{f.d.}}{\Longrightarrow} \quad \Big(e^{I^{(p)}(\psi^{(R)};\mu)-\frac{1}{2}\|\psi^{(R)}\|^2}\Big)_{p\in\mathbb{Z},\psi\in\mathcal{CL}_2} \quad \textit{as } m\to\infty.$$

PROOF. Let $\alpha \in \mathbb{T}^\infty_{\mu,\{N_m\}}$ and $\boldsymbol{\nu} \ll \boldsymbol{\mu}$. By Theorem 1.2

$$\Big(\tfrac{1}{\sqrt{N_m}}\sum_{i=1}^{N_m}\phi_k(x_i+p\alpha_i)\Big)_{p\in\mathbb{Z},k\in\mathbb{N}} \quad \text{under } \boldsymbol{\nu}$$
$$\overset{\text{f.d.}}{\Longrightarrow} \quad \Big(I^{(p)}(\phi_k;\mu)\Big)_{p\in\mathbb{Z},k\in\mathbb{N}} \quad \text{as } m\to\infty. \tag{2.3}$$

Also, by 1° in the proof of the implication "(A) $\Rightarrow$ (B)"

$$\lim_{m\to\infty}\tfrac{1}{N_m}\sum_{i=1}^{N_m}\phi_k(x_i+p\alpha_i)\phi_l(x_i+p\alpha_i) \;=\; \delta_{kl} \quad \text{in } \boldsymbol{\nu}. \tag{2.4}$$

For fixed $(c_1,\dots,c_R)\in\mathbb{R}^R$, set $z_{Ni} := \frac{1}{\sqrt{N}}\sum_{k=1}^R c_k\phi_k(x_i+p\alpha_i)$. Let $N \geq (2\sqrt{2}\sum_{k=1}^R |c_k|)^2$. Then $|z_{Ni}| \leq \frac{1}{2}$, and so, by (1.2)

$$\begin{aligned} 1+z_{Ni} &= e^{z_{Ni}-\frac{1}{2}z_{Ni}^2+r_{Ni}}, \\ |r_{Ni}| &= |r(z_{Ni})| \;\leq\; \tfrac{2}{3}|z_{Ni}|^3. \end{aligned}$$

This tells us that

$$\begin{aligned} &\left|\prod_{i=1}^N(1+z_{Ni}) - e^{\sum_{i=1}^N z_{Ni}-\frac{1}{2}\sum_{i=1}^N z_{Ni}^2}\right|^2 \\ &\quad\leq\; e^{\sum_{i=1}^N z_{Ni}-\frac{1}{2}\sum_{i=1}^N z_{Ni}^2}\Big(e^{\frac{2\sqrt{2}}{3}\frac{1}{\sqrt{N}}\sum_{k=1}^R|c_k|\sum_{i=1}^N z_{Ni}^2}-1\Big). \end{aligned}$$

By (2.3) and (2.4), the inequality above yields that

$$\left(\prod_{i=1}^{N_m}\Big(1+\tfrac{1}{\sqrt{N_m}}\sum_{k=1}^R c_k\phi_k(x_i+p\alpha_i)\Big)\right)_{p\in\mathbb{Z},(c_1,\dots,c_R)\in\mathbb{R}^R} \quad \text{under } \boldsymbol{\nu}$$

$$\overset{\text{f.d.}}{\Longrightarrow} \quad \left(e^{I^{(p)}\left(\sum_{k=1}^{R} c_k\phi_k;\mu\right)-\frac{1}{2}\|\sum_{k=1}^{R} c_k\phi_k\|^2} \right)_{p\in\mathbb{Z},(c_1,\ldots,c_R)\in\mathbb{R}^R} \qquad \text{as } m\to\infty.$$

But

$$1+Y_N(y\,;h^{\psi(R)}) \;=\; \prod_{i=1}^{N}\Big(1+\tfrac{1}{\sqrt{N}}\sum_{k=1}^{R}(\psi,\phi_k)\phi_k(y_i)\Big).$$

From this together with the above the assertion follows immediately. □

We state a main theorem in this section.

THEOREM 2.1. *Suppose that a sequence* $\{\mathbf{P}_i\}_{i=1}^{\infty}$ *satisfies the following:*

(a) $\mathbf{P}_i(dx) \ll dx$ $(^\forall i\in\mathbb{N})$.

(b) *There exists a subset* $\mathbb{N}_0$ *of* $\{1,2,3,\ldots\}$ *such that*

(b.i) $\prod_{i\in\mathbb{N}_0}\int_{\mathbb{T}}\sqrt{\frac{d\mathbf{P}_i}{dx}(x)}\,dx \,>\, 0,$

(b.ii) $\sup_{N\in\mathbb{N}}\frac{1}{\sqrt{N}}\,\#(\mathbb{N}\setminus\mathbb{N}_0)\cap\{1,\ldots,N\} \,=:\, C \,<\, \infty,$

(b.iii) $\lim_{N\to\infty}\frac{1}{\sqrt{N}}\sum_{i\in(\mathbb{N}\setminus\mathbb{N}_0)\cap\{1,\ldots,N\}}\left|\widehat{\frac{d\mathbf{P}_i}{dx}}(n)\right| \,=\, 0$ *for* $^\forall n\in\mathbb{Z}\setminus\{0\}$,

(b.iv) $\sup_{i\in\mathbb{N}\setminus\mathbb{N}_0}\sqrt{\int_{\mathbb{T}}(\frac{d\mathbf{P}_i}{dx}(x))^2dx} \,=:\, M \,<\, \infty.$

Then, for $^\forall\alpha\in\mathbb{T}^{\infty}_{\mu,\{N_m\}}$ *and* $^\forall\boldsymbol{\nu}\ll\boldsymbol{\mu}$

$$\Big(Y_{N_m}(\cdot+p\alpha;h)\Big)_{p\in\mathbb{Z},h=(h_n)\in H_{CM}} \qquad \textit{under } \boldsymbol{\nu}$$

$$\overset{\text{f.d.}}{\Longrightarrow} \quad \Big(\sum_{n=1}^{\infty}\tfrac{1}{n!}I_n^{(p)}(h_n;\mu)\Big)_{p\in\mathbb{Z},h=(h_n)\in H_{CM}} \qquad \textit{as } m\to\infty.$$

Here $I_n^{(p)}(h_n;\mu)$ *is an* n*-ple Wiener integral of* $h_n\in\mathcal{CSL}_2^n$ *with respect to* $I^{(p)}(\cdot\,;\mu)$.

COROLLARY 1. *For* $^\forall(h_n)_{n=1}^{\infty}$ *such that* $h_n\in\mathcal{CSL}_2^n$ $(n\geq 1)$

$$\Big(N_m^{-\frac{n}{2}}\sigma_n^{N_m}(\cdot+p\alpha;h_n)\Big)_{p\in\mathbb{Z},n\in\mathbb{N}} \;\overset{\text{f.d.}}{\Longrightarrow}\; \Big(\tfrac{1}{n!}I_n^{(p)}(h_n;\mu)\Big)_{p\in\mathbb{Z},n\in\mathbb{N}} \qquad \textit{as } m\to\infty.$$

REMARK 2.1. (i) The conditions on $\{\mathbf{P}_i\}_{i=1}^{\infty}$ in Theorem 2.1 are stronger than those in Theorem 1.2.
(ii) When $\mathbb{N}_0 = \mathbb{N}$, we can regard the conditions (b.ii), (b.iii) and (b.iv) as satisfied, and we understand as $C = M = 0$. In this case, by Kakutani dichotomy theorem, the condition (b.i) is equivalent to that $\prod_{i=1}^{\infty} \mathbf{P}_i \ll \mathbf{P}^{\infty}$.

For the proof of Theorem 2.1, we present two propositions:

PROPOSITION 1. (i) *For* $^{\forall} f \in \mathcal{CL}_2$

$$I_n^{(p)}(f^{\otimes n};\mu) \;=\; n!\|f\|^n H_n\Big(I^{(p)}(\tfrac{f}{\|f\|};\mu)\Big).$$

Here $(H_n)_{n=0}^{\infty}$ *are Hermite polynomials, i.e.,* $H_n(\xi) = \frac{(-1)^n}{n!} e^{\frac{\xi^2}{2}} \frac{d^n}{d\xi^n}\Big(e^{-\frac{\xi^2}{2}}\Big)$.
(ii) *For* $^{\forall} h \in \mathcal{CSL}_2^n$ *and* $^{\forall} k \in \mathcal{CSL}_2^m$

$$E\Big[I_n^{(p)}(h;\mu)\Big] \;=\; 0,$$

$$E\Big[I_n^{(p)}(h;\mu) I_m^{(p)}(k;\mu)\Big] \;=\; n!\,\delta_{nm}\,(h,k)_{\mathcal{L}_2^n}.$$

For the proof see [8].

PROPOSITION 2. *Suppose the conditions* (a) *and* (b) *in Theorem* 2.1. *Then there exists an increasing function* $\xi : [0,\infty) \to [0,\infty)$ *with* $\xi(0+) = \xi(0) = 0$ *such that*

$$\boldsymbol{\mu}\Big(|Y_N(\cdot + p\alpha\,; h)| > \eta\Big) \;\le\; \xi\Big(\frac{\|h\|_{H_{CM}}}{\eta}\Big)$$

for $^{\forall} N \in \mathbb{N}$, $^{\forall} p \in \mathbb{Z}$, $^{\forall} \alpha \in \mathbb{T}^{\infty}$, $^{\forall} h \in H_{CM}$ *and* $^{\forall} \eta > 0$. *Here* C *and* M *are constants in* (b.ii) *and* (b.iv), *respectively.*

PROOF. For simplicity set $\mathbb{N}_1 := \mathbb{N} \setminus \mathbb{N}_0$ and $\rho_i := \frac{d\mathbf{P}_i}{dx}$ $(i \in \mathbb{N})$. Let $N \in \mathbb{N}$ and $h = (h_n)_{n=1}^{\infty} \in H_{CM}$. We rewrite

$$Y_N(y;h) \;=\; \sum_{n=1}^{N} \Big(\frac{1}{\sqrt{N}}\Big)^n \frac{1}{n!} \sum_{\substack{i_1,\ldots,i_n \in \mathbb{N}_0 \cap \{1,\ldots,N\};\\ i_1,\ldots,i_n \text{ are distinct}}} h_n(y_{i_1},\ldots,y_{i_n})$$

$$+ \sum_{n=1}^{N} \Big(\tfrac{1}{\sqrt{N}}\Big)^n \sum_{r=1}^{n} \tfrac{1}{r!(n-r)!}$$
$$\times \sum_{\substack{i_1,\dots,i_{n-r} \in \mathbb{N}_0 \cap \{1,\dots,N\},\\ j_1,\dots,j_r \in \mathbb{N}_1 \cap \{1,\dots,N\};\\ i_1,\dots,i_{n-r} \text{ are distinct},\\ j_1,\dots,j_r \text{ are distinct}}} h_n(y_{i_1},\dots,y_{i_{n-r}},y_{j_1},\dots,y_{j_r}).$$

Let $\alpha \in \mathbb{T}^\infty$, $p \in \mathbb{N}$ and $\eta > 0$. Then, by Fubini's theorem

$$\begin{aligned}
&\boldsymbol{\mu}\Big(|Y_N(\cdot + p\alpha; h)| > \eta\Big)\\
&\le \Big(\prod_{k\in\mathbb{N}_0} \mathbf{P}_k\Big)\Bigg(\Big|\sum_{n=1}^{N}\Big(\tfrac{1}{\sqrt{N}}\Big)^n \tfrac{1}{n!} \sum_{\substack{i_1,\dots,i_n \in \mathbb{N}_0 \cap \{1,\dots,N\};\\ i_1,\dots,i_n \text{ are distinct}}}\\
&\qquad \times h_n(x_{i_1}+p\alpha_{i_1},\dots,x_{i_n}+p\alpha_{i_n})\Big| > \tfrac{\eta}{2}\Bigg)\\
&+ \mathbf{E}^{\prod_{k\in\mathbb{N}_0}\mathbf{P}_k}\Bigg[\Big(\prod_{k\in\mathbb{N}_1}\mathbf{P}_k\Big)\Bigg(\Big|\sum_{n=1}^{N}\Big(\tfrac{1}{\sqrt{N}}\Big)^n \sum_{r=1}^{n}\tfrac{1}{r!(n-r)!} \sum_{\substack{i_1,\dots,i_{n-r} \in \mathbb{N}_0 \cap \{1,\dots,N\},\\ j_1,\dots,j_r \in \mathbb{N}_1 \cap \{1,\dots,N\};\\ i_1,\dots,i_{n-r} \text{ are distinct},\\ j_1,\dots,j_r \text{ are distinct}}}\\
&\qquad \times h_n(x_{i_1}+p\alpha_{i_1},\dots,x_{i_{n-r}}+p\alpha_{i_{n-r}},\\
&\qquad\qquad x_{j_1}+p\alpha_{j_1},\dots,x_{j_r}+p\alpha_{j_r})\Big| > \tfrac{\eta}{2}\Bigg)\Bigg]\\
&=: \text{ the first term } + \text{ the second term.}
\end{aligned}$$

First we estimate the first term. By Chebyshev's inequality

$$\Big(\prod_{k\in\mathbb{N}_0} dx_k\Big)\Bigg(\Big|\sum_{n=1}^{N}\Big(\tfrac{1}{\sqrt{N}}\Big)^n \tfrac{1}{n!} \sum_{\substack{i_1,\dots,i_n \in \mathbb{N}_0 \cap \{1,\dots,N\};\\ i_1,\dots,i_n \text{ are distinct}}} \times h_n(x_{i_1}+p\alpha_{i_1},\dots,x_{i_n}+p\alpha_{i_n})\Big| > \tfrac{\eta}{2}\Bigg)$$

$$\leq \frac{2}{\eta}\left(\mathbf{E}^{\prod_{k\in\mathbb{N}_0} dx_k}\left[\sum_{1\leq n,m\leq N}\left(\frac{1}{\sqrt{N}}\right)^n\left(\frac{1}{\sqrt{N}}\right)^m \right.\right.$$
$$\times \frac{1}{n!}\frac{1}{m!}\sum_{\substack{i_1,\ldots,i_n\in\mathbb{N}_0\cap\{1,\ldots,N\};\\ i_1,\ldots,i_n \text{ are distinct}}}\ \sum_{\substack{j_1,\ldots,j_m\in\mathbb{N}_0\cap\{1,\ldots,N\};\\ j_1,\ldots,j_m \text{ are distinct}}}$$
$$\times h_n(x_{i_1}+p\alpha_{i_1},\ldots,x_{i_n}+p\alpha_{i_n})$$
$$\left.\left.\times h_m(x_{j_1}+p\alpha_{j_1},\ldots,x_{j_m}+p\alpha_{j_m})\right]\right)^{\frac{1}{2}}.$$

By the fact $\int_{\mathbb{T}} h_n(x_1,\ldots,x_{n-1},y)dy=0$, the right hand side equals

$$\frac{2}{\eta}\left(\sum_{n=1}^N\left(\frac{1}{N}\right)^n\sum_{\substack{i_1,\ldots,i_n\in\mathbb{N}_0\cap\{1,\ldots,N\};\\ i_1,\ldots,i_n \text{ are distinct}}}\frac{1}{n!}\|h_n\|^2\right)^{\frac{1}{2}},$$

and this is dominated by $\frac{2}{\eta}\Big(\sum_{n=1}^N \frac{1}{n!}\|h_n\|^2\Big)^{\frac{1}{2}} \leq \frac{2}{\eta}\|h\|_{H_{CM}}$. Hence it turns out that

$$\text{(2.5)}\quad \text{the first term} \leq \sup\left\{\Big(\prod_{k\in\mathbb{N}_0}\mathbf{P}_k\Big)(A);\Big(\prod_{k\in\mathbb{N}_0}dx_k\Big)(A)\leq\frac{2}{\eta}\|h\|_{H_{CM}}\right\}.$$

Next we estimate the second term. To do so, we temporarily denote by Φ the integrand in expectation $\mathbf{E}^{\prod_{k\in\mathbb{N}_0}\mathbf{P}_k}$. Since $0\leq\Phi\leq1$,

$$\text{(2.6)}\qquad \text{the second term} \leq \delta + \Big(\prod_{k\in\mathbb{N}_0}\mathbf{P}_k\Big)(\Phi>\delta)$$

for $^\forall\delta>0$. In the following we treat the right hand side.

By Chebyshev's inequality and then by Schwarz inequality

$$\Phi \leq \frac{2}{\eta}\sum_{n=1}^N\left(\frac{1}{\sqrt{N}}\right)^n\sum_{r=1}^n\frac{1}{r!(n-r)!}\sum_{\substack{j_1,\ldots,j_r\in\mathbb{N}_1\cap\{1,\ldots,N\};\\ j_1,\ldots,j_r \text{ are distinct}}}$$
$$\times\int_{\mathbb{T}^r}\Bigg|\sum_{\substack{i_1,\ldots,i_{n-r}\in\mathbb{N}_0\cap\{1,\ldots,N\};\\ i_1,\ldots,i_{n-r} \text{ are distinct}}} h_n(x_{i_1}+p\alpha_{i_1},\ldots,x_{i_{n-r}}+p\alpha_{i_{n-r}},$$
$$x_{j_1}+p\alpha_{j_1},\ldots,x_{j_r}+p\alpha_{j_r})\Bigg|$$

$$
\begin{aligned}
& \qquad\qquad\qquad \times \rho_{j_1}(x_{j_1}) \times \cdots \times \rho_{j_r}(x_{j_r}) dx_{j_1} \cdots dx_{j_r} \\
\le\ & \tfrac{2}{\eta} \sum_{n=1}^{N} \Big(\tfrac{1}{\sqrt{N}} \Big)^n \sum_{r=1}^{n} \tfrac{1}{r!(n-r)!} \sum_{\substack{j_1,\dots,j_r \in \mathbb{N}_1 \cap \{1,\dots,N\}; \\ j_1,\dots,j_r \text{ are distinct}}} \\
& \quad \times \Bigg\| \sum_{\substack{i_1,\dots,i_{n-r} \in \mathbb{N}_0 \cap \{1,\dots,N\}; \\ i_1,\dots,i_{n-r} \text{ are distinct}}} h_n(x_{i_1} + p\alpha_{i_1}, \dots, x_{i_{n-r}} + p\alpha_{i_{n-r}}, \cdot) \Bigg\|_{\mathcal{L}_2^r} \\
& \quad \times \|\rho_{j_1}\| \times \cdots \times \|\rho_{j_r}\| \\
\le\ & \tfrac{2}{\eta} \sum_{n=1}^{N} \sum_{r=1}^{n} \tfrac{(CM)^r}{r!(n-r)!} \Big(\tfrac{1}{\sqrt{N}} \Big)^{n-r} \\
& \quad \times \Bigg\| \sum_{\substack{i_1,\dots,i_{n-r} \in \mathbb{N}_0 \cap \{1,\dots,N\}; \\ i_1,\dots,i_{n-r} \text{ are distinct}}} h_n(x_{i_1} + p\alpha_{i_1}, \dots, x_{i_{n-r}} + p\alpha_{i_{n-r}}, \cdot) \Bigg\|_{\mathcal{L}_2^r} .
\end{aligned}
$$

Here we have used the conditions (b.ii) and (b.iv) in the last line. By this estimate, and then by Chebyshev's inequality and Schwarz one again

$$
\begin{aligned}
& \Big(\prod_{k \in \mathbb{N}_0} dx_k \Big) (\Phi > \delta) \\
\le\ & \tfrac{1}{\delta} \tfrac{2}{\eta} \sum_{n=1}^{N} \sum_{r=1}^{n} \tfrac{(CM)^r}{r!(n-r)!} \Big(\tfrac{1}{\sqrt{N}} \Big)^{n-r} \\
& \quad \times \Bigg(\mathbf{E}^{\prod_{k \in \mathbb{N}_0} dx_k} \Bigg[\int_{\mathbb{T}^r} \Bigg| \sum_{\substack{i_1,\dots,i_{n-r} \in \mathbb{N}_0 \cap \{1,\dots,N\}; \\ i_1,\dots,i_{n-r} \text{ are distinct}}} \\
& \qquad\qquad \times\ h_n(x_{i_1} + p\alpha_{i_1}, \dots, x_{i_{n-r}} + p\alpha_{i_{n-r}}, \\
& \qquad\qquad\qquad y_1, \dots, y_r) \Bigg|^2 dy_1 \cdots dy_r \Bigg] \Bigg)^{\frac{1}{2}} \\
=\ & \tfrac{1}{\delta} \tfrac{2}{\eta} \sum_{n=1}^{N} \sum_{r=1}^{n} \tfrac{(CM)^r}{r!(n-r)!} \Big(\tfrac{1}{\sqrt{N}} \Big)^{n-r} \Bigg(\sum_{\substack{i_1,\dots,i_{n-r} \in \mathbb{N}_0 \cap \{1,\dots,N\}; \\ i_1,\dots,i_{n-r} \text{ are distinct}}} (n-r)! \|h_n\|^2 \Bigg)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\le \tfrac{1}{\delta}\tfrac{2}{\eta}\sum_{n=1}^{N}\sum_{r=1}^{n}\tfrac{(CM)^r}{r!}\tfrac{\|h_n\|}{\sqrt{(n-r)!}}\\
&\le \tfrac{1}{\delta}\tfrac{2}{\eta}(e^{CM}-1)\sum_{m=0}^{\infty}\tfrac{1}{\sqrt{m!}}\,\|h\|_{H_{CM}}.
\end{aligned}
$$

Hence, combining this with (2.6) yields that

$$
\begin{aligned}
&\text{the second term}\\
&\le \delta+\sup\Big\{\Big(\prod_{k\in\mathbb{N}_0}\mathbf{P}_k\Big)(A);\Big(\prod_{k\in\mathbb{N}_0}dx_k\Big)(A)\\
&\qquad\qquad \le \tfrac{1}{\delta}\tfrac{2}{\eta}(e^{CM}-1)\sum_{m=0}^{\infty}\tfrac{1}{\sqrt{m!}}\,\|h\|_{H_{CM}}\Big\}.
\end{aligned}\tag{2.7}
$$

Now, letting $\delta=(\frac{2}{\eta}\|h\|_{H_{CM}})^{\frac{1}{2}}$ in (2.7), and then putting (2.5) and (2.7) we have

$$
\boldsymbol{\mu}\Big(|Y_N(\cdot+p\alpha;h)|>\eta\Big)\ \le\ \xi\Big(\frac{\|h\|_{H_{CM}}}{\eta}\Big),
$$

where

$$
\begin{aligned}
\xi(t)\ =\ &\sup\Big\{\Big(\prod_{k\in\mathbb{N}_0}\mathbf{P}_k\Big)(A);\Big(\prod_{k\in\mathbb{N}_0}dx_k\Big)(A)\le 2t\Big\}\\
&+\sqrt{2t}+\sup\Big\{\Big(\prod_{k\in\mathbb{N}_0}\mathbf{P}_k\Big)(A);\Big(\prod_{k\in\mathbb{N}_0}dx_k\Big)(A)\\
&\qquad\qquad \le (e^{CM}-1)\sum_{m=0}^{\infty}\tfrac{1}{\sqrt{m!}}\sqrt{2t}\Big\}.
\end{aligned}
$$

Since our condition (b.i) is equivalent to that $\prod_{k\in\mathbb{N}_0}\mathbf{P}_k \ll \prod_{k\in\mathbb{N}_0}dx_k$ by Kakutani dichotomy theorem, ξ satisfies $\xi(0+)=\xi(0)=0$, and thus this is the desired function. The proof is complete. □

PROOF OF THEOREM 2.1. Let $\alpha\in\mathbb{T}^{\infty}_{\mu,\{N_m\}}$ and $\boldsymbol{\nu}\ll\boldsymbol{\mu}$. Set $Z:=\frac{d\boldsymbol{\nu}}{d\boldsymbol{\mu}}$. Note that $Z\ge 0$ and $\mathbf{E}^{\boldsymbol{\mu}}[Z]=1$. The proof is done in two steps.

1° $\Big(1+Y_{N_m}(\cdot+p\alpha\,;h^{\psi})\Big)_{p\in\mathbb{Z},\psi\in\mathcal{CL}_2}$ *under* $\boldsymbol{\nu}$

$$
\overset{\text{f.d.}}{\Longrightarrow}\ \Big(e^{I^{(p)}(\psi;\mu)-\frac{1}{2}\|\psi\|^2}\Big)_{p\in\mathbb{Z},\psi\in\mathcal{CL}_2}\quad \textit{as } m\to\infty.
$$

PROOF. By Proposition 2 and (2.2),

$$\boldsymbol{\mu}\Big(|Y_N(\cdot+p\alpha\,;h^{\psi})-Y_N(\cdot+p\alpha\,;h^{\psi^{(R)}})|>\eta\Big)$$

$$\leq\ \xi\left(\mathrm{const}(CM,\|\psi\|)\frac{1}{\eta}\sqrt{\sum_{k=R+1}^{\infty}(\psi,\phi_k)^2}\right). \tag{2.8}$$

Clearly

$$E\Big[|I^{(p)}(\psi;\mu)-I^{(p)}(\psi^{(R)};\mu)|^2\Big]\ =\ \sum_{k=R+1}^{\infty}(\psi,\phi_k)^2. \tag{2.9}$$

Let $p_0\in\mathbb{Z}$, $\psi_1,\ldots,\psi_L\in\mathcal{CL}_2$ and $(a_{pl})_{\substack{1\leq p\leq K\\ 1\leq l\leq L}}\in\mathbb{R}^{KL}$. For ${}^{\forall}\varepsilon>0$

$$\begin{aligned}
&\Big|\mathbf{E}^{\boldsymbol{\nu}}\Big[e^{\sqrt{-1}\sum_{p=1}^{K}\sum_{l=1}^{L}a_{pl}(1+Y_{N_m}(\cdot+(p+p_0)\alpha;h^{\psi_l}))}\Big]\\
&\qquad -E\Big[e^{\sqrt{-1}\sum_{p=1}^{K}\sum_{l=1}^{L}a_{pl}\,e^{I^{(p+p_0)}(\psi_l;\mu)-\frac12\|\psi_l\|^2}}\Big]\Big|\\
\leq\ &\Big|\mathbf{E}^{\boldsymbol{\mu}}\Big[Z\Big(e^{\sqrt{-1}\sum_{p=1}^{K}\sum_{l=1}^{L}a_{pl}(1+Y_{N_m}(\cdot+(p+p_0)\alpha;h^{\psi_l}))}\\
&\qquad -e^{\sqrt{-1}\sum_{p=1}^{K}\sum_{l=1}^{L}a_{pl}(1+Y_{N_m}(\cdot+(p+p_0)\alpha;h^{\psi_l^{(R)}}))}\Big)\Big]\Big|\\
&+\Big|\mathbf{E}^{\boldsymbol{\nu}}\Big[e^{\sqrt{-1}\sum_{p=1}^{K}\sum_{l=1}^{L}a_{pl}(1+Y_{N_m}(\cdot+(p+p_0)\alpha;h^{\psi_l^{(R)}}))}\Big]\\
&\qquad -E\Big[e^{\sqrt{-1}\sum_{p=1}^{K}\sum_{l=1}^{L}a_{pl}\,e^{I^{(p+p_0)}(\psi_l^{(R)};\mu)-\frac12\|\psi_l^{(R)}\|^2}}\Big]\Big|\\
&+\Big|E\Big[e^{\sqrt{-1}\sum_{p=1}^{K}\sum_{l=1}^{L}a_{pl}\,e^{I^{(p+p_0)}(\psi_l^{(R)};\mu)-\frac12\|\psi_l^{(R)}\|^2}}\\
&\qquad -e^{\sqrt{-1}\sum_{p=1}^{K}\sum_{l=1}^{L}a_{pl}\,e^{I^{(p+p_0)}(\psi_l;\mu)-\frac12\|\psi_l\|^2}}\Big]\Big|\\
\leq\ &2\sup_{\boldsymbol{\mu}(B)\leq\delta_R(\varepsilon)}\mathbf{E}^{\boldsymbol{\mu}}[Z\,;B]+\varepsilon\\
&+\Big|\mathbf{E}^{\boldsymbol{\nu}}\Big[e^{\sqrt{-1}\sum_{p=1}^{K}\sum_{l=1}^{L}a_{pl}(1+Y_{N_m}(\cdot+(p+p_0)\alpha;h^{\psi_l^{(R)}}))}\Big]\\
&\qquad -E\Big[e^{\sqrt{-1}\sum_{p=1}^{K}\sum_{l=1}^{L}a_{pl}\,e^{I^{(p+p_0)}(\psi_l^{(R)};\mu)-\frac12\|\psi_l^{(R)}\|^2}}\Big]\Big|
\end{aligned}$$

$$+\left|E\left[e^{\sqrt{-1}\sum_{p=1}^{K}\sum_{l=1}^{L}a_{pl}\,e^{I^{(p+p_0)}(\psi_l^{(R)};\mu)-\frac{1}{2}\|\psi_l^{(R)}\|^2}}-e^{\sqrt{-1}\sum_{p=1}^{K}\sum_{l=1}^{L}a_{pl}\,e^{I^{(p+p_0)}(\psi_l;\mu)-\frac{1}{2}\|\psi_l\|^2}}\right]\right|.$$

Here $\delta_R(\varepsilon)$ in the first term is

$$\sup_{N\in\mathbb{N}}\boldsymbol{\mu}\Big(\big|\sum_{\substack{1\le p\le K\\1\le l\le L}}a_{pl}(Y_N(\cdot+(p+p_0)\alpha\,;h^{\psi_l})-Y_N(\cdot+(p+p_0)\alpha\,;h^{\psi_l^{(R)}}))\big|>\varepsilon\Big).$$

By (2.8) this converges to 0 as $R\to\infty$. Also, by (2.9) the fourth term $\to 0$ as $R\to\infty$, and by Claim 2.1 the third term $\to 0$ as $m\to\infty$. Hence we have

$$\limsup_{m\to\infty}\left|\mathbf{E}^{\boldsymbol{\nu}}\left[e^{\sqrt{-1}\sum_{p=1}^{K}\sum_{l=1}^{L}a_{pl}(1+Y_{N_m}(\cdot+(p+p_0)\alpha;h^{\psi_l}))}\right]-E\left[e^{\sqrt{-1}\sum_{p=1}^{K}\sum_{l=1}^{L}a_{pl}\,e^{I^{(p+p_0)}(\psi_l;\mu)-\frac{1}{2}\|\psi_l\|^2}}\right]\right|\quad\le\quad\varepsilon\quad\xrightarrow[\varepsilon\to0]{}\quad0,$$

which implies the assertion. □

2° Take $h^{(1)},\dots,h^{(L)}\in H_{CM}$ arbitrarily. By virtue of (2.1), for $^\forall\varepsilon>0$ and $1\le{}^\forall l\le L$ there exist $\psi_{l1},\dots,\psi_{lk_l}\in\mathcal{CL}_2$ and $t_{l1},\dots,t_{lk_l}\in\mathbb{R}$ such that $\|h^{(l)}-\sum_{i=1}^{k_l}t_{li}h^{\psi_{li}}\|_{H_{CM}}<\varepsilon$. By Propositions 2 and 1(ii)

$$\boldsymbol{\mu}\Big(|Y_N(\cdot+p\alpha\,;h^{(l)})-\sum_{i=1}^{k_l}t_{li}Y_N(\cdot+p\alpha\,;h^{\psi_{li}})|>\sqrt{\varepsilon}\Big)\quad\le\quad\xi(\sqrt{\varepsilon}),$$

$$E\Big[\big|\sum_{n=1}^{\infty}\tfrac{1}{n!}I_n^{(p)}\Big(\sum_{i=1}^{k_l}t_{li}\psi_{li}^{\otimes n};\mu\Big)-\sum_{n=1}^{\infty}\tfrac{1}{n!}I_n^{(p)}(h_n^{(l)};\mu)\big|^2\Big]\quad<\quad\varepsilon^2.$$

On the other hand, by Proposition 1(i)

$$e^{I^{(p)}(\psi;\mu)-\frac{1}{2}\|\psi\|^2}\quad=\quad1+\sum_{n=1}^{\infty}\tfrac{1}{n!}I_n^{(p)}(\psi^{\otimes n};\mu),\qquad{}^\forall\psi\in\mathcal{CL}_2,$$

and hence

$$\sum_{i=1}^{k_l}t_{li}\Big(e^{I^{(p)}(\psi_{li};\mu)-\frac{1}{2}\|\psi_{li}\|^2}-1\Big)\quad=\quad\sum_{n=1}^{\infty}\tfrac{1}{n!}I_n^{(p)}\Big(\sum_{i=1}^{k_l}t_{li}\psi_{li}^{\otimes n};\mu\Big).$$

Therefore, collecting these we have that for $^\forall(a_{pl})_{\substack{1\le p\le K\\1\le l\le L}} \in \mathbb{R}^{KL}$

$$
\begin{aligned}
&\Big|\mathbf{E}^{\boldsymbol{\nu}}\left[e^{\sqrt{-1}\sum_{p=1}^{K}\sum_{l=1}^{L} a_{pl}Y_{N_m}(\cdot+(p+p_0)\alpha;h^{(l)})}\right] \\
&\qquad - E\Big[e^{\sqrt{-1}\sum_{p=1}^{K}\sum_{l=1}^{L} a_{pl}\sum_{n=1}^{\infty}\frac{1}{n!}I_n^{(p+p_0)}(h_n^{(l)};\mu)}\Big]\Big| \\
\le\ & \Big|\mathbf{E}^{\boldsymbol{\mu}}\Big[Z\Big(e^{\sqrt{-1}\sum_{p=1}^{K}\sum_{l=1}^{L} a_{pl}Y_{N_m}(\cdot+(p+p_0)\alpha;h^{(l)})} \\
&\qquad - e^{\sqrt{-1}\sum_{p=1}^{K}\sum_{l=1}^{L} a_{pl}\sum_{i=1}^{k_l} t_{li}Y_{N_m}(\cdot+(p+p_0)\alpha;h^{\psi_{li}})}\Big)\Big]\Big| \\
&+ \Big|\mathbf{E}^{\boldsymbol{\nu}}\left[e^{\sqrt{-1}\sum_{p=1}^{K}\sum_{l=1}^{L}\sum_{i=1}^{k_l} a_{pl}t_{li}Y_{N_m}(\cdot+(p+p_0)\alpha;h^{\psi_{li}})}\right] \\
&\qquad - E\Big[e^{\sqrt{-1}\sum_{p=1}^{K}\sum_{l=1}^{L}\sum_{i=1}^{k_l} a_{pl}t_{li}\,(e^{I^{(p+p_0)}(\psi_{li};\mu)-\frac{1}{2}\|\psi_{li}\|^2}-1)}\Big]\Big| \\
&+ \Big|E\Big[e^{\sqrt{-1}\sum_{p=1}^{K}\sum_{l=1}^{L} a_{pl}\sum_{n=1}^{\infty}\frac{1}{n!}I_n^{(p+p_0)}(\sum_{i=1}^{k_l} t_{li}\psi_{li}^{\otimes n};\mu)} \\
&\qquad - e^{\sqrt{-1}\sum_{p=1}^{K}\sum_{l=1}^{L} a_{pl}\sum_{n=1}^{\infty}\frac{1}{n!}I_n^{(p+p_0)}(h_n^{(l)};\mu)}\Big]\Big| \\
\le\ & 2\sup_{\boldsymbol{\mu}(B)\le KL\xi(\sqrt{\varepsilon})}\mathbf{E}^{\boldsymbol{\mu}}[Z\,;B] + \max_{\substack{1\le p\le K\\1\le l\le L}}|a_{pl}|\,KL\sqrt{\varepsilon} \\
&+ \Big|\mathbf{E}^{\boldsymbol{\nu}}\left[e^{\sqrt{-1}\sum_{p=1}^{K}\sum_{l=1}^{L}\sum_{i=1}^{k_l} a_{pl}t_{li}Y_{N_m}(\cdot+(p+p_0)\alpha;h^{\psi_{li}})}\right] \\
&\qquad - E\Big[e^{\sqrt{-1}\sum_{p=1}^{K}\sum_{l=1}^{L}\sum_{i=1}^{k_l} a_{pl}t_{li}\,(e^{I^{(p+p_0)}(\psi_{li};\mu)-\frac{1}{2}\|\psi_{li}\|^2}-1)}\Big]\Big| \\
&+ \sum_{\substack{1\le p\le K\\1\le l\le L}}|a_{pl}|\varepsilon \\
\underset{\substack{\text{first } m\to\infty\\ \text{second } \varepsilon\to 0}}{\longrightarrow}\ & 0,
\end{aligned}
$$

which implies the conclusion of the theorem. □

3. Central Limit Theorem for Correlated Multiple Wiener Integrals

In this section we study the CLT for a sequence $\{I_n^{(p)}(\cdot;\mu)\}_{p=1}^{\infty}$.

We begin with the following. The system $\left(I^{(p)}(f)\right)_{p\in\mathbb{Z}, f\in\mathcal{CL}_2}$ in Theorem 1.2 is nondegenerate in the sense that for $^\forall$linearly independent $f_1,\dots,f_L \in$

$\mathcal{CL}_2$, ${}^\forall p_0 \in \mathbb{Z}$ and ${}^\forall K \in \mathbb{N}$, a random vector $\Big(I^{(p+p_0)}(f_i)\Big)_{1\le p\le K, 1\le i\le L}$ is nondegenerate Gaussian. This nondegeneracy is completely determined for general Gaussian system $\Big(I^{(p)}(f;\mu)\Big)_{p\in\mathbb{Z}, f\in\mathcal{CL}_2}$:

THEOREM 3.1. *The Gaussian system* $\Big(I^{(p)}(f;\mu)\Big)_{p\in\mathbb{Z}, f\in\mathcal{CL}_2}$ *is nondegenerate in the sense above, i.e., it holds that for* ${}^\forall$*linearly independent* $f_1,\dots,f_L \in \mathcal{CL}_2$, ${}^\forall p_0 \in \mathbb{Z}$ *and* ${}^\forall K \in \mathbb{N}$, *a random vector* $\Big(I^{(p+p_0)}(f_i;\mu)\Big)_{1\le p\le K, 1\le i\le L}$ *is nondegenerate Gaussian, if and only if* $\#\operatorname{supp}\mu = \infty$.

PROOF. First as for the " if " part. We suppose $\#\operatorname{supp}\mu = \infty$ and show the nondegeneracy of $\Big(I^{(p+p_0)}(f_i;\mu)\Big)_{1\le p\le K, 1\le i\le L}$ where $f_1,\dots,f_L \in \mathcal{CL}_2$ are linearly independent, $p_0 \in \mathbb{Z}$ and $K \in \mathbb{N}$. For this let $(\xi_{pi})_{1\le p\le K, 1\le i\le L} \in \mathbb{R}^{KL}$ be such that $\sum_{p=1}^{K}\sum_{i=1}^{L}\xi_{pi}I^{(p+p_0)}(f_i;\mu) = 0$. Then

$$\begin{aligned} 0 &= E\Big[\Big(\sum_{p=1}^{K}\sum_{i=1}^{L}\xi_{pi}I^{(p+p_0)}(f_i;\mu)\Big)^2\Big] \\ &= \sum_{|n|\ge 1}\int_{\mathbb{T}}\Big|\sum_{p=1}^{K}\sum_{i=1}^{L}\xi_{pi}\widehat{f_i}(n)e^{\sqrt{-1}\,2\pi npx}\Big|^2\mu(dx), \end{aligned}$$

and hence

$$\mu\Big(\{x\in\mathbb{T}; e^{\sqrt{-1}\,2\pi nx}\in Z(n)\}\Big) = 1, \qquad {}^\forall n\ge 1,$$

where

$$Z(n) := \Big\{z\in\mathbb{C}\,;\,\sum_{p=1}^{K}\Big(\sum_{i=1}^{L}\xi_{pi}\widehat{f_i}(n)\Big)z^p = 0\Big\}.$$

Since $\{x\in\mathbb{T}; e^{\sqrt{-1}\,2\pi nx}\in Z(n)\}$ is closed in $\mathbb{T}$, this implies

$$\operatorname{supp}\mu \subset \{x\in\mathbb{T}; e^{\sqrt{-1}\,2\pi nx}\in Z(n)\},$$

so from the assumption

$$\#\{x \in \mathbb{T}; e^{\sqrt{-1}\,2\pi n x} \in Z(n)\} \quad = \quad \infty.$$

But, by the fundamental theorem of algebra we observe that $\Big(\sum_{i=1}^{L} \xi_{pi} \widehat{f}_i(n)\Big)_{1\le p\le K} \neq 0$ implies $\# Z(n) \le K$. Consequently $\Big(\sum_{i=1}^{L} \xi_{pi} \widehat{f}_i(n)\Big)_{1\le p\le K} = 0$ $({}^\forall n \ge 1)$, i.e., $\sum_{i=1}^{L} \xi_{pi} f_i = 0$ $(1 \le {}^\forall p \le K)$. By the linear independence of $f_1, \ldots, f_L$, this implies $(\xi_{pi})_{1\le p\le K, 1\le i\le L} = 0$, and the nondegeneracy of $\Big(I^{(p+p_0)}(f_i;\mu)\Big)_{1\le p\le K, 1\le i\le L}$ is obtained.

Next as for the " only if " part. We suppose $\#\operatorname{supp}\mu < \infty$. Then there exist $0 \le \theta_1 < \cdots < \theta_m < 1$ and $a_1, \ldots, a_m > 0$ such that $\sum_{i=1}^{m} a_i = 1$ and $\mu = \sum_{i=1}^{m} a_i \delta_{\theta_i}$. For $N \in \mathbb{N}$ we define $(\beta_0, \ldots, \beta_{2Nm}) \in \mathbb{R}^{2Nm+1}$ as

$$\begin{aligned}\sum_{p=0}^{2Nm} \beta_p z^p &= \prod_{n=1}^{N}\prod_{i=1}^{m} (z - e^{\sqrt{-1}\,2\pi n\theta_i})(z - e^{-\sqrt{-1}\,2\pi n\theta_i}) \\ &= \prod_{n=1}^{N}\prod_{i=1}^{m} (z^2 - 2(\cos 2\pi n\theta_i) z + 1).\end{aligned}$$

By definition, for $f \in \mathcal{CL}_2$ such that $\widehat{f}(n) = 0$ $({}^\forall |n| \ge N+1)$

$$\begin{aligned}&E\Big[\Big(\sum_{p=1}^{2Nm+1} \beta_{p-1} I^{(p+p_0)}(f;\mu)\Big)^2\Big] \\ &= \sum_{1\le |n|\le N} |\widehat{f}(n)|^2 \sum_{i=1}^{m} a_i \Big|\sum_{p=0}^{2Nm} \beta_p (e^{\sqrt{-1}\,2\pi n\theta_i})^p\Big|^2 \\ &= 0,\end{aligned}$$

and also $(\beta_0, \ldots, \beta_{2Nm}) \neq 0$ because $\beta_0 = \beta_{2Nm} = 1$. These say the degeneracy of $\Big(I^{(p+p_0)}(f;\mu)\Big)_{1\le p\le 2Nm+1}$, so that $\#\operatorname{supp}\mu = \infty$ is necessary. $\Box$

For $h_k \in \mathcal{CSL}_2^k$ $(1 \le k \le L)$, $\left\{(I_1^{(p)}(h_1), \ldots, I_L^{(p)}(h_L))\right\}_{p\in\mathbb{Z}}$ is a sequence of i.i.d. random vectors with mean zero and finite covariance. The Lindeberg CLT says

$$\frac{1}{\sqrt{P}}\sum_{p=1}^{P}(I_1^{(p)}(h_1), \ldots, I_L^{(p)}(h_L)) \underset{P\to\infty}{\Longrightarrow} \mathfrak{N}\left(0, \begin{bmatrix} 1!\|h_1\|^2 & & 0 \\ & \ddots & \\ 0 & & L!\|h_L\|^2 \end{bmatrix}\right).$$

For a sequence $\{I_n^{(p)}(\cdot;\mu)\}_{p=1}^{\infty}$ of general multiple Wiener integrals this is also valid when μ is a " good " probability measure on $\mathbb{T}$. To see this, in the following we suppose that $\mu(dx) \ll dx$, i.e., $\mu(dx)$ is absolutely continuous relative to dx, and let $m := \frac{d\mu}{dx}$.

PROPOSITION 3. *For* $^\forall h_n \in \mathcal{CSL}_2^n$, $^\forall k_l \in \mathcal{CSL}_2^l$ *and* $^\forall p, q \in \mathbb{Z}$

$$\begin{aligned} &E\left[I_n^{(p)}(h_n;\mu)I_l^{(q)}(k_l;\mu)\right] \\ &= \begin{cases} n! \displaystyle\sum_{|i_1|,\ldots,|i_n|\ge 1} \widehat{h_n}(i_1,\ldots,i_n)\overline{\widehat{k_n}(i_1,\ldots,i_n)} \\ \qquad \times\, \widehat{\mu}(i_1(q-p))\cdots\widehat{\mu}(i_n(q-p)) \\ \hfill \text{if } n = l, \\ 0 \hfill \text{if } n \ne l. \end{cases} \end{aligned}$$

Here

$$\widehat{h_n}(i_1,\ldots,i_n) := \int_{\mathbb{T}^n} e^{-\sqrt{-1}\,2\pi(i_1x_1+\cdots+i_nx_n)}h_n(x_1,\ldots,x_n)dx_1\cdots dx_n.$$

PROOF. The proof is done in three steps.

1° *For* $^\forall p, q \in \mathbb{Z}$, $^\forall n, l \in \mathbb{N}$ *and* $^\forall f, g \in \mathcal{CL}_2$ *such that* $\|f\| = \|g\| = 1$

$$E\left[H_n(I^{(p)}(f;\mu))H_l(I^{(q)}(g;\mu))\right] = \delta_{nl}\frac{1}{n!}\left(E\left[I^{(p)}(f;\mu)I^{(q)}(g;\mu)\right]\right)^n.$$

PROOF. In case $p = q$ and $|(f,g)| = 1$ (i.e., $f = \pm g$), we have

$$\begin{aligned} \text{the left hand side} &= (\pm 1)^n \int_{\mathbb{R}} H_n(x)H_l(x)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \\ &= (\pm 1)^n \delta_{nl}\frac{1}{n!} = \text{the right hand side}. \end{aligned}$$

In case $p \neq q$ or $|(f,g)| < 1$, by Theorem 3.1 $(I^{(p)}(f;\mu), I^{(q)}(g;\mu))$ is nondegenerate Gaussian and its density is as follows:

$$P\Big((I^{(p)}(f;\mu), I^{(q)}(g;\mu)) \in dxdy\Big) = \tfrac{1}{2\pi}\tfrac{1}{\sqrt{1-c^2}} e^{-\frac{1}{2}\frac{1}{1-c^2}(x^2-2cxy+y^2)} dxdy,$$

where $c := E\Big[I^{(p)}(f;\mu) I^{(q)}(g;\mu)\Big]$. Without loss of generality let $n \leq l$. By the expression above

$$\begin{aligned}
&\text{the left hand side} \\
&= \int_{\mathbb{R}} \tfrac{1}{\sqrt{2\pi(1-c^2)}} e^{-\frac{x^2}{2(1-c^2)}} dx \int_{\mathbb{R}} H_l(y) H_n(x+cy) \tfrac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy.
\end{aligned}$$

To compute the y-integral, note that

$$\begin{aligned}
H_j(y) e^{-\frac{y^2}{2}} &= \tfrac{(-1)^j}{j!} \tfrac{d^j}{dy^j}\Big(e^{-\frac{y^2}{2}}\Big), \quad j \geq 0, \\
H_j'(y) &= H_{j-1}(y), \quad j \geq 1.
\end{aligned}$$

By integrating by parts n-times it turns out that

$$\int_{\mathbb{R}} H_l(y) H_n(x+cy) \tfrac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = \delta_{nl} \tfrac{c^n}{n!}.$$

Hence substituting this into the above we have the desired identity. □

2° *For* $^\forall n, l \in \mathbb{N}$ *and* $^\forall p, q \in \mathbb{Z}$

$$\begin{aligned}
&E\Big[I_n^{(p)}\big(\sum_{i:\text{finite sum}} c_i f_i^{\otimes n};\mu\big) I_l^{(q)}\big(\sum_{j:\text{finite sum}} d_j g_j^{\otimes l};\mu\big)\Big] \\
&= \begin{cases}
n! \sum_{|k_1|,\ldots,|k_n| \geq 1} \widehat{\sum_{i:\text{finite sum}} c_i f_i^{\otimes n}}(k_1,\ldots,k_n) & \\
\quad \times \widehat{\sum_{j:\text{finite sum}} d_j g_j^{\otimes n}}(k_1,\ldots,k_n) & \\
\quad \times \widehat{\mu}(k_1(q-p)) \cdots \widehat{\mu}(k_n(q-p)) & \text{if } n = l, \\
0 & \text{if } n \neq l.
\end{cases}
\end{aligned}$$

PROOF. Combining Proposition 1 and 1° we have that for $^\forall p,q\in\mathbb{Z}$, $^\forall n,l\in\mathbb{N}$ and $^\forall f,g\in\mathcal{CL}_2$

$$\begin{aligned}
&E\Big[I_n^{(p)}(f^{\otimes n};\mu)I_l^{(q)}(g^{\otimes l};\mu)\Big]\\
&\quad=\ \delta_{nl}n!\Big(E\Big[I^{(p)}(f;\mu)I^{(q)}(g;\mu)\Big]\Big)^n\\
&\quad=\ \delta_{nl}n!\sum_{|k_1|,\dots,|k_n|\ge 1}\widehat{f^{\otimes n}}(k_1,\dots,k_n)\overline{\widehat{g^{\otimes n}}(k_1,\dots,k_n)}\\
&\qquad\times\widehat{\mu}(k_1(q-p))\cdots\widehat{\mu}(k_n(q-p)).
\end{aligned}$$

From this the assertion follows immediately. □

3° Let $h_n,k_n\in\mathcal{CSL}_2^n$. From (2.1) we take, for $^\forall\varepsilon>0$, $\sum_{i:\text{finite sum}}c_if_i^{\otimes n}=:h_{n\varepsilon}$ and $\sum_{j:\text{finite sum}}d_jg_j^{\otimes n}=:k_{n\varepsilon}$ such that $\|h_n-h_{n\varepsilon}\|$, $\|k_n-k_{n\varepsilon}\|<\varepsilon$. By Proposition 1 this implies

$$\begin{aligned}
&\Big|E\Big[I_n^{(p)}(h_n;\mu)I_n^{(q)}(k_n;\mu)\Big]-E\Big[I_n^{(p)}(h_{n\varepsilon};\mu)I_n^{(q)}(k_{n\varepsilon};\mu)\Big]\Big|\\
&\quad\le\ n!(\|h_n\|+\|k_n\|+\varepsilon)\varepsilon.
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\Big|\sum_{|i_1|,\dots,|i_n|\ge 1}\widehat{h_n}(i_1,\dots,i_n)\overline{\widehat{k_n}(i_1,\dots,i_n)}\,\widehat{\mu}(i_1(q-p))\cdots\widehat{\mu}(i_n(q-p))\\
&-\sum_{|i_1|,\dots,|i_n|\ge 1}\widehat{h_{n\varepsilon}}(i_1,\dots,i_n)\overline{\widehat{k_{n\varepsilon}}(i_1,\dots,i_n)}\,\widehat{\mu}(i_1(q-p))\cdots\widehat{\mu}(i_n(q-p))\Big|\\
&\quad\le\ \varepsilon(\|h_n\|+\|k_n\|+\varepsilon).
\end{aligned}$$

Combining these with 2° we have

$$\begin{aligned}
&\Big|E\Big[I_n^{(p)}(h_n;\mu)I_n^{(q)}(k_n;\mu)\Big]\\
&-n!\sum_{|i_1|,\dots,|i_n|\ge 1}\widehat{h_n}(i_1,\dots,i_n)\overline{\widehat{k_n}(i_1,\dots,i_n)}\,\widehat{\mu}(i_1(q-p))\cdots\widehat{\mu}(i_n(q-p))\Big|\\
&\quad\le\ 2n!\varepsilon(\|h_n\|+\|k_n\|+\varepsilon)\ \underset{\varepsilon\to 0}{\longrightarrow}\ 0,
\end{aligned}$$

which is the conclusion of the proposition for $n=l$. The conclusion for $n\ne l$ will follow from the first part and 2°. □

CLAIM 3.1. *Suppose $m \in C(\mathbb{T})$ and let $h_n \in \mathcal{CSL}_2^n$.*

(i) $$E\Big[\Big(\tfrac{1}{\sqrt{P}}\sum_{p=1}^{P} I_n^{(p)}(h_n;\mu)\Big)^2\Big] \le \tfrac{\pi^2}{4} n! \|m\|_\infty^n \|h_n\|^2, \quad {}^\forall P \ge 1.$$

(ii) $$\lim_{P\to\infty} E\Big[\Big(\tfrac{1}{\sqrt{P}}\sum_{p=1}^{P} I_n^{(p)}(h_n;\mu)\Big)^2\Big]$$
$$= \begin{cases} g(0) & \text{if} \quad n=1, \\ n! \displaystyle\int_{\mathbb{T}^{n-1}} g(-t_1, t_1-t_2, \ldots, \\ \qquad\qquad t_{n-2}-t_{n-1}, t_{n-1}) dt_1 \cdots dt_{n-1} & \text{if} \quad n \ge 2. \end{cases}$$

Here

$$g(x_1,\ldots,x_n) := \sum_{k_1,\ldots,k_n \ge 1} \sum_{\varepsilon_1,\ldots,\varepsilon_n \in \{-1,1\}} |\widehat{h_n}(\varepsilon_1 k_1, \ldots, \varepsilon_n k_n)|^2$$
$$\times \tfrac{1}{k_1\cdots k_n} \sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} m\Big(\tfrac{\varepsilon_1 x_1 + j_1 - \frac{\varepsilon_1+1}{2}}{k_1}\Big) \times \cdots \times m\Big(\tfrac{\varepsilon_n x_n + j_n - \frac{\varepsilon_n+1}{2}}{k_n}\Big).$$

PROOF. Let $h_n \in \mathcal{CSL}_2^n$. By Proposition 3

$$E\Big[\Big(\sum_{p=1}^{P} I_n^{(p)}(h_n;\mu)\Big)^2\Big]$$
$$= n! \sum_{|i_1|,\ldots,|i_n| \ge 1} |\widehat{h_n}(i_1,\ldots,i_n)|^2$$
$$\times \int_{\mathbb{T}^n} \Big|\sum_{p=1}^{P} e^{\sqrt{-1}\,2\pi p(i_1x_1+\cdots+i_nx_n)}\Big|^2 m(x_1)\cdots m(x_n) dx_1\cdots dx_n$$
$$= n! \sum_{i_1,\ldots,i_n \ge 1} \sum_{\varepsilon_1,\ldots,\varepsilon_n \in \{-1,1\}} |\widehat{h_n}(\varepsilon_1 i_1,\ldots,\varepsilon_n i_n)|^2$$
$$\times \int_{\mathbb{T}^n} \Big|\sum_{p=1}^{P} e^{\sqrt{-1}\,2\pi p(\varepsilon_1 i_1x_1+\cdots+\varepsilon_n i_nx_n)}\Big|^2 m(x_1)\cdots m(x_n) dx_1\cdots dx_n.$$

The $(x_1,\dots,x_n)$-integral in the last line can be further computed as

$$\begin{aligned}
&\int_{\mathbb{T}^n}\Big|\sum_{p=1}^P e^{\sqrt{-1}\,2\pi p(\varepsilon_1 i_1 x_1+\cdots+\varepsilon_n i_n x_n)}\Big|^2 m(x_1)\cdots m(x_n)dx_1\cdots dx_n\\
&= \sum_{\substack{1\le j_1\le i_1\\ \cdots\\ 1\le j_n\le i_n}} \int_{\frac{j_1-1}{i_1}}^{\frac{j_1}{i_1}}\cdots\int_{\frac{j_n-1}{i_n}}^{\frac{j_n}{i_n}}\Big|\sum_{p=1}^P e^{\sqrt{-1}\,2\pi p(\varepsilon_1 i_1 x_1+\cdots+\varepsilon_n i_n x_n)}\Big|^2\\
&\qquad \times m(x_1)\cdots m(x_n)dx_1\cdots dx_n\\
&= \sum_{\substack{1\le j_1\le i_1\\ \cdots\\ 1\le j_n\le i_n}} \int_0^{\frac{1}{i_1}}\cdots\int_0^{\frac{1}{i_n}}\Big|\sum_{p=1}^P e^{\sqrt{-1}\,2\pi p\left(\varepsilon_1 i_1 x_1+\cdots+\varepsilon_n i_n x_n\right)}\Big|^2\\
&\qquad \times m(x_1+\tfrac{j_1-1}{i_1})\cdots m(x_n+\tfrac{j_n-1}{i_n})dx_1\cdots dx_n\\
&= \sum_{\substack{1\le j_1\le i_1\\ \cdots\\ 1\le j_n\le i_n}} \int_{\mathbb{T}^n}\Big|\sum_{p=1}^P e^{\sqrt{-1}\,2\pi p\left(x_1+\cdots+x_n\right)}\Big|^2\\
&\qquad \times m\Big(\frac{\varepsilon_1 x_1+j_1-\frac{\varepsilon_1+1}{2}}{i_1}\Big)\cdots m\Big(\frac{\varepsilon_n x_n+j_n-\frac{\varepsilon_n+1}{2}}{i_n}\Big)\frac{dx_1}{i_1}\cdots\frac{dx_n}{i_n},
\end{aligned}$$

so that

$$\begin{aligned}
&E\Big[\Big(\sum_{p=1}^P I_n^{(p)}(h_n;\mu)\Big)^2\Big]\\
&\quad = n!\int_{\mathbb{T}^n}\Big|\sum_{p=1}^P e^{\sqrt{-1}\,2\pi p(x_1+\cdots+x_n)}\Big|^2 g(x_1,\dots,x_n)dx_1\cdots dx_n.
\end{aligned}$$

From the definition of g it is easily seen that $g\ge 0$, $\int_{\mathbb{T}^n} g(x_1,\dots,x_n)dx_1\cdots dx_n=\|h_n\|^2<\infty$, and

$$g(x_1,\dots,x_k+1,\dots,x_n) = g(x_1,\dots,x_k,\dots,x_n), \tag{3.1}$$

$$g(-x_1,\dots,-x_n) = g(x_1,\dots,x_n). \tag{3.2}$$

By (3.1) we can check that

$$\int_{\mathbb{T}^n}\Big|\sum_{p=1}^P e^{\sqrt{-1}\,2\pi p(x_1+\cdots+x_n)}\Big|^2 g(x_1,\dots,x_n)dx_1\cdots dx_n$$

$$= \int_{\mathbb{T}} \Big| \frac{\sin \pi P x}{\sin \pi x} \Big|^2 \kappa(x) dx,$$

where

$$\kappa(x) \quad := \quad \begin{cases} g(x) & \text{if} \quad n = 1, \\ \displaystyle\int_{\mathbb{T}^{n-1}} g(x - t_1, t_1 - t_2, \ldots, \\ \qquad\qquad t_{n-2} - t_{n-1}, t_{n-1}) dt_1 \cdots dt_{n-1} & \text{if} \quad n \geq 2. \end{cases}$$

Consequently, combining two expressions above we have

$$E\Big[\Big(\tfrac{1}{\sqrt{P}} \sum_{p=1}^{P} I_n^{(p)}(h_n; \mu)\Big)^2\Big] \quad = \quad n! \tfrac{1}{P} \int_{\mathbb{T}} \Big| \frac{\sin \pi P x}{\sin \pi x} \Big|^2 \kappa(x) dx.$$

Now let $m \in C(\mathbb{T})$. Then $g \in C(\mathbb{T}^n)$ with $\|g\|_\infty \leq \|m\|_\infty^n \|h_n\|^2$, and hence $\kappa \in C(\mathbb{T})$ with $\|\kappa\|_\infty \leq \|m\|_\infty^n \|h_n\|^2$. Also, by (3.1) and (3.2), $\kappa(-x) = \kappa(x)$. From these facts

$$\begin{aligned} \tfrac{1}{P} \int_{\mathbb{T}} \Big| \frac{\sin \pi P x}{\sin \pi x} \Big|^2 \kappa(x) dx &= \tfrac{2}{P} \int_0^{\frac{1}{2}} \Big| \frac{\sin \pi P x}{\sin \pi x} \Big|^2 \kappa(x) dx \\ &= 2 \int_0^\infty \kappa(\tfrac{y}{P}) 1_{y < \frac{P}{2}} \Big| \frac{\pi \frac{y}{P}}{\sin \pi \frac{y}{P}} \Big|^2 \Big| \frac{\sin \pi y}{\pi y} \Big|^2 dy \\ &\leq 2\|\kappa\|_\infty (\tfrac{\pi}{2})^2 \int_0^\infty \Big| \frac{\sin \pi y}{\pi y} \Big|^2 dy \\ &\leq \tfrac{\pi^2}{4} \|m\|_\infty^n \|h_n\|^2. \end{aligned}$$

By putting the above the assertion (i) is obtained immediately. Also, by letting $P \to \infty$ the assertion (ii) follows from the Lebesgue convergence theorem. □

THEOREM 3.2. *Suppose $m \in C(\mathbb{T})$. Then $\{I^{(p)}(\cdot; \mu)\}_{p=1}^\infty$ satisfies the CLT in the sense that for $^\forall f_1, \ldots, f_L \in \mathcal{CL}_2$*

$$\tfrac{1}{\sqrt{P}} \sum_{p=1}^{P} (I^{(p)}(f_1; \mu), \ldots, I^{(p)}(f_L; \mu)) \underset{P \to \infty}{\Longrightarrow} \mathfrak{N}\Big(0, \Sigma(f_1, \ldots, f_L)\Big).$$

Here $\Sigma(f_1,\ldots,f_L)$ is a nonnegative symmetric matrix whose (i,j)-component is

$$\begin{aligned}\Sigma_{ij}(f_1,\ldots,f_L) &= \sum_{|n|\geq 1}\widehat{f_i}(n)\overline{\widehat{f_j}(n)}\frac{1}{|n|}\sum_{k=1}^{|n|}m(\tfrac{k-1}{|n|})\\ &= \sum_{n=1}^{\infty}2\mathrm{Re}(\widehat{f_i}(n)\overline{\widehat{f_j}(n)})\frac{1}{n}\sum_{k=1}^{n}m(\tfrac{k-1}{n}).\end{aligned}$$

Moreover, if m is positive, i.e., $\min_{x\in\mathbb{T}}m(x)>0$, then the limiting Gaussian $\mathfrak{N}\Big(0,\Sigma(f_1,\ldots,f_L)\Big)$ is nondegenerate in the sense that a matrix $\Sigma(f_1,\ldots,f_L)$ is nonsingular whenever $f_1,\ldots,f_L\in\mathcal{CL}_2$ are linearly independent.

Proof. Since, for $\xi=(\xi_1,\ldots,\xi_L)\in\mathbb{R}^L$

$$\begin{aligned}&E\Big[e^{\sqrt{-1}\sum_{i=1}^{L}\xi_i\frac{1}{\sqrt{P}}\sum_{p=1}^{P}I^{(p)}(f_i;\mu)}\Big]\\ &\quad= \exp\Big\{-\tfrac{1}{2}E\Big[\Big(\tfrac{1}{\sqrt{P}}\sum_{p=1}^{P}I^{(p)}(\sum_{i=1}^{L}\xi_i f_i;\mu)\Big)^2\Big]\Big\},\end{aligned}$$

Claim 3.1(ii) says

$$\begin{aligned}&\lim_{P\to\infty}E\Big[e^{\sqrt{-1}\sum_{i=1}^{L}\xi_i\frac{1}{\sqrt{P}}\sum_{p=1}^{P}I^{(p)}(f_i;\mu)}\Big]\\ &\quad= \exp\Big\{-\tfrac{1}{2}\sum_{|n|\geq 1}\Big|\sum_{i=1}^{L}\xi_i\widehat{f_i}(n)\Big|^2\frac{1}{|n|}\sum_{k=1}^{|n|}m(\tfrac{k-1}{|n|})\Big\},\end{aligned}$$

which implies the first part of the theorem. The second part will be clear from $\frac{1}{n}\sum_{k=1}^{n}m(\frac{k-1}{n})\geq\min_{x\in\mathbb{T}}m(x)$. $\square$

Let us proceed to the CLT for $\{I_n^{(p)}(\cdot;\mu)\}_{p=1}^{\infty}$. For this we define the following:

Definition 3.1. For a stationary sequence $\left\{(I^{(p)}(f;\mu))_{f\in\mathcal{CL}_2}\right\}_{p\in\mathbb{Z}}$ we define the α-mixing coefficients $(\alpha(N))_{N\geq 1}$ by

$$\alpha(N)$$
$$:= \sup\left\{|P(A\cap B)-P(A)P(B)|\,;\,\begin{array}{l}A\in\sigma(I^{(p)}(f;\mu);f\in\mathcal{CL}_2,p\leq 0)\\ B\in\sigma(I^{(p)}(f;\mu);f\in\mathcal{CL}_2,p\geq N+1)\end{array}\right\}.$$

Claim 3.2. *Suppose $m\in C(\mathbb{T})$ and positive. Then*

$$\alpha(N) \quad\leq\quad \frac{E_N(m)}{\min_{x\in\mathbb{T}} m(x)}.$$

Here

$$E_N(m) \quad := \quad \inf_{(a_n)_{|n|\leq N}\in\mathbb{C}^{2N+1}}\max_{x\in\mathbb{T}}\left|m(x)-\sum_{|n|\leq N}a_n e^{\sqrt{-1}\,2\pi nx}\right|.$$

Proof. The proof is done in exactly the same way as in Th.17.3.3 of [7]. We set

$$\mathcal{F}_a^b \quad := \quad \sigma(I^{(p)}(f;\mu);f\in\mathcal{CL}_2,a\leq p\leq b)$$

for $a,b\in\mathbb{Z}\cup\{\pm\infty\}$, $a<b$, and

$$\rho(N) \quad := \quad \sup\left\{\frac{|\mathrm{cov}(X,Y)|}{\sqrt{\mathrm{var}(X)}\sqrt{\mathrm{var}(Y)}}\,;\,X\in L_2(\mathcal{F}_{-\infty}^0),Y\in L_2(\mathcal{F}_{N+1}^\infty)\right\}.$$

From [9] this equals

$$\sup\left\{\frac{|E[XY]|}{\sqrt{E[X^2]}\sqrt{E[Y^2]}}\,;\,\begin{array}{l}X\in \text{l.s.}\{I^{(p)}(f;\mu);f\in\mathcal{CL}_2,p\leq 0\},\neq 0\\ Y\in \text{l.s.}\{I^{(p)}(f;\mu);f\in\mathcal{CL}_2,p\geq N+1\},\neq 0\end{array}\right\}$$

and satisfies $\alpha(N)\leq\rho(N)\leq 2\pi\alpha(N)$. In the following we treat $\rho(N)$.

We take arbitrarily X and Y from the linear spans above, respectively. Let

$$X \quad = \quad \sum_{p\leq 0,k\geq 1:\text{finite sum}} c_{pk}I^{(p)}(f_k;\mu),$$
$$Y \quad = \quad \sum_{q\geq N+1,l\geq 1:\text{finite sum}} d_{ql}I^{(q)}(g_l;\mu).$$

For $^\forall$ trigonometric polynomial $P_N(x) = \sum_{|j|\le N} a_j e^{\sqrt{-1}\,2\pi jx}$ of degree N we observe

$$
\begin{aligned}
|E[XY]| &= \Bigg| \sum_{|i|\ge 1} \int_{\mathbb{T}} \Big(\sum_{p\le 0, k\ge 1:\text{finite sum}} c_{pk} \widehat{f_k}(i) e^{\sqrt{-1}\,2\pi ipx} \Big) \\
&\qquad \times \Big(\sum_{q\ge N+1, l\ge 1:\text{finite sum}} d_{ql} \overline{\widehat{g_l}(i)} e^{-\sqrt{-1}\,2\pi iqx} \Big) m(x)dx \Bigg| \\
&= \Bigg| \sum_{|i|\ge 1} \int_{\mathbb{T}} \Big(\sum_{p\le 0, k\ge 1:\text{finite sum}} c_{pk} \widehat{f_k}(i) e^{\sqrt{-1}\,2\pi ipx} \Big) \\
&\qquad \times \Big(\sum_{q\ge N+1, l\ge 1:\text{finite sum}} d_{ql} \overline{\widehat{g_l}(i)} e^{-\sqrt{-1}\,2\pi iqx} \Big) \\
&\qquad \times (m(x) - P_N(x))dx \Bigg| \\
&\le \|m - P_N\|_\infty \sum_{|i|\ge 1} \sqrt{\int_{\mathbb{T}} \Big| \sum_{p\le 0, k\ge 1:\text{finite sum}} c_{pk} \widehat{f_k}(i) e^{\sqrt{-1}\,2\pi ipx} \Big|^2 dx} \\
&\qquad \times \sqrt{\int_{\mathbb{T}} \Big| \sum_{q\ge N+1, l\ge 1:\text{finite sum}} d_{ql} \overline{\widehat{g_l}(i)} e^{-\sqrt{-1}\,2\pi iqx} \Big|^2 dx} \\
&\le \frac{\|m - P_N\|_\infty}{\min_{y\in\mathbb{T}} m(y)} \\
&\qquad \times \sqrt{\sum_{|i|\ge 1} \int_{\mathbb{T}} \Big| \sum_{p\le 0, k\ge 1:\text{finite sum}} c_{pk} \widehat{f_k}(i) e^{\sqrt{-1}\,2\pi ipx} \Big|^2 m(x)dx} \\
&\qquad \times \sqrt{\sum_{|i|\ge 1} \int_{\mathbb{T}} \Big| \sum_{q\ge N+1, l\ge 1:\text{finite sum}} d_{ql} \overline{\widehat{g_l}(i)} e^{-\sqrt{-1}\,2\pi iqx} \Big|^2 m(x)dx} \\
&= \frac{\|m - P_N\|_\infty}{\min_{y\in\mathbb{T}} m(y)} \sqrt{E[X^2]} \sqrt{E[Y^2]}.
\end{aligned}
$$

This implies

$$\rho(N) \quad \leq \quad \frac{\|m - P_N\|_\infty}{\min_{y\in\mathbb{T}} m(y)},$$

and the assertion follows at once. □

Now we state the CLT for $\{I_n^{(p)}(\cdot;\mu)\}_{p=1}^\infty$.

THEOREM 3.3. *Suppose* $m \in C(\mathbb{T})$ *and positive. Then for* ${}^\forall(h_1, h_2, \dots, h_L) \in \mathcal{CL}_2 \times \mathcal{CSL}_2^2 \times \cdots \times \mathcal{CSL}_2^L$,

$$\tfrac{1}{\sqrt{P}} \sum_{p=1}^{P} (I_1^{(p)}(h_1;\mu), \dots, I_L^{(p)}(h_L;\mu)) \underset{P\to\infty}{\Longrightarrow} \mathfrak{N}\left(0, \begin{bmatrix} v_1 & & 0 \\ & \ddots & \\ 0 & & v_L \end{bmatrix}\right).$$

Here

$$v_k \;=\; v(h_k) \;:=\; \lim_{P\to\infty} E\Big[\Big(\tfrac{1}{\sqrt{P}} \sum_{p=1}^{P} I_k^{(p)}(h_k;\mu)\Big)^2\Big].$$

REMARK 3.1. Since, by Claim 3.1(ii), $v(h_k) \geq k!\Big(\min_{x\in\mathbb{T}} m(x)\Big)^k \times \|h_k\|^2$, the limiting Gaussian is nondegenerate when $h_1 \neq 0$, …, $h_L \neq 0$.

PROOF. Let $(h_1, h_2, \dots, h_L) \in \mathcal{CL}_2 \times \mathcal{CSL}_2^2 \times \cdots \times \mathcal{CSL}_2^L$. The proof is done in three steps.

1° Let $\varepsilon > 0$ be fixed arbitrarily. By (2.1), for each h_k we take $t_{k1}, \dots, t_{kn_k} \in \mathbb{R}$ and $f_{k1}, \dots, f_{kn_k} \in \mathcal{CL}_2$ with $\|f_{ki}\| = 1$ such that $\|h_k - \sum_{i=1}^{n_k} t_{ki} f_{ki}^{\otimes k}\| < \varepsilon$. For simplicity write $h_k^{(\varepsilon)} := \sum_{i=1}^{n_k} t_{ki} f_{ki}^{\otimes k}$. Claim 3.1(i) tells us that for ${}^\forall(\xi_1, \dots, \xi_L) \in \mathbb{R}^L$

$$\Big| E\Big[e^{\sqrt{-1}\sum_{k=1}^{L} \xi_k \frac{1}{\sqrt{P}} \sum_{p=1}^{P} I_k^{(p)}(h_k;\mu)}\Big] - E\Big[e^{\sqrt{-1}\sum_{k=1}^{L} \xi_k \frac{1}{\sqrt{P}} \sum_{p=1}^{P} I_k^{(p)}(h_k^{(\varepsilon)};\mu)}\Big]\Big|$$

$$\leq \quad \Big(\sum_{k=1}^{L} |\xi_k| (k!)^{\frac{1}{2}} \|m\|_\infty^{\frac{k}{2}}\Big) \tfrac{\pi}{2}\varepsilon,$$

$$\begin{aligned}
&\left|e^{-\frac{1}{2}\sum_{k=1}^{L}\xi_k^2 v(h_k)} - e^{-\frac{1}{2}\sum_{k=1}^{L}\xi_k^2 v(h_k^{(\varepsilon)})}\right| \\
&\quad \le \ \tfrac{1}{2}\Big(\sum_{k=1}^{L}\xi_k^2 k!\|m\|_\infty^k(2\|h_k\|+\varepsilon)\Big)\tfrac{\pi^2}{4}\varepsilon.
\end{aligned}$$

2° Let $(\xi_1,\ldots,\xi_L)\in\mathbb{R}^L$ be fixed and set

$$X_p \ := \ \sum_{k=1}^{L}\xi_k I_k^{(p)}(h_k^{(\varepsilon)};\mu), \quad p\in\mathbb{Z}.$$

$\{X_p\}_{p\in\mathbb{Z}}$ is a stationary sequence, and it satisfies

$$\sigma(X_p; a\le p\le b) \ \subset \ \sigma(I^{(p)}(f;\mu); f\in\mathcal{CL}_2, a\le p\le b),$$

because by Proposition 1

$$I_k^{(p)}(h_k^{(\varepsilon)};\mu) \ = \ \sum_{i=1}^{n_k} t_{ki} I_k^{(p)}(f_{ki}^{\otimes k};\mu) \ = \ \sum_{i=1}^{n_k} t_{ki}k! H_k(I^{(p)}(f_{ki};\mu)). \tag{3.3}$$

By Claim 3.2 the α-mixing coefficients $(\alpha(N))$ of $\{X_p\}_{p\in\mathbb{Z}}$ are estimated as

$$\begin{aligned}
\alpha(N) \ &:= \ \sup\left\{|P(A\cap B)-P(A)P(B)|;\ \begin{matrix} A\in\sigma(X_p; p\le 0), \\ B\in\sigma(X_p; p\ge N+1)\end{matrix}\right\} \\
&\le \ \frac{E_N(m)}{\min_{x\in\mathbb{T}} m(x)}, \quad N\ge 1.
\end{aligned}$$

Consequently $\{X_p\}_{p\in\mathbb{Z}}$ is α-mixing (or strongly mixing in terms of [7]).

Let us apply Th.18.4.2 of [7] for $\{X_p\}_{p\in\mathbb{Z}}$. To do so we need to check that a sequence $\{(\frac{1}{\sqrt{P}}\sum_{p=1}^{P} X_p)^2\}_{P=1}^{\infty}$ is uniformly integrable. But, by Lemma 3.4 of [17]

$$\lim_{P\to\infty} E\Big[\Big(\tfrac{1}{\sqrt{P}}\sum_{p=1}^{P} H_k(I^{(p)}(f;\mu))\Big)^4\Big] \ = \ 3v_k^2$$

where $f\in\mathcal{CL}_2$ with $\|f\| = 1$ and $v_k = v(\frac{1}{k!}f^{\otimes k}) = \lim_{P\to\infty} E[(\frac{1}{\sqrt{P}}\sum_{p=1}^{P} H_k (I^{(p)}(f;\mu)))^2] > 0$. This together with (3.3) clearly implies the L_4-boundedness of $\frac{1}{\sqrt{P}}\sum_{p=1}^{P} X_p$, $P\in\mathbb{N}$, so the uniform integrability above is valid.

Now, as mentioned earlier, we can apply Th.18.4.2 of [7] to have

$$\lim_{P\to\infty} E\Big[e^{\sqrt{-1}\sum_{k=1}^{L}\xi_k \frac{1}{\sqrt{P}}\sum_{p=1}^{P} I_k^{(p)}(h_k^{(\varepsilon)};\mu)}\Big] \;=\; e^{-\frac{1}{2}\sum_{k=1}^{L}\xi_k^2 v(h_k^{(\varepsilon)})}.$$

3° Collecting 1° and 2° yields

$$\lim_{P\to\infty} E\Big[e^{\sqrt{-1}\sum_{k=1}^{L}\xi_k \frac{1}{\sqrt{P}}\sum_{p=1}^{P} I_k^{(p)}(h_k;\mu)}\Big] \;=\; e^{-\frac{1}{2}\sum_{k=1}^{L}\xi_k^2 v(h_k)},$$

which is just the conclusion of the theorem. □

4. A Justification of the Claim of Sobol' et al

First let us state the claim of Sobol' et al ([13], [14]).

There are several kinds of deterministic sequences $\{x_p = (x_{1p},\dots,x_{Np})\}_{p=1}^{\infty}$ on $\mathbb{T}^N$, which are called low discrepancy sequences ([2]), having the following property: For $^{\forall}F:\mathbb{T}^N\to\mathbb{R}$ of finite variation

$$\frac{1}{P}\sum_{p=1}^{P} F(x_p) \;=\; \int_{\mathbb{T}^N} F(x)dx + O\Big(\frac{1}{P^{1-\varepsilon}}\Big) \qquad \text{as } P\to\infty \qquad (^{\forall}\varepsilon>0).$$

This convergence can be used for numerical integrations in $\mathbb{T}^N$, which is called the quasi Monte Carlo method. Since the usual Monte Carlo method converges at the rate of $O(\frac{1}{\sqrt{P}})$, this method is theoretically more effective.

However many authors have reported that practically the quasi Monte Carlo method does not converge so fast as it is expected, if the dimension N is very high. Among others there is the following claim of Sobol' et al:

Claim. *In high dimensions, the quasi Monte Carlo method seems to converge at the rate of exactly $O(\frac{1}{\sqrt{P}})$, if the integrands F depend equally on each coordinate.*

We try to give a probabilistic explanation to this claim. From Theorems 2.1 and 3.3 let us view the following generously.

Let $(h_1,\dots,h_L)\in\mathcal{CL}_2\times\cdots\times\mathcal{CSL}_2^L$ be such that $h_k\neq 0$ $(1\le k\le L)$. Corollary to Theorem 2.1 says that for $\alpha=(\alpha_1,\alpha_2,\dots)\in\mathbb{T}_\mu^\infty$

$$\Big(\frac{1}{\sqrt{N}}\sum_{1\le i_1\le N} h_1(x_{i_1}+p\alpha_{i_1}),$$
$$\dots, L!(\frac{1}{\sqrt{N}})^L \sum_{1\le i_1<\cdots<i_L\le N} h_L(x_{i_1}+p\alpha_{i_1},\dots,x_{i_L}+p\alpha_{i_L})\Big)_{p\in\mathbb{N}}$$

$$\xrightarrow{\text{f.d.}} \Big(I_1^{(p)}(h_1;\mu),\ldots,I_L^{(p)}(h_L;\mu)\Big)_{p\in\mathbb{N}} \quad \text{as } N\to\infty.$$

Consequently, for sufficiently large N we think that

$$\begin{aligned}&\Big(\tfrac{1}{\sqrt{N}}\sum_{1\le i_1\le N} h_1(x_{i_1}+p\alpha_{i_1}),\\ &\qquad \ldots, L!(\tfrac{1}{\sqrt{N}})^L \sum_{1\le i_1<\cdots<i_L\le N} h_L(x_{i_1}+p\alpha_{i_1},\ldots,x_{i_L}+p\alpha_{i_L})\Big)_{p\in\mathbb{N}}\\ &\qquad \text{“}=\text{”} \quad \Big(I_1^{(p)}(h_1;\mu),\ldots,I_L^{(p)}(h_L;\mu)\Big)_{p\in\mathbb{N}}.\end{aligned}$$

On the other hand, Theorem 3.3 says that

$$\begin{aligned}&\tfrac{1}{\sqrt{P}}\sum_{p=1}^{P}\Big(I_1^{(p)}(h_1;\mu),\ldots,I_L^{(p)}(h_L;\mu)\Big)\\ &\quad\Longrightarrow\quad \mathfrak{N}\left(0,\begin{bmatrix} v(h_1) & & 0\\ & \ddots & \\ 0 & & v(h_L)\end{bmatrix}\right) \text{ (nondegenerate Gaussian)} \quad \text{as } P\to\infty\end{aligned}$$

provided $m=\frac{d\mu}{dx}$ is continuous and positive. Hence combining these yields

$$\begin{aligned}&\tfrac{1}{\sqrt{P}}\sum_{p=1}^{P}\Big(\tfrac{1}{\sqrt{N}}\sum_{1\le i_1\le N} h_1(x_{i_1}+p\alpha_{i_1}),\\ &\qquad\qquad \ldots, L!(\tfrac{1}{\sqrt{N}})^L \sum_{1\le i_1<\cdots<i_L\le N} h_L(x_{i_1}+p\alpha_{i_1},\ldots,x_{i_L}+p\alpha_{i_L})\Big)\\ &\quad\Longrightarrow\quad \mathfrak{N}\left(0,\begin{bmatrix} v(h_1) & & 0\\ & \ddots & \\ 0 & & v(h_L)\end{bmatrix}\right) \text{ (nondegenerate Gaussian)} \quad \text{as } P\to\infty,\end{aligned}$$

so that for $k=1,\ldots,L$

$$\begin{aligned}&\tfrac{1}{P}\sum_{p=1}^{P}(\tfrac{1}{\sqrt{N}})^k \sum_{1\le i_1<\cdots<i_k\le N} h_k(x_{i_1}+p\alpha_{i_1},\ldots,x_{i_k}+p\alpha_{i_k})\\ &\quad\begin{cases} = & O\Big(\frac{1}{\sqrt{P}}\Big)\\ \neq & O\Big(\frac{1}{\sqrt{P}^{1+\varepsilon}}\Big) \ ({}^{\forall}\varepsilon>0)\end{cases} \quad \text{as } P\to\infty.\end{aligned}$$

This observation tells us that when $\alpha = (\alpha_1, \alpha_2, \ldots)$ is " regularly " distributed over the whole $\mathbb{T}$ (it need not be uniformly distributed), the claim of Sobol' et al " holds " for a low discrepancy sequence $\{(x_1 + p\alpha_1, \ldots, x_N + p\alpha_N)\}_{p=1}^{\infty}$ and an integrand $F(y_1, \ldots, y_N) = (\frac{1}{\sqrt{N}})^k \sum_{1 \le i_1 < \cdots < i_k \le N} h_k(y_{i_1}, \ldots, y_{i_k})$.

5. Concluding Remarks

Our interest in this paper is when the disappearance of dependency happens, in other words, how the distribution of $\alpha = (\alpha_1, \alpha_2, \ldots)$ in $\mathbb{T}$ influences the dependency. For this subject there are two works close to ours. We briefly introduce them to conclude this paper.

5.1. Fukuyama's work ([5], [6])

Let $\theta \in (1, \infty)$. We define $\boldsymbol{\varphi}_\theta : \mathbb{T} \to \mathbb{T}^\infty$ by

$$\boldsymbol{\varphi}_\theta(x) \quad = \quad \Big(\{\theta^{i-1}x\}\Big)_{i=1}^{\infty},$$

where $\{y\}$ denotes the fractional part of $y \in \mathbb{R}$. Let $\boldsymbol{\mu}$ be a probability measure on $\mathbb{T}^\infty$ induced by $\boldsymbol{\varphi}_\theta$, i.e., $\boldsymbol{\mu} = \mathbf{P} \circ \boldsymbol{\varphi}_\theta^{-1}$. Clearly $\boldsymbol{\mu}$ is singular relative to $\mathbf{P}^\infty$, and not a product probability measure. So it is not in a class of probability measures considered above. Also, for ${}^\forall\alpha \in \mathbb{T}$

$$\boldsymbol{\mu}\Big(\{x \in \mathbb{T}^\infty; x + \boldsymbol{\varphi}_\theta(\alpha) \in *\}\Big) \quad = \quad \boldsymbol{\mu}(*).$$

Let us state a result of Fukuyama. To do so, for $f \in \mathcal{CL}_2$ we set

$$D(f) \quad := \quad \sum_{k=0}^{\infty} \sqrt{\sum_{2^k \le |n| < 2^{k+1}} |\widehat{f}(n)|^2}.$$

By definition $\|f\| \le D(f)$, but $D(f) \in [0, \infty]$ (i.e., $D(f)$ is not necessarily convergent). Let μ be a probability measure on $\mathbb{T}$, $\{N_m\}_{m=1}^{\infty}$ a subsequence of $\mathbb{N}$ and $\alpha \in \mathbb{T}$ for which $\boldsymbol{\varphi}_\theta(\alpha) \in \mathbb{T}^\infty_{\mu, \{N_m\}}$, i.e.,

$$\tfrac{1}{N_m} \sum_{i=1}^{N_m} \delta_{\{\theta^{i-1}\alpha\}}(dx) \quad \Longrightarrow \quad \mu(dx) \quad \text{as } m \to \infty.$$

Then the following holds: For ${}^\forall f \in \mathcal{CL}_2$ with $D(f) < \infty$

$$\Big(\tfrac{1}{\sqrt{N_m}}\sum_{i=1}^{N_m} f(x_i + j\boldsymbol{\varphi}_\theta(\alpha)_i)\Big)_{j\in\mathbb{Z}} \quad\Longrightarrow\quad \mathfrak{N}(0,\Sigma(f)) \quad \text{as } m\to\infty.$$

Here $\Sigma(f)$ is an infinite nonnegative symmetric matrix whose (i,j)-component is given by the following manner: In case $\theta^r \notin \mathbb{Q}$ for ${}^\forall r \in \mathbb{N}$,

$$\Sigma_{ij}(f) \;=\; \int_{\mathbb{T}} \mu(dx)\int_{\mathbb{T}} f(t)f(t+|i-j|x)dt;$$

in case $\theta^r \in \mathbb{Q}$ for some $r \in \mathbb{N}$, by letting $s = \min\{n\in\mathbb{N};\theta^n\in\mathbb{Q}\}$ and writing $\theta^s = \frac{p}{q}$ to be irreducible

$$\Sigma_{ij}(f) \;=\; \sum_{k\in\mathbb{Z}}\int_{\mathbb{T}}\mu(dx)\int_{\mathbb{T}} f(p^{|k|}t)f(q^{|k|}(t+\widetilde{\mathrm{sgn}}(k)|i-j|x))dt,$$

where $\widetilde{\mathrm{sgn}}(k) := \begin{cases} 1 & k\ge 0 \\ -1 & k<0 \end{cases}$. Putting $\mu(dx) = dx$ in the expression above, we readily see

$$\Sigma_{ij}(f) \;=\; 0, \quad i\neq j,$$

and hence we have the disappearance of dependency whenever $\boldsymbol{\varphi}_\theta(\alpha) \in \mathbb{T}^\infty_{dx}$. This is valid at least for $f \in \mathcal{CL}_2$ with $D(f) < \infty$. For $\theta = 2$, Fukuyama remarks that the condition $D(f)<\infty$ is necessary for the CLT: $\frac{1}{\sqrt{N}}\sum_{i=1}^{N} f(x_i) \Rightarrow \mathfrak{N}(0,\sigma^2)$ as $N\to\infty$ (under $\boldsymbol{\mu} = \mathbf{P}\circ\boldsymbol{\varphi}_2^{-1}$). For this reason, when θ is in the second case above, this condition is the best possible for the disappearance of dependency. On the other hand, when θ is in the first case, is this really necessary? For, in this case, $\Sigma_{ij}(f) = \delta_{ij}\|f\|^2$. Anyway this question remains open.

5.2. Sugita's work ([15])

We first note that our and Fukuyama's works originate in [15].

Let $\boldsymbol{\mu} = \mathbf{P}\circ\boldsymbol{\varphi}_2^{-1}$. For $N\in\mathbb{N}$, we define a random variable $X^{(N)}$ on $(\mathbb{T}^\infty, \mathfrak{F}, \boldsymbol{\mu})$ by

$$X^{(N)}(x) \;:=\; \sum_{i=1}^{N} 1_{[\frac12,1)}(x_i) \pmod 2.$$

Clearly for $^\forall\alpha \in \mathbb{T}$, $^\forall N \geq 1$ and $^\forall j \in \mathbb{Z}$

$$\boldsymbol{\mu}\Big(X^{(N)}(\cdot + j\boldsymbol{\varphi}_2(\alpha)) = 1\Big) \;=\; \boldsymbol{\mu}\Big(X^{(N)}(\cdot + j\boldsymbol{\varphi}_2(\alpha)) = 0\Big) \;=\; \tfrac{1}{2}.$$

We have a question whether the disappearance of dependency happens as $N \to \infty$ for a stationary sequence $\Big(X^{(N)}(\cdot + j\boldsymbol{\varphi}_2(\alpha))\Big)_{j\in\mathbb{Z}}$. For this Sugita gives an affirmative answer: If, at least $\boldsymbol{\varphi}_2(\alpha) \in \mathbb{T}^\infty_{dx}$, in other words α is a dyadic normal number, a sequence $\Big(X^{(N)}(\cdot + j\boldsymbol{\varphi}_2(\alpha))\Big)_{j\in\mathbb{Z}}$ converges to the $\{0,1\}$-valued fair Bernoulli random variables in finite dimensional distribution as $N \to \infty$, i.e., for $^\forall j_1 < \cdots < j_k$ and $^\forall \varepsilon_1, \ldots, \varepsilon_k \in \{0,1\}$

$$\lim_{N\to\infty} \boldsymbol{\mu}\Big(X^{(N)}(\cdot + j_1\boldsymbol{\varphi}_2(\alpha)) = \varepsilon_1, \ldots, X^{(N)}(\cdot + j_k\boldsymbol{\varphi}_2(\alpha)) = \varepsilon_k\Big) \;=\; (\tfrac{1}{2})^k.$$

He says that this assumption on α is technical, and in fact the statement above will hold for every irrational number α. However, this is still open.

References

[1] Billingsley, P., *Probability and measure*, 3rd edition, John Wiley & Sons, 1995.

[2] Bouleau, N. and D. Lépingle, *Numerical methods for stochastic processes*, John Wiley & Sons, 1994.

[3] Dynkin, E. B. and A. Mandelbaum, Symmetric statistics, Poisson point processes, and multiple Wiener integrals, Ann. Statist. **11** (1983), 739–745.

[4] Fukuyama, K., The central limit theorem for Rademacher system, Proc. Japan Acad. **70** Ser. A (1994), 243–246.

[5] Fukuyama, K., The central limit theorem for Riesz-Raikov sums, Probab. Theory Relat. Fields **100** (1994), 57–75.

[6] Fukuyama, K., Riesz-Raikov sums and Weyl transform, Monte Carlo Methods and Appl. **2** (1996), 271–293.

[7] Ibragimov, I. A. and Yu. V. Linnik, *Independent and stationary sequences of random variables*, Wolters-Noordhoff, 1971.

[8] Itô, K., Multiple Wiener integral, J. Math. Soc. Japan **3** (1951), 157–169.

[9] Kolmogorov, A. N. and Yu. A. Rozanov, On strong mixing conditions for stationary Gaussian processes, Theory Probab. Appl. **5** (1960), 204–208.

[10] Kuipers, L. and H. Niederreiter, *Uniform distribution of sequences*, Interscience, 1974.

[11] McLeish, D. L., Dependent central limit theorems and invariance principles, Ann. Probab. **2** (1974), 620–628.

[12] Salem, R. and A. Zygmund, On lacunary trigonometric series I, Proc. Nat. Acad. Sci. USA **33** (1947), 333–338.
[13] Sobol', I. M., Turchaninov, V. I. and Yu. L. Levitan, *Quasirandom sequence generators*, Keldysh Institute of Applied Mathematics, Moscow, 1992.
[14] Sobol', I. M., *A primer for the Monte Carlo method*, CRC Press, Boca Raton/Ann Arbor/London/Tokyo, 1994.
[15] Sugita, H., Pseudo-random number generator by means of irrational rotation, Monte Carlo Methods and Appl. **1** (1995), 35–57.
[16] Sugita, H. and S. Takanobu, Limit theorem for symmetric statistics with respect to Weyl transformation: Disappearance of dependency, J. Math. Kyoto Univ. **38** (1998), 653–671.
[17] Sun, T.-C., A central limit theorem for non-linear functions of a normal stationary process, J. Math. Mech. **12** (1963), 945–977.
[18] Sun, T.-C., Some further results on central limit theorems for non-linear functions of a normal stationary process, J. Math. Mech. **14** (1965), 71–85.
[19] Walter, P., *An introduction to ergodic theory*, Springer, 1981.
[20] Zygmund, A., *Trigonometric series I*, Cambridge Univ. Press, Cambridge, 1959.

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