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Multi-dimensional transition layers for an exothermic reaction-diffusion system in long cylindrical domains

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Dedicated to Professor Junji Kato on his sixtieth birthday

Abstract. By using singular perturbation techniques, it is shown that an exothermal reaction-diffusion system with a small parameter in long cylindrical domains admits a family of transition layer solutions. The solutions exhibit spatial inhomogeneity in two directions, one in the axis of the cylinder and the other in the cross-section of the cylindrical domain. The profile of the solutions in the cross-sectional direction is determined by a family of solutions of a non-linear elliptic eigenvalue problem, called *the perturbed Gelfand problem.* On the other hand, the profile of the solutions in the axial direction of the cylindrical domain has a sharp transition layer. The stability analysis is also carried out for the equilibrium solutions, which reveals that a Hopf-bifurcation occurs as some control parameters are varied, exhibiting spatio-temporal oscillations.

1. Introduction

Variety of spatio and/or temporal patterns are observed in combsution processes with or without supply of fuel. In order to theoretically understand such pattern formations, several mathematical models have been proposed so far. Among them, we focus our attention on a thermal and diffusive equation which describes a single step exothermic reaction. The equation is given by the following two components system for the (nondimensional-

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ized) absolute temperature θ and the concentration of a reactant c :

$$(1.1)\qquad \begin{cases} \theta_t = \theta_{xx} + \Delta\theta + cf(\theta) \\ c_t = dc_{xx} + d\Delta c - \epsilon^2 cf(\theta) \end{cases}, \quad t > 0,\ (x, y) \in \Omega_L = (0, L) \times \Omega.$$

The operator Δ is the Laplacian operator in y, and Ω_L is a cylindrical domain with section Ω in $\mathbf{R}^N$ and the length L. The nonlinearity $f(\theta)$ takes the form

$$f(\theta) = \exp[\theta/(1 + \theta/\alpha)], \quad \alpha > 0,$$

which is called the Arrhenius rate in irreversible chemical reaction kinetics (see for instance, Frank-Kamenetzky [F]). The value of α in experiments varies from 5 to 100. The parameters d and ϵ are positive constants. In particular, ϵ is assumed to be sufficiently small but non zero (Sattinger [St]), which means that the thermal effect on the dynamics is extremely high.

The initial and boundary conditions are

$$(1.2)\qquad (\theta, c)(0, x, y) = (\theta_i, c_i)(x, y) \geq 0, \quad (x, y) \in \Omega_L,$$

$$(1.3)\qquad \begin{cases} (\theta, c)(t, x, y) = (\theta_0, c_0)(y), \quad t > 0, \quad (x, y) \in \Gamma_0 = \{x = 0, y \in \Omega\}, \\ (\theta, c)(t, x, y) = (\theta_1, c_1)(y), \quad t > 0, \quad (x, y) \in \Gamma_L = \{x = L, y \in \Omega\}, \end{cases}$$

$$(1.4)\qquad \begin{cases} \theta = 0, \quad d\partial c/\partial\nu = \epsilon^2 h(\beta - c), \quad t > 0, \\ (x, y) \in \Gamma = \{0 < x < L,\ y \in \partial\Omega\}, \end{cases}$$

where ν is the outward nomal unit vector on the boundary $\partial\Omega$. The condition on c in (1.4) indicates that the in- or out-flux of the reactant, with the flux rate $\epsilon^2 h$ through the boundary Γ, depends on the difference between β and the value of c on the boundary. We simply assume h and β are both positive constants. In particular, if $h = 0$, then there is no supply of reactant through the boundary.

The reaction kinetics of (1.1) is simple in the sense that the ordinary differential equation

$$\begin{cases} \theta_t = cf(\theta) \\ c_t = -\epsilon^2 cf(\theta) \end{cases}$$

exhibits monotone dynamics. Namely, the concentration c decreases to zero and the temperature θ increases to a certain positive value as the time t progresses to ∞. Therefore the source of spatio-temporal patterns observed in exothermic reactions, such as combustion processes, has to be sought somewhere else. The boundary conditions in (1.3)-(1.4) are one of the ways to make the system open. It is this openness of the system that makes the process capable of producing complicated spatio-temporal patterns as we will see in the sequel.

The purpose of this paper is to show the existence of a family of equilibrium solutions of (1.1)-(1.4) for sufficiently small $\epsilon > 0$, and to study the stability of the solutions. We also would like to clarify the effects of the domain shape of the cross-section Ω on the profile of the equilibrium solutions. It turns out that the structure of the global solution branch of a non-linear elliptic eigenvalue problem encodes the effect of the domain shape of Ω. Refer to the conditions **(H1)** through **(H5)** below.

When the domain size is small in the axial direction, one expects that the effects of the boundary conditions on Γ_0 and Γ_1 would strongly dominate the behaviour of the solutions. Our interest, therefore, is the question :

What would happen to the behaviour of the solutions
when the domain size is appropriately large ?

As a first step to answer this question, we consider the situation in which the cylindrical domain is long in the x-direction in the sense that :

(A1) $L = 1/\epsilon$.

Rescaling the x-variable by $x/L = \epsilon x$, the equation (1.1) now becomes :

$$(1.5) \quad \begin{cases} \theta_t = \epsilon^2\theta_{xx} + \Delta\theta + cf(\theta) \\ c_t = \epsilon^2 dc_{xx} + d\Delta c - \epsilon^2 cf(\theta) \end{cases}, \quad t > 0, \ (x, y) \in \Omega_1 = (0, 1) \times \Omega.$$

Correspondingly, the initial and boundary conditions (1.2)-(1.4) become :

$$(1.6) \qquad (\theta, c)(0, x, y) = (\theta_i, c_i)(x, y) \geq 0, \quad (x, y) \in \Omega_1,$$

$$(1.7) \quad \begin{cases} (\theta, c)(t, x, y) = (\theta_0, c_0)(y), \quad t > 0, \quad (x, y) \in \Gamma_0 = \{x = 0, y \in \Omega\}, \\ (\theta, c)(t, x, y) = (\theta_1, c_1)(y), \quad t > 0, \quad (x, y) \in \Gamma_1 = \{x = 1, y \in \Omega\}, \end{cases}$$

$$(1.8)\qquad \begin{cases} \theta = 0, \;\; d\partial c/\partial\nu = \epsilon^2 h(\beta - c), \;\; t > 0, \\ (x, y) \in \Gamma = \{0 < x < 1, \; y \in \partial\Omega\}. \end{cases}$$

Thus, the problem with which we are concerned in this paper is (1.5)-(1.8) with $\epsilon > 0$ being sufficiently small.

Before stating our main results, we consider the following boundary value problem with a parameter $\lambda > 0$:

$$(1.9)\qquad \Delta\phi + \lambda f(\phi) = 0, \;\; y \in \Omega, \qquad \phi = 0, \;\; y \in \partial\Omega.$$

This problem, called *a perturbed Gelfand problem,* has been extensively investigated by many authors. One fundamental result is

THEOREM 1 (Dancer [D]). *Let Ω be a ball in $\mathbf{R}^N$. For $\lambda > 0$, the problem* (1.9) *with $\alpha > 0$ has at least one and at most finitely many positive solutions $\phi(y; \lambda)$.*

In order to obtain qualitative properties of solutions to our problem, it is important to know how the number of positive solutions $\phi(y; \lambda)$ of (1.9) depends on λ. For a finite interval ($N = 1$) or a disk ($N = 2$), if $\alpha > 0$ is small, the solution branch is monotone increasing in $\lambda > 0$, that is, there is a unique solution $\phi(y; \lambda)$ for any fixed $\lambda > 0$, while if $\alpha > 0$ is rather large, the branch takes S shape structure (Figure 1), that is, there are three sub-branches, a small $(-)$, a middle (0), and a large $(+)$ ones with two turning points at $\lambda = \Lambda_0$ and $\lambda = \Lambda_1$. The corresponding solutions ϕ_-, ϕ_0 and ϕ_+ have the following properties :

(H1) *The problem* (1.9) *has exactly three sub-branches of solutions $\phi_+(y; \lambda)$, $\phi_-(y; \lambda)$ and $\phi_0(y; \lambda)$, whose domains of definition are, respectively,*

$$[\Lambda_0, \infty), \;\; [0, \Lambda_1], \;\; [\Lambda_0, \Lambda_1],$$

and $\phi_+(y; \Lambda_0) = \phi_0(y; \Lambda_0)$, $\phi_-(y; \Lambda_1) = \phi_0(y; \Lambda_1)$ for $y \in \Omega$.

(H2) *For each pair $0 < \lambda < \lambda' < \Lambda_1$ (or $\Lambda_0 < \lambda < \lambda'$), the following inequality*

$$\phi_-(y; \lambda) < \phi_-(y; \lambda'), \;\; y \in \Omega$$

$$(\; resp. \;\; \phi_+(y; \lambda) < \phi_+(y; \lambda'), \;\; y \in \Omega \;)$$

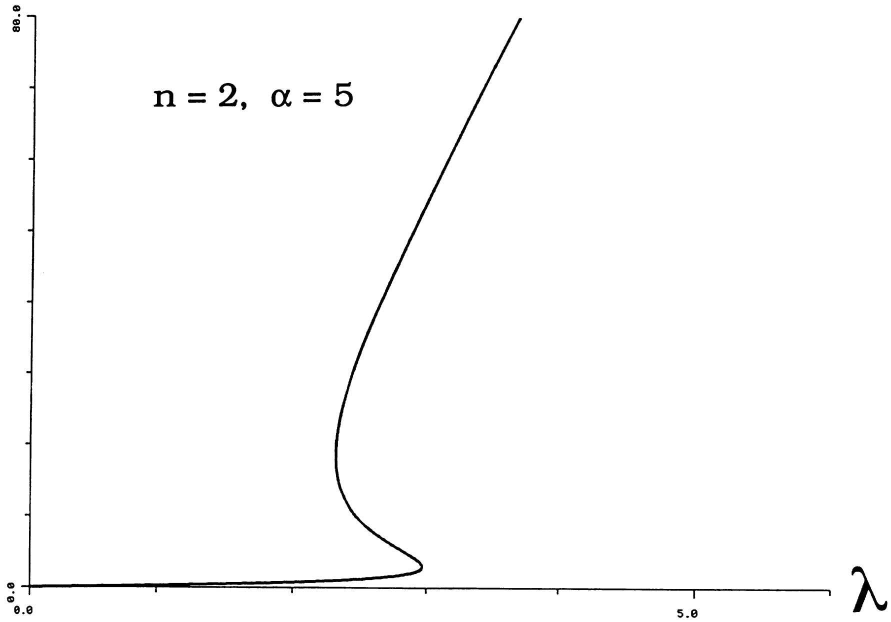

Figure 1. Global structure of the equilibrium solutions of the perturbed Gelfand problem in a 2-dimensional disk. $\alpha = 5$.

holds true.

(H3) *For* $\Lambda_0 < \lambda < \Lambda_1$, *one has the strict inequalities :*

$$\phi_-(y;\lambda) < \phi_0(y;\lambda) < \phi_+(y;\lambda), \quad y \in \Omega.$$

(H4) *The upper-branch* $\phi_+(\cdot,\lambda)$ *and the lower branch* $\phi_-(\cdot,\lambda)$ *are stable equilibria of the parabolic problem :*

$$\phi_t = \Delta\phi + \lambda f(\phi),\ t > 0,\ y \in \Omega \ \textit{and} \ \phi = 0,\ t > 0,\ y \in \partial\Omega,$$

while the middle branch $\phi_0(\cdot,\lambda)$ *is unstable with index 1.*

(H5) *For an energy functional* $H : [0,\infty) \times H_0^1(\Omega) \to \mathbf{R}$ *defined by*

$$H(\lambda, v) := \int_\Omega \left[|\nabla v|^2 - 2\lambda F(v)\right] dy, \quad F(v) := \int_0^v f(s)ds,$$

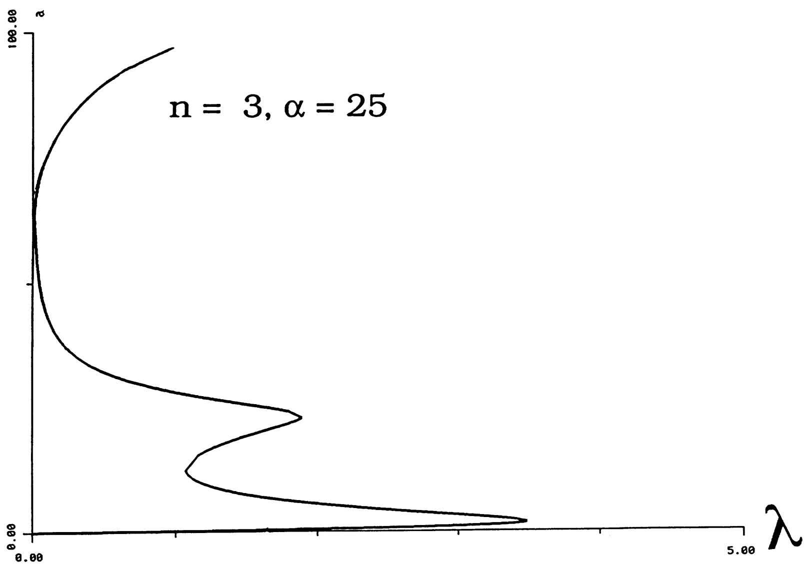

Figure 2. Global structure of the equilibrium solutions of the perturbed Gelfand problem in a 3-dimensional ball. $\alpha = 25$.

consider the difference :

$$J(\lambda) := H(\lambda, \phi_+(\cdot, \lambda)) - H(\lambda, \phi_-(\cdot, \lambda)).$$

Then $J'(\lambda) < 0$, *and there is a unique* $\lambda^* \in (\Lambda_0, \Lambda_1)$ *such that* $J(\lambda^*) = 0$.

On the other hand, in higher dimensions ($N \geq 3$), the situation is drastically different, depending on the value of α in the nonlinearity f. In fact, when $N = 3$, the global branch of solutions of (1.9) does not necessarily take S shape structure but exhibits double S shape or more complex ones for suitable values of α (Figure 2). The reader is refered to [BE].

Hereafter, we simply assume :

(A2) *The global solution branch of* (1.9) *takes the S-shape structure satisfying* **(H1)-(H5)** *in the above.*

In order to specify the situation further, we make two more assumptions, one on the boundary conditions in (1.7)-(1.8), and another on the structure of the global solution branch of (1.9).

(A3) (i) $(1/|\Omega|)\int_\Omega c_0(y)dy = \lambda_0 < \lambda^*,$

$$(1/|\Omega|)\int_\Omega c_1(y)dy = \lambda_1 > \lambda^*,$$

(ii) $\theta_0(y) = \phi_-(y;\lambda_0), \quad \theta_1(y) = \phi_+(y;\lambda_1),$

(iii) $\dfrac{\partial c_j(y)}{\partial \nu} = \epsilon^2 h(\beta - c_j(y)), \ \ y \in \partial\Omega, \ \ j = 0, 1,$

(iv) $c_j \in C^2(\bar{\Omega}), \quad ||c_j - \lambda_j||_{C^2(\bar{\Omega})} = O(\epsilon^2), \ \ j = 0, 1.$

(A4)
$$|\partial\Omega| h\Lambda_0 + \Lambda_0 \int_\Omega f(\phi_+(y;\Lambda_0))dy$$
$$> \ |\partial\Omega| h\Lambda_1 + \Lambda_1 \int_\Omega f(\phi_-(y;\Lambda_1))dy.$$

The first two conditions in assumption **(A3)** are not essential to the subsequent arguments. It only serves to simplify the presentation of the main result of our work. The conditions in **(A3)**(iii) are compatibility conditions on $\Gamma_j \cap \Gamma$. The conditions in **(A3)** (iv) are imposed to avoid arguments for boundary layers.

In order to explain the meaning of the condition **(A4)**, let us call $\phi_+(\cdot;\lambda)$ and $\phi_-(\cdot;\lambda)$, respectively, the hot state and the cold state corresponding to the uniform reactant concentration λ. From the conditions in **(A1)**, the cold state can not exist when the uniform reactant concentration is higher than Λ_1, while the hot state can not exist for the uniform reactant concetration lower than Λ_0. The condition **(A4)** is equivalent to

$$|\partial\Omega| h(\beta - \Lambda_0) - \Lambda_0 \int_\Omega f(\phi_+(y;\Lambda_0))dy$$
$$< \ |\partial\Omega| h(\beta - \Lambda_1) - \Lambda_1 \int_\Omega f(\phi_-(y;\Lambda_1))dy.$$

In both sides of the last inequality, the first term is the total supply of the reactant through the boundary and the second term represents the total amount of reactant consumed by the reaction. Therefore, each side in the above inequality represents the excess of the reactant inside the domain. The condition **(A4)** says that the excess at the upper limit of cold states exceeds that at the lower limit of the hot states. See [EM] for more detailed discussions.

The last property is certainly one of the effects of the cross-sectional domain shape Ω, as well as the rate h of the reactant supply through the boundary. By taking the rate constant h large enough, we can create a situation where the condition **(A4)** is violated. In such a case, the system will exhibit dynamical behaviors different from what will be explained in this work.

Throughout the remaining part of this paper, we understand that the constants h, λ_0 and λ_1 are fixed so that all the conditions in the above are satisfied.

We are now ready to state our main results.

THEOREM A. *Suppose that* **(A1)-(A4)** *are satisfied. There exists a constant* $l = l(\beta, d)$*, indicating a transition point, in the interval* $(0, 1)$ *such that for sufficiently small* $\epsilon > 0$*, there exists an* ϵ*-family of equilibrium solutions* $(\theta^\epsilon(x, y; \beta, d), c^\epsilon(x, y; \beta, d))$ *of* (1.5), (1.7), (1.8) *satisfying :*

(a) *for a function* $\lambda(x)$ *defined below,*

$$\lim_{\epsilon\to 0} c^\epsilon(x, y) = \lambda(x), \quad uniformly \ \ in \ \ \bar{\Omega}_1;$$

(b) *for any* $\delta > 0$,

$$\lim_{\epsilon\to 0} \theta^\epsilon(x, y) = \begin{cases} \phi_-(y; \lambda(x)), & 0 \le x \le l - \delta \\ \phi_+(y; \lambda(x)), & l + \delta \le x \le 1, \end{cases}$$

uniformly in x *and* $y \in \Omega$.

The function $\lambda(x)$ *is the solution of the following problem :*

$$0 = d\lambda_{xx} + h^*(\beta - \lambda) - G_*(\lambda), \quad 0 < x < 1,$$

$$\lambda(0) = \lambda_0, \quad \lambda(1) = \lambda_1,$$

$$\lambda(l) = \lambda^*, \quad \lambda_x(l - 0) = \lambda_x(l + 0),$$

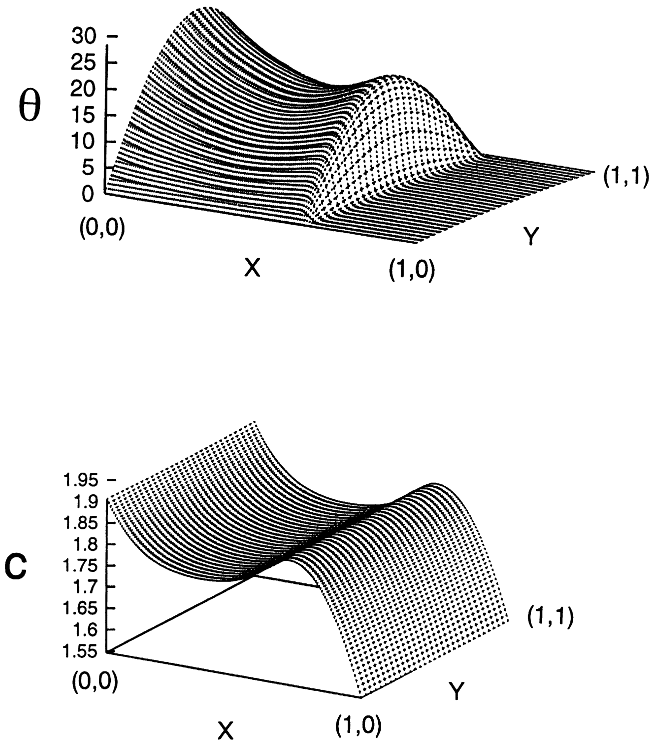

Figure 3. Spatial profile of $(\theta^\epsilon(x,y), c^\epsilon(x,y))$ with one internal layer for

$(x,y) \in \Omega_1 = \{0 < x < 1,\ \ 0 < y < 1\ \}$.

where the function G_* *and the constant* h^* *are defined by*

$$G_*(v) = \begin{cases} G_-(v) = (v/|\Omega|)\int_\Omega f(\phi_-(y;v))dy, & 0 \le v \le \lambda^*, \\ G_+(v) = (v/|\Omega|)\int_\Omega f(\phi_+(y;v))dy, & \lambda^* \le v < \infty, \end{cases}$$

and $h^* = h|\partial\Omega|/|\Omega|$, *respectively.*

The proof will be given in Section 3. From the profile of $\theta^\epsilon(x,y)$ (Figure 3), we call the solution $(c^\epsilon(x,y), \theta^\epsilon(x,y))$ an equilibrium solution with one layer. Theorem A clearly indicates that the profile of the equilibrium solution in Ω is approximately given by the solutions of *the perturbed Gelfand* *problem* (1.9). Although we restrict our consideration in Theorem A to equilibrium solutions with one internal layer, we can readily extend it to equilibrium solutions with multi-layers.

We next consider the stability of the equilibrium solutions obtained in Theorem A. In order to do so, we let constants $\beta_\pm > 0$ be defined by

$$\beta_\pm = \lambda^* + \frac{1}{h^*} G_\pm(\lambda^*).$$

Note that $\beta_- < \beta_+$. The stability properties of the solution $(\theta^\epsilon, c^\epsilon)$ is described in the following :

THEOREM B. *Suppose* **(A1)-(A4)** *are satisfied.*

(i) *For* $\beta \in (0, \beta_-] \cup [\beta_+, \infty)$, *the solution* $(\theta^\epsilon, c^\epsilon)$ *in Theorem A is stable for* $d > 0$.

(ii) *For* $\beta \in (\beta_-, \beta_+)$, *there exists a constant* $d_\epsilon(\beta) > 0$ *such that*

(a) *the solution* $(\theta^\epsilon, c^\epsilon)$ *in Theorem A is stable for* $d > d_\epsilon$ *;*

(b) *when* d *passes through* d_ϵ, *the solution undergoes a Hopf-bifurcation ;*

(c) *the limit* $d_* = \lim_{\epsilon \to 0} d_\epsilon$ *is characterized by*

$$d_*^{-1/2} = \int_{\lambda_0}^{\lambda^*} \left[2 \int_{\lambda}^{\lambda^*} g_-(s) ds \right]^{-1/2} d\lambda + \int_{\lambda^*}^{\lambda_1} \left[2 \int_{\lambda}^{\lambda^*} g_+(s) ds \right]^{-1/2} d\lambda,$$

where $g_\pm(s) = h^*(\beta - s) - G_\pm(s)$.

The proof will be given in Section 4. Figure 4 exhibits some results of our numerical simulation for the case $\Omega = (0,1)$, where the value of $\theta(t, x, 1/2)$ is collor-coded. It shows how the location of an oscillating layer in the periodic solution θ^ϵ behaves as time progresses when the parameter d decreases from the critical value of d_ϵ. The numerical simulation indicates that the Hopf-bifurcation in the theorem is a *super-critical* one, although we have not proved this theoretically. The global structure of solutions with respect to d is numerically shown in [MNS1].

An intuitive explanation of the result in Theorem B (ii) is as follows. For $\beta \in (\beta_-, \beta_+)$, we have

$$h^*(\beta - \lambda^*) - G_-(\lambda^*) > 0 > h^*(\beta - \lambda^*) - G_+(\lambda^*).$$

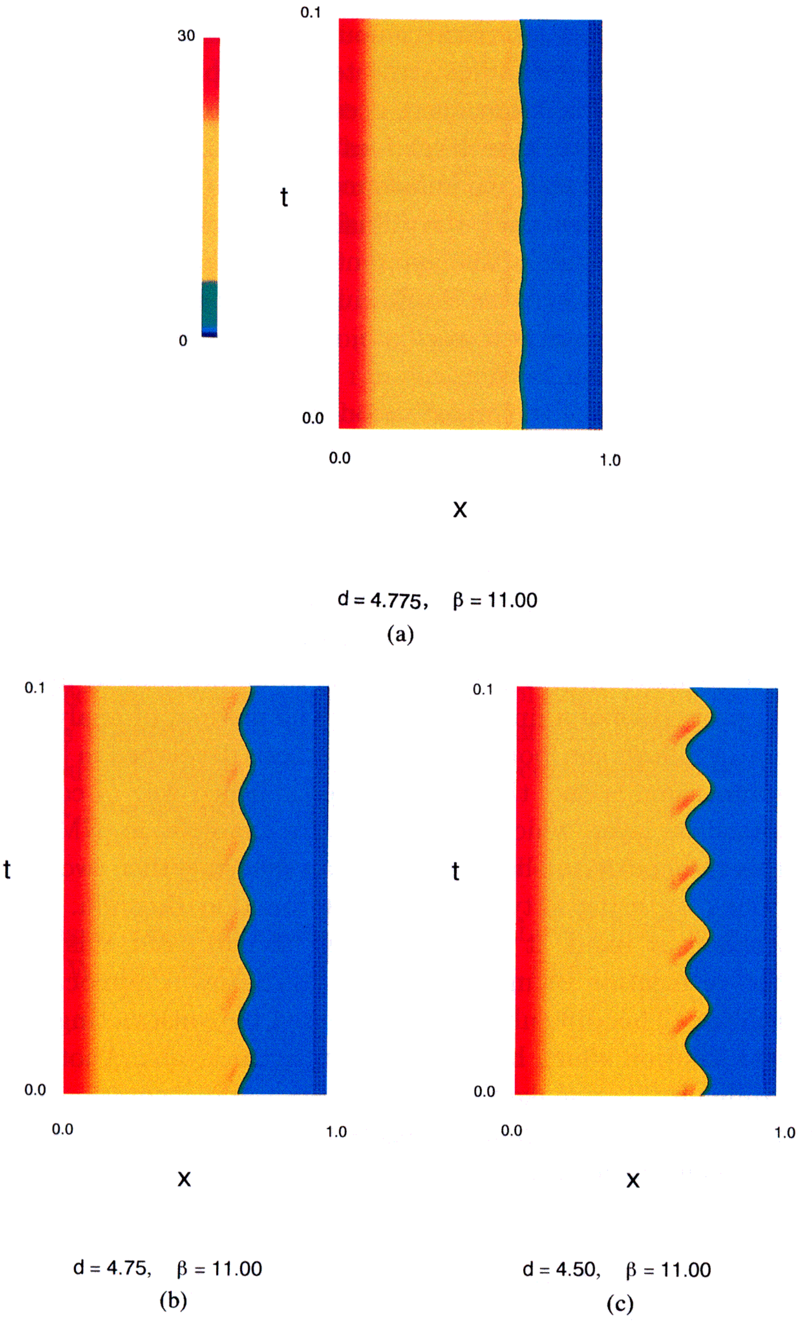

Figure 4. An oscillatiting layer position arising in $\theta^\epsilon(t, x, 1/2)$ for a periodic solution $\theta^\epsilon(t, x, y)$.

This means that there is an excess amount of the reactant for the cold state at $\lambda = \lambda^*$ and that the hot state tends to consume more reactant than can be supplied. Therefore, there occurs an imbalance of the reactant distribution near the transition layer location $x = l$. When the diffusion rate d of the reactant is large, the imbalance is adjusted due to the diffusion effect. When, on the other hand, the diffusion rate is lower than the critical value d_0, the diffusion effect of the reactant is no more capable of restoring the imbalance, and therefore, the equilibrium state can no longer be stable. As a result, the system starts to oscillate, as if, in an effort to settle down on a comfortable position.

We now briefly outline the organization of this paper.

Section 2 is devoted to the construction of approximate solutions. Basic ideas therein are straightforward generalization of singular perturbation expansions for one dimensional reaction-diffusion systems [Sk], although there are several new technical difficulties involved. Some of the difficulties are overcome by the beautiful results of Vega [V].

In Section 3, we prove that there is a true solution near the approximate solutions. To achieve this, a crucial point is to know detailed information on spectral properties of a linear operator. The method of analysis for the linear operator follows the footsteps of the methods developed in [Sk]. When the space dimension is one, there is a well-established way of constructing internal transition layer, which is employed, for example, in [NM]. The key point of this method is to show that one can glue together two boundary layer solutions by using a type of implicit function theorem. However, for our problem at hand, it is very difficult to carry out this procedure because the verification of matching condition poses a nonlocal nolinear elliptic problem. This difficulty is circumvented by constructing a smooth approximate solution which has internal transition layers. Another key to aviod the difficulty is to make the approximation too accurate. If one tries to make the approximation more accurate than ours, then one again faces the same difficulty as mentioned above. See **Remark 2** in Section 2.

Section 4 deals with the stability analysis of the equilibrium solutions. The analysis reveals that a Hopf-bifurcation occurs when the diffusion coefficient d of the reactant decreases. A similar type of bifurcation analysis was carried out in [NM] for a singularly perturbed reaction-diffusion system on one dimensional space. In [NM], the bifurcation parameter is a parameter controling the reaction rates of two reactants, while, in the present work, the bifurcation parameter is the diffusion rate of the concentration c. In the system treated in [NM], one reactant diffuses much slower and reacts much faster than the other, while in our problem diffusion rates of the two components are comparable. The nonlinearity in [NM] has a bistability properties of some sort. In the present paper, however, the nonlinearity is rather simple and monotone, exhibiting no apparent *bistability*. Our system, on the other hand, has an implicit *bistability* encoded in the global solution branch of *the perturbed Gelfand problem* (1.9). See **(H1)-(H5)**. This observation is one of the points we would like to strongly emphasize. In Section 5, we discuss several ramifications of the present work and its relation to a free boundary problem derived in [MSE]. It turns out that the free boundary problem captures the essential dynamics of the original problem (1.5)-(1.8), in the sense that the stability properties of the equilibrium solutions in Theorem A are obtained from studying the free boundary problem.

2. Approximation

In this section, we construct approximate solutions to the problem (1.5)-(1.8). It is convenient to rewrite the problem(1.5)-(1.8) in such a way that the boundary conditions for c in (1.8) become homogeneous. For this purpose, let g be a function satisfying

$$g \in C^2(\bar{\Omega}), \quad \partial g/\partial\nu = 1/d \text{ on } \partial\Omega.$$

One can easily construct such a function as above when the boundary $\partial\Omega$ is smooth. We now transform the variables (c, θ) to (v, u) by

$$v = \exp[\epsilon^2 hg(y)](c - \beta), \quad u = \theta,$$

then the equation (1.5) is recast as

$$\text{(2.1)} \qquad \begin{cases} u_t = \epsilon^2 u_{xx} + \Delta u + \{v \exp[-\epsilon^2 hg] + \beta\} f(u) \\ v_t = \epsilon^2 d v_{xx} + d\Delta v - \epsilon^2 E_1(u, v, \epsilon^2) + \epsilon^4 E_2(v), \end{cases}$$

where

$$E_1(u, v, \epsilon^2) = \{v + \beta \exp[\epsilon^2 hg]\} f(u) + 2dh\nabla v \cdot \nabla g + dhv\Delta g,$$

$$E_2(v) = dh^2 v|\nabla g|^2.$$

It is also more illuminating to decompose the v-variable into two parts :

$$v(x,y,t)=\bar{v}(x,y,t)+w(x,t),$$

where

$$w(x,t)=(1/|\Omega|)\int_{\Omega}v(x,y,t)dy, \qquad \int_{\Omega}\bar{v}(x,y,t)dy=0.$$

In terms of $(u,\bar{v},w)$ the equation (2.1) becomes

$$\tag{2.2} \begin{cases} u_t=\epsilon^2 u_{xx}+\Delta u+\{(v+w)\exp[-\epsilon^2 hg]+\beta\}f(u) \\ v_t=\epsilon^2 dv_{xx}+d\Delta v-\epsilon^2 R_1(u,v,w,\epsilon^2)+\epsilon^4 R_2(v,w) \\ \epsilon^{-2}w_t=dw_{xx}-A_1(u,v,w,\epsilon^2)+\epsilon^2 A_2(v,w), \end{cases}$$

where we dropped the bar from $\bar{v}$ for the sake of simplicity in notation. The functions $A_1,\ A_2$ and $R_1,\ R_2$ are given by

$$A_1(u,v,w,\epsilon^2)=(1/|\Omega|)\int_{\Omega}E_1(u,v+w,\epsilon^2)dy,$$

$$A_2(v,w)=(1/|\Omega|)\int_{\Omega}E_2(v+w)dy,$$

and

$$R_1(u,v,w,\epsilon^2)=E_1(u,v+w,\epsilon^2)-A_1(u,v,w,\epsilon^2),$$

$$R_2(v,w)=E_2(v+w)-A_2(v,w).$$

The equations in (2.2) are supplemented by the boundary conditions :

$$\tag{2.3} \begin{cases} (u,v)(t,0,y)=(\phi_-(y;\lambda_0),c_0^{\epsilon}(y)), \ \ t>0, \ \ y\in\Omega, \\ (u,v)(t,1,y)=(\phi_+(y;\lambda_1),c_1^{\epsilon}(y)), \ \ t>0, \ \ y\in\Omega, \\ w(0,t)=w_0^{\epsilon}, \quad w(1,t)=w_1^{\epsilon}, \\ u=0, d\partial v/\partial\nu=0, \quad t>0, \ \ (x,y)\in\Gamma, \end{cases}$$

where

$$w_j^\epsilon = (1/|\Omega|)\int_\Omega \exp[\epsilon^2 hg(y)](c_j(y)-\beta)dy,$$

$$c_j^\epsilon(y) = \exp[\epsilon^2 hg(y)](c_j(y)-\beta) - w_j^\epsilon$$

for $j = 0,\ 1$. It is easy to see that $w_j^\epsilon = \lambda_j - \beta + O(\epsilon^2)$ and $c_j^\epsilon = O(\epsilon^2)$.

REMARK 1. We note, in passing, that the second equation in (2.2) with the boundary condition $\partial v/\partial \nu = 0$ implies that $v(x,y,t) = O(\epsilon^2)$ uniformly in (x,y,t). Therefore, the essential dynamics of (2.2) is captured by a system of equations simpler than the original, such as

$$\begin{cases} u_t = \epsilon^2 u_{xx} + \Delta u + \{w+\beta\}f(u) \\ \epsilon^{-2} w_t = dw_{xx} - h^* w - \frac{w+\beta}{|\Omega|}\int_\Omega f(u(t,x,y))dy. \end{cases}$$

This system of equations looks more like a system of one-dimensinal equations. In fact, we showed in [MSE] that the equation (2.2) (or the problem (1.5)-(1.8)) could be approximated by one-dimensinal systems, at least qualitatively. A rigorous justification of reducing (2.2) to a one dimensinal system will be reported in a forthcoming paper. For now, one is advised to think of the simple version in the above, which contains all the essential ingredients of the original, in the subsequent analyses.

In order to prove Theorem A, we consider the following equation :

$$\begin{cases} 0 = \epsilon^2 u_{xx} + \Delta u + \{(v+w)\exp[-\epsilon^2 hg] + \beta\}f(u) \\ 0 = \epsilon^2 d v_{xx} + d\Delta v - \epsilon^2 R_1(u,v,w,\epsilon^2) + \epsilon^4 R_2(v,w) \\ 0 = dw_{xx} - A_1(u,v,w,\epsilon^2) + \epsilon^2 A_2(v,w) \end{cases} \tag{2.4}$$

under the boundary conditions (2.3). The purpose of the following subsections is to construct a C^2-approximate solution which satisfies the equation (2.4) within an order of ϵ^2.

REMARK 2. We emphasize that we do not attempt to make our approximation more accurate than the one given below. If one tries to construct approximate solutions with accuracy $O(\epsilon^k)$, $\ k \geq 3$, then, in order to make the approximation C^1 on $[0,1]\times\Omega$, one encounters the difficulty mentioned to in Introduction.

2.1. Outer solutions

We will approximate the solutions of (2.4), (2.3) in the regions where $\epsilon^2 u_{xx}$ is not significant (called outer regions). Let us substitute the formal expressions

$$u = U^0 + \epsilon U^1, \quad v = V^0 + \epsilon V^1, \quad w = W^0 + \epsilon W^1$$

into the equation (2.4). Equating the coefficient of ϵ^0 to zero, we obtain the following equations for (U^0, V^0, W^0).

$$\begin{cases} 0 = \Delta U^0 + (V^0 + W^0 + \beta) f(U^0) \\ 0 = d\Delta V^0 \\ 0 = dW_{xx} - A_1(U^0, V^0, W^0, 0). \end{cases} \tag{2.5}$$

The boundary conditions are :

$$\begin{cases} (U^0, V^0, W^0) = (\phi_-(y; \lambda_0), 0, \lambda_0 - \beta) \ \text{ on } \ \Gamma_0 \\ (U^0, V^0, W^0) = (\phi_+(y; \lambda_1), 0, \lambda_1 - \beta) \ \text{ on } \ \Gamma_1 \\ (U^0, \partial V^0/\partial\nu) = (0, 0) \quad \text{on } \ \Gamma. \end{cases} \tag{2.6}$$

The second equation in (2.5), the boundary conditions on Γ and the constraint

$$\int_\Omega V^0(x, y) dy = 0,$$

together imply that $V^0 \equiv 0$. Therefore the equation (2.5) reduces to

$$\begin{cases} 0 = \Delta U^0 + (W^0 + \beta) f(U^0) \\ 0 = dW^0_{xx} - h^* W^0 - \frac{W^0 + \beta}{|\Omega|} \int_\Omega f(U^0) dy. \end{cases} \tag{2.7}$$

The first equation in (2.7), under the boundary conditions $U^0(x, y) = 0, \quad y \in \partial\Omega$, has the three branches of solutions $\phi_j(y; W^0(x) + \beta), \ (j = -, +, 0)$. Due to the first and the second lines of (2.6), we can choose

$$U^0(x, y) = \begin{cases} \phi_-(y; W^0(x) + \beta), & 0 \le x \le l \\ \phi_+(y; W^0(x) + \beta), & l \le x \le 1 \end{cases}$$

for some $l \in (0,1)$, which is to be determined. If we now write as $\lambda(x) = W^0(x) + \beta$, then the second equation in (2.7) becomes :

$$0 = d\lambda_{xx} + h^*(\beta - \lambda) - \begin{cases} \frac{\lambda}{|\Omega|}\int_\Omega f(\phi_-(y;\lambda(x)))dy & 0 \le x \le l, \\ \frac{\lambda}{|\Omega|}\int_\Omega f(\phi_+(y;\lambda(x)))dy & l \le x \le 1. \end{cases}$$

Since the second derivative of the solutions of this equation necessarily has a jump discontinuity at $x = l$, we seek C^1-matched solutions. Namely, we pose the following problem :

$$\text{(2.8)} \qquad \begin{cases} 0 = d\lambda_{xx} + h^*(\beta - \lambda) - G_*(\lambda) \\ \lambda(0) = \lambda_0, \quad \lambda(1) = \lambda_1, \\ \lambda(l) = \lambda^*, \quad \lambda_x(l-0) = \lambda_x(l+0), \end{cases}$$

where the determination of the switching point $x = l$ is a part of the problem. The reason why we specify the value λ^* for $\lambda(x)$ at $x = l$ will become clear in the next subsection where we will construct an inner solution to smooth out the jump discontinuity in $U^0(x,y)$ at $x = l$.

For the problem (2.8), we have :

LEMMA 2.1. *For $\beta > 0$, there exist constants $d_0(\beta) < d_1(\beta)$ such that for $d \in (d_0, d_1)$, the problem* (2.8) *has a unique solution* $(\lambda(x,\beta,d), l(\beta,d))$.

The proof of this lemma will be given in Appendix.

The equation for (U^1, V^1, W^1) is

$$\text{(2.9)} \qquad \begin{aligned} &0 = \Delta U^1 + \lambda(x) f'(\phi_\pm)U^1 + \{V^1 + W^1\} f(\phi_\pm) \\ &0 = d\Delta V^1 \\ &0 = dW^1_{xx} - h^*W^1 - \frac{\lambda(x)}{|\Omega|}\int_\Omega f'(\phi_\pm)U^1 dy - \frac{1}{|\Omega|}\int_\Omega f(\phi_\pm)dy W^1. \end{aligned}$$

The second equation for V^1 in (2.9) and the boundary condition $\partial V^1/\partial\nu = 0$ on $\partial\Omega$, together with the constraint $\int_\Omega V(x,y)dy = 0$ for each $x \in [0,1]$, imply that $V^1 \equiv 0$. The first equation for U^1 then gives two branches of solutions :

$$\begin{aligned} U^{1,\pm} &= -W^1 \left[\Delta + \lambda(x) f'(\phi_\pm)\right]^{-1} f(\phi_\pm) \\ &= W^1 \frac{\partial \phi_\pm}{\partial \lambda}, \end{aligned}$$

where the existence of $[\Delta + \lambda(x) f'(\phi_\pm)]^{-1}$ is insured by **(H4)** in Section 1, and the identity

$$\left[\Delta + \lambda(x) f'(\phi_\pm)\right]^{-1} f(\phi_\pm) = -\frac{\partial \phi_\pm}{\partial \lambda}$$

is easily obtained from

$$\Delta \frac{\partial \phi_\pm}{\partial \lambda} + \lambda f'(\phi_\pm) \frac{\partial \phi_\pm}{\partial \lambda} + f(\phi_\pm) = 0.$$

Substituting $(U^{1,\pm}, V^1)$ into the third equation of (2.9), we obtain the following two equations for W^1 :

$$0 = dW^{1,-}_{xx} + g'_-(\lambda(x)) W^{1,-}$$

and

$$0 = dW^{1,+}_{xx} + g'_+(\lambda(x)) W^{1,+},$$

where $g_\pm(s) = h^*(\beta - s) - G_\pm(s)$. We consider the equation for $W^{1,-}$ (resp. $W^{1,+}$) on $(0, l)$ (resp. $(l, 1)$) under the boundary conditions

$$W^{1,-}(0) = 0, \quad W^{1,-}(l) = b,$$

$$(\text{resp. } W^{1,+}(1) = 0, \quad W^{1,+}(l) = b \ ),$$

where b is a constant to be specified when we consider inner solutions in the next subsection. The boundary values at $x = 0$ and $x = 1$ come from (2.3). Since the condition **(H2)** implies that ${g_\pm}' < 0$, the boundary value problems for $W^{1,\pm}$ are uniquely solvable. If we denote by $\lambda^{1,+}$ (resp. $\lambda^{1,-}$) the solution of

$$0 = d\lambda^{1,+}_{xx} + g'_+(\lambda(x))\lambda^{1,+}$$

$$(\text{resp. } 0 = d\lambda^{1,-}_{xx} + g'_-(\lambda(x))\lambda^{1,-} \ )$$

with the boundary conditions

$$\lambda^{1,+}(1) = 0, \ \ \lambda^{1,+}(l) = 1$$

$$(\text{resp. } \lambda^{1,-}(0) = 0, \ \ \lambda^{1,-}(l) = 1 \ ),$$

then we find $W^{1,\pm} = b\lambda^{1,\pm}$. One should also observe that the boundary values of $U^{1,-}$ on Γ_0 and the boundary values of $U^{1,+}$ on Γ_1 are both zero due to the facts $W^{1,-}(0) = 0$ and $W^{1,+}(1) = 0$. Therefore, the outer approximation we have obtained so far is :

$$U^\epsilon_{out} = \begin{cases} \phi_-(y;\lambda(x)) + \epsilon U^{1,-}(x,y), & 0 \le x \le l, \\ \phi_+(y;\lambda(x)) + \epsilon U^{1,+}(x,y), & l \le x \le 1, \end{cases}$$

$$V^\epsilon_{out} = 0, \quad 0 \le x \le 1,$$

$$W^\epsilon_{out} = \lambda(x) - \beta + \begin{cases} \epsilon W^{1,-}(x), & 0 \le x \le l, \\ \epsilon W^{1,+}(x), & l \le x \le 1. \end{cases}$$

Although it is possible to obtain higher order approximation to outer solutions, we would not do so because the second order approximation in the above will be sufficient for our purpose.

2.2. Inner solutions

The outer approximation in the previous subsection has a jump discontinuity at $x = l(\beta, d)$ in the u-component. To smooth this out, we introduce inner solutions in this subsection. One should also note that the boundary conditions on Γ_0 and Γ_1 in (2.3) are not satisfied by the outer approximation. It misses the boundary conditions within an order of ϵ^2. This discrepancy will be taken care of later on.

In order to smooth out the jump at $x = l$, let us introduce a stretched variable $\xi = (x - l)/\epsilon$. We also let

$$\Phi(y;\lambda) = \phi_+(y;\lambda) - \phi_-(y;\lambda),$$

and look for inner solutions in the form

$$\begin{aligned} u &= \Phi(y; \lambda(x) + \epsilon W^1(x) + \epsilon^2 w^2(\xi) + \epsilon^3 w^3(\xi))[z^0(\xi, y) + \epsilon z^1(\xi, y)] \\ &\quad + \phi_-(y; \lambda(x) + \epsilon W^1(x) + \epsilon^2 w^2(\xi) + \epsilon^3 w^3(\xi)), \\ v &= 0, \\ w &= W^0(x) + \epsilon W^1(x) + \epsilon^2 w^2(\xi) + \epsilon^3 w^3(\xi). \end{aligned}$$

Setting $x = \epsilon\xi + l$ in the above, we substitute the (u, v, w) into the equation (2.4). We then equate the coefficient of each power of ϵ to zero to obtain the equations for z^0, z^1, w^2, and w^3, successively. We could include the terms such as $\epsilon v^1(\xi), \epsilon^2 v^2(\xi), \epsilon w^1(\xi)$ in the above, but $\epsilon v^1(\xi), \epsilon w^1(\xi)$ turn out to be identically zero and v^2 will be taken care of when we show that there exists a genuine solution near the approximation later on (see Section 3). The equation for z^0 is

$$\begin{aligned} 0 &= \Phi(y;\lambda^*) z^0_{\xi\xi} + \Delta\{\Phi(y;\lambda^*) z^0 + \phi_-(y;\lambda^*)\} \\ &\quad + \lambda^* f(\Phi(y;\lambda^*) z^0 + \phi_-(y;\lambda^*)) \end{aligned}$$

with the boundary conditions

$$\lim_{\xi\to-\infty} z^0(\xi, y) = 0, \quad \lim_{\xi\to\infty} z^0(\xi, y) = 1.$$

If we set $u^0(\xi, y) = \Phi(y;\lambda^*) z^0 + \phi_-(y;\lambda^*)$, then u^0 satisfies

$$(2.10) \qquad \begin{cases} u^0_{\xi\xi} + \Delta u^0 + \lambda^* f(u^0) = 0, & (\xi, y) \in \mathbf{R} \times \Omega \\ u^0(\xi, y) = 0, & (\xi, y) \in \mathbf{R} \times \partial\Omega \\ u^0(-\infty, y) = \phi_-(y;\lambda^*), & u^0(\infty, y) = \phi_+(y;\lambda^*), \quad y \in \Omega. \end{cases}$$

For the problem (2.10), Vega [V] gives a beautiful result as follows.

THEOREM 2. *Under the conditions* **(H1)-(H5)** *with* $\lambda = \lambda^*$, *there is a solution* u^0 *of* (2.10), *which is unique up to phase shifts in* ξ*-variable. Moreover,*

$$u^0_\xi(\xi, y) > 0, \quad (\xi, y) \in \mathbf{R} \times \Omega,$$

and $u^0(\xi, y) - \phi_\pm(y; \lambda^*)$, $u^0_\xi(\xi, y)$ *and* $u^0_{\xi\xi}(\xi, y)$ *decay exponentially to zero in* $C^1(\bar{\Omega})$*-norm as* $\xi \to \pm\infty$.

We denote the unique solution of (2.10) by u^0, with an appropriately fixed phase. When we need to consider a solution of (2.10) obtained from u^0 by a phase shift γ, we use the symbol

$$u^{0,\gamma}(\xi, y) := u^0(\xi + \gamma, y).$$

Correspondingly, we use the symbol $z^{0,\gamma}$ for z^0 associated with $u^{0,\gamma}$. The phase shift γ will be determined by the solvability condition of the following equation for z^1 :

$$\begin{aligned}
0 = {} & \Phi(y; \lambda^*) z^1_{\xi\xi} + 2\Phi_\lambda(y; \lambda^*)\lambda_x(l) z^{0,\gamma}_\xi + \Phi_\lambda(y; \lambda^*) z^{0,\gamma}_{\xi\xi}\{\lambda_x(l)\xi + b\} \\
& + \Delta\Big[\Phi(y; \lambda^*) z^1 + \{\Phi_\lambda(y; \lambda^*) z^{0,\gamma} + \phi_{-\lambda}(y; \lambda^*)\}\{\lambda_x(l)\xi + b\}\Big] \\
& + \{\lambda_x(l)\xi + b\} f(u^{0,\gamma}) \\
& + \lambda^* f'(u^{0,\gamma})\Big[\Phi(y; \lambda^*) z^1 + \{\Phi_\lambda(y; \lambda^*) z^{0,\gamma} + \phi_{-\lambda}(y; \lambda^*)\}\{\lambda_x(l)\xi + b\}\Big],
\end{aligned}$$

where $\Phi_\lambda = \partial\Phi/\partial\lambda$ and $\phi_{-\lambda} = \partial\phi_-/\partial\lambda$. The boundary conditions are

$$\lim_{\xi \to \pm\infty} z^1(\xi, y) = 0, \quad \text{and} \quad z^1 = 0, \ \text{ on } \ \mathbf{R} \times \partial\Omega.$$

If we set $u^1(\xi, y) = \Phi(y; \lambda^*) z^1(\xi, y)$, then the last equation reduces to

$$0 = u^1_{\xi\xi} + \Delta u^1 + \lambda^* f'(u^{0,\gamma}) u^1 + p_1(\xi, y), \tag{2.11}$$

where

$$p_1(\xi,y) = 2\Phi_\lambda(y;\lambda^*)z_\xi^{0,\gamma}\lambda_x(l) + \Phi_\lambda(y;\lambda^*)z_{\xi\xi}^{0,\gamma}\{\lambda_x(l)\xi+b\}$$
$$+\left[\Delta\{\Phi_\lambda(y;\lambda^*)z^{0,\gamma}+\phi_{-\lambda}(y;\lambda^*)\}\right]\{\lambda_x(l)\xi+b\}$$
$$+\Big[f(u^{0,\gamma})+\lambda^* f'(u^{0,\gamma})\{\Phi_\lambda(y;\lambda^*)z^{0,\gamma}$$
$$+\phi_{-\lambda}(y;\lambda^*)\}\Big]\{\lambda_x(l)\xi+b\}.$$

Using the fact that the following limits are exponentially approached :

$$\lim_{\xi\to\pm\infty}(z_\xi^0, z_{\xi\xi}^0) = (0,0), \quad \lim_{\xi\to\infty} z^0 = 1, \quad \lim_{\xi\to-\infty} z^0 = 0,$$

and the identities

$$0 = \Delta\phi_{\pm\lambda}(y;\lambda^*) + \lambda^* f'(\phi_\pm(y;\lambda^*))\phi_{\pm\lambda}(y;\lambda^*) + f(\phi_\pm(y;\lambda^*)),$$

it is shown that the inhomogeneous term $p_1(\xi,y)$ decays exponentially to zero as $\xi\to\pm\infty$. As for the solvability of (2.11), we have the following result by employing the alternative theorem of Fredholm.

LEMMA 2.2. *The problem* (2.11) *is solvable if and only if the condition*

$$\int_{-\infty}^{\infty}\int_\Omega p_1(\xi,y)u_\xi^{0,\gamma}(\xi,y)d\xi dy = 0$$

is satisfied.

If we change variables by $\xi\to\xi-\gamma$, the solvability condition in the above is written as

$$-\gamma I_1 + I_0 = 0,$$

where I_1 and I_0 are respectively given by

$$I_1 = \lambda_x(l) \int_{-\infty}^{\infty} \int_{\Omega} f(u^0(\xi, y)) u_\xi^0(\xi, y) dy d\xi$$

$$+ \lambda_x(l) \int_{-\infty}^{\infty} \int_{\Omega} \Big[\Delta u_\lambda^*(\xi, y) + \lambda^* f'(u^0(\xi, y)) u_\lambda^*(\xi, y) \Big] u_\xi^0(\xi, y) dy d\xi$$

$$+ \lambda_x(l) \int_{-\infty}^{\infty} \int_{\Omega} \Phi_\lambda(y; \lambda^*) z_{\xi\xi}^0(\xi, y) u_\xi^0(\xi, y) dy d\xi,$$

where

$$u_\lambda^*(\xi, y) = \Phi_\lambda(y; \lambda^*) z^0(\xi, y) + \phi_{-\lambda}(y; \lambda^*).$$

I_1 can be simplified as :

$$I_1 = \lambda_x(l) \int_{\Omega} \left[F(\phi_+(y; \lambda^*)) - F(\phi_-(y; \lambda^*)) \right] dy.$$

This is obtained by integrating by parts with respect to y and using the equation

$$u_{\xi\xi\xi}^0 + \Delta u_\xi^0 + \lambda^* f'(u^0) u_\xi^0 = 0,$$

and again integrating by parts twice with respect to ξ as follows.

$$\int_{-\infty}^{\infty} \int_{\Omega} \Big[\Delta \left[\Phi_\lambda(y; \lambda^*) z^0(\xi, y) + \phi_{-\lambda}(y; \lambda^*) \right]$$

$$+ \lambda^* f'(u^0(\xi, y)) \Big] u_\xi^0(\xi, y) dy d\xi$$

$$= \int_{-\infty}^{\infty} \int_{\Omega} [\Delta u_\xi^0(\xi, y) + \lambda^* f'(u^0(\xi, y)) u_\xi^0(\xi, y)][\Phi_\lambda z^0 + \phi_{-\lambda}] dy d\xi$$

$$= -\int_{-\infty}^{\infty} \int_{\Omega} u_{\xi\xi\xi}^0 [\Phi_\lambda z^0 + \phi_{-\lambda}] dy d\xi = -\int_{-\infty}^{\infty} \int_{\Omega} \Phi_\lambda z_{\xi\xi}^0 u_\xi^0 dy d\xi.$$

By the same computation, I_0 is given as follows.

$$I_0 = 2\lambda_x(l) \int_{-\infty}^{\infty} \int_{\Omega} u_\xi^0(\xi, y) \Phi_\lambda(y; \lambda^*) z_\xi^0(\xi, y) d\xi dy$$

$$+ \lambda_x(l) \int_{-\infty}^{\infty} \int_{\Omega} u_\xi^0(\xi, y) \xi q_1(\xi, y) dy d\xi$$

$$+ b \int_{\Omega} \left[F(\phi_+(y; \lambda^*)) - F(\phi_-(y; \lambda^*)) \right] dy,$$

where

$$q_1(\xi, y) = \Phi_\lambda(y; \lambda^*) z_{\xi\xi}^0(\xi, y) + \Delta \left[\Phi_\lambda(y; \lambda^*) z^0(\xi, y) + \phi_{-\lambda}(y; \lambda^*) \right]$$
$$+ f(u^0(\xi, y)) + \lambda^* f'(u^0(\xi, y)) \left[\Phi_\lambda(y; \lambda^*) z^0(\xi, y) + \phi_{-\lambda}(y; \lambda^*) \right].$$

Therefore, if $\lambda_x(l) \neq 0$, we choose the phase shift to be $\gamma = I_0/I_1$. Note that

$$\gamma = \frac{1}{\lambda_x(l)} b + c_0,$$

where c_0 is a constant independent of b and γ. The constant b will be chosen later on. With this choice of the phase shift γ, we denote $u^{0,\gamma}$ simply by u^0. Then the problem (2.11) has a family of solutions

$$a u_\xi^0(\xi, y) + \bar{u}^1(\xi, y),$$

where $\bar{u}^1$ is a unique solution of (2.11) satsfying

$$\int_{-\infty}^{\infty} \int_{\Omega} u_\xi^0(\xi, y) \bar{u}^1(\xi, y) d\xi dy = 0.$$

The free parameter a in the above is determined by a solvability condition for the next level of approximation. Since higher order approximations are not necessary for our purpose in this paper, we simply set the free parameter $a = 0$.

If, on the other hand, $\lambda_x(l) = 0$, we choose $b = 0$ so that the solvability condition is satisfied, in which case we have $W^1(x) \equiv 0$, and $u^1 \equiv 0$ with the freedom of the phase shift still remaining.

The equation for w^2 reads

$$0 = dw^2_{\xi\xi} + dW^0_{xx}(l \pm 0) + h^*(\beta - \lambda^*) - (\lambda^*/|\Omega|) \int_\Omega f(u^0(\xi, y))dy.$$

It is here we need to solve the problem on each of the half lines $(-\infty, 0)$ and $(0, \infty)$. By using the identity

$$0 = dW^1_{xx}(l \pm 0) + h^*(\beta - \lambda^*) - \frac{\lambda^*}{|\Omega|} \int_\Omega f(\phi_\pm(y; \lambda^*))dy,$$

the solutions on each of the half lines are given by

$$dw^{2,\pm}(\xi) = \int_{\pm\infty}^{\xi} \int_{\pm\infty}^{\tau} (\lambda^*/|\Omega|) \int_\Omega \left[f(u^0(s, y)) - f(\phi_\pm(y; \lambda^*))\right] dyds d\tau,$$

which decay to zero exponentially as $\xi \to \pm\infty$. We now define a C^0-matched solution $w^2(\xi)$ by

$$w^2(\xi) = \begin{cases} w^{2,-}(\xi) - w^{2,-}(0), & \xi \le 0, \\ w^{2,+}(\xi) - w^{2,+}(0), & \xi \ge 0. \end{cases}$$

When $\lambda_x(l) \neq 0$, we choose the constant b so that

$$b\lambda_x^{1,-}(l - 0) + w^2_\xi(-0) = b\lambda_x^{1,+}(l + 0) + w^2_\xi(+0). \tag{2.12}$$

In order to show that b can be chosen so that (2.12) is satisfied, we argue as follows. By using the explict expressions for $w^2_\xi(\pm 0)$ above, we have

$$w^2_\xi(+0) - w^2_\xi(-0) = \frac{1}{d}\Bigg[\int_\infty^\gamma \frac{\lambda^*}{|\Omega|} \int_\Omega [f(u^0(s, y)) - f(\phi_+(y; \lambda^*))]dyds$$
$$- \int_{-\infty}^\gamma \frac{\lambda^*}{|\Omega|} \int_\Omega [f(u^0(s, y)) - f(\phi_-(y; \lambda^*))]dyds\Bigg]$$

with $\gamma = c_0 + b/\lambda_x(l)$. Now we have

$$\frac{\partial}{\partial\gamma}\big(w_\xi^2(+0) - w_\xi^2(-0)\big) = -\frac{1}{d}\frac{\lambda^*}{|\Omega|}\int_\Omega [f(\phi_+(y;\lambda^*)) - f(\phi_-(y;\lambda^*))]dy$$
$$= \frac{1}{d}(g_+(\lambda^*) - g_-(\lambda^*)),$$

which gives rise to

$$w_\xi^2(+0) - w_\xi^2(-0) = \frac{g_+(\lambda^*) - g_-(\lambda^*)}{d\lambda_x(l)}b + c_1$$

where c_1 is a constant independent of b and γ. Therefore one can uniquely choose b so as to satisfy (2.12), provided that

$$\lambda_x(l)\big(\lambda_x^{1,-}(l-0) - \lambda_x^{1,+}(l+0)\big) - \frac{g_+(\lambda^*) - g_-(\lambda^*)}{d} \neq 0$$

is true. To show this inequality, we use the differential equations $\lambda_x(x)$ and $\lambda^{1,\pm}(x)$ satisfy on $(0,l)$ and/or on $(l,1)$. Multiplying the equation of $\lambda^{1,+}$ (resp. $\lambda^{1,-}$) by λ_x and integrating on $(l,1)$ (resp. on $(0,l)$) by parts, it follows that

$$\lambda_x(l)\lambda_x^{1,+}(l+0) = \lambda_x(1)\lambda_x^{1,+}(1) - \frac{g_+(\lambda^*)}{d}$$

and

$$\lambda_x(l)\lambda_x^{1,-}(l-0) = \lambda_x(0)\lambda_x^{1,-}(0) - \frac{g_-(\lambda^*)}{d}.$$

Therefore,

$$\lambda_x(l)\big(\lambda_x^{1,-}(l-0) - \lambda_x^{1,+}(l+0)\big) - \frac{g_+(\lambda^*) - g_-(\lambda^*)}{d}$$
$$= \lambda_x(0)\lambda_x^{1,-}(0) - \lambda_x(1)\lambda_x^{1,+}(1) := W^*.$$

Now, from the proof in Appendix, $\lambda_x(0) > 0$, $\lambda_x(1) > 0$. Moreover, the differential equation for $\lambda^{1,\pm}$ together with $g'_\pm(\lambda(x)) < 0$ and $\lambda^{1,\pm}(l) = 1$ implies $\lambda_x^{1,-}(0) > 0$ and $\lambda_x^{1,+}(1) < 0$, which show $W^* > 0$. We can therefore choose b so that (2.12) is satisfied.

On the other hand, when $\lambda_x(l) = 0$, we have already chosen the constant $b = 0$. In this case, however, we still have the free parameter γ, the phase shift. We now choose it to satisfy

$$w^2_\xi(-0) = w^2_\xi(+0). \tag{2.13}$$

This is possible since

$$dw^2_\xi(-0) = \int_{-\infty}^{\gamma} (\lambda^*/|\Omega|) \int_\Omega \big[f(u^0(s,y)) - f(\phi_-(y;\lambda^*))\big]\, dyds > 0$$

is monotone increasing (note that the integrand is positive) from zero at $\xi = -\infty$ to ∞ at $\xi = \infty$, while

$$dw^2_\xi(+0) = \int_{\infty}^{\gamma} (\lambda^*/|\Omega|) \int_\Omega \big[f(u^0(s,y)) - f(\phi_+(y;\lambda^*))\big]\, dyds > 0$$

is monotone decreasing (note that the integrand is negative) from ∞ at $\xi = -\infty$ to zero at $\xi = \infty$. Therefore, there exists a unique value of γ for which the relation (2.13) is satisfied.

The equation for w^3 is

$$\begin{aligned} 0 = {} & dw^3_{\xi\xi} + dW^1_{xx}(l \pm 0) + d\lambda_{xxx}(l \pm 0)\xi \\ & - \frac{\lambda^*}{|\Omega|} \int_\Omega f'(u^0)\Big[u^1 + \{\Phi_\lambda z^0 + \phi_{-\lambda}\}\{\lambda_x(l)\xi + b\}\Big] dy \\ & - h^*\{\lambda_x(l)\xi + b\} - \frac{\lambda_x(l)\xi + b}{|\Omega|} \int_\Omega f(u^0) dy. \end{aligned}$$

When $\lambda_x(l) = 0$, we have $b = 0$, $W^1 \equiv 0$, $\lambda_{xxx}(l \pm 0) = 0$, and $u^1 \equiv 0$. Therefore w^3 is identically equal to zero. In this case, the condition (2.14) below is trivially satisfied.

When $\lambda_x(l) \neq 0$, we have to solve the equation for w^3 on each of the half lines $(-\infty, 0)$ and $(0, \infty)$. By using the relations

$$dW^1_{xx}(l \pm 0) = h^* W^1(l) + \frac{\lambda^*}{|\Omega|} \int_\Omega f'(\phi_\pm)\phi_{\pm\lambda} W^1(l) dy + \frac{W^1(l)}{|\Omega|} \int_\Omega f(\phi_\pm) dy$$

and

$$d\lambda_{xxx}(l \pm 0) = h^* \lambda_x(l) + \frac{\lambda^*}{|\Omega|} \int_\Omega f'(\phi_\pm)\phi_{\pm\lambda}\lambda_x(l)dy + \frac{\lambda_x(l)}{|\Omega|} \int_\Omega f(\phi_\pm)dy,$$

the equations for $w^{3,\pm}$ become

$$\begin{aligned} dw^{3,\pm}_{\xi\xi} = & \frac{\lambda^*}{|\Omega|} \int_\Omega f'(u^0(\xi,y))u^1(\xi,y)dy \\ & + \frac{\lambda^*}{|\Omega|} \int_\Omega \left[f'(u^0)\{\Phi_\lambda z^0 + \phi_{-\lambda}\} - f'(\phi_\pm)\phi_{\pm\lambda} \right] dy \{\lambda_x(l)\xi + b\} \\ & + \frac{\lambda_x(l)\xi + b}{|\Omega|} \int_\Omega \left[f(u^0) - f(\phi_\pm) \right] dy, \end{aligned}$$

which have unique solutions decaying exponentially at $\pm\infty$:

$$\begin{aligned} dw^{3,\pm}(\xi) = & \int_{\pm\infty}^{\xi} \int_{\pm\infty}^{\tau} \frac{\lambda^*}{|\Omega|} \int_\Omega f'(u^0(s,y))u^1(s,y)dyds d\tau \\ & + \int_{\pm\infty}^{\xi} \int_{\pm\infty}^{\tau} \frac{\lambda^*}{|\Omega|} \int_\Omega \Big[f'(u^0(s,y))\{\Phi_\lambda(s,y)z^0(s,y) + \phi_{-\lambda}(s,y)\} \\ & \qquad - f'(\phi_\pm(y;\lambda^*))\phi_{\pm\lambda}(y;\lambda^*) \Big] dy \{\lambda_x(l)s + b\} ds d\tau \\ & + \int_{\pm\infty}^{\xi} \int_{\pm\infty}^{\tau} \frac{\lambda_x(l)s + b}{|\Omega|} \int_\Omega \left[f(u^0(s,y)) - f(\phi_\pm(y;\lambda^*)) \right] dyds d\tau. \end{aligned}$$

We now define w^3 by

$$w^3(\xi) = \begin{cases} w^{3,-}(\xi) - w^{3,-}(0) - \xi w^{3,-}_\xi(0), & \xi \le 0, \\ w^{3,+}(\xi) - w^{3,+}(0) - \xi w^{3,+}_\xi(0), & \xi \ge 0. \end{cases}$$

Notice that $w^3(\xi)$ is C^1-matched at $\xi = 0$. Also, one can easily verify the following

$$W^1_{xx}(l-0) + w^{3,-}_{\xi\xi}(0) = W^1_{xx}(l+0) + w^{3,+}_{\xi\xi}(0) \tag{2.14}$$

by using the explicit forms of $w^{3,\pm}$ above and the C^2-property of u^0 and u^1.

Now let us define a tentative approximation $(U_T^\epsilon, W_T^\epsilon)$, which is valid on $[0,1]\times\Omega$, by

$$W_T^\epsilon(x) = W^0(x) + \epsilon W^1(x) + \epsilon^2 w^2(\frac{x-l}{\epsilon}) + \epsilon^3 w^3(\frac{x-l}{\epsilon}),$$

$$U_T^\epsilon(x,y) = \Phi(y; W_T^\epsilon(x))\left[z^0(\frac{x-l}{\epsilon}, y) + \epsilon z^1(\frac{x-l}{\epsilon}, y)\right] + \phi_-(y; W_T^\epsilon(x)).$$

These functions, W_T^ϵ and U_T^ϵ, are C^2-functions due to the relations (2.12)-(2.14). Moreover, W_T^ϵ is $O(\exp[-\delta/\epsilon])$-near of W_{out}^ϵ and U_T^ϵ is $O(\exp[-\delta/\epsilon])$-near of U_{out}^ϵ (with respect to the supremum norm) away from the transition position $\{l\}\times\Omega \in [0,1]\times\Omega$. This is true because of the exponentially decaying properties of w^2, w^3, z^0, and z^1. Therefore the approximation $(U_T^\epsilon, 0, W_T^\epsilon)$ satisfies (2.4) within an error bound of $O(\epsilon^2)$ in the outer region. We will now show that our approximation also satisfies (2.4) within an error bound $O(\epsilon^2)$ in the inner region. In fact, if we substitute the approximation into (2.4), written in terms of the stretched variable ξ and y, then the coefficients of ϵ^0 and ϵ^1 vanish in the resulting equations, due to the construction of $(U_T^\epsilon, W_T^\epsilon)$. Now the coefficient of ϵ^2 in the second equation of (2.4), the v-equation, is easily seen to be bounded uniformly in $(\xi, y) \in \mathbf{R}\times\bar{\Omega}$. After some computation, the coefficients of ϵ^2 in the u-component and the w-component in (2.4) are respectively given by

$$\begin{aligned}(2.15)\quad & \tilde{w}^2\Big[\tilde{L}^*[\Phi_\lambda(y;\lambda^*)z^0(\xi,y) + \phi_{-\lambda}(y;\lambda^*)] + f(u^0(\xi,y))\Big] \\ & + \frac{1}{2}(\tilde{w}^1)^2\Big[\tilde{L}^*[\Phi_{\lambda\lambda}z^0 + \phi_{-\lambda\lambda}] + 2f'(u^0)[\Phi_\lambda z^0 + \phi_{-\lambda}] \\ & \qquad + \lambda^* f''(u^0)[\Phi_\lambda z^0 + \phi_{-\lambda}]^2\Big] \\ & + \tilde{w}^1\tilde{L}^*[\Phi_\lambda z^1] + \frac{1}{2}\lambda^* f''(u^0)(u^1)^2 + \lambda^* f''(u^0)u^1[\Phi_\lambda z^0 + \phi_{-\lambda}]\tilde{w}^1 \\ & + \tilde{w}^1 f'(u^0)u^1 \\ & + [\Phi_\lambda z^0 + \phi_{-\lambda}]\tilde{w}^2_{\xi\xi} + 2[\Phi_\lambda z^0 + \phi_{-\lambda}]_\xi \tilde{w}^2_\xi - hg(y)W^0(l)f(u^0)\end{aligned}$$

$$+ 2[\Phi_\lambda z^1]_\xi \tilde{w}^1_\xi + 2[\Phi_{\lambda\lambda} z^0 + \phi_{-\lambda\lambda}]_\xi \tilde{w}^1 \tilde{w}^1_\xi + [\Phi_{\lambda\lambda} z^0 + \phi_{-\lambda\lambda}](\tilde{w}^1_\xi)^2,$$

and

$$\begin{aligned}
&- \frac{\xi^2}{2}\Bigg[\frac{W^0_{xx}(l)}{|\Omega|}\int_\Omega [f(u^0(\xi,y)) - f(\phi_\pm(y;\lambda^*))]dy \\
&\quad + 2\frac{(W^0_x(l))^2}{|\Omega|}\int_\Omega \big[f'(u^0(\xi,y))[\Phi_\lambda z^0 + \phi_{-\lambda}] - f'(\phi_\pm)\phi_{\pm\lambda}\big]dy \\
&\quad + \frac{\lambda^*(W^0_x(l))^2}{|\Omega|}\int_\Omega \big[f''(u^0)[\Phi_\lambda z^0 + \phi_{-\lambda}]^2 - f''(\phi_\pm)(\phi_{\pm\lambda})^2\big]dy \\
&\quad + \frac{\lambda^*(W^0_x(l))^2}{|\Omega|}\int_\Omega \big[f'(u^0)[\Phi_{\lambda\lambda} z^0 + \phi_{-\lambda\lambda}] - f'(\phi_\pm)\phi_{\pm\lambda\lambda}\big]dy \\
&\qquad + \frac{\lambda^* W^0_{xx}(l)}{|\Omega|}\int_\Omega \big[f'(u^0)[\Phi_\lambda z^0 + \phi_{-\lambda\lambda}] - f'(\phi_\pm)\phi_{\pm\lambda}\big]dy\Bigg]
\end{aligned}$$

(2.16)

$$\begin{aligned}
&- \xi\Bigg[\frac{W^1_x(l)}{|\Omega|}\int_\Omega [f(u^0(\xi,y)) - f(\phi_\pm(y;\lambda^*))]dy \\
&\quad + 2\frac{W^0_x(l)W^1(l)}{|\Omega|}\int_\Omega \big[f'(u^0(\xi,y))[\Phi_\lambda z^0 + \phi_{-\lambda}] - f'(\phi_\pm)\phi_{\pm\lambda}\big]dy \\
&\quad + \frac{\lambda^* W^0_x(l)W^1(l)}{|\Omega|}\int_\Omega \big[f''(u^0)[\Phi_\lambda z^0 + \phi_{-\lambda}]^2 - f''(\phi_\pm)(\phi_{\pm\lambda})^2\big]dy \\
&\quad + \frac{\lambda^* W^0_x(l)W^1(l)}{|\Omega|}\int_\Omega \big[f'(u^0)[\Phi_{\lambda\lambda} z^0 + \phi_{-\lambda\lambda}] - f'(\phi_\pm)\phi_{\pm\lambda\lambda}\big]dy \\
&\qquad + \frac{\lambda^* W^1_x(l)}{|\Omega|}\int_\Omega \big[f'(u^0)[\Phi_\lambda z^0 + \phi_{-\lambda\lambda}] - f'(\phi_\pm)\phi_{\pm\lambda}\big]dy\Bigg] \\
&- \frac{\lambda^* W^1(l)}{|\Omega|}\int_\Omega f'(u^0)dy - \frac{(W^1(l))^2}{|\Omega|}\int_\Omega f'(u^0(\xi,y))[\Phi_\lambda z^0 + \phi_{-\lambda\lambda}]dy \\
&- \frac{1}{2}\frac{\lambda^*(W^1(l))^2}{|\Omega|}\int_\Omega f''(u^0(\xi,y))[\Phi_\lambda z^0 + \phi_{-\lambda}]^2 dy
\end{aligned}$$

$$- \frac{hW^0(l)}{|\Omega|} \int_\Omega g(y) f(u^0) dy$$

$$- w^2(\xi) \Big[h^* + \frac{1}{|\Omega|} \int_\Omega f(u^0) dy + \frac{\lambda^*}{|\Omega|} \int_\Omega f'(u^0)[\Phi_\lambda z^0 + \phi_{-\lambda}] dy \Big]$$

$$- \frac{\lambda^* \tilde{w}^1}{|\Omega|} \int_\Omega f'(u^0(\xi, y)) \Phi_\lambda z^1 dy - \frac{\tilde{w}^1}{|\Omega|} \int_\Omega f'(u^0(\xi, y)) u^1 dy$$

$$- \frac{1}{2} \frac{\lambda^*}{|\Omega|} \int_\Omega f''(u^0(\xi, y))[(u^1)^2 + 2u^1[\Phi_\lambda z^0 + \phi_{-\lambda}] \tilde{w}^1] dy,$$

in which we used the following symbols ;

$$\tilde{L}^* u = u_{\xi\xi} + \Delta u + \lambda^* f'(u^0(\xi, y)) u$$

and

$$\tilde{w}^1 = W_x^0(l)\xi + W^1(l), \quad \tilde{w}^2 = \frac{\xi^2}{2} W_{xx}^0(l) + \xi W_x^1(l) + w^2(\xi).$$

By using again the exponetially decaying properties of z^0, z^1, w^2 together with the differntial equations $\phi_{\pm\lambda}$ and $\phi_{\pm\lambda\lambda}$ satisfy ;

$$\Delta\phi_\lambda + \lambda^* f'(\phi)\phi_\lambda + f(\phi) = 0,$$

$$\Delta\phi_{\lambda\lambda} + \lambda^* f'(\phi)\phi_{\lambda\lambda} + 2f'(\phi)\phi_\lambda + \lambda^* f''(\phi)(\phi_\lambda)^2 = 0,$$

one can readily verify that the quantities in (2.15) and (2.16) are bounded uniformly in $(\xi, y) \in \mathbf{R} \times \Omega$. Therefore, we conclude that our approximate solutions satisfy the equation (2.4) within an error bound of $O(\epsilon^2)$ measured by the supremum norm.

3. Proof of Theorem A

In this section, we will prove that there is a solution of (2.4) near $(U_T^\epsilon, 0, W_T^\epsilon)$. The construction of the approximation in the previous section shows

$$||\epsilon^2 (U_T^\epsilon)_{xx} + \Delta U_T^\epsilon + \{W_T^\epsilon \exp[-\epsilon^2 hg] + \beta\} f(U_T^\epsilon)||_{L^\infty} = O(\epsilon^2)$$

and

$$||(W_T^\epsilon)_{xx} - A_1(U_T^\epsilon, 0, W_T^\epsilon, \epsilon^2) + \epsilon^2 A_2(0, W_T^\epsilon)||_{L^\infty} = O(\epsilon^2).$$

In this sense, $(U_T^\epsilon, 0, W_T^\epsilon)$ is a good approximation. However, the boundary conditions on Γ_0 and Γ_1 are not satisfied. We now modify the approximation so that the boundary conditions are satisfied. Note, first of all, that $||c_j^\epsilon||_{C^2(\Omega)} = O(\epsilon^2)$. So, let us define V^ϵ by

$$V^\epsilon(x,y) = c_0^\epsilon(y) + x\{c_1^\epsilon(y) - c_0^\epsilon(y)\}.$$

We also define (U^ϵ, W^ϵ) by

$$W^\epsilon(x) = W_T^\epsilon(x) + (1-x)[w_0^\epsilon - W_T^\epsilon(0)] + x[w_1^\epsilon - W_T^\epsilon(1)],$$

$$U^\epsilon(x,y) = U_T^\epsilon(x) + (1-x)[\phi_-(y;\lambda_0) - U_T^\epsilon(0,y)] + x[\phi_+(y;\lambda_1) - U_T^\epsilon(1,y)].$$

One can easily verify $|w_0^\epsilon - W_T^\epsilon(0)| = O(\epsilon^2)$, $|w_1^\epsilon - W_T^\epsilon(1)| = O(\epsilon^2)$ as well as

$$||\phi_-(\cdot;\lambda_0) - U_T^\epsilon(0,\cdot)||_{C^2(\bar\Omega)} = O(\epsilon^2), \qquad ||\phi_+(\cdot;\lambda_1) - U_T^\epsilon(1,\cdot)||_{C^2(\bar\Omega)} = O(\epsilon^2).$$

We define R_j^ϵ $(j = 1,2,3)$, by

$$\begin{aligned} &R_1^\epsilon := \epsilon^2 U_{xx}^\epsilon + \Delta U^\epsilon + P_1(U^\epsilon, V^\epsilon, W^\epsilon, \epsilon^2), \\ (3.1)\qquad &R_2^\epsilon := \epsilon^2 dV_{xx}^\epsilon + \Delta dV^\epsilon + \epsilon^2 P_2(U^\epsilon, V^\epsilon, W^\epsilon, \epsilon^2), \\ &R_3^\epsilon := dW_{xx}^\epsilon + P_3(U^\epsilon, V^\epsilon, W^\epsilon, \epsilon^2), \end{aligned}$$

where

$$P_1(u,v,w,\epsilon^2) = \{(v+w)\exp[-\epsilon^2 hg] + \beta\}f(u),$$

$$P_2(u,v,w,\epsilon^2) = -R_1(u,v,w,\epsilon^2) + \epsilon^2 R_2(v,w),$$

$$P_3(u,v,w,\epsilon^2) = -A_1(u,v,w,\epsilon^2) + \epsilon^2 A_2(v,w).$$

Then, we find $R_j^\epsilon = O(\epsilon^2)$ $(j = 1,2,3)$.

Let us now look for a true solution of the problem (2.3)-(2.4) as the following type of perturbation from $(U^\epsilon, V^\epsilon, W^\epsilon)$:

$$u = U^\epsilon + p + U_\lambda^\epsilon r, \quad v = V^\epsilon + q, \quad w = W^\epsilon + r,$$

where

$$U_\lambda^\epsilon(x,y) = \Phi_\lambda(y; W_T^\epsilon(x))[z^0(\frac{x-l}{\epsilon}, y) + \epsilon z^1(\frac{x-l}{\epsilon}, y)] + \phi_{-\lambda}(y; W_T^\epsilon(x)).$$

The equation for (p,q,r) is

$$\text{(3.2)} \qquad \begin{cases} L_1^\epsilon p + L_2^\epsilon q + L_3^\epsilon r + R_1^\epsilon + N_1^\epsilon(p,q,r) = 0 \\ K_1^\epsilon q + \epsilon^2 K_2^\epsilon p + \epsilon^2 K_3^\epsilon r + R_2^\epsilon + \epsilon^2 N_2^\epsilon(p,q,r) = 0 \\ M_1^\epsilon r + M_2^\epsilon p + M_3^\epsilon q + R_3^\epsilon + N_3^\epsilon(p,q,r) = 0. \end{cases}$$

The boundary conditions for (p,q,r) are

$$\text{(3.3)} \quad \begin{cases} \text{(a)} \quad p = 0, \quad \text{on } \Gamma_0, \ \Gamma_1 \ \text{ and } \Gamma, \\ \text{(b)} \quad q = 0, \quad \text{on } \Gamma_0 \text{ and } \Gamma_1, \quad \partial q/\partial\nu = 0, \ \text{ on } \Gamma, \quad \int_\Omega q dy = 0, \\ \text{(c)} \quad r(0) = 0 = r(1). \end{cases}$$

In (3.2), $L_j^\epsilon, K_j^\epsilon$ and M_j^ϵ $(j = 1,2,3)$, are linear operators given by

$$L_1^\epsilon p = \epsilon^2 p_{xx} + \Delta p + \{(V^\epsilon + W^\epsilon)\exp[-\epsilon^2 hg] + \beta\} f'(U^\epsilon) p,$$

$$L_2^\epsilon q = \exp[-\epsilon^2 hg] f(U^\epsilon) q,$$

$$L_3^\epsilon r = L_1^\epsilon[U_\lambda^\epsilon r] + \exp[-\epsilon^2 hg] f(U^\epsilon) r,$$

$$K_1^\epsilon q = \epsilon^2 q_{xx} + d\Delta q + \epsilon^2 \frac{\partial P_2}{\partial v}(U^\epsilon, V^\epsilon, W^\epsilon, \epsilon^2) q,$$

$$K_2^\epsilon p = \frac{\partial P_2}{\partial u}(U^\epsilon, V^\epsilon, W^\epsilon, \epsilon^2) p,$$

$$K_3^\epsilon r = \Big[\frac{\partial P_2}{\partial u}(U^\epsilon, V^\epsilon, W^\epsilon, \epsilon^2) U_\lambda^\epsilon + \frac{\partial P_2}{\partial w}(U^\epsilon, V^\epsilon, W^\epsilon, \epsilon^2)\Big] r,$$

$$M_1^\epsilon r = dr_{xx} + \Big[\frac{\partial P_3}{\partial u}(U^\epsilon, V^\epsilon, W^\epsilon, \epsilon^2)U_\lambda^\epsilon + \frac{\partial P_3}{\partial w}(U^\epsilon, V^\epsilon, W^\epsilon, \epsilon^2)\Big] r,$$

$$M_2^\epsilon p = \frac{\partial P_3}{\partial u}(U^\epsilon, V^\epsilon, W^\epsilon, \epsilon^2)p,$$

$$M_3^\epsilon q = \frac{\partial P_3}{\partial v}(U^\epsilon, V^\epsilon, W^\epsilon, \epsilon^2)q.$$

$N_j^\epsilon(p,q,r)$ $(j = 1,2,3)$, are nonlinear terms in (p,q,r) of order $O(|p|^2 + |q|^2 + |r|^2)$. In this section, we will solve the problem (3.2)-(3.3). The idea of solution method is very simple, although we have to work hard to actually cary it out.

We will first solve the second and the third equations of (3.2) in (q,r) as a function of p. Substituting it into the first equation of (3.2), we obtain a single equation for p. Analyzing the linear part of the resulting single equation, we will show the solvability of the problem (3.2) by a Liapunov-Shcmidt type of procedure. This will be done in subsection 3.2, after we establish preliminary lemmas in the following subsection 3.1.

3.1. Spectral Analysis

Let us define $H_D^2(0,1)$ by $H_D^2(0,1) = \{r \in H^2(0,1); r(0) = 0 = r(1)\}$ and denote by $H_{D,N}^2$ the set of functions in $H^2(\Omega_1)$ satisfying the boundary conditions in (3.3 (b)).

LEMMA 3.1. (i) *The operator* $M_1^\epsilon : H_D^2(0,1) \to L^2(0,1)$ *has an inverse*

$$(M_1^\epsilon)^{-1} : L^2(0,1) \to H_D^2(0,1),$$

which is bounded by a constant independent of small $\epsilon > 0$. *Consequently,*

$$(M_1^\epsilon)^{-1} : C^0(0,1) \to C^0(0,1)$$

is bounded by a constant independent of small $\epsilon > 0$.

(ii) *The operator* $K_1^\epsilon : H_{D,N}^2 :\to L^2(\Omega_1)$ *is invertible. The inverse*

$$(K_1^\epsilon)^{-1} : L^2(\Omega_1) \to L^2(\Omega_1)$$

is bounded by a constant independent of small $\epsilon > 0$. *Moreover,* $(K_1^\epsilon)^{-1}$ *maps* $C^0(\Omega_1)$ *into itself and is bounded by a constant independent of small* $\epsilon > 0$.

Proof. (i) The operator M_1^ϵ is written as

$$M_1^\epsilon r = M_1^* r + a^\epsilon(x) r,$$

where

$$M_1^* r = d r_{xx} - h^* r - G'_*(\lambda(x)) r$$

and $||a^\epsilon(x)||_{L^\infty(0,1)} = O(\epsilon)$. Since M_1^* is invertible, the conclusion of the lemma directly follows.

(ii) Since the operator K_1^ϵ has the form

$$K_1^\epsilon q = \epsilon^2 d q_{xx} + d\Delta q + \epsilon^2 a^\epsilon(x,y) q,$$

where $a^\epsilon(x,y)q$ is a differential operator of first order with bounded coefficients, it suffices to show that the first eigenvalue of $\epsilon^2 d q_{xx} + d\Delta q$ with the boundary conditions in (3.3 (b)) is bounded away from zero by a negative constant. This is easily seen true because the first eigenvalue of the Laplacian is negative.

In order to prove the second statement, let us consider the equation

$$K_1^\epsilon q = b,$$

where $b \in C^0(\Omega_1)$ and $\int_\Omega b(x,y)dy = 0,\ x \in [0,1]$. By using the L^p-theory [GT, Theorem 9.15 and Lemma 9.16] together with the result in the above, one can conclude that the equation $K_1^\epsilon q = b$ has a unique solution which belongs to $W^{2,p}(\Omega_1)$ for $p \geq 2$. We have $q \in C^0(\Omega_1)$ by the imbedding theorm [GT, p 158] for sufficiently large p. Therefore, $(K_1^\epsilon)^{-1}$ maps $C^0(\Omega_1)$ into itself. We now show that the bound of $(K_1^\epsilon)^{-1}$ is bounded for small $\epsilon > 0$. (The following argument is suggested by [T].) If this were not true, we can find sequences $\{\epsilon_j\}$, $\{b_j\}$ and $\{q_j\}$ such that

$$\epsilon_j \to 0,\ \ ||b_j||_{C^0(\Omega_1)} \to 0,\ \text{ as }\ j \to \infty, \quad ||q_j||_{C^0(\Omega_1)} = 1, \quad \text{and}$$

$$K_1^{\epsilon_j} q_j = b_j.$$

The equation is rewritten in terms of a stretched variable $\xi = (x - x_0)/\epsilon$, $x_0 \in (0,1)$ as

$$d\tilde{q}_{j\xi\xi} + d\Delta\tilde{q}_j + \epsilon^2 \tilde{a}^\epsilon(\xi, y)\tilde{q}_j = \tilde{b}_j.$$

Choosing an appropriate convergent subsequence of $\{\tilde{q}_j\}$ by the usual compactness argument and calling the subsequence $\{\tilde{q}_j\}$ again, one can show that as $j \to \infty$ $\tilde{q}_j$ approaches a solution of the problem

$$d\tilde{q}_{\xi\xi} + d\Delta\tilde{q} = 0, \quad \tilde{q}(\pm\infty, y) = 0,\ y \in \Omega, \quad \frac{\partial \tilde{q}}{\partial \nu}(\xi, y) = 0,\ \xi \in \mathbf{R},\ \ y \in \partial\Omega,$$

which has a unique solution $\tilde{q} \equiv 0$. Since this is true for each chioce of $x_0 \in (0,1)$, we contradict the fact $||q_j||_{C^0(\Omega_1)} = 1$. □

LEMMA 3.2. (i) *The first eigenvalue μ_0^ϵ of the operator L_1^ϵ is of order ϵ and has the following limiting behavior :*

$$\lim_{\epsilon \to 0} \frac{\mu_0^\epsilon}{\epsilon} = \hat{\mu}_0^* := -\frac{\lambda_x(l)}{\kappa^2} \int_\Omega [F(\phi_+(y;\lambda^*)) - F(\phi_-(y;\lambda^*))]\, dy,$$

where $\kappa > 0$ is given by

$$\kappa^2 = \int_{-\infty}^{\infty} \int_\Omega \left[u_\xi^0(\xi, y)\right]^2 dy d\xi.$$

(ii) *There is a constant $\rho_0 > 0$ such that μ_0^ϵ is the only eigenvalue of L_1^ϵ in the half line $[-\rho_0, \infty)$.*

(iii) *The L^2-normalized first eigenfunction ψ_0^ϵ of the operator L_1^ϵ has the limiting behavior :*

$$\lim_{\epsilon \to 0} \sqrt{\epsilon}\psi_0^\epsilon(\epsilon\xi + l, y) = \frac{1}{\kappa} u_\xi^0(\xi, y)$$

C^2-uniformly on compact subsets of $\mathbf{R} \times \bar{\Omega}$.

(iv) *If we denote by $\bar{L}_1^\epsilon$ the restriction of L_1^ϵ to the orthogonal complement $[\psi_0^\epsilon]^\perp$ of the span of ψ_0^ϵ, then $\bar{L}_1^\epsilon$ maps $[\psi_0^\epsilon]^\perp \cap C^0(\Omega_1)$ into itself and is bounded by a constant independent of small $\epsilon > 0$.*

PROOF. Let us consider a scaled version of L_1^ϵ. Define $\tilde{L}_1^\epsilon$ by

$$\tilde{L}_1^\epsilon \tilde{p} := \tilde{p}_{\xi\xi} + \Delta\tilde{p} + \{(\tilde{V}^\epsilon + \tilde{W}^\epsilon)\exp[-\epsilon^2 hg] + \beta\} f'(\tilde{U}^\epsilon)\tilde{p},$$

where the functions with tilde such as $\tilde{p}$ are considered as a function of the stretched variable $\xi = (x-l)/\epsilon$ and y. While the eigenvalues of $\tilde{L}_1^\epsilon$ are the same as those of L_1^ϵ, the eigenfunctions $\tilde{\psi}_j^\epsilon$ of $\tilde{L}_1^\epsilon$ and the eigenfunctions ψ_j^ϵ of L_1^ϵ are related as follows :

$$\tilde{\psi}_j^\epsilon(\xi, y) = \sqrt{\epsilon}\psi_j^\epsilon(\epsilon\xi + l, y).$$

Now one should observe that the operator $\tilde{L}_1^\epsilon$ approaches $\tilde{L}_1^*$ defined by

$$\tilde{L}_1^*\tilde{p} := \tilde{p}_{\xi\xi} + \Delta\tilde{p} + \lambda^* f'(u^0(\xi, y))\tilde{p},$$

C^2-uniformly on compact sets of $\mathbf{R} \times \Omega$ as $\epsilon \to 0$. The limiting operator $\tilde{L}_1^*$ has the following propeties : (a) zero is a simple eigenvalue with a corresponding eigenfunction being $u_\xi^0(\xi, y)$, and (b) there exists a positive constant, say ρ_0, such that zero is the only point of the spectra of $\tilde{L}_1^*$ in the half interval $[-\rho_0, \infty)$. These facts are easily reduced from a Vega's result [V], saying that $u_\xi^0(\xi, y) > 0$, and the fact that the essential spectrum of the self-adjoint operator $\tilde{L}_1^*$ is bounded away from zero.

By using a standard compactness argument, one can show that the eigenpairs of $\tilde{L}_1^\epsilon$ converge to those of $\tilde{L}_1^*$ as $\epsilon \to 0$. Of course, some of the eigenvalues of $\tilde{L}_1^\epsilon$ approach the continuous spectrum of $\tilde{L}_1^*$. Therefore, parts (ii) and (iii) of the lemma have been established, as well as the fact that $\mu_0^\epsilon \to 0$.

In order to prove the full statement of part (i), let us introduce a test function ψ^ϵ defined by

$$\psi^\epsilon(x, y) = \frac{u_\xi^0((x-l)/\epsilon, y)}{||u_\xi^0((\cdot - l)/\epsilon, \cdot)||_{L^2(\Omega_1)}} = \frac{1}{\sqrt{\epsilon}\kappa} u_\xi^0(\frac{x-l}{\epsilon}, y) + Exp,$$

where Exp stands for a quantity of order $\exp[-\delta/\epsilon]$ for some constant $\delta > 0$. Multiplying the relation

$$L_1^\epsilon \psi_0^\epsilon = \mu_0^\epsilon \psi_0^\epsilon$$

by ψ^ϵ and integrating by parts over Ω_1, we obtain

$$\mu_0^\epsilon \int_0^1 \int_\Omega \psi^\epsilon \psi_0^\epsilon dy d\xi$$

$$= \frac{1}{\sqrt{\epsilon}\kappa}\int_0^1\int_\Omega [(V^\epsilon + W^\epsilon)f'(U^\epsilon) - \lambda^* f'(u^0(\frac{x-l}{\epsilon}, y))]$$

$$\times u_\xi^0(\frac{x-l}{\epsilon}, y)\psi_0^\epsilon(x,y)dydx + Exp$$

$$= \frac{1}{\kappa}\int_{-l/\epsilon}^{(1-l)/\epsilon}\int_\Omega [W^\epsilon(\epsilon\xi + l)f'(\tilde{U}^\epsilon) - \lambda^* f'(u^0(\xi, y))]$$

$$\times u_\xi^0(\xi, y)\tilde{\psi}_0^\epsilon(\xi, y)dyd\xi + O(\epsilon^2).$$

Note that

$$\tilde{\psi}_0^\epsilon(\xi, y) = \sqrt{\epsilon}\psi_0^\epsilon(\epsilon\xi + l, y)$$

is employed in the last equality above. By using the expansion

$$W^\epsilon(\epsilon\xi + l)f'(\tilde{U}^\epsilon) - \lambda^* f'(u^0(\xi, y))$$

$$= \epsilon\Big[u^1(\xi, y) + \{\Phi_\lambda z^0(\xi, y) + \phi_{-\lambda}\}\{\lambda_x(l)\xi + b\}\Big] + O(\epsilon^2)$$

and the fact

$$\lim_{\epsilon\to 0}\tilde{\psi}_0^\epsilon(\xi, y) = \lim_{\epsilon\to 0}\sqrt{\epsilon}\psi_0^\epsilon(\epsilon\xi + l, y) = \frac{1}{\kappa}u_\xi^0(\xi, y),$$

we obtain the limit

$$\lim_{\epsilon\to 0}\frac{\mu_0^\epsilon}{\epsilon} = \frac{1}{\kappa^2}\int_{-\infty}^{\infty}\int_\Omega \lambda^* f''(u_\xi^0)\Big[u^1(\xi, y) + \{\Phi_\lambda z^0(\xi, y) + \phi_{-\lambda}\}\{\lambda_x(l)\xi + b\}\Big]$$

$$\times (u_\xi^0(\xi, y))^2 dyd\xi$$

$$+ \frac{1}{\kappa^2}\int_{-\infty}^{\infty}\int_\Omega \{\lambda_x(l)\xi + b\}f'(u_\xi^0)(u_\xi^0(\xi, y))^2 dyd\xi.$$

For the sake of simplicity, let us use the symbols $A = \lambda_x(l)\xi + b$ and

$$B = u^1(\xi, y) + \{\Phi_\lambda z^0(\xi, y) + \phi_{-\lambda}\}\{\lambda_x(l)\xi + b\}.$$

Note that $A_\xi = \lambda_x(l)$ and $A_{\xi\xi} = 0$. Let us now compute the integral by using integration by parts :

$$\begin{aligned}
(3.4) \qquad & \int_{-\infty}^{\infty}\int_\Omega Af'(u^0)(u^0_\xi)^2 dyd\xi + \int_{-\infty}^{\infty}\int_\Omega \lambda^* f''(u^0)B(u^0_\xi)^2 dyd\xi \\
= & -\int_{-\infty}^{\infty}\int_\Omega f(u^0)[\lambda_x(l)u^0_\xi + Au^0_{\xi\xi}]dyd\xi \\
& -\int_{-\infty}^{\infty}\int_\Omega \lambda^* f'(u^0)[B_\xi u^0_\xi + Bu^0_{\xi\xi}]dyd\xi \\
= & -\lambda_x(l)\int_\Omega \Big[F(\phi_+(y;\lambda^*)) - F(\phi_-(y;\lambda^*))\Big]dy \\
& -\int_{-\infty}^{\infty}\int_\Omega f(u^0)Au^0_{\xi\xi}dyd\xi \\
& + \int_{-\infty}^{\infty}\int_\Omega \lambda^* f(u^0)B_{\xi\xi}dyd\xi - \int_{-\infty}^{\infty}\int_\Omega \lambda^* f'(u^0)Bu^0_{\xi\xi}dyd\xi.
\end{aligned}$$

By using $\lambda^* f(u^0) = -u^0_{\xi\xi} - \Delta u^0$ and integrating twice by parts in ξ and in y, the third term in (3.4) is rewritten as

$$\int_{-\infty}^{\infty}\int_\Omega \lambda^* f(u^0)B_{\xi\xi}dyd\xi = -\int_{-\infty}^{\infty}\int_\Omega B_{\xi\xi}u^0_{\xi\xi}dyd\xi - \int_{-\infty}^{\infty}\int_\Omega (\Delta B)u^0_{\xi\xi}dyd\xi.$$

Therefore, the equation (3.4) continues as

$$\begin{aligned}
(3.4) = & -\lambda_x(l)\int_\Omega [F(\phi_+(y;\lambda^*)) - F(\phi_-(y;\lambda^*))] \\
& - \int_{-\infty}^{\infty}\int_\Omega \Big[B_{\xi\xi} + \Delta B + \lambda^* f'(u^0)B + Af(u^0)\Big]u^0_{\xi\xi} \\
= & -\lambda_x(l)\int_\Omega \Big[F(\phi_+(y;\lambda^*)) - F(\phi_-(y;\lambda^*))\Big].
\end{aligned}$$

In the last equality, we used the fact

$$B_{\xi\xi} + \Delta B + \lambda^* f'(u^0)B + Af(u^0) = 0,$$

which is equivalent to the equation (2.11) for $u^1(\xi, y)$ appeared in Section2.2. This completes the proof of part (i) of Lemma3.2.

(iv) The proof is similar to that for K_1^ϵ in Lemma 3.1 (ii). By the L^p-theory as in the proof of Lemma 3.1 (ii), $L_1^\epsilon p = b$ has a unique solution in $C^0(\Omega_1)$ for each $b \in C^0(\Omega_1)$. Therefore $(\bar{L}_1^\epsilon)^{-1}$ maps $[\psi_0^\epsilon]^\perp \cap C^0(\Omega_1)$ into itself. To show that the norm of $(\bar{L}_1^\epsilon)^{-1}$ is bounded for small $\epsilon > 0$, we argue by contradiction. Thus assume that the norm of $(\bar{L}_1^\epsilon)^{-1}$ is unbounded as $\epsilon \to 0$. Then we can find sequences $\{\epsilon_j\}$, $\{p_j\}$ and $\{b_j\}$ such that $\epsilon_j \to 0$, $||b_j||_{C^0(\Omega_1)} \to 0$, $||p_j||_{C^0(\Omega_1)} = 1$, as $j \to \infty$, and $L_1^{\epsilon_j} p_j = b_j$, where $p_j \perp \psi_0^{\epsilon_j}$ and $b_j \perp \psi_0^{\epsilon_j}$. Rewriting the last equation in terms of a stretched variable $\xi = (x - x_0)/\epsilon$, we obtain the equation

$$\tilde{p}_{j\xi\xi} + \Delta\tilde{p}_j + \{(\tilde{V}^\epsilon + \tilde{W}^\epsilon)\exp[-\epsilon^2 hg] + \beta\} f'(U^\epsilon)\tilde{p}_j = \tilde{b}_j.$$

If $x_0 = l$, then, as $j \to \infty$, $\tilde{p}_j$ approaches a solution of

$$\tilde{p}_{\xi\xi} + \Delta\tilde{p} + \lambda^* f'(u^0(\xi, y))\tilde{p} = 0$$

with the boundary conditions

$$\tilde{p}(\pm\infty, y) = 0,\ y \in \Omega \ \text{ and } \ \tilde{p}(\xi, y) = 0,\ \xi \in \mathbf{R},\ y \in \partial\Omega.$$

Since the latter problem has a unique family of solutions $\alpha u_\xi^0(\xi, y)$, $\alpha \in \mathbf{R}$, we have $\tilde{p}_j \to \alpha u_\xi^0(\xi, y)$ as $j \to \infty$. On the other hand, the fact $p_j \in [\psi_0^\epsilon]^\perp$ implies

$$0 = \int_{-l/\epsilon}^{(1-l)/\epsilon} \int_\Omega \tilde{p}_j \tilde{\psi}_0^\epsilon(\xi, y) dy d\xi \ \to \ \alpha\kappa$$

as $j \to \infty$, where we used Lemma 3.2 (iii). Therefore $\alpha = 0$ and we conclude $\tilde{p}_j \to 0$.

On the other hand, if $x_0 \neq l$, then the equation for $\tilde{p}_j$ approaches

$$\tilde{p}_{\xi\xi} + \Delta\tilde{p} + \lambda(x_0) f'(\phi_\pm(y; \lambda(x_0))\tilde{p} = 0$$

with the boundary conditions in the above. If we denote by $(\mu_0, \Psi(y))$ the first eigenpair of the following eigenvalue problem

$$\Delta\Psi + \lambda(x_0) f'(\phi_\pm(y; \lambda(x_0))\Psi = \mu\Psi,\ y \in \Omega \ \text{ and } \ \Psi = 0,\ y \in \partial\Omega,$$

then the condition **(H4)** in Section 1 implies that $\mu_0 < 0$, and that Ψ is of constant sign, say positive. Now for each positive (resp. negative) α, $\alpha\Psi$ is an upper (resp. a lower) solution of

$$\tilde{p}_{\xi\xi} + \Delta\tilde{p} + \lambda(x_0)f'(\phi_\pm(y;\lambda(x_0))\tilde{p} = 0.$$

Therefore, for each $\alpha \geq 0$, we have $|\tilde{p}(\xi, y)| \leq \alpha\Psi(\xi, y)$ and hence $\tilde{p} \equiv 0$. In this way, we contradict the condition $||p_j||_{C^0(\Omega_1)} = 1$. This completes the proof of Lemma 3.2. $\square$

LEMMA 3.3. (i) *For each $r \in H^2_D(0,1)$, the estimate*

$$||L_3^\epsilon r||_{L^2(\Omega_1)} = O(\sqrt{\epsilon})||r||_{H^2(0,1)}$$

is valid.

(ii) *For $p \in L^2(\Omega_1)$, the following estimate is true :*

$$||L_3^\epsilon \left[(M_1^\epsilon)^{-1} M_2^\epsilon p\right] ||_{L^2(\Omega_1)} = O(\sqrt{\epsilon})||p||_{L^2(\Omega_1)}.$$

(iii) *Let Q^ϵ be the orthogonal projection onto $[\psi_0^\epsilon]^\perp$. Then for $p \in C^0(\Omega_1)$ we have*

$$||Q^\epsilon L_3^\epsilon \Big[(M_1^\epsilon)^{-1} M_2^\epsilon p\Big] ||_{C^0(\Omega_1)} = o(1)||p||_{C^0(\Omega_1)},$$

where $o(1)$ is a quantity which goes to zero as $\epsilon \to 0$.

PROOF. (i) Notice that

$$\begin{aligned} L_3^\epsilon r &= L_1^\epsilon(U_\lambda^\epsilon r) + \exp[-\epsilon^2 hg]f(U^\epsilon)r \\ &= \{L_1^\epsilon(U_\lambda^\epsilon) + \exp[-\epsilon^2 hg]f(U^\epsilon)\}r \\ &\quad + \ \epsilon^2\{2(U_\lambda^\epsilon)_x r_x + U_\lambda^\epsilon r_{xx}\}. \end{aligned}$$

Using the fact

$$\Delta\phi_{\pm\lambda}(\cdot;\lambda) + \lambda f'(\phi_\pm)\phi_{\pm\lambda}(\cdot;\lambda) + f(\phi_\pm) = 0,$$

one can easily show the estimate

$$||L_1^\epsilon(U_\lambda^\epsilon) + \exp[-\epsilon^2 hg]f(U^\epsilon)||_{L^2(\Omega_1)} = O(\sqrt{\epsilon}).$$

Therefore the statement (i) follows.

(ii) Apply the argument in the above to $(M_1^\epsilon)^{-1}M_2^\epsilon p$ and use the fact

$$\begin{aligned}
||\left[(M_1^\epsilon)^{-1}M_2^\epsilon p\right]_{xx}||_{L^2(0,1)} &= O(||p||_{L^2(\Omega_1)}),\\
||\left[(M_1^\epsilon)^{-1}M_2^\epsilon p\right]_{x}||_{L^2(0,1)} &= O(||p||_{L^2(\Omega_1)}).
\end{aligned}$$

(iii) When $p \in C^0(\Omega_1)$ one can prove, by using the second statement in Lemma 3.1 (i), that

$$\begin{aligned}
||\left[(M_1^\epsilon)^{-1}M_2^\epsilon p\right]_{xx}||_{C^0(0,1)} &= O(||p||_{C^0(\Omega_1)}),\\
||\left[(M_1^\epsilon)^{-1}M_2^\epsilon p\right]_{x}||_{C^0(0,1)} &= O(||p||_{C^0(\Omega_1)}).
\end{aligned}$$

One can therefore estimate $L_3^\epsilon r$, where $r = (M_1^\epsilon)^{-1}M_2^\epsilon p$, as

$$\begin{aligned}
Q^\epsilon L_3^\epsilon r = Q^\epsilon &\Big[\{L_1^\epsilon(U_\lambda^\epsilon) + \exp[-\epsilon^2 hg]f(U^\epsilon)\}r\Big]\\
&+ \ \epsilon^2 Q^\epsilon\Big[\{2(U_\lambda^\epsilon)_x r_x + U_\lambda^\epsilon r_{xx}\}\Big]\\
&= Q^\epsilon\Big[\{L_1^\epsilon(U_\lambda^\epsilon) + \exp[-\epsilon^2 hg]f(U^\epsilon)\}r\Big] + O(\epsilon^2)||p||_{C^0(\Omega_1)}.
\end{aligned}$$

In order to estimate the first term in the last equation, let us compute

$$I^\epsilon(x,y) := L_1^\epsilon(U_\lambda^\epsilon) + \exp[-\epsilon^2 hg]f(U^\epsilon)$$

explicitly. By an elementary computation, we obtain

$$\begin{aligned}
I^\epsilon(x,y) = \Phi_\lambda z^0_{\xi\xi} &+ \Delta\big[\Phi_\lambda z^0 + \phi_{-\lambda}\big] + \lambda(x) f'(\Phi z^0 + \phi_-)\big[\Phi_\lambda z^0 + \phi_{-\lambda}\big]\\
&+ f(\Phi z^0 + \phi_-) + O(\epsilon),
\end{aligned}$$

where $\Phi,\ \Phi_\lambda,\ \phi_-$ are evaluated at $(y;\lambda(x))$ and $z^0,\ z^0_{\xi\xi}$ are evaluated at $((x-l)/\epsilon, y)$, and $O(\epsilon)$ is of this order with respect to the supremum norm.

For $|x-l| > \sqrt{\epsilon}$, the following estimates are valid :

$$|z^0_{\xi\xi}(\frac{x-l}{\epsilon}, y)| = O(e^{-\delta/\sqrt{\epsilon}}), \quad |x-l| \geq \sqrt{\epsilon},$$

$$|z^0(\frac{x-l}{\epsilon}, y) - 1| = O(e^{-\delta/\sqrt{\epsilon}}), \quad x > l + \sqrt{\epsilon},$$

$$|z^0(\frac{x-l}{\epsilon}, y)| = O(e^{-\delta/\sqrt{\epsilon}}), \quad x < l - \sqrt{\epsilon}.$$

By using

$$\Delta\phi_{\pm\lambda}(\cdot;\lambda) + \lambda f'(\phi_\pm)\phi_{\pm\lambda}(\cdot;\lambda) + f(\phi_\pm) = 0,$$

we have $I^\epsilon(x,y) = O(\epsilon)$, for $|x-l| \geq \sqrt{\epsilon}$.

On the other hand, for $|x-l| \leq \sqrt{\epsilon}$, we express I^ϵ in terms of the stretched variable $\xi = (x-l)/\epsilon$. It has the form

$$\begin{aligned} I^\epsilon(\xi, y) = {} & \Phi_\lambda(y;\lambda^*) z^0_{\xi\xi}(\xi, y) + \Delta\big[\Phi_\lambda(y;\lambda^*) z^0 + \phi_{-\lambda}(y;\lambda^*)\big] \\ & + \lambda^* f'(u^0(\xi,y))\big[\Phi_\lambda(y;\lambda^*) z^0(\xi,y) + \phi_{-\lambda}(y;\lambda)\big] \\ & + f(u^0(\xi,y)) + O(\sqrt{\epsilon}), \end{aligned}$$

where, again, $O(\sqrt{\epsilon})$ is of this order when measured by the supremum norm. We now use the following claim :

Claim: There exists a constant k such that

$$\begin{aligned} k u^0_\xi(\xi, y) = {} & \Phi_\lambda(y;\lambda^*) z^0_{\xi\xi}(\xi, y) + \Delta\big[\Phi_\lambda(y;\lambda^*) z^0 + \phi_{-\lambda}(y;\lambda)\big] \\ & + \lambda^* f'(u^0(\xi,y))\big[\Phi_\lambda(y;\lambda^*) z^0(\xi,y) + \phi_{-\lambda}(y;\lambda)\big] + f(u^0(\xi,y)). \end{aligned}$$

Accepting the claim for the moment, we have

$$I^\epsilon(x,y) = O(\sqrt{\epsilon}) + k u^0_\xi(\frac{x-l}{\epsilon}, y)$$

for $(x,y) \in \Omega_1$, which leads us to the estimate

$$
\begin{aligned}
&|Q^\epsilon[I^\epsilon(x,y)r(x)]| \\
&\quad = O(\sqrt{\epsilon})||p||_{C^0} \\
&\qquad + |kr(l)| \cdot |\Big[u_\xi^0(\frac{x-l}{\epsilon}, y) - \int_\Omega \int_0^1 u_\xi^0(\frac{x-l}{\epsilon}, y)\psi_0^\epsilon(x,y)dxdy \cdot \psi_0^\epsilon(x,y)\Big]|.
\end{aligned}
$$

By using Lemma 3.2 (iii), the limit of the quantity inside the square brackets is computed as follows.

$$
\begin{aligned}
&\lim_{\epsilon \to 0}\Big[u_\xi^0(\frac{x-l}{\epsilon}, y) - \int_\Omega \int_0^1 u_\xi^0(\frac{x-l}{\epsilon}, y)\psi_0^\epsilon(x,y)dxdy \cdot \psi_0^\epsilon(x,y)\Big] \\
&= \lim_{\epsilon \to 0}\Big[u_\xi^0(\xi, y) - \int_\Omega \int_{-l/\epsilon}^{(1-l)/\epsilon} u_\xi^0(s,y)\sqrt{\epsilon}\psi_0^\epsilon(\epsilon s + l, y)dsdy \cdot \sqrt{\epsilon}\psi_0^\epsilon(\epsilon\xi + l, y)\Big] \\
&= u_\xi^0(\xi, y) - \frac{1}{\kappa^2}\int_\Omega \int_{-\infty}^{\infty} [u_\xi^0(s,y)]^2 dsdy \cdot u_\xi^0(\xi, y) = 0.
\end{aligned}
$$

This completes the proof. $\square$

PROOF OF CLAIM. This is a consequence of the results by Vega [V]. Under the conditions **(H1)-(H5)**, he proved that the nonlinear eigenvalue problem

$$
u_{\xi\xi} + cu_\xi + \Delta u + \lambda f(u) = 0, \quad (\xi, y) \in \mathbf{R} \times \Omega,
$$
$$
u(\xi, y) = 0, \;\; (\xi, y) \in \mathbf{R} \times \partial\Omega, \quad u(\pm\infty, y) = \phi_\pm(y;\lambda), \;\; y \in \Omega,
$$

has a solution $(c,u) = (c(\lambda), u(\xi, y; \lambda))$ which is unique up to phase shifts in ξ-variable for each $\lambda \in [\Lambda_0, \Lambda_1]$. Moreover, the solution $(c(\lambda), u(\xi, y; \lambda))$ is differentiable in $\lambda \in (\Lambda_0, \Lambda_1)$. Differentiating the equation with respect to λ at λ^* and using the fact $c(\lambda^*) = 0$, we obtain

$$
(u_\lambda)_{\xi\xi} + +\Delta u_\lambda + \lambda^* f'(u)u_\lambda + f(u) = -c'(\lambda^*)u_\xi.
$$

By uniqueness of the solution, we have $\Phi_\lambda z^0 + \phi_{-\lambda} = u_\lambda$ for $\lambda = \lambda^*$. This completes the proof of the claim. $\square$

LEMMA 3.4. (i) *For each $\theta(x) \in H^1(0,1)$, we have the equality*

$$\lim_{\epsilon\to 0}\langle \theta, f(U^\epsilon)\psi_0^\epsilon/\sqrt{\epsilon}\rangle_{\Omega_1}$$
$$= \frac{1}{\kappa}\int_\Omega [F(\phi_+(y;\lambda^*)) - F(\phi_-(y;\lambda^*))]\, dy \cdot \theta(l),$$

where the symbol $\langle\cdot,\cdot\rangle_{\Omega_1}$ means L^2-inner product between functions on Ω_1.

(ii) *As $\epsilon \to 0$ the function $(M_1^\epsilon)^{-1}M_2^\epsilon\psi_0^\epsilon/\sqrt{\epsilon}$ converges to a function z^* in $H^1(0,1)$ which satisfies*

$$d\{z_x^*(1)\theta(1) - z_x^*(0)\theta(0)\} - d\langle z_x^*, \theta_x\rangle_{(0,1)} + \langle {g'}_* z^*, \theta\rangle_{(0,1)}$$
$$= \frac{-\theta(l)}{|\Omega|\kappa}\int_\Omega \lambda^*\,[f(\phi_+(y;\lambda^*)) - f(\phi_-(y;\lambda^*))]\, dy,$$

for each $\theta(x) \in H^1(0,1)$. The symbol $\langle\cdot,\cdot\rangle_{(0,1)}$ stands for the L^2-inner product between functions on the interval $(0,1)$.

(iii) *As $\epsilon \to 0$, we have the following estimates :*

$$||(M_1^\epsilon)^{-1}M_2^\epsilon\psi_0^\epsilon||_{H^1(0,1)} = O(\sqrt{\epsilon}),$$
$$||M_2^\epsilon(M_1^\epsilon)^{-1}U_\lambda^\epsilon\psi_0^\epsilon||_{H^1(0,1)} = O(\sqrt{\epsilon}),$$
$$||(M_1^\epsilon)^{-1}M_2^\epsilon f(U^\epsilon)\psi_0^\epsilon||_{H^1(0,1)} = O(\sqrt{\epsilon}),$$
$$||M_2^\epsilon(M_1^\epsilon)^{-1}f(U^\epsilon)\psi_0^\epsilon||_{H^1(0,1)} = O(\sqrt{\epsilon}).$$

PROOF. (i) We prove the statement for $\theta(x) \in C^1(0,1)$. Then the density of $C^1(0,1)$ in $H^1(0,1)$ completes the proof. Noting the properties

$$\lim_{\epsilon\to 0}\tilde{\theta}(\xi) = \theta(l), \quad \lim_{\epsilon\to 0}\tilde{U}^\epsilon(\xi,y) = u^0(\xi,y)$$

and

$$\lim_{\epsilon\to 0}\tilde{\psi}_0^\epsilon(\xi,y) = \frac{1}{\kappa}u_\xi^0(\xi,y),$$

one can compute the integral

$$
\begin{aligned}
&\langle \theta, f(U^\epsilon)\psi_0^\epsilon/\sqrt{\epsilon}\rangle_{\Omega_1} \\
&= \int_{-l/\epsilon}^{(1-l)/\epsilon}\int_\Omega \tilde{\theta}(\xi) f(\tilde{U}^\epsilon)\sqrt{\epsilon}\psi_0^\epsilon(\epsilon\xi + l, y)dyd\xi,
\end{aligned}
$$

which converges, as $\epsilon \to 0$, to

$$
\begin{aligned}
&\frac{1}{\kappa}\int_\Omega \int_{-\infty}^{\infty} \theta(l) f(u^0(\xi, y))u_\xi^0(\xi, y)d\xi dy \\
&= \frac{\theta(l)}{\kappa}\int_\Omega \left[F(\phi_+(y;\lambda^*) - F(\phi_-(y;\lambda^*))\right] dy.
\end{aligned}
$$

(ii) If we denote by z^ϵ the function $(M_1^\epsilon)^{-1}M_2^\epsilon\psi_0^\epsilon/\sqrt{\epsilon}$, then it satisfies

$$
M_1^\epsilon z^\epsilon = M_2^\epsilon \psi_0^\epsilon/\sqrt{\epsilon}.
$$

We write this equation in weak form and apply Lax-Milgram Theorem to the left side of the resulting equation. Then M_1^ϵ, as a mapping from $H^1(0,1)$ to $[H^1(0,1)]'$ (the dual space of $H^1(0,1)$), has a bounded inverse. On the other hand, from the proof of part (i), it follows that the right hand side of the resulting equation converges to Dirac's delta-function at $x = l$ in $[H^1(0,1)]'$. It is then easy to show that z^ϵ is a Cauchy sequence in $H^1(0,1)$ as $\epsilon \to 0$. Therefore z^ϵ converges to z^* in $H^1(0,1)$. The equation which is satisfied by z^* is obtained by taking limits ($\epsilon \to 0$) in the weak form of the differential equation.

(iii) This follows immediately from the proof of part (i). $\square$

3.2. Existence of solutions

By using Lemma 3.1, the second and the third equations in (3.2) are solvable in (q, r) :

$$
\begin{aligned}
q &= -\epsilon^2(K_1^\epsilon)^{-1}p + O(\epsilon^2) + O(\epsilon^2)p + O(|p|^2), \\
r &= -(M_1^\epsilon)^{-1}M_2^\epsilon p + O(\epsilon^2) + O(\epsilon^2)p + O(|p|^2).
\end{aligned}
$$

Substituting these into the first equation of (3.2), we obtain

$$(3.5) \qquad L_1^\epsilon p - L_3^\epsilon[(M_1^\epsilon)^{-1}M_2^\epsilon p] + O(\epsilon^2)p + O(\epsilon^2) + O(|p|^2) = 0,$$

where $O(\epsilon^2)$ means a quantity of order ϵ^2 measured in L^∞-norm. This notation will be used in this sense below, unless stated otherwise.

Let us decompose p as a sum : $p = a\sqrt{\epsilon}\psi_0^\epsilon + p_1$, where

$$\langle \psi_0^\epsilon, p_1\rangle_{\Omega_1} = 0, \quad a \in \mathbf{R}.$$

Let $\bar{L}_1^\epsilon$ be the restriction of L_1^ϵ to the orthogonal completement of the span of ψ_0^ϵ, $[\psi_0^\epsilon]^\perp$. We denote by Q^ϵ the orthogonal projection onto $[\psi_0^\epsilon]^\perp$. According to the decomposition of p in the above, the equation (3.5) splits into two equations

$$(3.6) \qquad \mu_0^\epsilon a - \frac{1}{\sqrt{\epsilon}}\langle L_3^\epsilon\left[(M_1^\epsilon)^{-1}M_2^\epsilon(a\psi_0^\epsilon + p_1)\right], \ \psi_0^\epsilon\rangle_{\Omega_1} + \frac{1}{\sqrt{\epsilon}}B^\epsilon(a, p_1) = 0,$$

$$(3.7) \qquad \bar{L}_1^\epsilon p_1 - Q^\epsilon L_3^\epsilon\left[(M_1^\epsilon)^{-1}M_2^\epsilon p_1\right]$$
$$= \sqrt{\epsilon}aQ^\epsilon L_3^\epsilon\left[(M_1^\epsilon)^{-1}M_2^\epsilon\psi_0^\epsilon\right] + O(\epsilon^2)(1 + O(|p|)) + O(|p|^2),$$

where

$$B^\epsilon(a, p_1) = O(\epsilon^{5/2}) + O(\epsilon^2|p|^2) + \langle N_1^\epsilon(a\sqrt{\epsilon}\psi_0^\epsilon + p_1, q, r), \ \psi_0^\epsilon \ \rangle_{\Omega_1}$$

with $q = O(\epsilon^2(1 + |p|)^2)$ and $r = O(|p|^2)$. The reason why we have $O(\epsilon^{5/2})$-term in the expression of $B^\epsilon(a, p_1)$ is:

$$\langle \ R_1^\epsilon, \psi_0^\epsilon \ \rangle_{\Omega_1} = \sqrt{\epsilon}\langle \ R_1^\epsilon, \psi_0^\epsilon/\sqrt{\epsilon} \ \rangle_{\Omega_1} = \sqrt{\epsilon}O(\epsilon^2).$$

By using Lemma 3.2 (iv) and Lemma 3.3 (iii), the equation (3.7) is soluble in p_1 :

$$p_1^\epsilon(a) = \sqrt{\epsilon}(\bar{L}_1^\epsilon)^{-1}L_3^\epsilon\left[(M_1^\epsilon)^{-1}M_2^\epsilon\psi_0^\epsilon\right]a + O(\epsilon^2) + O(|a|^2),$$

and hence we have

$$p_1^\epsilon(a) = O(\epsilon)a + O(\epsilon^2) + O(|a|^2).$$

Substituting this into (3.6), one obtains a scalar equation in (a, ϵ) :

$$B_0(\epsilon) + B_1(\epsilon)a + B_2(\epsilon, a) = 0, \tag{3.8}$$

where $B_0(\epsilon) = O(\epsilon^2)$,

$$\begin{aligned} B_1(\epsilon) = \mu_0^\epsilon &- \langle L_3^\epsilon \left[(M_1^\epsilon)^{-1} M_2^\epsilon \psi_0^\epsilon\right], \ \psi_0^\epsilon \rangle_{\Omega_1} (1 + O(\sqrt{\epsilon})) \\ &- \langle L_3^\epsilon \{(M_1^\epsilon)^{-1} M_2^\epsilon (\bar{L}_1^\epsilon)^{-1} L_3^\epsilon \left[(M_1^\epsilon)^{-1} M_2^\epsilon \psi_0^\epsilon\right]\}, \ \psi_0^\epsilon \rangle_{\Omega_1} \\ &\times (1 + O(\sqrt{\epsilon})) \end{aligned} \tag{3.9}$$

and $B_2(\epsilon, a)$ is such that

$$\lim_{\epsilon \to 0} \left. \frac{\partial^2 B_2(\epsilon, a)}{\partial a^2} \right|_{a=0} = 0.$$

The fact that the last limit is zero follows from the same line of argument as in [Sk, p. 37]. In fact one can argue as follows.

$$\begin{aligned} \left. \frac{\partial^2 B_2(\epsilon, a)}{\partial a^2} \right|_{a=0} &= \frac{1}{\sqrt{\epsilon}} \int_0^1 \int_\Omega \Big[(V^\epsilon + W^\epsilon) \exp[-\epsilon^2 h g] + \beta \Big] \\ &\qquad \times f''(U^\epsilon)(\sqrt{\epsilon}\psi_0^\epsilon)^2 \cdot \psi_0^\epsilon dy dx \\ &= \frac{\lambda^*}{\sqrt{\epsilon}} \int_\Omega \int_{-l/\epsilon}^{(1-l)/\epsilon} f''(u^*(\xi, y))(\sqrt{\epsilon}\psi_0^\epsilon)^3 \sqrt{\epsilon} d\xi \, dy + O(\epsilon) \\ &= \lambda^* \int_\Omega \int_{-l/\epsilon}^{(1-l)/\epsilon} f''(u^*(\xi, y))(\sqrt{\epsilon}\psi_0^\epsilon)^3 \, d\xi \, dy + O(\epsilon) \\ &\to \frac{\lambda^*}{\kappa^3} \int_\Omega \int_{-\infty}^{\infty} f''(u^*(\xi, y))(u_\xi^*(\xi, y))^3 \, d\xi \, dy \quad (\text{ as } \ \epsilon \to 0 \text{ }). \end{aligned}$$

The last integral is zero. To show this, we use integration by parts, together with the decay property of u_ξ^* as $\xi \to \pm\infty$ as follows.

$$\lambda^* \int_\Omega \int_{-\infty}^{\infty} f''(u^*(\xi, y))(u_\xi^*(\xi, y))^3 \, d\xi \, dy$$

$$= -2\lambda^* \int_\Omega \int_{-\infty}^{\infty} f'(u^*(\xi,y)) u^*_\xi u^*_{\xi\xi}\, d\xi\, dy = 2\lambda^* \int_\Omega \int_{-\infty}^{\infty} f(u^*) u^*_{\xi\xi\xi}\, d\xi\, dy$$

$$= -2 \int_\Omega \int_{-\infty}^{\infty} [u^*_{\xi\xi} + \Delta u^*] u^*_{\xi\xi\xi}\, d\xi\, dy \quad (u^*_{\xi\xi} + \Delta u^* + \lambda^* f(u^*) = 0 \text{ is used})$$

$$= -\int_\Omega \int_{-\infty}^{\infty} \left[(u^*_{\xi\xi})^2\right]_\xi d\xi\, dy + 2\int_{-\infty}^{\infty} \int_\Omega \nabla u^* \cdot \nabla u^*_{\xi\xi\xi}\, dy\, d\xi$$

$$= -2 \int_\Omega \int_{-\infty}^{\infty} \nabla u^*_\xi \cdot \nabla u^*_{\xi\xi}\, d\xi\, dy = -\int_\Omega \int_{-\infty}^{\infty} \left(|\nabla u^*_\xi|^2\right)_\xi d\xi\, dy = 0$$

The solvability of (3.8) is equivalent to the existence of solutions for the problem (3.2)-(3.3).

Lemma 3.5. *The following limit exists and is nonzero.*

$$\lim_{\epsilon\to 0} \frac{B_1(\epsilon)}{\epsilon} = \hat\rho^*$$

$$= \frac{d\{z^*_x(1)\lambda_x(1) - z^*_x(0)\lambda_x(0)\}}{G_+(\lambda^*) - G_-(\lambda^*)} \frac{1}{\kappa^2} \int_\Omega [F(\phi_+(y;\lambda)) - F(\phi_-(y;\lambda))]\, dy < 0.$$

We will prove this lemma after we show the solvability of (3.8). Rescaling a in (3.8) as $a = \epsilon\hat a$, it reduces to

$$\frac{1}{\epsilon^2} B_0(\epsilon) + \frac{1}{\epsilon} B_1(\epsilon)\hat a + \frac{1}{\epsilon^2} B_2(\epsilon, \epsilon\hat a) = 0. \tag{3.10}$$

By using Lemma 3.5 and the implicit function theorem, it is easy to see that (3.10) is uniquely solvable near $(\hat a, \epsilon) = (-b_0/\hat\rho^*, 0)$ (with $b_0 = \lim_{\epsilon\to 0} B_0(\epsilon)/\epsilon^2$) as $\hat a = \hat a(\epsilon)$. Therefore, we have proven the existence of an ϵ-family of equilibrium solutions of (2.2)-(2.3).

Proof of Lemma 3.5. We will show the following two statements :

$$\begin{aligned} &\lim_{\epsilon\to 0} \frac{1}{\epsilon} \langle L^\epsilon_3 \left[(M^\epsilon_1)^{-1} M^\epsilon_2 \psi^\epsilon_0\right],\ \psi^\epsilon_0 \rangle_{\Omega_1} \\ &= \lim_{\epsilon\to 0} \langle L^\epsilon_3 \left[(M^\epsilon_1)^{-1} M^\epsilon_2 \psi^\epsilon_0/\sqrt{\epsilon}\right],\ \psi^\epsilon_0/\sqrt{\epsilon}\ \rangle_{\Omega_1} \end{aligned} \tag{3.11}$$

$$= \hat{\mu}_0^* - \frac{d\{z_x^*(1)\lambda_x(1) - z_x^*(0)\lambda_x(0)\}}{G_+(\lambda^*) - G_-(\lambda^*)}\frac{1}{\kappa^2}$$

$$\times \int_\Omega \Big[F(\phi_+(y;\lambda)) - F(\phi_-(y;\lambda))\Big]dy,$$

and

$$\langle L_3^\epsilon\{(M_1^\epsilon)^{-1}M_2^\epsilon(\bar{L}_1^\epsilon)^{-1}L_3^\epsilon\left[(M_1^\epsilon)^{-1}M_2^\epsilon\psi_0^\epsilon\right]\},\ \psi_0^\epsilon\rangle_{\Omega_1} = O(\epsilon^{3/2}). \tag{3.12}$$

To show (3.11), we note that

$$\langle L_3^\epsilon\left[(M_1^\epsilon)^{-1}M_2^\epsilon\psi_0^\epsilon/\sqrt{\epsilon}\right],\ \psi_0^\epsilon/\sqrt{\epsilon}\,\rangle_{\Omega_1}$$

$$= \langle U_\lambda^\epsilon\left[(M_1^\epsilon)^{-1}M_2^\epsilon\psi_0^\epsilon/\sqrt{\epsilon}\right],\ L_1^\epsilon\psi_0^\epsilon/\sqrt{\epsilon}\,\rangle_{\Omega_1}$$

$$+ \langle\left[(M_1^\epsilon)^{-1}M_2^\epsilon\psi_0^\epsilon/\sqrt{\epsilon}\right],\ \exp[-\epsilon^2 hg]f(U^\epsilon)\psi_0^\epsilon/\sqrt{\epsilon}\,\rangle_{\Omega_1}$$

$$= \mu_0^\epsilon\langle U_\lambda^\epsilon\left[(M_1^\epsilon)^{-1}M_2^\epsilon\psi_0^\epsilon/\sqrt{\epsilon}\right],\ \psi_0^\epsilon/\sqrt{\epsilon}\,\rangle_{\Omega_1}$$

$$+ \langle\left[(M_1^\epsilon)^{-1}M_2^\epsilon\psi_0^\epsilon/\sqrt{\epsilon}\right],\ \exp[-\epsilon^2 hg]f(U^\epsilon)\psi_0^\epsilon/\sqrt{\epsilon}\,\rangle_{\Omega_1}.$$

Therefore the limit on the left side of (3.11) is evaluated as

$$\lim_{\epsilon\to 0}\langle z^\epsilon, f(U^\epsilon)\psi_0^\epsilon/\sqrt{\epsilon}\rangle_{\Omega_1} = \frac{z^*(l)}{\kappa}\int_\Omega\Big[F(\phi_+(y;\lambda^*)) - F(\phi_-(y;\lambda^*))\Big]dy$$

from Lemma 3.4 (i). We will now compute the value of $z^*(l)$ by using Lemma 3.4 (ii). Take the test function $\theta(x) = \lambda_x(x)$ in Lemma 3.4 (ii) to obtain

$$d\{z_x^*(1)\lambda_x(1) - z_x^*(0)\lambda_x(0)\} - d\langle z_x^*, \lambda_{xx}\rangle_{(0,1)} + \langle g'_* z^*, \lambda_x\rangle_{(0,1)} \tag{3.13}$$

$$= \frac{-\lambda_x(l)\lambda^*}{|\Omega|\kappa}\int_\Omega\left[f(\phi_+(y;\lambda^*)) - f(\phi_-(y;\lambda^*))\right]dy.$$

Since $d(\lambda_x)_{xx} + g'_*\lambda_x = 0$ for $x \neq l$ and the boundary conditions $z^*(0) = 0 = z^*(1)$ are satisfied, integration by parts leads us to

$$\langle g'_* z^*, \lambda_x\rangle_{(0,1)} = \langle z^*, g'_*\lambda_x\rangle_{(0,1)} = -d\langle z^*, \lambda_{xxx}\rangle_{(0,1)}$$

$$= -dz^*(l)[\lambda_{xx}(l-0) - \lambda_{xx}(l+0)] + d\langle z_x^*, \lambda_{xx}\rangle_{(0,1)}.$$

Substituting this into (3.13), we obtain

$$
\begin{aligned}
& dz^*(l)[\lambda_{xx}(l-0) - \lambda_{xx}(l+0)] \\
&= d\{z_x^*(1)\lambda_x(1) - z_x^*(0)\lambda_x(0)\} \\
&\quad + \frac{\lambda_x(l)\lambda^*}{|\Omega|\kappa} \int_\Omega [f(\phi_+(y;\lambda^*)) - f(\phi_-(y;\lambda^*))]\, dy.
\end{aligned}
$$

By using the identity

$$
d\lambda_{xx}(l \pm 0) + h^*(\beta - \lambda^*) - \frac{\lambda^*}{|\Omega|} \int_\Omega f(\phi_\pm(y;\lambda^*))dy = 0,
$$

we obtain

$$
\begin{aligned}
& d\{\lambda_{xx}(l-0) - \lambda_{xx}(l+0)\} \\
&= -\frac{\lambda^*}{|\Omega|} \int_\Omega [f(\phi_+(y;\lambda^*)) - f(\phi_-(y;\lambda^*))]\, dy \\
&= -[G_+(\lambda^*) - G_-(\lambda^*)],
\end{aligned}
$$

and

$$
z^*(l) = -\frac{\lambda_x(l)}{\kappa} - \frac{d\{z_x^*(1)\lambda_x(1) - z_x^*(0)\lambda_x(0)\}}{G_+(\lambda^*) - G_-(\lambda^*)}.
$$

Therefore, recalling the value of $\hat{\mu}_0^*$ from Lemma 3.2 (i), this completes the proof of (3.11).

We now prove (3.12). Recalling the definition of L_3^ϵ, we have

$$
\begin{aligned}
(3.14)\quad & \langle L_3^\epsilon\{(M_1^\epsilon)^{-1} M_2^\epsilon (\bar{L}_1^\epsilon)^{-1} L_3^\epsilon \left[(M_1^\epsilon)^{-1} M_2^\epsilon \psi_0^\epsilon\right]\},\ \psi_0^\epsilon\rangle_{\Omega_1} \\
&= \langle U_\lambda^\epsilon\{(M_1^\epsilon)^{-1} M_2^\epsilon (\bar{L}_1^\epsilon)^{-1} L_3^\epsilon \left[(M_1^\epsilon)^{-1} M_2^\epsilon \psi_0^\epsilon\right]\},\ L_1^\epsilon\psi_0^\epsilon\rangle_{\Omega_1} \\
&\quad + \langle\{(M_1^\epsilon)^{-1} M_2^\epsilon (\bar{L}_1^\epsilon)^{-1} L_3^\epsilon \left[(M_1^\epsilon)^{-1} M_2^\epsilon \psi_0^\epsilon\right]\},\ f(U^\epsilon)\psi_0^\epsilon\rangle_{\Omega_1} + O(\epsilon^2) \\
&\qquad (\text{ by using } L_1^\epsilon\psi_0^\epsilon = \mu_0^\epsilon\psi_0^\epsilon = O(\epsilon)\psi_0^\epsilon, \ ) \\
&= O(\epsilon)\langle(\bar{L}_1^\epsilon)^{-1} L_3^\epsilon \left[(M_1^\epsilon)^{-1} M_2^\epsilon \psi_0^\epsilon\right],\ M_2^\epsilon (M_1^\epsilon)^{-1} U_\lambda^\epsilon \psi_0^\epsilon\rangle_{\Omega_1}
\end{aligned}
$$

$$+\,\langle(\bar{L}_1^\epsilon)^{-1}L_3^\epsilon\left[(M_1^\epsilon)^{-1}M_2^\epsilon\psi_0^\epsilon\right],\ M_2^\epsilon(M_1^\epsilon)^{-1}f(U^\epsilon)\psi_0^\epsilon\rangle_{\Omega_1}+O(\epsilon^2).$$

By using the same argument as in the proof of Lemma 3.3, and noting the fact

$$||\left[(M_1^\epsilon)^{-1}M_2^\epsilon\psi_0^\epsilon\right]_{xx}||_{L^2(0,1)}=O(1)$$

together with the estimates in Lemma 3.4 (iii), we obtain

$$||L_3^\epsilon\left[(M_1^\epsilon)^{-1}M_2^\epsilon\psi_0^\epsilon\right]||_{L^2(\Omega_1)}=O(\epsilon).$$

This and the estimates in Lemma 3.4 (iii) applied in (3.14) establish the estimate (3.12).

Recalling the asymptotic behavior of μ_0^ϵ from Lemma 3.2 (i), (3.11) and (3.12) prove Lemma 3.5, except that the limit is non zero. To show that the limit is nonzero, it suffices to show $z_x^*(0)>0$ and $z_x^*(1)<0$, since $\lambda_x(0)>0$ and $\lambda_x(1)>0$ hold from the construction of $\lambda(x)$. See Appendix. Let us, first of all, observe that z^* satisfies

$$dz_{xx}^*+g'_*(\lambda(x))z^*=0$$

on $(0,l)$ and $(l,1)$, together with the boundary conditions $z^*(0)=0=z^*(1)$ and $z^*(l-0)=z^*(l+0)$. Moreover, the proof of Lemma 3.4 implies that $M_2^\epsilon\psi_0^\epsilon/\sqrt{\epsilon}$ approaches a nonzero constant times Dirac's delta function at $x=l$. Therefore the proof of Lemma 3.4 (ii) shows that z^* can not be identically equal to zero on $(0,1)$. This fact is not trivial, although we have already computed the value $z^*(l)$ in the proof of the statement (3.11). Since $g'_*(\lambda(x))<0$ on $[0,1]$, the derivative z_x^* is monotone increasing (resp. decreasing) on $(0,l)$ and on $(l,1)$ with a jump discontinuity at $x=l$, if $z_x^*(0)>0$, $z_x^*(1)<0$ (resp. $z_x^*(0)<0$, $z_x^*(1)>0$). Therefore we will show that

$$z_x^*(l-0)-z_x^*(l+0)>0,$$

which implies $z_x(0)>0$ and $z_x(1)<0$. For this purpose, we take the test function $\theta(x)=z^*(x)$ in Lemma 3.4 (b). By using the differential equation for z^* on $(0,l)$ and $(l,1)$, and integrating by parts as before, we obtain

$$d\{z_x^*(l-0)-z_x^*(l+0)\}=\frac{\lambda^*}{|\Omega|\kappa}\int_\Omega[f(\phi_+(y;\lambda^*))-f(\phi_-(y;\lambda^*))]\,dy>0.$$

Therefore the limit value $\hat{\rho}^*$ in Lemma 3.5 is negative. This completes the proof of Lemma 3.5. □

So far, we have proven the existence of an ϵ-family of equilibrium solutions of (2.2)-(2.3), near the approximate solution constructed in Section2. We denote the solution by $(u^\epsilon, v^\epsilon, w^\epsilon)$ which is expressed as

$$u^\epsilon = U^\epsilon + p^\epsilon + U^\epsilon_\lambda r^\epsilon, \quad v^\epsilon = q^\epsilon, \quad w^\epsilon = W^\epsilon + r^\epsilon,$$

where $(p^\epsilon, q^\epsilon, r^\epsilon)$ is the solution of (3.2). We will now show that the L^∞-norm of $(p^\epsilon, q^\epsilon, r^\epsilon)$ goes to zero as $\epsilon \to 0$. Recall that

$$p^\epsilon = \epsilon^{3/2}\hat{a}(\epsilon)\psi^\epsilon_0 + p^\epsilon_1(\epsilon\hat{a}(\epsilon)).$$

We know from Lemma 3.2 (iii) that $\sqrt{\epsilon}\psi^\epsilon_0$ is bounded as $\epsilon \to 0$, and hence that $||\epsilon^{3/2}\hat{a}(\epsilon)\psi^\epsilon_0||_{L^\infty} = O(\epsilon)$. Therefore we conclude that $||p^\epsilon||_{L^\infty} = O(\epsilon)$. Since q^ϵ and r^ϵ are expressed in terms of p^ϵ as in the second and the third lines in the beginning of subsection 3.2, we also have proved $||q^\epsilon||_{L^\infty} = O(\epsilon^2)$ and $||r^\epsilon||_{L^\infty} = O(\epsilon)$.

Therefore, recalling the construction of the approximate solution (U^ϵ, W^ϵ), we complete the proof of Theorem A.

4. Proof of Theorem B

We study the stability property of the solution $(u^\epsilon, v^\epsilon, w^\epsilon)$. For this purpose, we linearize the equation (2.2) around the solution $(u^\epsilon, v^\epsilon, w^\epsilon)$ and study the eigenvalue problem associated with it. It turns out convenient to use another time scale τ which is related to the original time scale t by $\tau = \epsilon^2 t$. According to the existence proof, it is also convenient to look for the eigenfunctions in the following form :

$$u = p + U^\epsilon_\lambda r, \quad v = q, \quad w = r.$$

Then the equation for (p, q, r) is

$$\begin{cases} L^\epsilon_1 p + L^\epsilon_2 q + L^\epsilon_3 r = \epsilon^2\rho p - \epsilon^2\rho U^\epsilon_\lambda r \\ K^\epsilon_1 q + \epsilon^2 K^\epsilon_2 p + \epsilon^2 K^\epsilon_3 r = \epsilon^2 \rho q \\ M^\epsilon_1 r + M^\epsilon_2 p + M^\epsilon_3 q = \rho r, \end{cases} \tag{4.1}$$

where the operators $L_j^\epsilon, K_j^\epsilon, M_j^\epsilon$ $(j = 1, 2, 3)$, are given by the same formulae as those appeared in the previous section, except that we now replace the approximate solution $(U^\epsilon, V^\epsilon, W^\epsilon)$ by the true solution $(u^\epsilon, v^\epsilon, w^\epsilon)$ in the formulae. This abuse of notation should not cause any problem in the subsequent argument. Moreover, one should observe that Lemma 3.1 through Lemma 3.5 remain valid for the linear operators in this section. We therefore equote these lemmas frequently in this section.

We now start to solve (4.1). Notice that there is a positive constant ρ_0 such that the eigenvalues of $\bar{L}_1^\epsilon$, $K_1^\epsilon, M_1^\epsilon$ are contained in the interval $(-\infty, -\rho_0]$. In order to study the stability property of the solution $(u^\epsilon, v^\epsilon, w^\epsilon)$, we only need to examine the eigenvalues of (4.1) in the region

$$\Re\rho > -\rho_0.$$

In the sequel, we always assume that ρ satisfies this condition.

By using Lemma 3.1, we can solve the second and the third equations in (4.1) as

$$q = \epsilon^2\Big[I - \epsilon^2(K_1^\epsilon - \epsilon^2\rho)^{-1}K_3^\epsilon(M_1^\epsilon - \rho)^{-1}M_3^\epsilon\Big]^{-1}$$
$$\times (K_1^\epsilon - \epsilon^2\rho)^{-1}\Big[-K_2^\epsilon p + K_3^\epsilon(M_1^\epsilon - \rho)^{-1}M_2^\epsilon p\Big]$$

and

$$r = \Big[I - \epsilon^2(M_1^\epsilon - \rho)^{-1}M_3^\epsilon(K_1^\epsilon - \epsilon^2\rho)^{-1}K_3^\epsilon\Big]^{-1}$$
$$\times (M_1^\epsilon - \rho)^{-1}\Big[-M_2^\epsilon p + \epsilon^2 M_3^\epsilon(K_1^\epsilon - \epsilon^2\rho)^{-1}K_2^\epsilon p\Big].$$

For short, we write these as

$$q = \epsilon^2 K_\rho^\epsilon p, \quad r = M_\rho^\epsilon p.$$

We now substitute these into the first equation of (4.1). The resulting equation is

$$(L_1^\epsilon - \epsilon^2\rho)p + (L_3^\epsilon - \epsilon^2\rho U_\lambda^\epsilon)M_\rho^\epsilon p + \epsilon^2 L_2^\epsilon K_\rho^\epsilon p = 0.$$

We decompose p as $p = a\psi_0^\epsilon + p_1$, where as before $a \in \mathbf{R}$ and $\langle \psi_0^\epsilon, p_1 \rangle = 0$. Under this deocomposition, the last equation now splits into two parts :

$$(4.2) \qquad \begin{aligned} 0 = a(\mu_0^\epsilon - \epsilon^2\rho) &+ \langle U_\lambda^\epsilon M_\rho^\epsilon \psi_0^\epsilon \ , \ (\mu_0^\epsilon - \epsilon^2\rho)\psi_0^\epsilon \rangle \\ &+ a\langle \left[M_\rho^\epsilon + \epsilon^2 K_\rho^\epsilon \right] \psi_0^\epsilon \ , \ \exp[-\epsilon^2 hg] f(u^\epsilon)\psi_0^\epsilon \ \rangle \\ &+ \langle U_\lambda^\epsilon M_\rho^\epsilon p_1 \ , \ (\mu_0^\epsilon - \epsilon^2\rho)\psi_0^\epsilon \rangle \\ &+ \langle \left[M_\rho^\epsilon + \epsilon^2 K_\rho^\epsilon \right] p_1 \ , \ \exp[-\epsilon^2 hg] f(u^\epsilon)\psi_0^\epsilon \ \rangle \end{aligned}$$

and

$$(4.3) \qquad \begin{aligned} &(\bar{L}_1^\epsilon - \epsilon^2\rho)p_1 + Q^\epsilon (L_3^\epsilon - \epsilon^2\rho U_\lambda^\epsilon) M_\rho^\epsilon p_1 + \epsilon^2 Q^\epsilon L_2^\epsilon K_\rho^\epsilon p_1 \\ = \ & - aQ^\epsilon \Big[L_3^\epsilon M_\rho^\epsilon \psi_0^\epsilon - \epsilon^2 \rho U_\lambda^\epsilon M_\rho^\epsilon \psi_0^\epsilon + \epsilon^2 L_2^\epsilon K_\rho^\epsilon \psi_0^\epsilon \Big]. \end{aligned}$$

By Lemma 3.3 (i), we have

$$||L_3^\epsilon M_\rho^\epsilon p_1||_{L^2(\Omega_1)} = O(\sqrt{\epsilon})||M_\rho^\epsilon p_1||_{H^2(0,1)} = O(\sqrt{\epsilon})||p_1||_{L^2(\Omega_1)},$$

which enables us to solve (4.3) in p_1 as

$$p_1 = aL_\rho^\epsilon \psi_0^\epsilon.$$

By using the same argument as in the proof of Lemma 3.3 (i), the principal part of the right hand side of (4.3) is estimated as follows :

$$\begin{aligned} ||L_3^\epsilon M_\rho^\epsilon \psi_0^\epsilon||_{L^2(\Omega_1)} &\le ||\{L_1^\epsilon(U_\lambda^\epsilon) + \exp[-\epsilon^2 hg] f(u^\epsilon)\} M_\rho^\epsilon \psi_0^\epsilon||_{L^2(\Omega_1)} \\ &\quad + \epsilon^2 ||2(U_\lambda^\epsilon)_x (M_\rho^\epsilon \psi_0^\epsilon)_x + U_\lambda^\epsilon (M_\rho^\epsilon \psi_0^\epsilon)_{xx}||_{L^2(\Omega_1)} \\ &\le O(\sqrt{\epsilon})||M_\rho^\epsilon \psi_0^\epsilon||_{H^1(0,1)} + O(\epsilon)||M_\rho^\epsilon \psi_0^\epsilon||_{H^2(0,1)}. \end{aligned}$$

Moreover, by the same line of arguments as in the proof of Lemma 3.4 (ii), one can prove that $||M_\rho^\epsilon \psi_0^\epsilon||_{H^1(0,1)} = O(\sqrt{\epsilon})$, which implies $||L_3^\epsilon M_\rho^\epsilon \psi_0^\epsilon||_{L^2(\Omega_1)} = O(\epsilon)$. We therefore have $||L_\rho^\epsilon \psi_0^\epsilon||_{L^2(\Omega_1)} = O(\epsilon)$, and hence $||p_1^\epsilon||_{L^2(\Omega_1)} = aO(\epsilon)$.

Substituting $p_1 = aL^\epsilon_\rho \psi^\epsilon_0$ into (4.2), one obtains the following equation for ρ to satisfy as an eigenvalue of (4.1) :

$$(4.4) \qquad 0 = (\mu^\epsilon_0 - \epsilon^2\rho)C^\epsilon(\rho) + \langle M^\epsilon_\rho \psi^\epsilon_0 \ , \ \exp[-\epsilon^2 hg] f(u^\epsilon)\psi^\epsilon_0 \rangle$$
$$+ \langle \Big[M^\epsilon_\rho L^\epsilon_\rho + \epsilon^2 K^\epsilon_\rho L^\epsilon_\rho + \epsilon^2 K^\epsilon_\rho \Big] \psi^\epsilon_0 \ , \ \exp[-\epsilon^2 hg] f(u^\epsilon)\psi^\epsilon_0 \rangle,$$

where $C^\epsilon(\rho)$ is given by

$$C^\epsilon(\rho) = 1 + \langle \Big[M^\epsilon_\rho + M^\epsilon_\rho L^\epsilon_\rho \Big] \psi^\epsilon_0 \ , \ U^\epsilon_\lambda \psi^\epsilon_0 \ \rangle.$$

Note that $C^\epsilon(\rho) = 1 + O(\epsilon)$ follows from the same argument as in the proof of Lemma 3.4. Dividing by ϵ the both side of equation (4.4), one obtains

$$(4.5) \qquad \hat{\mu}^\epsilon_0 - \epsilon\rho = D^\epsilon(\rho),$$

where $\hat{\mu}^\epsilon_0 = \mu^\epsilon_0/\epsilon$, and $D^\epsilon(\rho)$ is given by

$$-C^\epsilon(\rho)D^\epsilon(\rho) = \langle M^\epsilon_\rho \psi^\epsilon_0/\sqrt{\epsilon} \ , \ \exp[-\epsilon^2 hg] f(u^\epsilon)\psi^\epsilon_0/\sqrt{\epsilon} \ \rangle$$
$$+ \langle \Big[M^\epsilon_\rho L^\epsilon_\rho + \epsilon^2 K^\epsilon_\rho L^\epsilon_\rho + \epsilon^2 K^\epsilon_\rho \Big] \psi^\epsilon_0/\sqrt{\epsilon} \ ,$$
$$\exp[-\epsilon^2 hg] f(u^\epsilon)\psi^\epsilon_0/\sqrt{\epsilon} \ \rangle.$$

The principal part $D^\epsilon_1(\rho)$ of $D^\epsilon(\rho)$ is

$$D^\epsilon_1(\rho) = \langle (M^\epsilon_1 - \rho)^{-1} M^\epsilon_2 \psi^\epsilon_0/\sqrt{\epsilon} \ , \ \exp[-\epsilon^2 hg] f(u^\epsilon)\psi^\epsilon_0/\sqrt{\epsilon} \ \rangle.$$

Note that $D^\epsilon_1(\rho)$, as well as $D^\epsilon(\rho)$, is an analytic function of ρ which assumes real values for real ρ, and that $D^\epsilon(\rho)$ (resp. $D^\epsilon_1(\rho)$) can be written as

$$D^\epsilon(\rho) = A^\epsilon(\rho_R, \rho_I^2) - \sqrt{-1}\rho_I B^\epsilon(\rho_R, \rho_I^2)$$
$$(\text{ resp. } \ D^\epsilon_1(\rho) = A^\epsilon_1(\rho_R, \rho_I^2) - \sqrt{-1}\rho_I B^\epsilon_1(\rho_R, \rho_I^2) \ \ ),$$

where $\rho = \rho_R + \sqrt{-1}\rho_I$ with $\rho_R, \ \rho_I \in \mathbf{R}$ and $A^\epsilon, \ B^\epsilon$ (resp. $A^\epsilon_1, \ B^\epsilon_1$) are real valued functions.

LEMMA 4.1. *For ρ with $\Re\rho > -\rho_0$, A^ϵ and B^ϵ have the following properties :*

(a) $A^\epsilon(\rho_R, \rho_I^2) > 0$, *and* $B^\epsilon(\rho_R, \rho_I^2) > 0$*;*

(b) $\lim_{\rho_R\to\infty} A^\epsilon(\rho_R, \rho_I^2) = 0$, *and* $\lim_{|\rho_I|\to\infty} B^\epsilon(\rho_R, \rho_I^2) = 0$*;*

(c) $\partial A^\epsilon(\rho_R, \rho_I^2)/\partial\rho_R < 0$, *and* $\lim_{\rho_R\to\infty} \partial A^\epsilon(\rho_R, \rho_I^2)/\partial\rho_R = 0$*;*

(d) $\lim_{\epsilon\to 0} A^\epsilon(0,0) = \hat{\mu}_0^* - \hat{\rho}^* > 0$, *where* μ_0^* *and* $\hat{\rho}^*$*, respectively, are those appreaed in Lemma 3.2 (i) and Lemma 3.5 ;*

(e) *There exists* $c_0 > 0$ *such that* $B^\epsilon(0,0) \geq c_0$ *for* $\epsilon > 0$ *small ;*

(f) $\partial B^\epsilon(\rho_R, \rho_I^2)/\partial(\rho_I^2) < 0$, *and* $\lim_{|\rho_I|\to\infty} \partial B^\epsilon(\rho_R, \rho_I^2)/\partial(\rho_I^2) = 0$.

PROOF. Since the essential feature of the proof for A^ϵ and B^ϵ is the same as that for A_1^ϵ and B_1^ϵ, we will show the statements of the lemma for A_1^ϵ and B_1^ϵ.

Let $\{\nu_n^\epsilon; \varphi_n^\epsilon\}$ be the eigen-pairs of M_1^ϵ. Lemma 3.1 implies that

$$-\rho_0 > \nu_0^\epsilon > \nu_1^\epsilon > \cdots.$$

By using the Fourier expansion, D_1^ϵ is written as

$$D_1^\epsilon(\rho) = \sum_{n\geq 0} \frac{\langle M_2^\epsilon \psi_0^\epsilon/\sqrt{\epsilon}\ ,\ \varphi_n^\epsilon\ \rangle\ \langle \exp[-\epsilon^2 hg] f(u^\epsilon)\psi_0^\epsilon/\sqrt{\epsilon}\ ,\ \varphi_n^\epsilon\ \rangle}{\nu_n^\epsilon - \rho}.$$

Therefore we have

$$\begin{aligned} &A_1^\epsilon(\rho_R, \rho_I^2) \\ &= \sum_{n\geq 0} \frac{\langle -M_2^\epsilon \psi_0^\epsilon/\sqrt{\epsilon}\ ,\ \varphi_n^\epsilon\ \rangle\ \langle \exp[-\epsilon^2 hg] f(u^\epsilon)\psi_0^\epsilon/\sqrt{\epsilon}\ ,\ \varphi_n^\epsilon\ \rangle(\rho_R - \nu_n^\epsilon)}{(\rho_R - \nu_n^\epsilon)^2 + \rho_I^2}, \end{aligned}$$

$$B_1^\epsilon(\rho_R, \rho_I^2) = \sum_{n\geq 0} \frac{\langle -M_2^\epsilon \psi_0^\epsilon/\sqrt{\epsilon}\ ,\ \varphi_n^\epsilon\ \rangle\ \langle \exp[-\epsilon^2 hg] f(u^\epsilon)\psi_0^\epsilon/\sqrt{\epsilon}\ ,\ \varphi_n^\epsilon\ \rangle}{(\rho_R - \nu_n^\epsilon)^2 + \rho_I^2}.$$

Recalling now that

$$-M_2^\epsilon p = \frac{1}{|\Omega|}\int_\Omega \Big[v^\epsilon + w^\epsilon + \beta\exp[\epsilon^2 hg]\Big] f'(u^\epsilon) p dy,$$

Lemma 3.2 (iii) implies

$$\lim_{\epsilon\to 0}\langle -M_2^\epsilon\psi_0^\epsilon/\sqrt{\epsilon}\ ,\ \varphi_n^\epsilon\ \rangle = \frac{\lambda^*}{\kappa}\int_\Omega [f(\phi_+(y;\lambda^*)) - f(\phi_-(y;\lambda^*))]\,dy\cdot\varphi_n^*(l)$$

and

$$\begin{aligned}&\lim_{\epsilon\to 0}\langle \exp[-\epsilon^2 hg] f(u^\epsilon)\psi_0^\epsilon/\sqrt{\epsilon}\ ,\ \varphi_n^\epsilon\ \rangle\\ &= \frac{1}{\kappa}\int_\Omega [F(\phi_+(y;\lambda^*)) - F(\phi_-(y;\lambda^*))]\,dy\cdot\varphi_n^*(l),\end{aligned}$$

where $\varphi_n^* = \lim_{\epsilon\to 0}\varphi_n^\epsilon$. Therefore for each $n\geq 0$,

$$\langle -M_2^\epsilon\psi_0^\epsilon/\sqrt{\epsilon}\ ,\ \varphi_n^\epsilon\ \rangle\ \langle \exp[-\epsilon^2 hg] f(u^\epsilon)\psi_0^\epsilon/\sqrt{\epsilon}\ ,\ \varphi_n^\epsilon\ \rangle$$

is positive. Now all the statements, except for (c), in the lemma follow immediately from the expressions of A_1^ϵ and B_1^ϵ. To show the statement (c), note that

$$A_1^\epsilon(0,0) = \langle\ (M_1^\epsilon)^{-1}M_2^\epsilon\psi_0^\epsilon/\sqrt{\epsilon}\ ,\ \exp[-\epsilon^2 hg] f(u^\epsilon)\psi_0^\epsilon/\sqrt{\epsilon}\ \rangle$$

whose limit as $\epsilon\to 0$ has been computed in the proof of Lemma 3.5. This completes the proof of Lemma 4.1. □

By using Lemma 4.1, we can now analyse the equation (4.5), which is written as

$$(4.6)\qquad \begin{cases} (a) & \hat{\mu}_0^\epsilon - \epsilon\rho_R = A^\epsilon(\rho_R,\rho_I^2),\\ (b) & \rho_I\,\epsilon = \rho_I B^\epsilon(\rho_R,\rho_I^2).\end{cases}$$

PROOF OF THEOREM B (i). Recall, from Lemma 3.2 (i), that

$$sign(\hat{\mu}_0^*) = -sign(\lambda_x(l)).$$

From the result in Appendix, we know that $\lambda_x(l)\geq 0$ for $\beta\in[0,\beta_-]\cup[\beta_+,\infty)$, and therefore $\hat{\mu}_0^\epsilon$ is non-negative for small $\epsilon>0$. If (4.5) has a solution with non-negative real parts, $\rho_R\geq 0$, then the right hand side of (4.6-a) is positive (Lemma 4.1), while left hand side is negative for $\epsilon > 0$ small. Therefore (4.1) cannot have eigenvalues with non-negative real part. $\square$

Proof of Theorem B (ii). Let us first show that the equation (4.5) has a pair of pure imaginary solutions with non-zero imaginary part for some $d > 0$. If $\rho_I \neq 0$ then (4.6-b) with $\rho_R = 0$ is written as : $\epsilon = B^\epsilon(0, \rho_I^2)$. Because of Lemma 4.1 (b), (e) and (f), there exists a unique $\rho_I(\epsilon) > 0$ such that $\epsilon = B^\epsilon(0, \rho_I(\epsilon)^2)$. Note that $\rho_I(\epsilon) \to \infty$ as $\epsilon \to 0$. Substituting this into (4.6-a) (with $\rho_R = 0$), we now have to find d such that

$$\hat{\mu}_0^\epsilon = A^\epsilon(0, \rho_I(\epsilon)^2)$$

is satisfied. We know that

$$\lim_{\epsilon \to 0} \hat{\mu}_0^\epsilon = \hat{\mu}_0^* = -k\lambda_x(l)$$

with some $k > 0$. Moreover, the result in Appendix shows that

$$\frac{d}{d(d)}\lambda_x(l(\beta, d); \beta, d) < 0,$$

when $\lambda_x(l)$ is near zero. Therefore there exists $d = d_\epsilon(\beta)$ for which

$$\hat{\mu}_0^\epsilon = A^\epsilon(0, \rho_I(\epsilon)^2)$$

holds true. We will now show that the solution $(\rho_R, \rho_I) = (0, \rho_I(\epsilon))$ can be extended to a family of solutions $(\rho_R^d(\epsilon), \rho_I^d(\epsilon))$ for d near $d_\epsilon(\beta)$, and that

$$\frac{d}{d(d)}\rho_R^d(\epsilon)|_{d = d_\epsilon(\beta)} < 0.$$

Now consider the equation

$$\text{(4.7)} \qquad \begin{cases} \hat{\mu}_0^\epsilon = \epsilon\rho_R + A^\epsilon(\rho_R, \rho_I^2), \\ \epsilon = B^\epsilon(\rho_R, \rho_I^2). \end{cases}$$

The Jacobian, say Jac, of the right hand side of (4.7) with respect to (ρ_R, ρ_I^2) at $d = d_\epsilon(\beta)$ is given by

$$\begin{aligned} Jac = \Big[\epsilon + \frac{\partial A^\epsilon}{\partial \rho_R}(0, \rho_I(\epsilon)^2)\Big] \frac{\partial B^\epsilon}{\partial(\rho_I^2)}(0, \rho_I(\epsilon)^2) \\ - \frac{\partial A^\epsilon}{\partial(\rho_I^2)}(0, \rho_I(\epsilon)^2) \frac{\partial B^\epsilon}{\partial \rho_R}(0, \rho_I(\epsilon)^2). \end{aligned}$$

By using the Cauchy-Riemann equations

$$\frac{\partial A^\epsilon}{\partial \rho_R} = \frac{\partial(-\rho_I B^\epsilon)}{\partial \rho_I}, \qquad \frac{\partial A^\epsilon}{\partial \rho_I} = -\frac{\partial(-\rho_I B^\epsilon)}{\partial \rho_R},$$

and the identities

$$\frac{\partial}{\partial \rho_I} = 2\rho_I \frac{\partial}{\partial(\rho_I^2)}, \quad \epsilon = B^\epsilon(0, \rho_I(\epsilon)^2),$$

we obtain

$$\begin{aligned} \frac{\partial A^\epsilon}{\partial \rho_R}(0, \rho_I(\epsilon)^2) &= -\epsilon - 2\rho_I(\epsilon)^2 \frac{\partial B^\epsilon}{\partial(\rho_I^2)}(0, \rho_I(\epsilon)^2), \\ \frac{\partial A^\epsilon}{\partial(\rho_I^2)}(0, \rho_I(\epsilon)^2) &= \frac{1}{2}\frac{\partial B^\epsilon}{\partial \rho_R}(0, \rho_I(\epsilon)^2). \end{aligned} \tag{4.8}$$

The relations in (4.8) immediately imply

$$Jac = -2\rho_I(\epsilon)^2 \Big[\frac{\partial B^\epsilon}{\partial(\rho_I^2)}(0, \rho_I(\epsilon)^2)\Big]^2 - \frac{1}{2}\Big[\frac{\partial B^\epsilon}{\partial \rho_R}(0, \rho_I(\epsilon)^2)\Big]^2 < 0.$$

Applying the implicit function theorem to (4.7) around $(\rho_R, \rho_I, d) = (0, \rho_I(\epsilon)^2, d_\epsilon(\beta))$, we obtain a family of eigenvalues for (4.1) as

$$\rho^d(\epsilon) = \rho_R^d(\epsilon) + \sqrt{-1}\rho_I^d(\epsilon).$$

In order to compute $d\rho_R^d(\epsilon)/d(d)$, differentiate the relations in (4.7) with respect to d. The resulting equations are

$$\frac{d}{d(d)}\Big(\hat{\mu}_0^\epsilon\Big) = \epsilon \frac{d\rho_R^d}{d(d)} + \frac{\partial A^\epsilon}{\partial \rho_R} \cdot \frac{d\rho_R^d}{d(d)} + \frac{\partial A^\epsilon}{\partial(\rho_I^2)} \cdot \frac{d(\rho_I^d)^2}{d(d)}$$

and

$$0 = \frac{\partial B^\epsilon}{\partial \rho_R} \cdot \frac{d\rho_R^d}{d(d)} + \frac{\partial B^\epsilon}{\partial (\rho_I^2)} \cdot \frac{d(\rho_I^d)^2}{d(d)}.$$

Substituting the last equation into the first equation and using (4.8), we arrive at the following expression :

$$\frac{d}{d(d)}\Big(\hat{\mu}_0^\epsilon\Big) = -\left[2\rho_I(\epsilon)^2\Big[\frac{\partial B^\epsilon}{\partial(\rho_I^2)}\Big]^2 + \Big[\frac{\partial B^\epsilon}{\partial \rho_R}\Big]^2\right]\Big[\frac{\partial B^\epsilon}{\partial(\rho_I^2)}\Big]^{-1}\frac{d\rho_R^d}{d(d)}.$$

Since

$$\frac{d}{d(d)}\Big(\hat{\mu}_0^\epsilon\Big) < 0, \quad \frac{\partial B^\epsilon}{\partial(\rho_I^2)} < 0$$

for small $\epsilon > 0$, we finally conclude that for small $\epsilon > 0$

$$\frac{d}{d(d)}\Big(\hat{\mu}_0^\epsilon\Big)\Big|_{d=d_\epsilon(\beta)} < 0.$$

Therefore a Hopf-bifurcation takes place when d passes $d_\epsilon(\beta)$.

To show Theorem B (ii-c), note that

$$\hat{\mu}_0^\epsilon(d_\epsilon(\beta)) = A^\epsilon(0, \rho_I(\epsilon)^2) \quad \text{and} \quad \rho_I(\epsilon) \to \infty \;\; \text{as} \;\; \epsilon \to 0.$$

Therefore, as $\epsilon \to 0$,

$$\hat{\mu}_0^*(d_*) = \lim_{\epsilon \to 0} A^\epsilon(0, \rho_I(\epsilon)^2) = 0.$$

Since $\hat{\mu}_0^*(d) = -k\lambda_x(l(\beta, d))$ for some $k > 0$, d_* is the value of d for which $\lambda_x(l(\beta, d)) = 0$ is satisfied. The result in Appendix completes the proof of Theorem B (ii-c). □

5. Concluding remarks

We have discussed in this paper the existence and stability of equilibrium solutions with one internal layer. However, we should note that it is possible for equilibrium solutions with multi layers to exist depending upon the values of β and d. This result will be reported in [MNS2].

For a simple case, if we take the symmetric boundary conditions

$$c_0(y) = c_1(y), \ \ \theta_0(y) = \theta_1(y) = \phi_+(y;\lambda_1) \ \ (\text{ or } \ \phi_-(y;\lambda_0) \ ), \ \ y \in \Omega,$$

one expect that any equilibrium solution is symmetric at $x = 1/2$, which possesses two internal layers because of the symmetric property of the system (1.5). Under this situation, it is numerically shown that when d decreases, the symmetric solution becomes unstable through Hopf-bifurcation. Now the following question arises : Is the bifurcating periodic solution symmetric or anti-symmetric with respect to $x = 1/2$? Intuitively, one could speculate that any periodic solution is symmetric with respect to $x = 1/2$. However, this is not necessarily true. The dependency of solutions on β and d is rather complicated, as is seen from Figure 5, exhibiting a variety of spatio-temporal patterns. A detailed study of this situation will be reported in [MNS2].

As the analyses in the previous sections show, it is technically complicated to treat the problem (1.5)-(1.8) directly. Therefore, from the practical view point of studying dynamical behaviour of (1.1), it is desirable to have a simple model which captures essential dynamics of (1.1). By using the fast-slow dynamics method, we derived in [MSE] a free boundary problem associated with the problem (1.5)-(1.8). Although the derivation of the free boundary problem in [MSE] is based on reasonable arguments, it has not been fully justified in a rigorous manner. In this section, we will show that the essential ingredients in Theorems A and B in the above are encoded in the free boundary problem.

In order to write down the free boundary problem, we need to use the following lemma.

LEMMA 5.1 (Vega [V]). *Consider the following nonlinear eigenvalue problem for a fixed* $\lambda \in (\Lambda_0, \Lambda_1)$ *in the cylindrical domain* $\Omega_\infty = \{(z, y) \in \mathbf{R} \times \Omega\}$ *:*

$$-\mu u_z = u_{zz} + \Delta u + \lambda f(u), \ \ (z, y) \in \Omega_\infty$$

with the boundary conditions

$$u(-\infty, y) = \phi_-(y;\lambda), \ \ u(\infty, y) = \phi_+(y;\lambda), \ \ y \in \Omega, \ \ u = 0, z \in \mathbf{R}, \ y \in \partial\Omega.$$

Then there is a unique $\mu(\lambda)$ *such that the problem above has a solution* $u(z, y;\lambda)$ *which is monotone in* z.

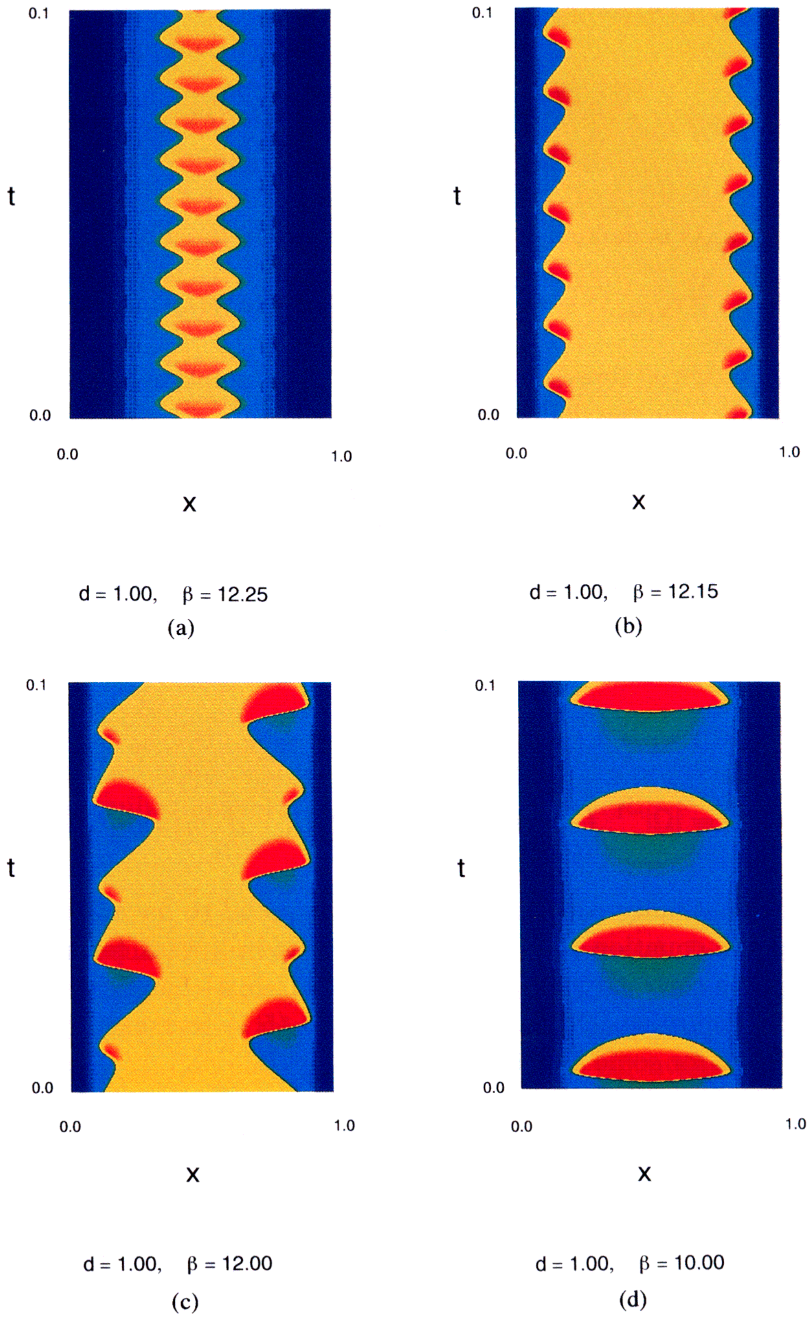

Figure 5. A variety of spatio-temporal patterns under symmetric boundary conditions $\theta = 0$ and $c = 0$ on $\Gamma_0 \cup \Gamma_1$.

The free boundary problem is given by :

$$\epsilon^{-2}\partial\lambda/\partial t = d\lambda_{xx} + h^*(\beta - \lambda) - \mathcal{F}(x, s, \lambda) \tag{5.1}$$

$$\epsilon^{-1}ds/dt = \mu(\lambda(t, s)), \quad t > 0, \tag{5.2}$$

where $\mathcal{F}(x, s, \lambda)$ is defined by using the *Heaviside function* H as follows.

$$\mathcal{F}(x, s, \lambda) = G_-(\lambda)H(s - x) + G_+(\lambda)H(x - s).$$

The variable $\lambda(t, x)$ represents the average of the solutions $c(t, x, y)$ of (1.5) in y-direction over the domain Ω :

$$\lambda(t, x) = \frac{1}{|\Omega|}\int_\Omega c(t, x, y)dy,$$

and the variable s signifies the location of the layer position in the θ-component of the solution of (1.5). Initial and boundary conditions for (λ, s) are :

$$\lambda(0, x) = \lambda_i(x), \quad 0 < x < 1, \quad s(0) = s_i, \quad 0 < s_i < 1, \tag{5.3}$$

$$\lambda(t, 0) = |\Omega|^{-1}\int_\Omega c_0(y)dy, \quad \lambda(t, 1) = |\Omega|^{-1}\int_\Omega c_1(y)dy, \quad t > 0. \tag{5.4}$$

As for the continuity condition on λ, it is required to have $\lambda \in C^1([0, 1])$. Therefore the equation (5.1) should be written in a weak form. The well-posedness for such a type of problem as (5.1)-(5.4) has been estabilshed in [HNM]. Following [HNM], we understand (5.1) to mean the following identity holds for some $T > 0$:

$$\begin{aligned}\epsilon^{-2}\int_0^T \langle\lambda_t, a\rangle dt = &\int_0^T\int_0^1 \{-d\lambda_x\, a_x + h^*(\beta - \lambda)\, a\,\}dxdt \\ &- \int_0^T\int_0^1 \mathcal{F}(x, s, \lambda)\, a\; dxdt,\end{aligned}$$

where $a \in L^2(0, T; H_0^1(0, 1))$ and $\langle\,\cdot, \cdot\,\rangle$ denotes the duality pairing between $[H_0^1(0, 1)]'$ (the dual space of $H_0^1(0, 1)$) and $H_0^1(0, 1)$.

In order to study how the problem (5.1)-(5.4) is related to Theorems A and B, we need another lemma.

LEMMA 5.2. (i) *The value of travelling wave-speed* μ *is zero at* $\lambda = \lambda^*$ *:* $\mu(\lambda^*) = 0$.

(ii) *The derivative of* $\mu(\lambda^*)$ *at* $\lambda = \lambda^*$ *is given by*

$$\mu'(\lambda^*) = -\frac{1}{\kappa^2}\int_\Omega \Big[F(\phi_+(y;\lambda^*)) - F(\phi_-(y;\lambda^*))\Big]dy,$$

where

$$\kappa^2 = \int_{-\infty}^{\infty}\int_\Omega \Big[u_z(z,y;\lambda^*)\Big]^2 dydz,$$

and

$$F(u) = \int_0^u f(\tau)d\tau.$$

PROOF. Part (i) follows immediately from Lemma 5.1 and **(H-5)** in Section 1. Part (ii) is obtained by integrating the equation in Lemma 5.1 over $\mathbf{R}\times\Omega$. □

Because of Lemma 5.2 (i), one easily finds that the problem (5.1)-(5.4) has a unique equilibrium solution

$$(\lambda, s) = (\bar\lambda, l),$$

where $\bar\lambda$ is the equilibrium solution appeared in Theorem A. In order to distinguish it from the dynamical variable λ in (5.1)and (5.2), we use this symbol in the following part of this section. We now linearize (5.1)-(5.4) around the equilibrium solution and consider the associated eigenvalue problem :

$$\epsilon\rho s = \mu'(\lambda^*)\bar\lambda_x(l)s + \mu'(\lambda^*)\lambda(l), \tag{5.5}$$

$$\begin{aligned}\rho\int_0^1 \lambda a dx = &-d\int_0^1 \lambda_x a_x dx - \int_0^1 \Big[h^*\lambda + \frac{\partial\mathcal{F}}{\partial\lambda}(x,l,\bar\lambda)\lambda\Big]adx \\ &+\frac{\lambda^*}{|\Omega|}\int_\Omega \Big[f(\phi_+(y;\lambda^*)) - f(\phi_-(y;\lambda^*))\Big]dy\cdot a(l)s.\end{aligned}\tag{5.6}$$

When we linearize (5.1)-(5.4) in the above, we rescaled the time by $t \to t/\epsilon^2$.

The equation (5.6) is written as :

$$\langle (M-\rho)\lambda, a\rangle = -f^*\langle \delta_l, a\rangle \cdot s, \quad a \in H_0^1(0,1),$$

where δ_l is the Dirac's delta function at $x = l$,

$$f^* = \frac{\lambda^*}{|\Omega|}\int_\Omega \Big[f(\phi_+(y;\lambda^*)) - f(\phi_-(y;\lambda^*))\Big]dy,$$

and $M : H_0^1(0,1) \ \to\ [H_0^1(0,1)]'$ is defined by

$$M\lambda = d\lambda_{xx} - h^*\lambda - \frac{\partial \mathcal{F}}{\partial \lambda}(x,l,\bar{\lambda})\lambda.$$

There is a constant $\rho_0 > 0$ such that for ρ with $\Re\rho > -\rho_0$, the operator $M-\rho$ has a bounded inverse

$$(M-\rho)^{-1} : [H_0^1(0,1)]' \ \to\ H_0^1(0,1).$$

Therefore (5.6) is uniquely solved as $\lambda(x) = -sf^*\Big[(M-\rho)^{-1}\delta_l\Big](x)$. Substituting this into (5.5), we obtain the equation for ρ to be an eigenvalue of (5.5)-(5.6) :

$$\mu'(\lambda^*)\bar{\lambda}_x(l) - \epsilon\rho = f^*\Big[(M-\rho)^{-1}\delta_l\Big](l). \tag{5.7}$$

Recalling the values of $\mu'(\lambda^*)$ and $\hat{\mu}_0^*$, respectively, from Lemma 5.2 (ii) and Lemma 3.2 (i), we obtain $\mu'(\lambda^*)\bar{\lambda}_x(l) = \hat{\mu}_0^*$. Moreover, one can show that the right hand side of (5.7) is the limit of D^ϵ (from (4.5) in the previous section) as $\epsilon \to 0$. Therefore (5.7) is equivalent to

$$\begin{aligned} \mu'(\lambda^*)\bar{\lambda}_x(l) - \epsilon\rho_R &= A^*(\rho_R, \rho_I^2) \\ \epsilon\rho_I &= \rho_I B^*(\rho_R, \rho_I^2), \end{aligned} \tag{5.8}$$

which is precisely the limiting (as $\epsilon \to 0$) equation of (4.6). The analysis of (5.8) is the same as (even simpler than) that of (4.6).

6. Appendix

A proof of Lemma 2.1, along with several properties of the solution $\lambda(x;\beta,d)$ of the problem (2.8), will be given in this appendix.

Let us rewrite the problem (2.8) as follows.

$$\begin{cases} 0 = d\lambda_{xx} + h^*\beta - H(\lambda) \\ \lambda(0) = \lambda_0, \quad \lambda(1) = \lambda_1, \\ \lambda(l) = \lambda^*, \quad \lambda_x(l-0) = \lambda_x(l+0) \end{cases}$$

where $H(v) = h^*v + G_*(v)$. We will also use $H_\pm(v) = h^*v + G_\pm(v)$, and $H^*_\pm = H_\pm(\lambda^*)$, $H^0_+ = H_+(\Lambda_0)$ and $H^1_- = H_-(\Lambda_1)$.

Due to the condition **(A4)**, we have the following inequalities.

$$H^*_- < H^1_- < H^0_+ < H^*_+.$$

There are two cases to deal with :

Case I. $\beta \le H^*_-/h^* = \beta_-$ or $\beta \ge H^*_+/h^* = \beta_+$,

Case II. $H^*_-/h^* < \beta < H^*_+/h^*$.

To solve the problem (2.8), we use the phase plane method. Consider the phase planes of the following problems :

$$\text{(L)} \quad 0 = v^-_{\xi\xi} + h^*\beta - H_-(v^-)$$

$$\text{(R)} \quad 0 = v^+_{\xi\xi} + h^*\beta - H_+(v^+),$$

where $\xi = x/\sqrt{d}$. Let A be the point (λ^*, p) on the v-v_ξ phase plane. We denote by B (resp. C) the point on the line $v = \lambda_0$ (resp. $v = \lambda_1$) which also lies on the negative (resp. positive) orbit of A under the flow generated by (L) (resp. (R)). We denote by $l_0(p,\beta)$ (or $l_1(p,\beta)$) the time spent between A and B (resp. C). In order to solve (2.8), we need to find $p = p(\beta, d)$ such that

$$l_2(p,\beta) := l_0(p,\beta) + l_1(p,\beta) = \frac{1}{\sqrt{d}}.$$

Once we find $p = p(\beta, d)$ which satisfies this equation, then our solution $\lambda(x; \beta, d)$ of (2.8) is given by

$$\lambda(x) = \begin{cases} v^-(x/\sqrt{d}; p), & 0 \le x \le l(\beta, d), \\ v^+(x/\sqrt{d}; p), & l(\beta, d) \le x \le 1, \end{cases}$$

where $l(\beta, d) = \sqrt{d} l_0(p(\beta, d), \beta)$ and $v^-(\xi; p)$ (or $v^+(\xi; p)$) is a unique solution of (L) (resp. (R)) with an initial value $(v, v_\xi) = (\lambda^*, p)$.

We first deal with Case I. In this case, we always have $p > p_\beta$, where

$$p_\beta = \sqrt{-2 \int_{\lambda^-(\beta)}^{\lambda^*} g_-(s) ds}, \quad \beta \le \beta_-,$$

$$p_\beta = \sqrt{-2 \int_{\lambda^+(\beta)}^{\lambda^*} g_+(s) ds}, \quad \beta \ge \beta_+,$$

where $\lambda^\pm(\beta)$ is a unique solution of $h^* \beta = H_\pm(\lambda)$. l_0 and l_1 are respectively given by

$$l_0(p, \beta) = \int_{\lambda_0}^{\lambda^*} \frac{dv}{\sqrt{p^2 + 2 \int_v^{\lambda^*} g_-(s) ds}},$$

$$l_1(p, \beta) = \int_{\lambda^*}^{\lambda_1} \frac{dv}{\sqrt{p^2 + 2 \int_v^{\lambda^*} g_+(s) ds}}.$$

Since one can easily find that $\partial l_0 / \partial p < 0$ and $\partial l_1 / \partial p < 0$, l_2 is monotone decreasing in p for each fixed β. Moreover, $\lim_{p \to p_\beta} l_2(p, \beta) = \infty$, $\lim_{p \to \infty} l_2(p, \beta) = 0$. Therefore for each $d > 0$, $l_2(p, \beta) = 1/\sqrt{d}$ has a unique solution $p = p(\beta, d)$. This completes the proof of Lemma 2.1 for Case I.

We now deal with Case II. There are three sub-cases to be considered :

$$\text{(II-a)} \quad \frac{H_-^*}{h^*} < \beta \le \frac{H_-^1}{h^*},$$

$$\text{(II-b)} \quad \frac{H_-^1}{h^*} < \beta < \frac{H_+^0}{h^*},$$

$$\text{(II-c)}\ \frac{H_+^0}{h^*} \le \beta < \frac{H_+^*}{h^*}.$$

We present a treatment only for sub-case (II-a). The other cases can be analysed in the same way as we do for (II-a). We note that the value p of the point $A = (\lambda^*, p)$ in the phase plane may become negative in the present case.

For $p \ge 0$, the function l_2 is given by the same formula as in Case I, and therefore it is monotone decreasing in p and $l_2(p,\beta) = 1/\sqrt{d}$ has a unique solution. Moreover, when $p = 0$, the corresponding value for d is given by

$$d_0^{-1/2} = l_0(0,\beta) + l_1(0,\beta),$$

which is the limiting value appeared in Theorem B (ii-c).

On the other hand, when $p < 0$, the functions l_0 and l_1 are given by

$$l_0(p,\beta) = \int_{\lambda_0}^{\lambda^*} \frac{dv}{\sqrt{p^2 + 2\int_v^{\lambda^*} g_-(s)ds}} + 2\int_{\lambda^*}^{r_0(p,\beta)} \frac{dv}{\sqrt{p^2 + 2\int_v^{\lambda^*} g_-(s)ds}},$$

where $r_0(p,\beta)$ is a unique solution of

$$p^2 + 2\int_{r_0}^{\lambda^*} g_-(s)ds = 0$$

and

$$l_1(p,\beta) = \int_{\lambda^*}^{\lambda_1} \frac{dv}{\sqrt{p^2 + 2\int_v^{\lambda^*} g_+(s)ds}} + 2\int_{r_1(p,\beta)}^{\lambda^*} \frac{dv}{\sqrt{p^2 + 2\int_v^{\lambda^*} g_+(s)ds}},$$

where $r_1(p,\beta)$ is a unique solution of

$$p^2 + 2\int_{r_1}^{\lambda^*} g_+(s)ds = 0.$$

By an elementary computation after suitable change of variables and using ${g_\pm}' < 0$, one can show that $\partial l_2/\partial p < 0$. Therefore the equation $l_2(p,\beta) = 1/\sqrt{d}$ has a unique solution and the proof of Lemma 2.1 has been completed.

Differentiating $l_2(p(\beta,d),\beta)=1/\sqrt{d}$ with respect to d, we obtain

$$\frac{\partial}{\partial d}p(\beta,d)=-\frac{1}{2}\Big(\frac{\partial l_2}{\partial p}\Big)^{-1}d^{-3/2}>0.$$

Since

$$\lambda_x(l(\beta,d);\beta,d)=-\frac{1}{2}d^{-3/2}v_\xi^{\pm}(l_0(p(\beta,d)),\beta)=-\frac{1}{2}d^{-3/2}p(\beta,d)$$

is available, we can compute the derivative

$$\begin{aligned}\frac{d}{d(d)}\lambda_x(l(\beta,d);\beta,d)&=\frac{3}{4}d^{-5/2}p(\beta,d)-\frac{1}{2}d^{-3/2}\frac{\partial}{\partial d}p(\beta,d)\\&=\frac{3}{4}d^{-5/2}p(\beta,d)+\frac{1}{4}d^{-3}\Big(\frac{\partial l_2}{\partial p}\Big)^{-1}<0\end{aligned}$$

when $p(\beta,d)$ is near zero. This fact is used in the proof of Theorem B (ii).

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