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A construction of the fundamental solution for Schrödinger equations

By Naoto KUMANO-GO

Abstract. In this paper, applying the skip method in Fujiwara [3] to the theory of multi-products of Fourier integral operators in Kitada and H.Kumano-go [6], we give a construction of the fundamental solution for the Cauchy problem of a pseudo-differential equation of Schrödinger's type. We regard this construction as a multi-product of Fourier integral operators, and investigate the pointwise convergence of the phase function and that of the symbol. Here we use neither the solution of the Hamilton-Jacobi equation in [6] nor the action of the classical orbit in [2],[4].

0. Introduction

Let L_h be a Schrödinger operator defined by $L_h \equiv i\partial_t + K_h(t, X, D_x)$ on $[0, T]$ with a parameter $0 < h < 1$. For sufficiently small T_0 $(0 < T_0 \leq T)$, let $U_h(t, s)$ $(0 \leq s \leq t \leq T_0)$ be the fundamental solution for the operator L_h such that

$$(0.1) \qquad \begin{cases} L_h U_h(t, s) = 0 & \text{on } [s, T_0] \\ \quad U_h(s, s) = I & (0 \leq s \leq T_0)\,. \end{cases}$$

Noting that

$$(0.2) \qquad K_h(t, X, D_x)u(x) \equiv \int e^{ix\cdot\xi} K_h(t, x, \xi)\hat{u}(\xi)đ\xi \quad (u \in \mathcal{S}, \quad đ\xi \equiv (2\pi)^{-n} d\xi)\,,$$

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one may ask whether

$$U_h(t,s)u(x) = \int e^{ix\cdot\xi} e^{i\int_s^t K_h(\tau,x,\xi)d\tau} \hat{u}(\xi) đ\xi\,?$$

Of course, the answer is " No ". However, this simple idea is our starting point. In what follows, we set

$$e^{i\int_s^t K_h(\tau)d\tau}(X,D_x)u(x) \equiv \int e^{ix\cdot\xi} e^{i\int_s^t K_h(\tau,x,\xi)d\tau} \hat{u}(\xi) đ\xi \quad (u \in \mathcal{S})\,. \tag{0.3}$$

By the result of Kitada and H.Kumano-go [6], we have the following:

Let $H_h(t,x,\xi)$ be a real-valued function on $[0,T]\times \boldsymbol{R}^{2n}$ with a parameter $0<h<1$, which has a continuous derivative $\partial_\xi^\alpha D_x^\beta H_h(t,x,\xi)$ on $[0,T]\times\boldsymbol{R}^{2n}$ for any α,β, satisfying

$$\begin{aligned} &|\partial_\xi^\alpha D_x^\beta \{h^{\rho-\delta} H_h(t,h^\delta x, h^{-\rho}\xi)\}| \\ &\qquad \le \begin{cases} C_{\alpha,\beta}\langle x;\xi\rangle^{2-|\alpha+\beta|} & (|\alpha+\beta|\le 1) \\ C_{\alpha,\beta} & (|\alpha+\beta|\ge 2)\,, \end{cases} \end{aligned} \tag{0.4}$$

on $[0,T]\times\boldsymbol{R}^{2n}$. Here $C_{\alpha,\beta}$ is a constant independent of a parameter $0<h<1$.

Let $W_h(t,x,\xi)$ be a complex-valued function on $[0,T]\times\boldsymbol{R}^{2n}$ with a parameter $0<h<1$, which has a continuous derivative $\partial_\xi^\alpha D_x^\beta W_h(t,x,\xi)$ on $[0,T]\times\boldsymbol{R}^{2n}$ for any α,β, satisfying

$$|\partial_\xi^\alpha D_x^\beta\{W_h(t,h^\delta x,h^{-\rho}\xi)\}| \le C'_{\alpha,\beta} \tag{0.5}$$

on $[0,T]\times\boldsymbol{R}^{2n}$. Here $C'_{\alpha,\beta}$ is a constant independent of a parameter $0<h<1$.

Set

$$\begin{cases} K_h(t,x,\xi) \equiv H_h(t,x,\xi) + W_h(t,x,\xi) \\ L_h \equiv i\partial_t + K_h(t,X,D_x)\,. \end{cases} \tag{0.6}$$

Then, using the solution $\phi_h^\star(t,s)(x,\xi)$ of the Hamilton-Jacobi equation

$$\begin{cases} -\partial_t\phi_h^\star(t,s)(x,\xi) + H_h(t,x,\nabla_x\phi_h^\star(t,s)(x,\xi)) = 0 \\ \qquad\qquad \text{on } [s,T_0]\times\boldsymbol{R}^{2n} \\ \phi_h^\star(s,s)(x,\xi) = x\cdot\xi\,, \end{cases} \tag{0.7}$$

we can construct $U_h(t,s)$ from the Fourier integral operator $I(\phi_h^\star(t,s))$ with the phase function $\phi_h^\star(t,s)(x,\xi)$ and the symbol 1, by the method of the successive approximation and by H.Kumano-go-Taniguchi theorem. Especially, in the case $W_h(t,x,\xi)\equiv 0$, we have

$$U_h(t,s)=\lim_{|\varDelta_{t,s}|\to 0} I(\phi_h^\star(t,t_1))I(\phi_h^\star(t_1,t_2))\cdots I(\phi_h^\star(t_\nu,s))\,, \tag{0.8}$$

where $\varDelta_{t,s}:(T_0\ge)\,t\equiv t_0\ge t_1\ge\cdots\ge t_\nu\ge t_{\nu+1}\equiv s\,(\ge 0)$ is an arbitrary division of interval [s,t] into subintervals, and $|\varDelta_{t,s}|\equiv \max_{1\le j\le \nu+1}|t_{j-1}-t_j|$.

Now, in this case $W_h(t,x,\xi)\equiv 0$, by applying Fujiwara's skip method in [3] to Kitada and H.Kumano-go's theory, we can construct $U_h(t,s)$ directly with (0.8). Here the following facts are important:

(A) $\phi_h^\star(t,\theta)\#\phi_h^\star(\theta,s)=\phi_h^\star(t,s)\quad (T_0\ge t\ge\theta\ge s\ge 0)$

(B) There exist a constant C_l and $r_h(t,\theta,s)(x,\xi)\in\mathbb{B}^0_{\rho,\delta}(h)$ $(T_0\ge t\ge\theta\ge s\ge 0)$ such that

$$I(\phi_h^\star(t,\theta))I(\phi_h^\star(\theta,s))=I(\phi_h^\star(t,s))+r_h(t,\theta,s)(\phi_h^\star(t,s);X,D_x)$$

and

$$|r_h(t,\theta,s)|_l^{(0)}\le C_l|t-\theta||\theta-s|\,.$$

The arguments about the solution of the Hamilton-Jacobi equation in Kitada and H.Kumano-go [6], however, seem to be too technical. Then, can we construct $U_h(t,s)$ in a simple style without the solution of the Hamilton-Jacobi equation ? On the other hand, we can rewrite (0.8) with

$$\begin{aligned}U_h(t,s)=\lim_{|\varDelta_{t,s}|\to 0} & e^{i(\phi_h^\star(t,t_1)(x,\xi)-x\cdot\xi)}(X,D_x)e^{i(\phi_h^\star(t_1,t_2)(x,\xi)-x\cdot\xi)}(X,D_x)\\ &\cdots e^{i(\phi_h^\star(t_\nu,s)(x,\xi)-x\cdot\xi)}(X,D_x)\,.\end{aligned}$$

Furthermore, $\phi_h^\star(t_{j-1},t_j)(x,\xi)-x\cdot\xi$ is approximated to $\int_{t_j}^{t_{j-1}}H_h(\tau,x,\xi)d\tau$. Then, can we construct $U_h(t,s)$ from $e^{i\int_{t_j}^{t_{j-1}}H_h(\tau)d\tau}(X,D_x)$? Our answer is to get the following expression for general cases including $W_h(t,x,\xi)\not\equiv 0$:

$$\begin{aligned}U_h(t,s)=\lim_{|\varDelta_{t,s}|\to 0} & e^{i\int_{t_1}^{t}K_h(\tau)d\tau}(X,D_x)e^{i\int_{t_2}^{t_1}K_h(\tau)d\tau}(X,D_x)\\ &\cdots e^{i\int_{s}^{t_\nu}K_h(\tau)d\tau}(X,D_x)\,.\end{aligned} \tag{0.9}$$

Here we consider (A) and (B). If we replace $\phi_h^\star(t,s)(x,\xi)$ by $x \cdot \xi + \int_s^t H_h(\tau,x,\xi)d\tau$, then (A) does not hold any longer and (B) requires some improvement. Therefore, in order to get (0.9), we have to make the best of the theory of multi-products of phase functions in Kitada and H.Kumano-go [6].

Main Theorems

The main theorems for a Schrödinger equation are Theorem 5.1, 5.3 and 5.4. For the details, see Section 5. Roughly speaking, Theorem 5.1, 5.3 and 5.4 are as follows:

Let $H_h(t,x,\xi)$ be a real-valued function on $[0,T]\times\boldsymbol{R}^{2n}$ with a parameter $0<h<1$, which has a continuous derivative $\partial_\xi^\alpha D_x^\beta H_h(t,x,\xi)$ on $[0,T]\times\boldsymbol{R}^{2n}$ for any α,β, satisfying

$$(0.10)\qquad \begin{aligned} &|\partial_\xi^\alpha D_x^\beta\{h^{\rho-\delta}H_h(t,h^\delta x,h^{-\rho}\xi)\}| \\ &\qquad\le \begin{cases} C_{\alpha,\beta}\langle x;\xi\rangle^{2-|\alpha+\beta|} & (|\alpha+\beta|\le 1) \\ C_{\alpha,\beta} & (|\alpha+\beta|\ge 2)\,, \end{cases}\end{aligned}$$

on $[0,T]\times\boldsymbol{R}^{2n}$. Here $C_{\alpha,\beta}$ is a constant independent of a parameter $0<h<1$.

Let $W_h(t,x,\xi)$ be a complex-valued function on $[0,T]\times\boldsymbol{R}^{2n}$ with a parameter $0<h<1$, which has a continuous derivative $\partial_\xi^\alpha D_x^\beta W_h(t,x,\xi)$ on $[0,T]\times\boldsymbol{R}^{2n}$ for any α,β, satisfying

$$(0.11)\qquad \begin{cases} |\partial_\xi^\alpha D_x^\beta\{W_h(t,h^\delta x,h^{-\rho}\xi)\}| \le \begin{cases} C'_{0,0}\langle x;\xi\rangle & (\alpha=\beta=0) \\ C'_{\alpha,\beta} & (|\alpha+\beta|\ge 1) \end{cases} \\ Im\{W_h(t,h^\delta x,h^{-\rho}\xi)\}\ge -C'_{0,0}\,, \end{cases}$$

on $[0,T]\times\boldsymbol{R}^{2n}$. Here $C'_{\alpha,\beta}$ is a constant independent of a parameter $0<h<1$.

Set

$$(0.12)\qquad \begin{cases} K_h(t,x,\xi)\equiv H_h(t,x,\xi)+W_h(t,x,\xi) \\ L_h\equiv i\partial_t+K_h(t,X,D_x)\,. \end{cases}$$

Furthermore, let T_0 $(0<T_0\le T)$ be sufficiently small.

Theorem 5.1.

For any $\varDelta_{t_0,t_{\nu+1}} : (T_0 \geq)\, t_0 \geq t_1 \geq \cdots \geq t_{\nu+1}\, (\geq 0)$*, there exists a Fourier integral operator* $p_h(\varDelta_{t_0,t_{\nu+1}})(\phi_h(\varDelta_{t_0,t_{\nu+1}}); X, D_x)$ *with a phase function* $\phi_h(\varDelta_{t_0,t_{\nu+1}})(x,\xi)$ *and a symbol* $p_h(\varDelta_{t_0,t_{\nu+1}})(x,\xi)$ *such that*

$$\begin{aligned} (0.13) \qquad & p_h(\varDelta_{t_0,t_{\nu+1}})(\phi_h(\varDelta_{t_0,t_{\nu+1}}); X, D_x) \\ & = e^{i\int_{t_1}^{t_0} K_h(\tau)d\tau}(X, D_x) e^{i\int_{t_2}^{t_1} K_h(\tau)d\tau}(X, D_x) \\ & \quad \cdots e^{i\int_{t_{\nu+1}}^{t_\nu} K_h(\tau)d\tau}(X, D_x)\,. \end{aligned}$$

Furthermore, the set $\{\phi_h(\varDelta_{t_0,t_{\nu+1}})\}_{\varDelta_{t_0,t_{\nu+1}}}$ *is bounded in* $\mathbb{B}^{\delta-\rho,2}_{\rho,\delta}(h)$*, and the set* $\{p_h(\varDelta_{t_0,t_{\nu+1}})\}_{\varDelta_{t_0,t_{\nu+1}}}$ *is bounded in* $\mathbb{B}^{0}_{\rho,\delta}(h)$*.*

Theorem 5.3.

For any $(T_0 \geq)\, t \geq s\, (\geq 0)$*, there exist a phase function* $\phi^\star_h(t,s)(x,\xi)$ *and a symbol* $p^\star_h(t,s)(x,\xi)$ *such that* $\phi_h(\varDelta_{t,s})$ *uniformly converges to* $\phi^\star_h(t,s)$ *in* $\mathbb{B}^{\delta-\rho,2}_{\rho,\delta}(h)$ *as* $|\varDelta_{t,s}|$ *tends to* 0*, and* $p_h(\varDelta_{t,s})$ *uniformly converges to* $p^\star_h(t,s)$ *in* $\mathbb{B}^{0,1}_{\rho,\delta}(h)$ *as* $|\varDelta_{t,s}|$ *tends to* 0*.*

Furthermore, it follows that

$$\begin{aligned} (0.14) \qquad & e^{i\phi^\star_h(t,s)(x,\xi) - ix'\cdot\xi} p^\star_h(t,s)(x,\xi) \\ & = \lim_{|\varDelta_{t,s}|\to 0} \mathrm{O_s}\text{—}\iint\cdots\iint \exp\Bigg(\sum_{j=1}^{\nu+1}\Big(i(x^{j-1}-x^j)\cdot\xi^j \\ & \quad + i\int_{t_j}^{t_{j-1}} K_h(\tau, x^{j-1}, \xi^j)d\tau\Big)\Bigg) dx^1 đ\xi^1 \cdots dx^\nu đ\xi^\nu\,, \end{aligned}$$

where $\varDelta_{t,s} : (T_0 \geq)\, t \equiv t_0 \geq t_1 \geq \cdots \geq t_{\nu+1} \equiv s\, (\geq 0)$*,* $x^0 \equiv x$*,* $x^{\nu+1} \equiv x'$ *and* $\xi^{\nu+1} \equiv \xi$*.*

Theorem 5.4.

The Fourier integral operator $U_h(t,s) \equiv p^\star_h(t,s)(\phi^\star_h(t,s); X, D_x)$ *with the phase function* $\phi^\star_h(t,s)(x,\xi)$ *and the symbol* $p^\star_h(t,s)(x,\xi)$ *in Theorem 5.3, has the following form:*

$$\begin{aligned} (0.15) \qquad U_h(t,s) & = \lim_{|\varDelta_{t,s}|\to 0} e^{i\int_{t_1}^{t} K_h(\tau)d\tau}(X, D_x) e^{i\int_{t_2}^{t_1} K_h(\tau)d\tau}(X, D_x) \\ & \quad \cdots e^{i\int_{s}^{t_\nu} K_h(\tau)d\tau}(X, D_x)\,, \end{aligned}$$

where $\Delta_{t,s} : (T_0 \geq)\, t \equiv t_0 \geq t_1 \geq \cdots \geq t_\nu \geq t_{\nu+1} \equiv s\, (\geq 0)$.

Furthermore, $U_h(t,s)$ *is the fundamental solution for the operator* L_h.

REMARK. An example of a Schrödinger operator L_h:

$$
\begin{aligned}
(0.17)\qquad L_h \equiv i\partial_t + \frac{1}{h}\Bigg\{ \sum_{j,k=1}^{n} &\Big(a_{j,k}(t)\Big(\frac{h\partial_{x_j}}{i}\Big)\Big(\frac{h\partial_{x_k}}{i}\Big) \\
&+ b_{j,k}(t)x_j\Big(\frac{h\partial_{x_k}}{i}\Big) + c_{j,k}(t)x_jx_k\Big) \\
&+ \sum_{j=1}^{n}\Big(a_j(t)\Big(\frac{h\partial_{x_j}}{i}\Big) + b_j(t)x_j\Big) + c(t,x)\Bigg\} \\
&+ d(t,x).
\end{aligned}
$$

Here $a_{j,k}(t)$, $b_{j,k}(t)$, $c_{j,k}(t)$, $a_j(t)$ and $b_j(t)$ are real-valued continuous functions on $[0,T]$. $c(t,x)$ is a real-valued function on $[0,T] \times \boldsymbol{R}^{2n}$ which has a continuous and bounded derivative $D_x^\beta c(t,x)$ on $[0,T] \times \boldsymbol{R}^{2n}$ for any β. $d(t,x)$ is a complex-valued function on $[0,T] \times \boldsymbol{R}^{2n}$, which has a continuous and bounded derivative $D_x^\beta d(t,x)$ on $[0,T] \times \boldsymbol{R}^{2n}$ for any $\beta \neq 0$, and whose imaginary part $Im\{d(t,x)\}$ is lower bounded on $[0,T] \times \boldsymbol{R}^{2n}$.

The Plan of This Paper

The plan of this paper is as follows.

Section 1. We introduce the classes of symbols $\mathbb{B}^{m}_{\rho,\delta}(h)$, $\mathbb{B}^{m,\lambda}_{\rho,\delta}(h)$ and the families of pseudo-differential operators $\boldsymbol{B}^{m}_{\rho,\delta}(h)$, $\boldsymbol{B}^{m,\lambda}_{\rho,\delta}(h)$ in Kitada and H.Kumano-go [6].

Section 2. We define the class of phase functions $\mathbb{P}_{\rho,\delta}(\tau, h, \{\kappa_l\}_{l=0}^{\infty})$ and investigate the properties of the multi-products of phase functions in more detail. It is important that the multi-product of phase functions $(\phi_{1,h}\#\phi_{2,h}\#\cdots\#\phi_{\nu+1,h})(x,\xi)$ is ‘ close ’ to $x\cdot\xi + \sum_{j=1}^{\nu+1} J_{j,h}(x,\xi)$ where $J_{j,h}(x,\xi) \equiv \phi_{j,h}(x,\xi) - x\cdot\xi$. This section plays an essential role to prove the pointwise convergence of the phase function in Section 5.

Section 3. We introduce the families of Fourier integral operators $\boldsymbol{B}^{m}_{\rho,\delta}(\phi_h)$, $\boldsymbol{B}^{m,\lambda}_{\rho,\delta}(\phi_h)$ in Kitada and H.Kumano-go [6].

Section 4. We apply Fujiwara's skip method in [3] to Kitada and H.Kumano-go's theory of multi-products of Fourier integral operators, and investigate some properties. This section plays an essential role to prove the convergence of the symbol in Section 5.

Section 5. We give a construction of the fundamental solution for the Cauchy problem of a pseudo-differential equation of Schrödinger's type. We regard this construction as a multi-product of Fourier integral operators, and investigate the pointwise convergence of the phase function and that of the symbol.

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Notation

For $x=(x_1,\dots,x_n)\in \boldsymbol{R}^n_x$, $\xi=(\xi_1,\dots,\xi_n)\in \boldsymbol{R}^n_\xi$ and multi-indices of non-negative integers $\alpha=(\alpha_1,\dots,\alpha_n)$, $\beta=(\beta_1,\dots,\beta_n)$, we employ the usual notation:

$$|\alpha|=\alpha_1+\cdots+\alpha_n,\quad |\beta|=\beta_1+\cdots+\beta_n,$$

$$x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n,\quad \langle x\rangle=(1+|x|^2)^{1/2},\quad \langle x;\xi\rangle=(1+|x|^2+|\xi|^2)^{1/2},$$

$$\partial_{\xi_j}=\frac{\partial}{\partial\xi_j},\quad D_{x_j}=-i\frac{\partial}{\partial x_j},\quad \partial_\xi^\alpha=\partial_{\xi_1}^{\alpha_1}\cdots\partial_{\xi_n}^{\alpha_n},\quad D_x^\beta=D_{x_1}^{\beta_1}\cdots D_{x_n}^{\beta_n}.$$

$\mathcal{S}$ denotes the Schwartz space of rapidly decreasing C^∞-functions on $\boldsymbol{R}^n$. $\mathcal{B}(\boldsymbol{R}^{2n})$ denotes the set of C^∞-functions on $\boldsymbol{R}^{2n}$ whose derivatives are all bounded. $\mathcal{B}^{1,\infty}(\boldsymbol{R}^{2n})$ denotes the set of C^∞-functions $g(x,\xi)$ on $\boldsymbol{R}^{2n}$ whose derivatives $\partial_\xi^\alpha D_x^\beta g(x,\xi)$ are bounded for $|\alpha+\beta|\ge 1$. For $u\in S$, $f\in\mathcal{B}(\boldsymbol{R}^{2n})$ and $g\in\mathcal{B}^{1,\infty}(\boldsymbol{R}^{2n})$, we define semi-norms $|u|_{l,\mathcal{S}}$, $|f|_{l,\mathcal{B}}$ and $|g|_{l,\mathcal{B}^{1,\infty}}(l=0,1,2,\dots)$, respectively by

$$|u|_{l,\mathcal{S}}\equiv\max_{k+|\alpha|\le l}\sup_x|\langle x\rangle^k\partial_x^\alpha u(x)|\quad (l=0,1,2,\dots),$$

$$|f|_{l,\mathcal{B}} \equiv \max_{|\alpha+\beta|\le l} \sup_{x,\xi} |\partial_\xi^\alpha D_x^\beta f(x,\xi)| \quad (l=0,1,2,\dots)\,,$$

and

$$|g|_{l,\mathcal{B}^{1,\infty}} \equiv \begin{cases} \sup_{x,\xi} |g(x,\xi)|/\langle x;\xi\rangle \quad (l=0) \\ \sup_{x,\xi} |g(x,\xi)|/\langle x;\xi\rangle + \max_{1\le|\alpha+\beta|\le l} \sup_{x,\xi} |\partial_\xi^\alpha D_x^\beta g(x,\xi)| \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (l=1,2,\dots)\,. \end{cases}$$

Then, $\mathcal{S}$, $\mathcal{B}(\boldsymbol{R}^{2n})$ and $\mathcal{B}^{1,\infty}(\boldsymbol{R}^{2n})$ are Fréchet spaces with these semi-norms.

The Fourier transform $\hat{u}(\xi) \equiv \mathcal{F}[u](\xi)$ and the inverse Fourier transform $\overline{\mathcal{F}}[v](x)$ are defined respectively by

$$\begin{cases} \hat{u}(\xi) \equiv \mathcal{F}[u](\xi) \equiv \int e^{-ix\cdot\xi} u(x)dx \quad (u\in\mathcal{S}) \\ \overline{\mathcal{F}}[v](x) \equiv \int e^{ix\cdot\xi} v(\xi)\bar{d}\xi \quad (v\in\mathcal{S}), \end{cases}$$

where $\bar{d}\xi \equiv (2\pi)^{-n}d\xi$. $\mathrm{O_s}$—$\iint \cdot dy\bar{d}\eta$ denotes the usual oscillatory integrals. See [7].

1. Pseudo-Differential Operators

In this section, we introduce the basic properties of the pseudo-differential operators defined by Kitada and H.Kumano-go in [6]. For the details, see [6]. In what follows, for simplicity, let $n_0 \equiv 2[n/2+1]$.

Definition 1.1 (A Family of Pseudo-Differential Operators $\boldsymbol{B}^m_{\rho,\delta}(h)$).

(1) We say that a family $\{p_h\}_{0<h<1}$ of C^∞-functions $p_h \equiv p_h(x,\xi,x',\xi',x'')$ belongs to the class $\{\mathbb{B}^m_{\rho,\delta}(h)\}_{0<h<1}(m\in\boldsymbol{R},\ 0\le\delta\le\rho\le 1)$ of symbols, if p_h satisfies

$$\sup_{0<h<1} \sup_{x,\xi,x',\xi',x''} \frac{|\partial_\xi^\alpha D_x^\beta \partial_{\xi'}^{\alpha'} D_{x'}^{\beta'} D_{x''}^{\beta''} p_h(x,\xi,x',\xi',x'')|}{h^{m+\rho|\alpha+\alpha'|-\delta|\beta+\beta'+\beta''|}} < \infty\,, \tag{1.1}$$

for any $\alpha,\beta,\alpha',\beta',\beta''$. We write $\{p_h\}_{0<h<1} \in \{\mathbb{B}^m_{\rho,\delta}(h)\}_{0<h<1}$, or simply $p_h \in \mathbb{B}^m_{\rho,\delta}(h)$.

(2) For $p_h(x,\xi,x',\xi',x'') \in \mathbb{B}^m_{\rho,\delta}(h)$, we define a family $\{P_h\}_{0<h<1}$ of pseudo-differential operators $P_h \equiv p_h(X, D_x, X', D_{x'}, X'')$ by

$$\begin{aligned}(1.2)\qquad & p_h(X,D_x,X',D_{x'},X'')u(x)\\ &\equiv \mathrm{O_s}\text{—}\iiiint e^{-i(y^1\cdot\eta^1+y^2\cdot\eta^2)}p_h(x,\eta^1,x+y^1,\eta^2,x+y^1+y^2)\\ &\quad\times u(x+y^1+y^2)dy^2 đ\eta^2 dy^1 đ\eta^1 \quad (u\in\mathcal{S})\,.\end{aligned}$$

We write $\{P_h\}_{0<h<1} \in \{\boldsymbol{B}^m_{\rho,\delta}(h)\}_{0<h<1}$, or simply $P_h \in \boldsymbol{B}^m_{\rho,\delta}(h)$.

Remark.

1°. For $p_h \in \mathbb{B}^m_{\rho,\delta}(h)$, we define semi-norms $|p_h|^{(m)}_l (l = 0,1,2,\dots)$ by

$$(1.3)\qquad |p_h|^{(m)}_l \equiv \max_{|\alpha+\beta+\alpha'+\beta'+\beta''|\le l}\ \sup_{0<h<1}\ \sup_{x,\xi,x',\xi',x''}\left(\frac{|\partial^\alpha_\xi D^\beta_x \partial^{\alpha'}_{\xi'} D^{\beta'}_{x'} D^{\beta''}_{x''} p_h(x,\xi,x',\xi',x'')|}{h^{m+\rho|\alpha+\alpha'|-\delta|\beta+\beta'+\beta''|}}\right).$$

Then $\mathbb{B}^m_{\rho,\delta}(h)$ is a Fréchet space with these semi-norms.

2°. For $p_h(x,\xi,x',\xi',x'') \in \mathbb{B}^m_{\rho,\delta}(h)$, we define $\widetilde{p}_h$ by

$$(1.4)\qquad \widetilde{p}_h(x,\xi,x',\xi',x'') \equiv h^{-m}p_h(h^\delta x, h^{-\rho}\xi, h^\delta x', h^{-\rho}\xi', h^\delta x'')\,.$$

Then we have

$$(1.5)\qquad \widetilde{p}_h \in \mathbb{B}^0_{0,0}(h),$$

and

$$(1.6)\qquad |\widetilde{p}_h|^{(0)}_l(\text{ in } \mathbb{B}^0_{0,0}(h)) = |p_h|^{(m)}_l(\text{ in } \mathbb{B}^m_{\rho,\delta}(h))\,.$$

3°. Symbols $p_h(x,\xi), p_h(\xi,x') \in \mathbb{B}^m_{\rho,\delta}(h)$ are often called *single symbols*.

4°. For symbols $p_h(x,\xi), p_h(\xi,x'), p_h(x,\xi,x') \in \mathbb{B}^m_{\rho,\delta}(h)$, we have the representation formulae:

$$(1.7)\qquad p_h(X,D_x)u(x) = \int e^{ix\cdot\xi}p_h(x,\xi)\hat{u}(\xi)đ\xi \quad (u\in\mathcal{S})\,,$$

$$(1.8)\qquad \mathcal{F}[p_h(D_x,X')u](\xi) = \int e^{-ix'\cdot\xi}p_h(\xi,x')u(x')dx' \quad (u\in\mathcal{S})\,,$$

$$\begin{aligned}(1.9)\qquad & p_h(X,D_x,X')u(x)\\ &= \mathrm{O_s}\text{—}\iint e^{i(x-x')\cdot\xi}p_h(x,\xi,x')u(x')dx'đ\xi \quad (u\in\mathcal{S})\,.\end{aligned}$$

Definition 1.2 (A Family of Pseudo-Differential Operators $\boldsymbol{B}^{m,\lambda}_{\rho,\delta}(h)$).

(1) We say that a family $\{p_h\}_{0<h<1}$ of C^∞-functions $p_h \equiv p_h(x,\xi)$ belongs to the class $\{\mathbb{B}^{m,\lambda}_{\rho,\delta}(h)\}_{0<h<1}$ ($m, \lambda \in \boldsymbol{R}$, $0 \le \delta \le \rho \le 1$) of symbols, if $p_h(x,\xi)\langle h^{-\delta}x; h^\rho \xi\rangle^{-\lambda}$ belongs to $\mathbb{B}^{m}_{\rho,\delta}(h)$.

We write $\{p_h\}_{0<h<1} \in \{\mathbb{B}^{m,\lambda}_{\rho,\delta}(h)\}_{0<h<1}$, or simply $p_h \in \mathbb{B}^{m,\lambda}_{\rho,\delta}(h)$.

(2) For $p_h(x,\xi) \in \mathbb{B}^{m,\lambda}_{\rho,\delta}(h)$, we define a family $\{P_h\}_{0<h<1}$ of pseudo-differential operators $P_h \equiv p_h(X, D_x)$ by

$$p_h(X,D_x)u(x) \equiv \int e^{ix\cdot\xi} p_h(x,\xi)\hat{u}(\xi) đ\xi \quad (u \in \mathcal{S})\,. \tag{1.10}$$

We write $\{P_h\}_{0<h<1} \in \{\boldsymbol{B}^{m,\lambda}_{\rho,\delta}(h)\}_{0<h<1}$, or simply $P_h \in \boldsymbol{B}^{m,\lambda}_{\rho,\delta}(h)$.

Remark.

1°. For $p_h \in \mathbb{B}^{m,\lambda}_{\rho,\delta}(h)$, we define semi-norms $|p_h|^{(m,\lambda)}_l (l = 0, 1, 2, \dots)$ by

$$|p_h|^{(m,\lambda)}_l \equiv \max_{|\alpha+\beta|\le l} \sup_{0<h<1} \sup_{x,\xi} \frac{|\partial^\alpha_\xi D^\beta_x\{p_h(x,\xi)\langle h^{-\delta}x; h^\rho\xi\rangle^{-\lambda}\}|}{h^{m+\rho|\alpha|-\delta|\beta|}}\,. \tag{1.11}$$

Then $\mathbb{B}^{m,\lambda}_{\rho,\delta}(h)$ is a Fréchet space with these semi-norms.

2°. For $p_h(x,\xi) \in \mathbb{B}^{m,\lambda}_{\rho,\delta}(h)$, we define $\widetilde{p}_h$ by

$$\widetilde{p}_h(x,\xi) \equiv h^{-m} p_h(h^\delta x, h^{-\rho}\xi)\,. \tag{1.12}$$

Then we have

$$\widetilde{p}_h \in \mathbb{B}^{0,\lambda}_{0,0}(h)\,, \tag{1.13}$$

and

$$|\widetilde{p}_h|^{(0,\lambda)}_l (\text{ in } \mathbb{B}^{0,\lambda}_{0,0}(h)) = |p_h|^{(m,\lambda)}_l (\text{ in } \mathbb{B}^{m,\lambda}_{\rho,\delta}(h))\,. \tag{1.14}$$

Proposition 1.3 (Continuity on $\mathcal{S}$).

(1) *There exist a constant $M_{m,h,l}$(depending only on m, h) and integers l', l'' such that*

$$
\begin{aligned}
(1.15)\qquad & |p_h(X, D_x, X', D_{x'}, X'')u|_{l,\mathcal{S}} \leq M_{m,h,l}|p_h|_{l'}^{(m)}|u|_{l'',\mathcal{S}} \\
& (p_h(x,\xi,x',\xi',x'') \in \mathbb{B}^m_{\rho,\delta}(h),\, u \in \mathcal{S})\,.
\end{aligned}
$$

(2) *There exist a constant $M_{m,\lambda,h,l}$(depending only on m, λ, h) and integers l'_λ, l''_λ(depending only on λ) such that*

$$
\begin{aligned}
(1.16)\qquad & |p_h(X, D_x)u|_{l,\mathcal{S}} \leq M_{m,\lambda,h,l}|p_h|_{l'_\lambda}^{(m,\lambda)}|u|_{l''_\lambda,\mathcal{S}} \\
& (p_h(x,\xi) \in \mathbb{B}^{m,\lambda}_{\rho,\delta}(h),\, u \in \mathcal{S})\,.
\end{aligned}
$$

Proof. See [6]. □

Proposition 1.4 (Multi-products).

There exist constants A and C_l, satisfying the following:

For $p_{j,h}(x,\xi,x') \in \mathbb{B}^{m_j}_{\rho,\delta}(h)$ ($\nu = 1, 2, \ldots,\ j = 1, 2, \ldots, \nu+1$), define $q_{\nu+1,h}(x,\xi,x')$ by

$$
\begin{aligned}
(1.17)\qquad q_{\nu+1,h}(x,\xi,x') \equiv \mathrm{O_s}\text{—}\iint \cdots \iint & \exp(-i\sum_{j=1}^{\nu} y^j \cdot \eta^j) \\
\times \prod_{j=1}^{\nu} & p_{j,h}(x+\bar{y}^{j-1}, \xi+\eta^j, x+\bar{y}^j) \\
& \cdot p_{\nu+1,h}(x+\bar{y}^\nu, \xi, x')dy^\nu đ\eta^\nu \cdots dy^1 đ\eta^1 \\
(\bar{y}^0 \equiv 0,\quad \bar{y}^j \equiv y^1 + \cdots + y^j,\quad & j = 1, 2, \ldots, \nu)\,.
\end{aligned}
$$

Then it follows that

$$
(1.18)\qquad q_{\nu+1,h}(x,\xi,x') \in \mathbb{B}^{\bar{m}_{\nu+1}}_{\rho,\delta}(h)\,,
$$

$$
\begin{aligned}
(1.19)\qquad & q_{\nu+1,h}(X, D_x, X') \\
& = p_{1,h}(X, D_x, X')p_{2,h}(X, D_x, X') \cdots p_{\nu+1,h}(X, D_x, X')\,,
\end{aligned}
$$

and

$$|q_{\nu+1,h}|_l^{(\bar{m}_{\nu+1})} \le C_l A^{\nu+1} \max_{l_1+l_2+\cdots+l_{\nu+1}\le l} \prod_{j=1}^{\nu+1} |p_{j,h}|_{l_j+3n_0}^{(m_j)}, \tag{1.20}$$

where $\bar{m}_{\nu+1} \equiv m_1 + m_2 + \cdots + m_{\nu+1}$.

Proof. See [6]. □

Proposition 1.5 (Simplified Symbols).
There exists a constant C_l *satisfying the following:*
For $p_h(x,\xi,x',\xi',x'') \in \mathbb{B}^m_{\rho,\delta}(h)$, *define* $p_{h,L}(x,\xi,x')$ *and* $p_{h,R}(x,\xi,x')$, *respectively by*

$$\begin{cases} p_{h,L}(x,\xi,x') \equiv \mathrm{O_s}\text{—}\iint e^{-iy\cdot\eta} p_h(x,\xi+\eta,x+y,\xi,x')dy đ\eta \\ p_{h,R}(x,\xi,x') \equiv \mathrm{O_s}\text{—}\iint e^{-iy\cdot\eta} p_h(x,\xi,x'+y,\xi-\eta,x')dy đ\eta\,. \end{cases} \tag{1.21}$$

Then it follows that

$$p_{h,L}(x,\xi,x'), p_{h,R}(x,\xi,x') \in \mathbb{B}^m_{\rho,\delta}(h)\,, \tag{1.22}$$

$$p_h(X,D_x,X',D_{x'},X'') = p_{h,L}(X,D_x,X') = p_{h,R}(X,D_x,X')\,, \tag{1.23}$$

and

$$|p_{h,L}|_l^{(m)}, |p_{h,R}|_l^{(m)} \le C_l |p_h|_{l+2n_0}^{(m)}\,. \tag{1.24}$$

Proof. See [6]. □

Proposition 1.6 (The Inverse of $(I - p_h(X,D_x,X'))$).
Let the constant A *be the same as that in Proposition 1.4. There exists a constant* C_l *satisfying the following:*
If $p_h(x,\xi,x') \in \mathbb{B}^0_{\rho,\delta}(h)$ *satisfies* $|p_h|_{3n_0}^{(0)} \le 1/(2A)$, *there exists* $q_h(x,\xi,x') \in \mathbb{B}^0_{\rho,\delta}(h)$ *such that*

$$\begin{aligned} I &= q_h(X,D_x,X')(I - p_h(X,D_x,X')) \\ &= (I - p_h(X,D_x,X'))q_h(X,D_x,X')\,, \end{aligned} \tag{1.25}$$

and

$$|q_h|_l^{(0)} \le C_l\{\max(1, |p_h|_{l+3n_0}^{(0)})\}^l . \tag{1.26}$$

PROOF. See [6]. □

PROPOSITION 1.7 ($\boldsymbol{L}^2$-Boundedness).
There exists a constant C such that

$$\begin{aligned} &\|p_h(X, D_x, X')u\|_{\boldsymbol{L}^2} \le Ch^m |p_h|_{3n_0}^{(m)} \|u\|_{\boldsymbol{L}^2} \\ &(p_h(x,\xi,x') \in \mathbb{B}_{\rho,\delta}^m(h),\ u \in \mathcal{S}) . \end{aligned} \tag{1.27}$$

PROOF. See [6] and [7]. □

DEFINITION 1.8 (Hilbert Spaces, $\boldsymbol{H}_{0,h}$, $\boldsymbol{H}_{2,h}$).

(1) We define the Hilbert space $\boldsymbol{H}_{0,h}$ as the completion of $\mathcal{S}$ with respect to the norm

$$\|u\|_{0,h} \equiv h^{\rho-\delta}\|u\|_{\boldsymbol{L}^2} . \tag{1.28}$$

(2) We define the Hilbert space $\boldsymbol{H}_{2,h}$ as the completion of $\mathcal{S}$ with respect to the norm

$$\|u\|_{2,h} \equiv \left(\sum_{|\alpha+\beta|\le 2} \|(h^{-\delta}x)^\alpha (h^\rho D_x)^\beta u(x)\|_{\boldsymbol{L}^2}^2 \right)^{1/2} . \tag{1.29}$$

2. Phase Functions

In this section, we define phase functions and investigate the properties of multi-products of phase functions in more detail. This section plays an essential role to prove the pointwise convergence of the phase function in Section 5. The definition of phase functions below is a little different from that in Kitada and H.Kumano-go [6]. This definition is essential in this paper.

DEFINITION 2.1 (A Family of Phase Functions $\mathbb{P}_{\rho,\delta}(\tau, h, \{\kappa_l\}_{l=0}^{\infty})$).

For $\tau \geq 0$ and a sequence of positive constants $\{\kappa_l\}_{l=0}^{\infty}(\kappa_0 = 1,\, \kappa_l \leq \kappa_{l+1})$, we say that a family $\{\phi_h\}_{0<h<1}$ of real-valued C^{∞}-functions $\phi_h(x,\xi)$ belongs to the class $\{\mathbb{P}_{\rho,\delta}(\tau, h, \{\kappa_l\}_{l=0}^{\infty})\}_{0<h<1}$ of phase functions, if $J_h(x,\xi) \equiv \phi_h(x,\xi) - x\cdot\xi$ satisfies

$$
(2.1)\quad \sup_{0<h<1}\Bigg(\sum_{|\alpha+\beta|\leq 1}\sup_{x,\xi}\frac{|\partial_\xi^\alpha D_x^\beta\{h^{\rho-\delta}J_h(h^\delta x, h^{-\rho}\xi)\}|}{\langle x;\xi\rangle^{2-|\alpha+\beta|}} + \sum_{2\leq|\alpha+\beta|\leq 2+l}\sup_{x,\xi}|\partial_\xi^\alpha D_x^\beta\{h^{\rho-\delta}J_h(h^\delta x, h^{-\rho}\xi)\}|\Bigg) \leq \kappa_l\tau\,,
$$

for any $l = 0, 1, 2, \ldots$.

We write $\{\phi_h\}_{0<h<1} \in \{\mathbb{P}_{\rho,\delta}(\tau, h, \{\kappa_l\}_{l=0}^{\infty})\}_{0<h<1}$, or simply $\phi_h \in \mathbb{P}_{\rho,\delta}(\tau, h, \{\kappa_l\}_{l=0}^{\infty})$. Furthermore, for simplicity, we sometimes omit $\{\kappa_l\}_{l=0}^{\infty}$ and write simply $\mathbb{P}_{\rho,\delta}(\tau, h)$.

REMARK. For $\phi_h \in \mathbb{P}_{\rho,\delta}(\tau, h, \{\kappa_l\}_{l=0}^{\infty})$, we define $\widetilde{\phi}_h$ and $\widetilde{J}_h$ respectively by

$$
(2.2)\quad \begin{cases} \widetilde{\phi}_h(x,\xi) \equiv h^{\rho-\delta}\phi_h(h^\delta x, h^{-\rho}\xi) \\ \widetilde{J}_h(x,\xi) \equiv h^{\rho-\delta}J_h(h^\delta x, h^{-\rho}\xi)\,. \end{cases}
$$

Then we have

$$
(2.3)\quad \widetilde{\phi}_h \in \mathbb{P}_{0,0}(\tau, h, \{\kappa\}_{l=0}^{\infty})\,.
$$

PROPOSITION 2.2.

Let $\phi_{j,h} \in \mathbb{P}_{\rho,\delta}(\tau_j, h)(j = 1, 2, \ldots)$ *and* $\sum_{j=1}^{\infty}\tau_j < 1/2$.

Then, for any $\nu = 1, 2, 3, \ldots,$ $0 < h < 1$ *and* $(x,\xi) \in \boldsymbol{R}^{2n}$, *the solution* $\{X_{\nu,h}^j, \Xi_{\nu,h}^j\}_{j=1}^{\nu}(x,\xi)$ $(\in \boldsymbol{R}^{2\nu n})$ *of the equation*

$$
(2.4)\quad \begin{cases} X_{\nu,h}^j = \nabla_\xi\phi_{j,h}(X_{\nu,h}^{j-1}, \Xi_{\nu,h}^j) \\ \Xi_{\nu,h}^j = \nabla_x\phi_{j+1,h}(X_{\nu,h}^j, \Xi_{\nu,h}^{j+1}) \\ (j = 1, \ldots, \nu, \quad X_{\nu,h}^0 \equiv x, \quad \Xi_{\nu,h}^{\nu+1} \equiv \xi)\,, \end{cases}
$$

and the solution $\{\widetilde{X}^j_{\nu,h}, \widetilde{\Xi}^j_{\nu,h}\}_{j=1}^{\nu}(x,\xi)$ $(\in \boldsymbol{R}^{2\nu n})$ *of the equation*

$$(2.5)\qquad \begin{cases} \widetilde{X}^j_{\nu,h} = \nabla_\xi \widetilde{\phi}_{j,h}(\widetilde{X}^{j-1}_{\nu,h}, \widetilde{\Xi}^j_{\nu,h}) \\ \widetilde{\Xi}^j_{\nu,h} = \nabla_x \widetilde{\phi}_{j+1,h}(\widetilde{X}^j_{\nu,h}, \widetilde{\Xi}^{j+1}_{\nu,h}) \\ (j = 1, \dots, \nu, \quad \widetilde{X}^0_{\nu,h} \equiv x, \quad \widetilde{\Xi}^{\nu+1}_{\nu,h} \equiv \xi)\,, \end{cases}$$

exist uniquely. Furthermore, it follows that

$$(2.6)\qquad \begin{cases} X^j_{\nu,h}(x,\xi), \widetilde{X}^j_{\nu,h}(x,\xi) \in C^\infty(\boldsymbol{R}^{2n}) \quad (j = 0, 1, \dots, \nu) \\ \Xi^j_{\nu,h}(x,\xi), \widetilde{\Xi}^j_{\nu,h}(x,\xi) \in C^\infty(\boldsymbol{R}^{2n}) \quad (j = 1, 2, \dots, \nu+1)\,, \end{cases}$$

and

$$(2.7)\qquad \begin{cases} \widetilde{X}^j_{\nu,h}(x,\xi) = h^{-\delta} X^j_{\nu,h}(h^\delta x, h^{-\rho}\xi) \quad (j = 0, 1, \dots, \nu) \\ \widetilde{\Xi}^j_{\nu,h}(x,\xi) = h^{\rho} \Xi^j_{\nu,h}(h^\delta x, h^{-\rho}\xi) \quad (j = 1, 2, \dots, \nu+1)\,. \end{cases}$$

PROOF. See [6]. □

DEFINITION 2.3 (Multi-products of Phase Functions).
Let $\phi_{j,h} \in \mathbb{P}_{\rho,\delta}(\tau_j, h)(j = 1, 2, \dots)$ and $\sum_{j=1}^{\infty} \tau_j < 1/2$.
For $\nu = 1, 2, \dots$, we define the $\#$-$(\nu+1)$ products

$$(2.8)\qquad \begin{cases} \Phi_{\nu+1,h} \equiv \phi_{1,h} \# \phi_{2,h} \# \cdots \# \phi_{\nu+1,h} \\ \widetilde{\Phi}_{\nu+1,h} \equiv \widetilde{\phi}_{1,h} \# \widetilde{\phi}_{2,h} \# \cdots \# \widetilde{\phi}_{\nu+1,h}\,, \end{cases}$$

respectively by

$$(2.9)\qquad \begin{cases} \Phi_{\nu+1,h}(x,\xi) \equiv \sum_{j=1}^{\nu+1} \phi_{j,h}(X^{j-1}_{\nu,h}, \Xi^j_{\nu,h}) - \sum_{j=1}^{\nu} X^j_{\nu,h} \cdot \Xi^j_{\nu,h} \\ \widetilde{\Phi}_{\nu+1,h}(x,\xi) \equiv \sum_{j=1}^{\nu+1} \widetilde{\phi}_{j,h}(\widetilde{X}^{j-1}_{\nu,h}, \widetilde{\Xi}^j_{\nu,h}) - \sum_{j=1}^{\nu} \widetilde{X}^j_{\nu,h} \cdot \widetilde{\Xi}^j_{\nu,h}\,. \end{cases}$$

REMARK. Set

$$(2.10)\qquad \begin{cases} J_{j,h}(x,\xi) \equiv \phi_{j,h}(x,\xi) - x\cdot\xi \quad (j = 1, 2, \dots, \nu+1) \\ \widetilde{J}_{j,h}(x,\xi) \equiv \widetilde{\phi}_{j,h}(x,\xi) - x\cdot\xi \quad (j = 1, 2, \dots, \nu+1)\,, \end{cases}$$

and

$$\text{(2.11)}\qquad \begin{cases} \boldsymbol{J}_{\nu+1,h}(x,\xi) \equiv \Phi_{\nu+1,h}(x,\xi) - x\cdot\xi \\ \widetilde{\boldsymbol{J}}_{\nu+1,h}(x,\xi) \equiv \widetilde{\Phi}_{\nu+1,h}(x,\xi) - x\cdot\xi\,. \end{cases}$$

Then we get

$$\text{(2.12)}\qquad \begin{cases} \widetilde{\boldsymbol{J}}_{\nu+1,h}(x,\xi) = \sum_{j=1}^{\nu+1} \widetilde{J}_{j,h}(\widetilde{X}^{j-1}_{\nu,h}, \widetilde{\Xi}^{j}_{\nu,h}) \\ \qquad\qquad - \sum_{j=1}^{\nu} (\widetilde{X}^{j}_{\nu,h} - \widetilde{X}^{j-1}_{\nu,h})\cdot(\widetilde{\Xi}^{j}_{\nu,h} - \xi) \\ \nabla_{(x,\xi)}\widetilde{\boldsymbol{J}}_{\nu+1,h}(x,\xi) = \sum_{j=1}^{\nu+1} (\nabla_{(x,\xi)}\widetilde{J}_{j,h})(\widetilde{X}^{j-1}_{\nu,h}, \widetilde{\Xi}^{j}_{\nu,h})\,. \end{cases}$$

Furthermore, using (2.7), we have

$$\text{(2.13)}\qquad \widetilde{\Phi}_{\nu+1,h}(x,\xi) = h^{\rho-\delta}\Phi_{\nu+1,h}(h^{\delta}x, h^{-\rho}\xi)\,.$$

Proposition 2.4.

(1) *Let* $\phi_{j,h} \in \mathbb{P}_{\rho,\delta}(\tau_j, h)(j = 1,2,\dots)$ *and* $\sum_{j=1}^{\infty} \tau_j \le 1/4$.

Then it follows that

(2.14)

$$\begin{cases} |\widetilde{X}^{j}_{\nu,h} - \widetilde{X}^{j-1}_{\nu,h}|, |\widetilde{\Xi}^{j-1}_{\nu,h} - \widetilde{\Xi}^{j}_{\nu,h}| \le 2\tau_j\langle x;\xi\rangle \\ 2^{-1}\langle x;\xi\rangle \le \langle x + \theta(\widetilde{X}^{j-1}_{\nu,h} - x); \xi + \theta(\widetilde{\Xi}^{j}_{\nu,h} - \xi)\rangle \le 2\langle x;\xi\rangle \\ |f(x + \theta(\widetilde{X}^{j-1}_{\nu,h} - x), \xi + \theta(\widetilde{\Xi}^{j}_{\nu,h} - \xi))| \le |f|_{0,\mathcal{B}} \\ |g(x + \theta(\widetilde{X}^{j-1}_{\nu,h} - x), \xi + \theta(\widetilde{\Xi}^{j}_{\nu,h} - \xi))| \le 2|g|_{0,\mathcal{B}^{1,\infty}}\langle x;\xi\rangle \\ \begin{pmatrix} \nu = 1,2,\dots,\quad j = 1,2,\dots,\nu+1,\quad |\theta|\le 1,\quad 0<h<1, \\ f(x,\xi)\in\mathcal{B}(\boldsymbol{R}^{2n}),\quad g(x,\xi)\in\mathcal{B}^{1,\infty}(\boldsymbol{R}^{2n}),\quad \widetilde{X}^{\nu+1}_{\nu,h}\equiv\widetilde{X}^{\nu}_{\nu,h},\quad \widetilde{\Xi}^{0}_{\nu,h}\equiv\widetilde{\Xi}^{1}_{\nu,h} \end{pmatrix}. \end{cases}$$

(2) *There exists a constant* c_0' *satisfying the following:*

Let $\phi_{j,h} \in \mathbb{P}_{\rho,\delta}(\tau_j, h)(j = 1,2,\dots)$ *and* $\sum_{j=1}^{\infty} \tau_j \le 1/4$.

Then it follows that

$$(2.15)\quad \begin{cases} |\overrightarrow{\nabla}_x(\widetilde{X}^j_{\nu,h} - \widetilde{X}^{j-1}_{\nu,h})|, |\overrightarrow{\nabla}_x(\widetilde{\Xi}^{j-1}_{\nu,h} - \widetilde{\Xi}^j_{\nu,h})| \le c'_0\tau_j \\ |\overrightarrow{\nabla}_\xi(\widetilde{X}^j_{\nu,h} - \widetilde{X}^{j-1}_{\nu,h})|, |\overrightarrow{\nabla}_\xi(\widetilde{\Xi}^{j-1}_{\nu,h} - \widetilde{\Xi}^j_{\nu,h})| \le c'_0\tau_j \\ (\nu = 1, 2, \dots, \quad j = 1, 2, \dots, \nu+1, \quad 0 < h < 1, \\ \qquad\qquad \widetilde{X}^{\nu+1}_{\nu,h} \equiv \widetilde{X}^\nu_{\nu,h}, \quad \widetilde{\Xi}^0_{\nu,h} \equiv \widetilde{\Xi}^1_{\nu,h}). \end{cases}$$

(3) *For any* $\{\kappa_l\}_{l=0}^\infty(\kappa_0 = 1, \kappa_l \le \kappa_{l+1})$, *there exist constants* c'_l(*depending only on* $\kappa_1, \kappa_2, \dots, \kappa_{l-1}, \kappa_l$) *and* c''_{l-1}, c'''_{l-1}(*depending only on* $\kappa_1, \kappa_2, \dots, \kappa_{l-1}$), *satisfying the following:*

Let $\phi_{j,h} \in \mathbb{P}_{\rho,\delta}(\tau_j, h, \{\kappa_l\}_{l=0}^\infty)$ $(j = 1, 2, \dots)$ *and* $\sum_{j=1}^\infty \tau_j \le 1/4$.

Then it follows that

$$(2.16)\quad \begin{cases} |\partial_\xi^\alpha D_x^\beta \overrightarrow{\nabla}_x(\widetilde{X}^j_{\nu,h} - \widetilde{X}^{j-1}_{\nu,h})|, |\partial_\xi^\alpha D_x^\beta \overrightarrow{\nabla}_x(\widetilde{\Xi}^{j-1}_{\nu,h} - \widetilde{\Xi}^j_{\nu,h})| \le c'_{|\alpha+\beta|}\tau_j \\ |\partial_\xi^\alpha D_x^\beta \overrightarrow{\nabla}_\xi(\widetilde{X}^j_{\nu,h} - \widetilde{X}^{j-1}_{\nu,h})|, |\partial_\xi^\alpha D_x^\beta \overrightarrow{\nabla}_\xi(\widetilde{\Xi}^{j-1}_{\nu,h} - \widetilde{\Xi}^j_{\nu,h})| \le c'_{|\alpha+\beta|}\tau_j \\ |\partial_\xi^\alpha D_x^\beta f(x + \theta(\widetilde{X}^{j-1}_{\nu,h} - x), \xi + \theta(\widetilde{\Xi}^j_{\nu,h} - \xi))| \le c''_{|\alpha+\beta|-1}|f|_{|\alpha+\beta|,\mathcal{B}} \\ |\partial_\xi^\alpha D_x^\beta g(x + \theta(\widetilde{X}^{j-1}_{\nu,h} - x), \xi + \theta(\widetilde{\Xi}^j_{\nu,h} - \xi))| \le c'''_{|\alpha+\beta|-1}|g|_{|\alpha+\beta|,\mathcal{B}^{1,\infty}} \\ \begin{pmatrix} \nu = 1, 2, \dots, \quad j = 1, 2, \dots, \nu+1, \quad |\theta| \le 1, \quad |\alpha+\beta| \ge 1, \quad 0 < h < 1, \\ f(x,\xi) \in \mathcal{B}(\boldsymbol{R}^{2n}), \quad g(x,\xi) \in \mathcal{B}^{1,\infty}(\boldsymbol{R}^{2n}), \quad \widetilde{X}^{\nu+1}_{\nu,h} \equiv \widetilde{X}^\nu_{\nu,h}, \quad \widetilde{\Xi}^0_{\nu,h} \equiv \widetilde{\Xi}^1_{\nu,h} \end{pmatrix}. \end{cases}$$

Proof. By induction. See [6].□

Proposition 2.5.

For any $\{\kappa_l\}_{l=0}^\infty(\kappa_0 = 1, \kappa_l \le \kappa_{l+1})$, *there exist a constant* $c_0 > 1$(*independent of* $\{\kappa_l\}_{l=0}^\infty$) *and a sequence of positive constants* $\{\kappa'_l\}_{l=0}^\infty(\kappa'_0 = 1, \kappa'_l \le \kappa'_{l+1}, \kappa_l \le \kappa'_l)$ (*depending only on* $\kappa_1, \kappa_2, \dots, \kappa_l$), *satisfying the following:*

(1) *Let* $\phi_{j,h} \in \mathbb{P}_{\rho,\delta}(\tau_j, h, \{\kappa_l\}_{l=0}^\infty)(j = 1, 2, \dots)$ *and* $\sum_{j=1}^\infty \tau_j \le 1/4$.

Then, for any $\nu = 1, 2, \dots$, *it follows that*

$$(2.17)\quad \begin{cases} \Phi_{\nu+1,h} \equiv (\phi_{1,h}\#\phi_{2,h}\#\cdots\#\phi_{\nu+1,h}) \in \mathbb{P}_{\rho,\delta}(c_0\bar{\tau}_{\nu+1}, h, \{\kappa'_l\}_{l=0}^\infty) \\ \widetilde{\Phi}_{\nu+1,h} \equiv (\widetilde{\phi}_{1,h}\#\widetilde{\phi}_{2,h}\#\cdots\#\widetilde{\phi}_{\nu+1,h}) \in \mathbb{P}_{0,0}(c_0\bar{\tau}_{\nu+1}, h, \{\kappa'_l\}_{l=0}^\infty), \end{cases}$$

where $\bar{\tau}_{\nu+1} \equiv \tau_1 + \tau_2 + \cdots + \tau_{\nu+1}$.

(2) *Let* $\phi_{j,h} \in \mathbb{P}_{\rho,\delta}(\tau_j, h, \{\kappa_l\}_{l=0}^{\infty})(j = 1, 2, \dots)$ *and* $\sum_{j=1}^{\infty} \tau_j \le 1/(4c_0)$.

Then, for any $\nu = 1, 2, \dots,$ *it follows that*

$$\begin{aligned}\Phi_{\nu+2,h} &\equiv (\phi_{1,h}\#\phi_{2,h}\#\cdots\#\phi_{\nu+1,h}\#\phi_{\nu+2,h}) \\ &= (\phi_{1,h}\#\phi_{2,h}\#\cdots\#\phi_{\nu+1,h})\#\phi_{\nu+2,h} \\ &= \phi_{1,h}\#(\phi_{2,h}\#\cdots\#\phi_{\nu+1,h}\#\phi_{\nu+2,h})\,.\end{aligned} \tag{2.18}$$

Furthermore, if $\{X_h, \Xi_h\}(x, \xi)$ *is the solution of the equations*

$$\left\{\begin{array}{l} X_h = \nabla_\xi \Phi_{\nu+1,h}(x, \Xi_h) \\ \Xi_h = \nabla_x \phi_{\nu+2,h}(X_h, \xi)\,, \end{array}\right. \tag{2.19}$$

then it follows that

$$X^j_{\nu+1,h}(x,\xi) = \left\{\begin{array}{ll} X^j_{\nu,h}(x, \Xi_h) & (j = 1, 2, \dots, \nu) \\ X_h(x, \xi) & (j = \nu + 1)\,, \end{array}\right. \tag{2.20}$$

and

$$\Xi^j_{\nu+1,h}(x,\xi) = \left\{\begin{array}{ll} \Xi^j_{\nu,h}(x, \Xi_h) & (j = 1, 2, \dots, \nu) \\ \Xi_h(x, \xi) & (j = \nu + 1)\,. \end{array}\right. \tag{2.21}$$

Proof. See [6]. □

Lemma 2.6.

(1) *Let* $\phi_{j,h} \in \mathbb{P}_{\rho,\delta}(\tau_j, h)(j = 1, 2, \dots)$ *and* $\sum_{j=1}^{\infty} \tau_j \le 1/4$.

Then it follows that

$$\begin{aligned}&\langle x + \theta(\widetilde{X}^{j-1}_{\nu,h} - x) + \theta' y; \xi + \theta(\widetilde{\Xi}^{j}_{\nu,h} - \xi) + \theta''\eta\rangle^{\pm 1} \\ &\qquad \le 4\langle y; \eta\rangle\langle x; \xi\rangle^{\pm 1} \\ &(\nu = 1, 2, \dots, \quad j = 1, 2, \dots, \nu + 1, \\ &\qquad\qquad |\theta|, |\theta'|, |\theta''| \le 1, \quad 0 < h < 1)\,.\end{aligned} \tag{2.22}$$

(2) *For any* $\{\kappa_l\}_{l=0}^{\infty}(\kappa_0 = 1,\ \kappa_l \le \kappa_{l+1})$, $\lambda \in \boldsymbol{R}$, α' *and* β', *there exists a constant* $C_{\lambda,\alpha',\beta',l-1}$(*depending only on* $\lambda, \alpha', \beta', \kappa_1, \kappa_2, \dots, \kappa_{l-1}$) *satisfying the following:*

Let $\phi_{j,h} \in \mathbb{P}_{\rho,\delta}(\tau_j, h, \{\kappa_l\}_{l=0}^{\infty})$ $(j = 1, 2, \dots)$ *and* $\sum_{j=1}^{\infty} \tau_j \le 1/4$. *Then it follows that*

$$
\begin{aligned}
(2.23)\quad & |\partial_\xi^\alpha D_x^\beta \partial_\eta^{\alpha'} D_y^{\beta'} \langle x + \theta(\widetilde{X}_{\nu,h}^{j-1} - x) + \theta' y; \xi + \theta(\widetilde{\Xi}_{\nu,h}^{j} - \xi) + \theta'' \eta\rangle^\lambda| \\
&\quad \le C_{\lambda,\alpha',\beta',|\alpha+\beta|-1} \langle y; \eta\rangle^{|\lambda|} \langle x; \xi\rangle^\lambda \\
&\quad (\nu = 1, 2, \dots, \quad j = 1, 2, \dots, \nu + 1, \\
&\qquad\qquad |\theta|, |\theta'|, |\theta''| \le 1, \quad 0 < h < 1).
\end{aligned}
$$

Proof. By induction. Note Proposition 2.4. □

For $\phi_{j,h} \in \mathbb{P}_{\rho,\delta}(\tau_j, h)(j = 1, 3)$ and $\phi_{2,\vartheta,h} \in \mathbb{P}_{\rho,\delta}(\tau_2, h)(0 \le \vartheta \le 1)$ with $\sum_{j=1}^{3} \tau_j < 1/2$, let $\{\widetilde{X}_{2,\vartheta,h}^j, \widetilde{\Xi}_{2,\vartheta,h}^j\}_{j=1}^2(x, \xi)$ be the solution of the equations

$$
(2.24)\qquad
\begin{cases}
\widetilde{X}_{2,\vartheta,h}^1 = \nabla_\xi \widetilde{\phi}_{1,h}(\widetilde{X}_{2,\vartheta,h}^0, \widetilde{\Xi}_{2,\vartheta,h}^1), \\
\widetilde{\Xi}_{2,\vartheta,h}^1 = \nabla_x \widetilde{\phi}_{2,\vartheta,h}(\widetilde{X}_{2,\vartheta,h}^1, \widetilde{\Xi}_{2,\vartheta,h}^2) \\
\widetilde{X}_{2,\vartheta,h}^2 = \nabla_\xi \widetilde{\phi}_{2,\vartheta,h}(\widetilde{X}_{2,\vartheta,h}^1, \widetilde{\Xi}_{2,\vartheta,h}^2), \\
\widetilde{\Xi}_{2,\vartheta,h}^2 = \nabla_x \widetilde{\phi}_{3,h}(\widetilde{X}_{2,\vartheta,h}^2, \widetilde{\Xi}_{2,\vartheta,h}^3) \\
(\widetilde{X}_{2,\vartheta,h}^0 \equiv x, \quad \widetilde{\Xi}_{2,\vartheta,h}^3 \equiv \xi, \quad 0 \le \vartheta \le 1).
\end{cases}
$$

Furthermore, let $\widetilde{\phi}_{2,\vartheta,h}(x, \xi)$ be continuously differentiable on $0 \le \vartheta \le 1$. Set

$$
(2.25)\qquad
\begin{cases}
\widetilde{J}_{2,\vartheta,h}(x, \xi) \equiv \widetilde{\phi}_{2,\vartheta,h}(x, \xi) - x \cdot \xi \\
\widetilde{\boldsymbol{J}}_{3,\vartheta,h}(x, \xi) \equiv (\widetilde{\phi}_{1,h} \# \widetilde{\phi}_{2,\vartheta,h} \# \widetilde{\phi}_{3,h})(x, \xi) - x \cdot \xi\,.
\end{cases}
$$

Then we get the following lemma.

Lemma 2.7.

For any $\{\kappa_l\}_{l=0}^{\infty}(\kappa_0 = 1,\ \kappa_l \le \kappa_{l+1})$ *and* $\{\chi_l\}_{l=0}^{\infty}(0 \le \chi_l \le \chi_{l+1})$, *there exist a constant* C_l *and a sequence of positive constants* $\{\chi'_l\}_{l=0}^{\infty}(0 \le \chi'_l \le \chi'_{l+1},\ \chi_l \le \chi'_l)$ *(depending only on* $\kappa_1, \kappa_2, \dots, \kappa_l$ *and* $\chi_0, \chi_1, \dots, \chi_l$*), satisfying the following:*

Let $\phi_{j,h} \in \mathbb{P}_{\rho,\delta}(\tau_j, h, \{\kappa_l\}_{l=0}^{\infty})(j = 1, 3)$, $\phi_{2,\vartheta,h} \in \mathbb{P}_{\rho,\delta}(\tau_2, h, \{\kappa_l\}_{l=0}^{\infty})(0 \le \vartheta \le 1)$ *and* $\sum_{j=1}^{3} \tau_j \le 1/4$, *and let* $\partial_\vartheta \widetilde{J}_{2,\vartheta,h}(x,\xi)$ *satisfy*

$$\tag{2.26} \begin{cases} |\partial_\vartheta \widetilde{J}_{2,\vartheta,h}(x,\xi)| \le \chi_0 (\tau_2)^2 \langle x;\xi\rangle^2 \\ |\partial_\xi^\alpha D_x^\beta \nabla_{(x,\xi)} \partial_\vartheta \widetilde{J}_{2,\vartheta,h}(x,\xi)| \le \chi_{|\alpha+\beta|} (\tau_2)^2 \langle x;\xi\rangle \\ (0 < h < 1, \quad 0 \le \vartheta \le 1)\,. \end{cases}$$

Then it follows that

$$\tag{2.27} \begin{aligned} &|\partial_\xi^\alpha D_x^\beta \partial_\vartheta \widetilde{X}^j_{2,\vartheta,h}|,\ |\partial_\xi^\alpha D_x^\beta \partial_\vartheta \widetilde{\Xi}^j_{2,\vartheta,h}| \le C_{|\alpha+\beta|} (\tau_2)^2 \langle x;\xi\rangle \\ &(j = 1, 2, \quad 0 < h < 1, \quad 0 \le \vartheta \le 1)\,, \end{aligned}$$

and

$$\tag{2.28} \begin{cases} |\partial_\vartheta \widetilde{\boldsymbol{J}}_{3,\vartheta,h}(x,\xi)| \le \chi'_0 (\tau_2)^2 \langle x;\xi\rangle^2 \\ |\partial_\xi^\alpha D_x^\beta \nabla_{(x,\xi)} \partial_\vartheta \widetilde{\boldsymbol{J}}_{3,\vartheta,h}(x,\xi)| \le \chi'_{|\alpha+\beta|} (\tau_2)^2 \langle x;\xi\rangle \\ (0 < h < 1, \quad 0 \le \vartheta \le 1)\,. \end{cases}$$

Proof. By induction. Note Proposition 2.4 and Lemma 2.6. □

Let $\phi_{j,h} \in \mathbb{P}_{\rho,\delta}(\tau_j, h)(j = 1, 2, \dots)$ and $\sum_{j=1}^{\infty} \tau_j < 1/2$.

For $\nu = 1, 2, \dots$ and $0 \le \vartheta \le 1$, set

$$\tag{2.29} \begin{cases} \Phi_{\nu+1,\vartheta,h}(x,\xi) \equiv \vartheta \Phi_{\nu+1,h}(x,\xi) + (1-\vartheta)\Big(x\cdot\xi + \sum_{j=1}^{\nu+1} J_{j,h}(x,\xi)\Big) \\ \widetilde{\Phi}_{\nu+1,\vartheta,h}(x,\xi) \equiv \vartheta \widetilde{\Phi}_{\nu+1,h}(x,\xi) + (1-\vartheta)\Big(x\cdot\xi + \sum_{j=1}^{\nu+1} \widetilde{J}_{j,h}(x,\xi)\Big)\,, \end{cases}$$

and

$$\tag{2.30} \begin{cases} \boldsymbol{J}_{\nu+1,\vartheta,h}(x,\xi) \equiv \Phi_{\nu+1,\vartheta,h}(x,\xi) - x\cdot\xi \\ \widetilde{\boldsymbol{J}}_{\nu+1,\vartheta,h}(x,\xi) \equiv \widetilde{\Phi}_{\nu+1,\vartheta,h}(x,\xi) - x\cdot\xi\,. \end{cases}$$

Then we have

$$
\text{(2.31)} \qquad \begin{cases} \Phi_{\nu+1,1,h}(x,\xi) = \Phi_{\nu+1,h}(x,\xi), \\ \Phi_{\nu+1,0,h}(x,\xi) = x\cdot\xi + \sum_{j=1}^{\nu+1} J_{j,h}(x,\xi) \\ \widetilde{\Phi}_{\nu+1,1,h}(x,\xi) = \widetilde{\Phi}_{\nu+1,h}(x,\xi), \\ \widetilde{\Phi}_{\nu+1,0,h}(x,\xi) = x\cdot\xi + \sum_{j=1}^{\nu+1} \widetilde{J}_{j,h}(x,\xi)\,, \end{cases}
$$

and

$$
\text{(2.32)} \qquad \widetilde{\Phi}_{\nu+1,\vartheta,h}(x,\xi) = h^{\rho-\delta}\Phi_{\nu+1,\vartheta,h}(h^{\delta}x, h^{-\rho}\xi)\,.
$$

Furthermore, we get the following lemma.

LEMMA 2.8.
For any $\{\kappa_l\}_{l=0}^{\infty}(\kappa_0 = 1,\ \kappa_l \le \kappa_{l+1})$*, there exist a constant* $c_0 > 1$*(independent of* $\{\kappa_l\}_{l=0}^{\infty}$*) and sequences of positive constants* $\{\kappa'_l\}_{l=0}^{\infty}(\kappa'_0 = 1,\ \kappa'_l \le \kappa'_{l+1},\ \kappa_l \le \kappa'_l)$*,* $\{\chi_l\}_{l=0}^{\infty}(0 \le \chi_l \le \chi_{l+1})$*(depending only on* $\kappa_1, \kappa_2, \ldots, \kappa_l$*), satisfying the following:*

Let $\phi_{j,h} \in \mathbb{P}_{\rho,\delta}(\tau_j, h, \{\kappa_l\}_{l=0}^{\infty})(j = 1, 2, \ldots)$ *and* $\sum_{j=1}^{\infty} \tau_j \le 1/4$*.*

Then it follows that

$$
\text{(2.33)} \qquad \begin{cases} \Phi_{\nu+1,\vartheta,h} \in \mathbb{P}_{\rho,\delta}(c_0\bar{\tau}_{\nu+1}, h, \{\kappa'_l\}_{l=0}^{\infty}) \\ \widetilde{\Phi}_{\nu+1,\vartheta,h} \in \mathbb{P}_{0,0}(c_0\bar{\tau}_{\nu+1}, h, \{\kappa'_l\}_{l=0}^{\infty})\,, \end{cases}
$$

and

$$
\text{(2.34)} \qquad \begin{cases} |\partial_{\vartheta}\widetilde{\boldsymbol{J}}_{\nu+1,\vartheta,h}(x,\xi)| \le \chi_0(\bar{\tau}_{\nu+1})^2\langle x;\xi\rangle^2 \\ |\partial_{\xi}^{\alpha}D_x^{\beta}\nabla_{(x,\xi)}\partial_{\vartheta}\widetilde{\boldsymbol{J}}_{\nu+1,\vartheta,h}(x,\xi)| \le \chi_{|\alpha+\beta|}(\bar{\tau}_{\nu+1})^2\langle x;\xi\rangle \\ (\nu = 1, 2, \ldots,\quad 0 \le \vartheta \le 1,\quad 0 < h < 1)\,, \end{cases}
$$

where $\bar{\tau}_{\nu+1} \equiv \tau_1 + \tau_2 + \cdots + \tau_{\nu+1}$*.*

PROOF. From (2.12), we have

$$\partial_\vartheta \widetilde{\boldsymbol{J}}_{\nu+1,\vartheta,h}(x,\xi) = -\sum_{j=1}^{\nu}(\widetilde{X}^j_{\nu,h} - \widetilde{X}^{j-1}_{\nu,h}) \cdot (\widetilde{\Xi}^j_{\nu,h} - \xi)$$
$$+\sum_{j=1}^{\nu+1}\int_0^1 (\overrightarrow{\nabla}_{(x,\xi)}\widetilde{J}_{j,h})(x+\theta(\widetilde{X}^{j-1}_{\nu,h} - x), \xi+\theta(\widetilde{\Xi}^j_{\nu,h} - \xi))d\theta$$
$$\cdot \begin{pmatrix} \widetilde{X}^{j-1}_{\nu,h} - x \\ \widetilde{\Xi}^j_{\nu,h} - \xi \end{pmatrix}, \tag{2.35}$$

and

$$\nabla_{(x,\xi)}\partial_\vartheta \widetilde{\boldsymbol{J}}_{\nu+1,\vartheta,h}(x,\xi)$$
$$= \sum_{j=1}^{\nu+1}\int_0^1 (\overrightarrow{\nabla}_{(x,\xi)}\nabla_{(x,\xi)}\widetilde{J}_{j,h})(x+\theta(\widetilde{X}^{j-1}_{\nu,h} - x), \xi+\theta(\widetilde{\Xi}^j_{\nu,h} - \xi))d\theta$$
$$\cdot \begin{pmatrix} \widetilde{X}^{j-1}_{\nu,h} - x \\ \widetilde{\Xi}^j_{\nu,h} - \xi \end{pmatrix}. \tag{2.36}$$

Using Proposition 2.4, we get (2.34). □

3. Fourier Integral Operators

In this section, we introduce the basic properties of the Fourier integral operators defined by Kitada and H.Kumano-go in [6]. For the details, see [6]. In what follows, for simplicity, let $n_0 \equiv 2[n/2+1]$.

DEFINITION 3.1 (Fourier Integral Operators $\boldsymbol{B}^m_{\rho,\delta}(\phi_h), \boldsymbol{B}^m_{\rho,\delta}(\phi^*_h)$).

(1) For $0 \le \tau < 1$, $\phi_h \in \mathbb{P}_{\rho,\delta}(\tau, h)$ and $p_h(x,\xi) \in \mathbb{B}^m_{\rho,\delta}(h)$, we define a family of Fourier integral operators $p_h(\phi_h; X, D_x)$ by

$$p_h(\phi_h; X, D_x)u(x) = \int e^{i\phi_h(x,\xi)} p_h(x,\xi)\hat{u}(\xi) đ\xi \quad (u \in \mathcal{S}). \tag{3.1}$$

We write $\{p_h(\phi_h; X, D_x)\}_{0<h<1} \in \{\boldsymbol{B}^m_{\rho,\delta}(\phi_h)\}_{0<h<1}$, or simply $p_h(\phi_h; X, D_x) \in \boldsymbol{B}^m_{\rho,\delta}(\phi_h)$. We sometimes use this definition for $0 \le \tau < \infty$. (See Theorem 4.1.)

(2) For $0 \leq \tau < 1$, $\phi_h \in \mathbb{P}_{\rho,\delta}(\tau, h)$ and $q_h(\xi, x') \in \mathbb{B}^m_{\rho,\delta}(h)$, we define a family of conjugate Fourier integral operators $q_h(\phi_h^*; D_x, X')$ by

$$\mathcal{F}[q_h(\phi^*; D_x, X')u](\xi) = \int e^{-i\phi_h(x',\xi)} q_h(\xi, x')u(x')dx' \quad (u \in \mathcal{S}). \tag{3.2}$$

We write $\{q_h(\phi_h^*; D_x, X')\}_{0<h<1} \in \{\boldsymbol{B}^m_{\rho,\delta}(\phi_h^*)\}_{0<h<1}$, or simply $q_h(\phi_h^*; D_x, X') \in \boldsymbol{B}^m_{\rho,\delta}(\phi_h^*)$.

Definition 3.2 (Fourier Integral Operators $\boldsymbol{B}^{m,\lambda}_{\rho,\delta}(\phi_h)$).

For $0 \leq \tau < 1$, $\phi_h \in \mathbb{P}_{\rho,\delta}(\tau, h)$ and $p_h(x, \xi) \in \mathbb{B}^{m,\lambda}_{\rho,\delta}(h)$, we define a family of Fourier integral operators $p_h(\phi_h; X, D_x)$ by

$$p_h(\phi_h; X, D_x)u(x) = \int e^{i\phi_h(x,\xi)} p_h(x, \xi)\hat{u}(\xi)đ\xi \quad (u \in \mathcal{S}). \tag{3.3}$$

We write $\{p_h(\phi_h; X, D_x)\}_{0<h<1} \in \{\boldsymbol{B}^{m,\lambda}_{\rho,\delta}(\phi_h)\}_{0<h<1}$, or simply $p_h(\phi_h; X, D_x) \in \boldsymbol{B}^{m,\lambda}_{\rho,\delta}(\phi_h)$. We sometimes use this definition for $0 \leq \tau < \infty$. (See Theorem 4.2.)

In the following proposition, it is a revised point that $M_{m,h,l}, M_{m,\lambda,h,l}$, l' and l'' are independent of the choice of phase functions as well as the choice of symbols.

Proposition 3.3 (Continuity on $\mathcal{S}$).

(1) *For any $0 \leq \tau_0 < 1$ and $\{\kappa_l\}_{l=0}^{\infty}(\kappa_0 = 1,\ \kappa_l \leq \kappa_{l+1})$, there exist a constant $M_{m,h,l}$(depending only on $m, h, \tau_0, \{\kappa_l\}_{l=0}^{\infty}(\kappa_0 = 1,\ \kappa_l \leq \kappa_{l+1})$) and integers l', l'' such that*

$$\begin{cases} |p_h(\phi_h; X, D_x)u|_{l,\mathcal{S}} \leq M_{m,h,l}|p_h|^{(m)}_{l'}|u|_{l'',\mathcal{S}} \\ |q_h(\phi_h^*; D_x, X')u|_{l,\mathcal{S}} \leq M_{m,h,l}|q_h|^{(m)}_{l'}|u|_{l'',\mathcal{S}} \\ (0 \leq \tau \leq \tau_0,\ \phi_h \in \mathbb{P}_{\rho,\delta}(\tau, h, \{\kappa_l\}_{l=0}^{\infty}), \\ p_h(x, \xi), q_h(\xi, x') \in \mathbb{B}^m_{\rho,\delta}(h),\ u \in \mathcal{S}) . \end{cases} \tag{3.4}$$

(2) *For any $0 \leq \tau_0 < 1$ and $\{\kappa_l\}_{l=0}^{\infty}(\kappa_0 = 1,\ \kappa_l \leq \kappa_{l+1})$, there exist a constant $M_{m,\lambda,h,l}$(depending only on $m, \lambda, h, \tau_0, \{\kappa_l\}_{l=0}^{\infty}(\kappa_0 = 1,\ \kappa_l \leq \kappa_{l+1})$) and integers l'_λ, l''_λ(depending only on λ) such that*

$$|p_h(\phi_h; X, D_x)u|_{l,\mathcal{S}} \leq M_{m,\lambda,h,l}|p_h|^{(m,\lambda)}_{l'_\lambda}|u|_{l''_\lambda,\mathcal{S}} \tag{3.5}$$
$$(0 \leq \tau \leq \tau_0,\ \phi_h \in \mathbb{P}_{\rho,\delta}(\tau, h, \{\kappa_l\}_{l=0}^{\infty}),\ p_h(x, \xi) \in \mathbb{B}^{m,\lambda}_{\rho,\delta}(h),\ u \in \mathcal{S}) .$$

PROOF. See [6]. □

PROPOSITION 3.4 ($\boldsymbol{B}^{m}_{\rho,\delta}(\phi_h)\circ\boldsymbol{B}^{m'}_{\rho,\delta}(\phi^*_h)$, $\boldsymbol{B}^{m'}_{\rho,\delta}(\phi^*_h)\circ\boldsymbol{B}^{m}_{\rho,\delta}(\phi_h) \subset \boldsymbol{B}^{m+m'}_{\rho,\delta}(h)$).
For any $0 \le \tau_0 < 1$ *and* $\{\kappa_l\}_{l=0}^{\infty}(\kappa_0 = 1, \kappa_l \le \kappa_{l+1})$, *there exists a constant* C_{l+2n_0}(*depending only on* $\tau_0, \kappa_1, \kappa_2, \dots, \kappa_{l+2n_0}$) *satisfying the following:*

Let $0 \le \tau \le \tau_0$, $\phi_h \in \mathbb{P}_{\rho,\delta}(\tau, h, \{\kappa_l\}_{l=0}^{\infty})$, $p_h(x,\xi) \in \boldsymbol{B}^{m}_{\rho,\delta}(h)$ *and* $q_h(\xi, x') \in \boldsymbol{B}^{m'}_{\rho,\delta}(h)$. *Then there exist* $r_h(x,\xi), r'_h(\xi,x') \in \boldsymbol{B}^{m+m'}_{\rho,\delta}(h)$ *such that*

$$\begin{cases} p_h(\phi_h; X, D_x) q_h(\phi^*_h; D_x, X') = r_h(X, D_x) \\ q_h(\phi^*_h; D_x, X') p_h(\phi_h; X, D_x) = r'_h(D_x, X')\,, \end{cases} \tag{3.6}$$

and

$$|r_h|_l^{(m+m')}, |r'_h|_l^{(m+m')} \le C_{l+2n_0} |p_h|^{(m)}_{l+2n_0} |q_h|^{(m')}_{l+2n_0}\,. \tag{3.7}$$

PROOF. See [6]. □

PROPOSITION 3.5 ($\boldsymbol{B}^{m}_{\rho,\delta}(h)\circ\boldsymbol{B}^{m'}_{\rho,\delta}(\phi_h)$, $\boldsymbol{B}^{m'}_{\rho,\delta}(\phi_h)\circ\boldsymbol{B}^{m}_{\rho,\delta}(h) \subset \boldsymbol{B}^{m+m'}_{\rho,\delta}(\phi_h)$).
For any $0 \le \tau_0 < 1$ *and* $\{\kappa_l\}_{l=0}^{\infty}(\kappa_0 = 1, \kappa_l \le \kappa_{l+1})$, *there exists a constant* C_{l+2n_0-1}(*depending only on* $\tau_0, \kappa_1, \kappa_2, \dots, \kappa_{l+2n_0-1}$) *satisfying the following:*

Let $0 \le \tau \le \tau_0$, $\phi_h \in \mathbb{P}_{\rho,\delta}(\tau, h, \{\kappa_l\}_{l=0}^{\infty})$, $p_h(x,\xi) \in \boldsymbol{B}^{m}_{\rho,\delta}(h)$ *and* $q_h(x,\xi) \in \boldsymbol{B}^{m'}_{\rho,\delta}(h)$. *Then there exist* $r_h(x,\xi), r'_h(x,\xi) \in \boldsymbol{B}^{m+m'}_{\rho,\delta}(h)$ *such that*

$$\begin{cases} p_h(X, D_x) q_h(\phi_h; X, D_x) = r_h(\phi_h; X, D_x) \\ q_h(\phi_h; X, D_x) p_h(X, D_x) = r'_h(\phi_h; X, D_x)\,, \end{cases} \tag{3.8}$$

and

$$|r_h|_l^{(m+m')}, |r'_h|_l^{(m+m')} \le C_{l+2n_0-1} |p_h|^{(m)}_{l+2n_0} |q_h|^{(m')}_{l+2n_0}\,. \tag{3.9}$$

PROOF. See [6]. □

PROPOSITION 3.6 ($\boldsymbol{B}^{m}_{\rho,\delta}(h)\circ\boldsymbol{B}^{m'}_{\rho,\delta}(\phi_h^*)$, $\boldsymbol{B}^{m'}_{\rho,\delta}(\phi_h^*)\circ\boldsymbol{B}^{m}_{\rho,\delta}(h)\subset\boldsymbol{B}^{m+m'}_{\rho,\delta}(\phi_h^*)$).
For any $0\le\tau_0<1$ *and* $\{\kappa_l\}_{l=0}^{\infty}(\kappa_0=1,\,\kappa_l\le\kappa_{l+1})$*, there exists a constant* C_{l+2n_0-1}*(depending only on* $\tau_0,\kappa_1,\kappa_2,\dots,\kappa_{l+2n_0-1}$*) satisfying the following:*

Let $0\le\tau\le\tau_0$*,* $\phi_h\in\mathbb{P}_{\rho,\delta}(\tau,h,\{\kappa_l\}_{l=0}^{\infty})$*,* $p_h(\xi,x')\in\boldsymbol{B}^{m}_{\rho,\delta}(h)$ *and* $q_h(\xi,x')\in\boldsymbol{B}^{m'}_{\rho,\delta}(h)$*. Then there exist* $r_h(\xi,x'),r'_h(\xi,x')\in\boldsymbol{B}^{m+m'}_{\rho,\delta}(h)$ *such that*

$$
\begin{cases} p_h(D_x,X')q_h(\phi_h^*;D_x,X')=r_h(\phi_h^*;D_x,X') \\ q_h(\phi_h^*;D_x,X')p_h(D_x,X')=r'_h(\phi_h^*;D_x,X')\,, \end{cases}
\tag{3.10}
$$

and

$$
|r_h|_l^{(m+m')},|r'_h|_l^{(m+m')}\le C_{l+2n_0-1}|p_h|_{l+2n_0}^{(m)}|q_h|_{l+2n_0}^{(m')}\,.
\tag{3.11}
$$

PROOF. See [6]. □

PROPOSITION 3.7 ($\boldsymbol{L}^2$-Boundedness).
For any $0\le\tau_0<1$ *and* $\{\kappa_l\}_{l=0}^{\infty}(\kappa_0=1,\,\kappa_l\le\kappa_{l+1})$*, there exists a constant* C_{5n_0}*(depending only on* $\tau_0,\kappa_1,\kappa_2,\dots,\kappa_{5n_0}$*) such that*

$$
\begin{cases} \|p_h(\phi_h;X,D_x)u\|_{\boldsymbol{L}^2}\le C_{5n_0}h^m|p_h|_{5n_0}^{(m)}\|u\|_{\boldsymbol{L}^2} \\ \|q_h(\phi_h^*;D_x,X')u\|_{\boldsymbol{L}^2}\le C_{5n_0}h^m|q_h|_{5n_0}^{(m)}\|u\|_{\boldsymbol{L}^2} \\ (0\le\tau\le\tau_0,\ \phi_h\in\mathbb{P}_{\rho,\delta}(\tau,h,\{\kappa_l\}_{l=0}^{\infty}), \\ \qquad p_h(x,\xi),q_h(\xi,x')\in\mathbb{B}^{m}_{\rho,\delta}(h),\ u\in\mathcal{S})\,. \end{cases}
\tag{3.12}
$$

PROOF. See [6]. □

PROPOSITION 3.8 (Inverse of $I(\phi_h),I(\phi_h^*)$).
For any $\{\kappa_l\}_{l=0}^{\infty}(\kappa_0=1,\,\kappa_l\le\kappa_{l+1})$*, there exist constants* $0<\tau_0<1$*(depending only on* $\kappa_1,\kappa_2,\dots,\kappa_{5n_0}$*) and* C_{l+7n_0}*(depending only on* $\kappa_1,\kappa_2,\dots,\kappa_{l+7n_0}$*), satisfying the following:*

Let $\phi_h\in\mathbb{P}_{\rho,\delta}(\tau,h,\{\kappa_l\}_{l=0}^{\infty})$ *and* $0\le\tau\le\tau_0$*.*

Then there exist $r_h(\xi, x'), r'_h(x,\xi) \in \mathbb{B}^0_{\rho,\delta}(h)$ *such that*

$$\begin{cases} I(\phi_h) r_h(\phi_h^*; D_x, X') = r_h(\phi_h^*; D_x, X') I(\phi_h) = I \\ I(\phi_h^*) r'_h(\phi_h; X, D_x) = r'_h(\phi_h; X, D_x) I(\phi_h^*) = I\,, \end{cases} \tag{3.13}$$

and

$$|r_h|_l^{(0)}, |r'_h|_l^{(0)} \le C_{l+7n_0}\,. \tag{3.14}$$

Proof. See [6]. □

4. Multi-Products of Fourier Integral Operators

In this section, we apply Fujiwara's skip method in [3] to Kitada and H.Kumano-go's theory of multi-products of Fourier integral operators. This section plays an essential role to prove the pointwise convergence of the symbol in Section 5.

Theorem 4.1 ($\boldsymbol{B}^{m_1}_{\rho,\delta}(\phi_{1,h}) \circ \boldsymbol{B}^{m_2}_{\rho,\delta}(\phi_{2,h}) \subset \boldsymbol{B}^{m_1+m_2}_{\rho,\delta}(\phi_{1,h}\#\phi_{2,h})$).
For any $\{\kappa_l\}_{l=0}^\infty(\kappa_0 = 1,\ \kappa_l \le \kappa_{l+1})$, *there exist constants* $C_{3l+2n+1}$(*depending only on* $\kappa_1, \kappa_2, \ldots, \kappa_{3l+2n+1}$) *and* $C'_{3l+2n+3}$(*depending only on* $\kappa_1, \kappa_2, \ldots, \kappa_{3l+2n+3}$), *satisfying the following:*

Let $\phi_{j,h} \in \mathbb{P}_{\rho,\delta}(\tau_j, h, \{\kappa_l\}_{l=0}^\infty)$, $p_{j,h}(x,\xi) \in \mathbb{B}^{m_j}_{\rho,\delta}(h)(j = 1,2)$, $\tau_1 + \tau_2 \le 1/4$, *and* $\{X_h, \Xi_h\}(x,\xi)$ *be the solution of the equations*

$$\begin{cases} X_h = \nabla_\xi \phi_{1,h}(x, \Xi_h) \\ \Xi_h = \nabla_x \phi_{2,h}(X_h, \xi)\,. \end{cases} \tag{4.1}$$

Then there exist $q_h(x,\xi), r_h(x,\xi) \in \mathbb{B}^{m_1+m_2}_{\rho,\delta}(h)$ *such that*

$$q_h(x,\xi) = p_{1,h}(x, \Xi_h(x,\xi)) p_{2,h}(X_h(x,\xi),\xi) + r_h(x,\xi)\,, \tag{4.2}$$

$$p_{1,h}(\phi_{1,h}; X, D_x) p_{2,h}(\phi_{2,h}; X, D_x) = q_h(\phi_{1,h}\#\phi_{2,h}; X, D_x)\,, \tag{4.3}$$

$$|q_h|_l^{(m_1+m_2)} \le C_{3l+2n+1} |p_{1,h}|^{(m_1)}_{3l+2n+1} |p_{2,h}|^{(m_2)}_{3l+2n+1}\,, \tag{4.4}$$

and

$$|r_h|_l^{(m_1+m_2)} \le C'_{3l+2n+3}(\tau_1|p_{1,h}|_{3l+2n+3}^{(m_1)} + \max_{1\le j\le n}|\partial_{\xi_j}p_{1,h}|_{3l+2n+3}^{(m_1+\rho)}) \times (\tau_2|p_{2,h}|_{3l+2n+3}^{(m_2)} + \max_{1\le j\le n}|D_{x_j}p_{2,h}|_{3l+2n+3}^{(m_2-\delta)})\,. \tag{4.5}$$

Proof.

(I) For $u\in\mathcal{S}$, we can write

$$p_{1,h}(\phi_{1,h};X,D_x)p_{2,h}(\phi_{2,h};X,D_x)u(x) = \int e^{i(\phi_{1,h}\#\phi_{2,h})(x,\xi)}q_h(x,\xi)\hat{u}(\xi)đ\xi\,, \tag{4.6}$$

where

$$\psi_h(x,\xi',x',\xi)\equiv\phi_{1,h}(x,\xi')-x'\cdot\xi'+\phi_{2,h}(x',\xi)-(\phi_{1,h}\#\phi_{2,h})(x,\xi)\,, \tag{4.7}$$

and

$$q_h(x,\xi)\equiv \mathrm{O_s}\text{—}\iint e^{i\psi_h(x,\xi',x',\xi)}p_{1,h}(x,\xi')p_{2,h}(x',\xi)dx'đ\xi'\,. \tag{4.8}$$

Set

$$\begin{cases} \widetilde{q}_h(x,\xi)\equiv h^{-(m_1+m_2)}q_h(h^\delta x,h^{-\rho}\xi)\\ \widetilde{p}_{j,h}(x,\xi)\equiv h^{-m_j}p_{j,h}(h^\delta x,h^{-\rho}\xi)\in\mathbb{B}^0_{0,0}(h)\quad(j=1,2)\\ \widetilde{\psi}_h(x,\xi',x',\xi)\equiv h^{\rho-\delta}\psi_h(h^\delta x,h^{-\rho}\xi',h^\delta x',h^{-\rho}\xi)\,. \end{cases} \tag{4.9}$$

By a change of variables: $(x,\xi,x',\xi')\to(h^\delta x,h^{-\rho}\xi,h^\delta x',h^{-\rho}\xi')$, we have

$$\widetilde{q}_h(x,\xi)=h^{-2n\sigma}\mathrm{O_s}\text{—}\iint e^{ih^{-2\sigma}\widetilde{\psi}_h(x,\xi',x',\xi)} \cdot\widetilde{p}_{1,h}(x,\xi')\widetilde{p}_{2,h}(x',\xi)dx'đ\xi'\,, \tag{4.10}$$

where $\sigma\equiv(\rho-\delta)/2$. Furthermore, set

$$\begin{cases} \widetilde{X}_h(x,\xi)\equiv h^{-\delta}X_h(h^\delta x,h^{-\rho}\xi)\\ \widetilde{\Xi}_h(x,\xi)\equiv h^{\rho}\Xi_h(h^\delta x,h^{-\rho}\xi)\\ \widetilde{J}_{j,h}(x,\xi)\equiv h^{\rho-\delta}\phi_{j,h}(h^\delta x,h^{-\rho}\xi)-x\cdot\xi\quad(j=1,2)\,. \end{cases} \tag{4.11}$$

By a change of variables: $x'=\widetilde{X}_h(x,\xi)+h^\sigma y$, $\xi'=\widetilde{\Xi}_h(x,\xi)+h^\sigma\eta$, we have

$$\begin{aligned}(4.12)\qquad \widetilde{q}_h(x,\xi)=\mathrm{O_s}\text{—}\iint e^{i\widetilde{\varphi}_h(y,\eta;x,\xi)}\widetilde{p}_{1,h}(x,\widetilde{\Xi}_h(x,\xi)+h^\sigma\eta)\\ \cdot\widetilde{p}_{2,h}(\widetilde{X}_h(x,\xi)+h^\sigma y,\xi)dy\,đ\eta\,,\end{aligned}$$

where

$$(4.13)\qquad \widetilde{\varphi}_h(y,\eta;x,\xi)\equiv h^{-2\sigma}\widetilde{\psi}_h(x,\widetilde{\Xi}_h(x,\xi)+h^\sigma\eta,\widetilde{X}_h(x,\xi)+h^\sigma y,\xi)\,.$$

Setting

$$(4.14)\quad \begin{cases}\widetilde{A}_h(\eta;x,\xi)\equiv\int_0^1(1-\theta)(\overrightarrow{\nabla}_\xi\nabla_\xi\widetilde{J}_{1,h})(x,\widetilde{\Xi}_h(x,\xi)+\theta h^\sigma\eta)d\theta\eta\cdot\eta\\ \widetilde{B}_h(y;x,\xi)\equiv\int_0^1(1-\theta)(\overrightarrow{\nabla}_x\nabla_x\widetilde{J}_{2,h})(\widetilde{X}_h(x,\xi)+\theta h^\sigma y,\xi)d\theta y\cdot y\,,\end{cases}$$

we can write

$$(4.15)\qquad \widetilde{\varphi}_h(y,\eta;x,\xi)=-y\cdot\eta+\widetilde{A}_h(\eta;x,\xi)+\widetilde{B}_h(y;x,\xi)\,.$$

(II) For $0\le\vartheta\le1$, set

$$(4.16)\qquad \widetilde{\varphi}_{\vartheta,h}(y,\eta;x,\xi)\equiv-y\cdot\eta+\widetilde{A}_h(\vartheta\eta;x,\xi)+\widetilde{B}_h(y;x,\xi)\,.$$

Then we have

$$(4.17)\qquad \begin{cases}\widetilde{\varphi}_{1,h}(y,\eta;x,\xi)=\widetilde{\varphi}_h(y,\eta;x,\xi)\\ \widetilde{\varphi}_{0,h}(y,\eta;x,\xi)=-y\cdot\eta+\widetilde{B}_h(y;x,\xi)\,.\end{cases}$$

Furthermore, we get

$$(4.18)\qquad \begin{cases}\nabla_\eta\widetilde{A}_h(\eta;x,\xi)=\int_0^1(\overrightarrow{\nabla}_\xi\nabla_\xi\widetilde{J}_{1,h})(x,\widetilde{\Xi}_h(x,\xi)+\theta h^\sigma\eta)d\theta\eta\\ \nabla_y\widetilde{B}_h(y;x,\xi)=\int_0^1(\overrightarrow{\nabla}_x\nabla_x\widetilde{J}_{2,h})(\widetilde{X}_h(x,\xi)+\theta h^\sigma y,\xi)d\theta y\,,\end{cases}$$

and

$$(4.19)\qquad \begin{cases}\nabla_\eta\widetilde{\varphi}_{\vartheta,h}(y,\eta;x,\xi)=-y+\vartheta(\nabla_\eta\widetilde{A}_h)(\vartheta\eta;x,\xi)\\ \nabla_y\widetilde{\varphi}_{\vartheta,h}(y,\eta;x,\xi)=-\eta+(\nabla_y\widetilde{B}_h)(y;x,\xi)\,.\end{cases}$$

By Proposition 2.4, there exists a constant C_l''(depending only on $\kappa_1, \kappa_2, \dots, \kappa_l$) such that

$$\left\{\begin{array}{l} |\partial_\xi^\alpha D_x^\beta \partial_\eta^{\alpha'} (\nabla_\eta \widetilde{A}_h)(\vartheta\eta; x,\xi)| \le C''_{|\alpha+\beta+\alpha'|}\tau_1\langle\eta\rangle \quad (0 \le \vartheta \le 1) \\ |\partial_\xi^\alpha D_x^\beta D_y^{\beta'} (\nabla_y \widetilde{B}_h)(y; x,\xi)| \le C''_{|\alpha+\beta+\beta'|}\tau_2\langle y\rangle\,, \end{array}\right. \tag{4.20}$$

and

$$\left\{\begin{array}{l} 3^{-1}\langle y;\eta\rangle \le \langle \nabla_{(y,\eta)}\widetilde{\varphi}_{\vartheta,h}(y,\eta;x,\xi)\rangle \le 3\langle y;\eta\rangle \\ |\partial_\xi^\alpha D_x^\beta \partial_\eta^{\alpha'} D_y^{\beta'} \nabla_{(y,\eta)}\widetilde{\varphi}_{\vartheta,h}(y,\eta;x,\xi)| \le C''_{|\alpha+\beta+\alpha'+\beta'|}\langle y;\eta\rangle \\ |\partial_\xi^\alpha D_x^\beta \partial_\eta^{\alpha'} D_y^{\beta'} \widetilde{\varphi}_{\vartheta,h}(y,\eta;x,\xi)| \le C''_{|\alpha+\beta+\alpha'+\beta'|}\langle y;\eta\rangle^2 \\ (0 \le \vartheta \le 1)\,. \end{array}\right. \tag{4.21}$$

(III) Setting $f(\vartheta; y,\eta;x,\xi) \equiv e^{i\widetilde{\varphi}_{\vartheta,h}(y,\eta;x,\xi)}\widetilde{p}_{1,h}(x, \widetilde{\Xi}_h + h^\sigma(\vartheta\eta))\widetilde{p}_{2,h}(\widetilde{X}_h + h^\sigma y, \xi)$ and using the formula: $f(1) = f(0) + \int_0^1 f'(\vartheta)d\vartheta$, we have

$$\begin{aligned} \widetilde{q}_h(x,\xi) &= \widetilde{p}_{1,h}(x,\widetilde{\Xi}_h) \\ &\quad \cdot \mathrm{O_s}\text{—}\iint e^{-iy\cdot\eta} e^{i\widetilde{B}_h(y;x,\xi)}\widetilde{p}_{2,h}(\widetilde{X}_h + h^\sigma y,\xi)dyđ\eta \\ &\quad + \mathrm{O_s}\text{—}\iint\int_0^1 f'(\vartheta;y,\eta;x,\xi)d\vartheta dy đ\eta \\ &= \widetilde{p}_{1,h}(x,\widetilde{\Xi}_h)\widetilde{p}_{2,h}(\widetilde{X}_h,\xi) + \widetilde{r}_h(x,\xi)\,, \end{aligned} \tag{4.22}$$

where

$$\widetilde{r}_h(x,\xi) \equiv \int_0^1 \widetilde{r}_{\vartheta,h}(x,\xi)d\vartheta\,, \tag{4.23}$$

and

$$\begin{aligned} \widetilde{r}_{\vartheta,h}(x,\xi) \equiv \mathrm{O_s}\text{—}\iint &(e^{-iy\cdot\eta}\eta) \\ &\cdot \Big(i(\nabla_\eta\widetilde{A}_h)(\vartheta\eta)\widetilde{p}_{1,h}(x,\widetilde{\Xi}_h + h^\sigma(\vartheta\eta)) \\ &\qquad + h^\sigma(\nabla_\xi\widetilde{p}_{1,h})(x,\widetilde{\Xi}_h + h^\sigma(\vartheta\eta))\Big) \\ &\times e^{i\widetilde{A}_h(\vartheta\eta)} e^{i\widetilde{B}_h(y)}\widetilde{p}_{2,h}(\widetilde{X}_h + h^\sigma y,\xi)dy đ\eta\,. \end{aligned} \tag{4.24}$$

Integrating by parts, we get

$$\widetilde{r}_{\vartheta,h}(x,\xi) = \mathrm{O_s}\text{—}\iint e^{i\widetilde{\varphi}_{\vartheta,h}(y,\eta;x,\xi)}\widetilde{a}_{\vartheta,h}(y,\eta;x,\xi)dy đ\eta\,, \tag{4.25}$$

where

$$\begin{aligned}\widetilde{a}_{\vartheta,h}(y,\eta;x,\xi) \equiv \Big(& i(\nabla_\eta \widetilde{A}_h)(\vartheta\eta)\widetilde{p}_{1,h}(x,\widetilde{\Xi}_h + h^\sigma(\vartheta\eta)) \\ &+ h^\sigma(\nabla_\xi \widetilde{p}_{1,h})(x,\widetilde{\Xi}_h + h^\sigma(\vartheta\eta))\Big) \\ &\cdot \Big((\nabla_y \widetilde{B}_h)(y)\widetilde{p}_{2,h}(\widetilde{X}_h + h^\sigma y,\xi) \\ &+ h^\sigma(-i\nabla_x \widetilde{p}_{2,h})(\widetilde{X}_h + h^\sigma y,\xi)\Big).\end{aligned} \tag{4.26}$$

(IV) Set

$$\begin{aligned}&e^{i\widetilde{\varphi}_{\vartheta,h}(y,\eta;x,\xi)}\widetilde{a}_{\vartheta,h,\alpha,\beta}(y,\eta;x,\xi) \\ &\qquad\equiv \partial_\xi^\alpha D_x^\beta\Big(e^{i\widetilde{\varphi}_{\vartheta,h}(y,\eta;x,\xi)}\widetilde{a}_{\vartheta,h}(y,\eta;x,\xi)\Big).\end{aligned} \tag{4.27}$$

By (4.20), (4.21) and Proposition 2.4, there exists a constant C_l'''(depending only on $\kappa_1,\kappa_2,\ldots,\kappa_l$) such that

$$\begin{aligned}|\partial_\xi^\gamma D_x^\mu \partial_\eta^{\alpha'} D_y^{\beta'}\widetilde{a}_{\vartheta,h,\alpha,\beta}(y,\eta;x,\xi)| &\le C'''_{|\alpha+\beta+\alpha'+\beta'+\gamma+\mu|}\langle y;\eta\rangle^{2|\alpha+\beta|+2} \\ &\times (\tau_1|\widetilde{p}_{1,h}|^{(0)}_{|\alpha+\beta+\alpha'+\gamma+\mu|} + \max_{1\le j\le n}|\partial_{\xi_j}\widetilde{p}_{1,h}|^{(0)}_{|\alpha+\beta+\alpha'+\gamma+\mu|}) \\ &\times (\tau_2|\widetilde{p}_{2,h}|^{(0)}_{|\alpha+\beta+\beta'+\gamma+\mu|} + \max_{1\le j\le n}|D_{x_j}\widetilde{p}_{2,h}|^{(0)}_{|\alpha+\beta+\beta'+\gamma+\mu|}) \\ &\qquad (0\le\vartheta\le 1)\,.\end{aligned} \tag{4.28}$$

Hence, for $l' = 2|\alpha+\beta| + 2n + 3$, we can write

$$\begin{aligned}\partial_\xi^\alpha D_x^\beta \widetilde{r}_{\vartheta,h}(x,\xi) &= \mathrm{O_s}\text{—}\iint e^{i\widetilde{\varphi}_{\vartheta,h}(y,\eta;x,\xi)}\widetilde{a}_{\vartheta,h,\alpha,\beta}(y,\eta;x,\xi)dy đ\eta \\ &= \iint e^{i\widetilde{\varphi}_{\vartheta,h}}(1 + i\nabla_{(y,\eta)}\cdot\nabla_{(y,\eta)}\widetilde{\varphi}_{\vartheta,h})^{l'} \\ &\quad\cdot \langle\nabla_{(y,\eta)}\widetilde{\varphi}_{\vartheta,h}\rangle^{-2l'}\widetilde{a}_{\vartheta,h,\alpha,\beta}(y,\eta;x,\xi)dy đ\eta\,,\end{aligned} \tag{4.29}$$

and there exists a constant $C'_{3l+2n+3}$(depending only on $\kappa_1, \kappa_2, \dots, \kappa_{3l+2n+3}$) such that

$$(4.30)\quad |\widetilde{r}_{\vartheta,h}|_l^{(0)} \le C'_{3l+2n+3}(\tau_1|\widetilde{p}_{1,h}|_{3l+2n+3}^{(0)} + \max_{1\le j\le n}|\partial_{\xi_j}\widetilde{p}_{1,h}|_{3l+2n+3}^{(0)}) \times (\tau_2|\widetilde{p}_{2,h}|_{3l+2n+3}^{(0)} + \max_{1\le j\le n}|D_{x_j}\widetilde{p}_{2,h}|_{3l+2n+3}^{(0)}) \quad (0\le\vartheta\le 1)\,.$$

From (4.23), we have

$$(4.31)\quad |\widetilde{r}_h|_l^{(0)} \le C'_{3l+2n+3}(\tau_1|\widetilde{p}_{1,h}|_{3l+2n+3}^{(0)} + \max_{1\le j\le n}|\partial_{\xi_j}\widetilde{p}_{1,h}|_{3l+2n+3}^{(0)}) \times (\tau_2|\widetilde{p}_{2,h}|_{3l+2n+3}^{(0)} + \max_{1\le j\le n}|D_{x_j}\widetilde{p}_{2,h}|_{3l+2n+3}^{(0)})\,.$$

Furthermore, we can get the estimate for $\widetilde{q}_h(x,\xi)$ in a way similar to this. □

THEOREM 4.2 ($\boldsymbol{B}_{\rho,\delta}^{m_1,\lambda_1}(\phi_{1,h})\circ\boldsymbol{B}_{\rho,\delta}^{m_2,\lambda_2}(\phi_{2,h}) \subset \boldsymbol{B}_{\rho,\delta}^{m_1+m_2,\lambda_1+\lambda_2}(\phi_{1,h}\#\phi_{2,h})$). *For any $\lambda_1,\lambda_2\in\boldsymbol{R}$ and $\{\kappa_l\}_{l=0}^\infty(\kappa_0=1,\ \kappa_l\le\kappa_{l+1})$, there exists a constant $C_{\lambda_1,\lambda_2,l_{\lambda_1,\lambda_2}}$(depending only on $\lambda_1,\lambda_2,\kappa_1,\kappa_2,\dots,\kappa_{l_{\lambda_1,\lambda_2}}$) satisfying the following:*

Let $\phi_{j,h}\in\mathbb{P}_{\rho,\delta}(\tau_j,h,\{\kappa_l\}_{l=0}^\infty)$, $p_{j,h}(x,\xi)\in\mathbb{B}_{\rho,\delta}^{m_j,\lambda_j}(h)(j=1,2)$ and $\tau_1+\tau_2\le 1/4$. Then there exists $q_h(x,\xi)\in\mathbb{B}_{\rho,\delta}^{m_1+m_2,\lambda_1+\lambda_2}(h)$ such that

$$(4.32)\qquad p_{1,h}(\phi_{1,h};X,D_x)p_{2,h}(\phi_{2,h};X,D_x) = q_h(\phi_{1,h}\#\phi_{2,h};X,D_x)\,,$$

and

$$(4.33)\qquad |q_h|_l^{(m_1+m_2,\lambda_1+\lambda_2)} \le C_{\lambda_1,\lambda_2,l_{\lambda_1,\lambda_2}}|p_{1,h}|_{l_{\lambda_1,\lambda_2}}^{(m_1,\lambda_1)}|p_{2,h}|_{l_{\lambda_1,\lambda_2}}^{(m_2,\lambda_2)}\,,$$

where $l_{\lambda_1,\lambda_2}\equiv 3l+2n+1+[|\lambda_1|+|\lambda_2|]$.

PROOF.

(I) is the same as that in the proof of Theorem 4.1 except for the following modification:

$$(4.34)\qquad \widetilde{p}_{j,h}(x,\xi)\equiv h^{-m_j}p_{j,h}(h^\delta x,h^{-\rho}\xi)\in\mathbb{B}_{0,0}^{0,\lambda_j}(h)\quad(j=1,2)\,.$$

(II) By Proposition 2.4, there exists a constant C'_l(depending only on κ_1, $\kappa_2,\dots,\kappa_l$) such that

$$\left\{\begin{array}{l} 3^{-1}\langle y;\eta\rangle \le \langle \nabla_{(y,\eta)}\widetilde{\varphi}_h(y,\eta;x,\xi)\rangle \le 3\langle y;\eta\rangle \\ |\partial_\xi^\alpha D_x^\beta \partial_\eta^{\alpha'} D_y^{\beta'} \nabla_{(y,\eta)}\widetilde{\varphi}_h(y,\eta;x,\xi)| \le C'_{|\alpha+\beta+\alpha'+\beta'|}\langle y;\eta\rangle \\ |\partial_\xi^\alpha D_x^\beta \partial_\eta^{\alpha'} D_y^{\beta'} \widetilde{\varphi}_h(y,\eta;x,\xi)| \le C'_{|\alpha+\beta+\alpha'+\beta'|}\langle y;\eta\rangle^2 . \end{array}\right. \tag{4.35}$$

(III) Set

$$\widetilde{p}^{\diamond}_{j,h}(x,\xi) \equiv \widetilde{p}_{j,h}(x,\xi)\langle x;\xi\rangle^{-\lambda_j} \in \mathbb{B}^0_{0,0}(h) \quad (j=1,2)\,. \tag{4.36}$$

Then we have

$$\widetilde{q}_h(x,\xi) = \mathrm{O_s}\text{—}\iint e^{i\widetilde{\varphi}_h(y,\eta;x,\xi)}\widetilde{a}_h(y,\eta;x,\xi)dy đ\eta\,, \tag{4.37}$$

where

$$\begin{aligned} \widetilde{a}_h(y,\eta;x,\xi) \equiv{}& \widetilde{p}^{\diamond}_{1,h}(x,\widetilde{\Xi}_h + h^\sigma\eta)\widetilde{p}^{\diamond}_{2,h}(\widetilde{X}_h + h^\sigma y,\xi) \\ &\times \langle x;\widetilde{\Xi}_h + h^\sigma\eta\rangle^{\lambda_1}\langle \widetilde{X}_h + h^\sigma y;\xi\rangle^{\lambda_2}\,. \end{aligned} \tag{4.38}$$

(IV) Set

$$e^{i\widetilde{\varphi}_h(y,\eta;x,\xi)}\widetilde{a}_{h,\alpha,\beta}(y,\eta;x,\xi) \equiv \partial_\xi^\alpha D_x^\beta\Big(e^{i\widetilde{\varphi}_h(y,\eta;x,\xi)}\widetilde{a}_h(y,\eta;x,\xi)\Big). \tag{4.39}$$

By (4.35), Proposition 2.4 and Lemma 2.6, there exists a constant $C''_{\lambda_1,\lambda_2,l}$ (depending only on $\lambda_1,\lambda_2,\kappa_1,\kappa_2,\dots,\kappa_l$) such that

$$\begin{aligned} &|\partial_\xi^\gamma D_x^\mu \partial_\eta^{\alpha'} D_y^{\beta'}\widetilde{a}_{h,\alpha,\beta}(y,\eta;x,\xi)| \\ &\le C''_{\lambda_1,\lambda_2,|\alpha+\beta+\alpha'+\beta'+\gamma+\mu|}\langle y;\eta\rangle^{|\lambda_1|+|\lambda_2|+2|\alpha+\beta|}\langle x;\xi\rangle^{\lambda_1+\lambda_2} \\ &\quad \times |\widetilde{p}^{\diamond}_{1,h}|^{(0)}_{|\alpha+\beta+\alpha'+\gamma+\mu|}|\widetilde{p}^{\diamond}_{2,h}|^{(0)}_{|\alpha+\beta+\beta'+\gamma+\mu|}\,. \end{aligned} \tag{4.40}$$

Hence, for $l' = 2|\alpha+\beta| + 2n + 1 + [|\lambda_1| + |\lambda_2|]$, we can write

$$\begin{aligned} \partial_\xi^\alpha D_x^\beta \widetilde{q}_h(x,\xi) &= \mathrm{O_s}\text{—}\iint e^{i\widetilde{\varphi}_h(y,\eta;x,\xi)}\widetilde{a}_{h,\alpha,\beta}(y,\eta;x,\xi)dy đ\eta \\ &= \iint e^{i\widetilde{\varphi}_h}(1 + i\nabla_{(y,\eta)}\cdot\nabla_{(y,\eta)}\widetilde{\varphi}_h)^{l'} \\ &\qquad \cdot \langle \nabla_{(y,\eta)}\widetilde{\varphi}_h\rangle^{-2l'}\widetilde{a}_{h,\alpha,\beta}(y,\eta;x,\xi)dy đ\eta \end{aligned} \tag{4.41}$$

and there exists a constant $C_{\lambda_1,\lambda_2,l_{\lambda_1,\lambda_2}}$ (depending only on $\lambda_1, \lambda_2, \kappa_1, \kappa_2, \ldots, \kappa_{l_{\lambda_1,\lambda_2}}$) such that

$$|\widetilde{q}_h|_l^{(0,\lambda_1+\lambda_2)} \leq C_{\lambda_1,\lambda_2,l_{\lambda_1,\lambda_2}} |\widetilde{p}_{1,h}^{\diamond}|_{l_{\lambda_1,\lambda_2}}^{(0)} |\widetilde{p}_{2,h}^{\diamond}|_{l_{\lambda_1,\lambda_2}}^{(0)} . \tag{4.42}$$

Therefore, we can get (4.33). □

Theorem 4.3 (H.Kumano-go-Taniguchi theorem).

For any $\{\kappa_l\}_{l=0}^{\infty} (\kappa_0 = 1,\, \kappa_l \leq \kappa_{l+1})$, *there exist constants* $0 < \tau_0' < 1/(4c_0)$ *(depending only on* $\kappa_1, \kappa_2, \ldots, \kappa_{5n_0}$*) and* C_{3l+23n_0}*(depending only on* $\kappa_1, \kappa_2, \ldots, \kappa_{3l+23n_0}$*) satisfying the following:*

Let $\phi_{j,h} \in \mathbb{P}_{\rho,\delta}(\tau_j, h, \{\kappa_l\}_{l=0}^{\infty})$, $p_{j,h}(x,\xi) \in \mathbb{B}_{\rho,\delta}^{m_j}(h) (j = 1, 2, \ldots)$ *and* $\sum_{j=1}^{\infty} \tau_j \leq \tau_0'$.

Then, for any $\nu = 1, 2, \ldots,$ *there exists* $r_{\nu+1,h}(x,\xi) \in \mathbb{B}_{\rho,\delta}^{\bar{m}_{\nu+1}}(h)$ *such that*

$$\begin{aligned} &r_{\nu+1,h}(\phi_{1,h}\#\phi_{2,h}\#\cdots\#\phi_{\nu+1,h}; X, D_x) \\ &\quad = p_{1,h}(\phi_{1,h}; X, D_x) p_{2,h}(\phi_{2,h}; X, D_x) \cdots p_{\nu+1,h}(\phi_{\nu+1,h}; X, D_x) , \end{aligned} \tag{4.43}$$

and

$$|r_{\nu+1,h}|_l^{(\bar{m}_{\nu+1})} \leq (C_{3l+23n_0})^{\nu+1} \prod_{j=1}^{\nu+1} |p_{j,h}|_{l+11n_0}^{(m_j)} , \tag{4.44}$$

where $\bar{m}_{\nu+1} \equiv m_1 + m_2 + \cdots + m_{\nu+1}$.

Proof.

(I) By Proposition 2.5(1), there exist a constant $c_0 > 1$(independent of $\{\kappa_l\}_{l=0}^{\infty}$) and a sequence of positive constants $\{\kappa_l'\}_{l=0}^{\infty} (\kappa_0' = 1,\, \kappa_l' \leq \kappa_{l+1}',\, \kappa_l \leq \kappa_l')$(depending only on $\kappa_1, \kappa_2, \ldots, \kappa_l$), satisfying the following:

For any $\phi_{j,h} \in \mathbb{P}_{\rho,\delta}(\tau_j, h, \{\kappa_l\}_{l=0}^{\infty}) (j = 1, 2, \ldots)$ with $\sum_{j=1}^{\infty} \tau_j \leq 1/4$, it follows that

$$\Phi_{j,h} = (\phi_{1,h}\#\phi_{2,h}\#\cdots\#\phi_{j,h}) \in \mathbb{P}_{\rho,\delta}(c_0\bar{\tau}_j, h, \{\kappa_l'\}) . \tag{4.45}$$

(II) By Proposition 3.8, there exists a constant $0 < \tau_0 < 1/(4c_0)$(depending only on $\kappa'_1, \kappa'_2, \dots, \kappa'_{5n_0}$) satisfying the following:

For any $\phi_{j,h} \in \mathbb{P}_{\rho,\delta}(\tau_j, h, \{\kappa_l\}_{l=0}^{\infty})(j = 1, 2, \dots)$ with $\sum_{j=1}^{\infty} \tau_j \le \tau_0/c_0(\equiv \tau'_0)$, there exist $s_{j,h}(\xi, x'), t_{j,h}(\xi, x') \in \mathbb{B}^0_{\rho,\delta}(h)$ such that

$$\left\{ \begin{array}{l} I(\phi_{j,h}) s_{j,h}(\phi^*_{j,h}; D_x, X') = I \\ t_{j,h}(\Phi^*_{j,h}; D_x, X') I(\Phi_{j,h}) = I\,. \end{array} \right. \tag{4.46}$$

(III) For any $j = 1, 2, \dots$, set

$$\begin{aligned} q_{j,h}(X, D_x) \equiv & \Big(I(\Phi_{j-1,h}) I(\phi_{j,h}) \Big) \\ & \circ \Big(\Big(s_{j,h}(\phi^*_{j,h}; D_x, X') p_{j,h}(\phi_{j,h}; X, D_x) \Big) t_{j,h}(\Phi^*_{j,h}; D_x, X') \Big)\,, \end{aligned} \tag{4.47}$$

where $\Phi_{0,h} \equiv x \cdot \xi$. Then we have

$$\begin{aligned} & r_{\nu+1,h}(\phi_{1,h} \# \phi_{2,h} \# \cdots \# \phi_{\nu+1,h}; X, D_x) \\ & = p_{1,h}(\phi_{1,h}; X, D_x) p_{2,h}(\phi_{2,h}; X, D_x) \cdots p_{\nu+1,h}(\phi_{\nu+1,h}; X, D_x) \\ & = q_{1,h}(X, D_x) q_{2,h}(X, D_x) \cdots q_{\nu+1,h}(X, D_x) I(\Phi_{\nu+1,h})\,. \end{aligned} \tag{4.48}$$

By Proposition 3.5, 1.4, 3.4, 3.6 and Theorem 4.1, we get (4.44). $\square$

Definition 4.4.

For $\tau \ge 0$ and a sequence of positive constants $\{\kappa_l\}_{l=0}^{\infty}(\kappa_0 = 1,\ \kappa_l \le \kappa_{l+1})$, we say that a family $\{\omega_h\}_{0<h<1}$ of complex-valued C^∞-functions $\omega_h(x, \xi)$ belongs to the class $\{\Omega_{\rho,\delta}(\tau, h, \{\kappa_l\}_{l=0}^{\infty})\}_{0<h<1}$, if $\widetilde{\omega}_h(x, \xi) \equiv \omega_h(h^\delta x, h^{-\rho}\xi)$ satisfies

$$\left\{ \begin{array}{l} \inf\limits_{0<h<1} \inf\limits_{x,\xi} Im\{\widetilde{\omega}_h(x, \xi)\} \ge -\kappa_0 \tau \\ \sup\limits_{0<h<1} \left(\sup\limits_{x,\xi} \frac{|\widetilde{\omega}_h(x,\xi)|}{\langle x;\xi \rangle} + \sum\limits_{1 \le |\alpha+\beta| \le 1+l} \sup\limits_{x,\xi} |\partial^\alpha_\xi D^\beta_x \widetilde{\omega}_h(x, \xi)| \right) \le \kappa_l \tau\,, \end{array} \right. \tag{4.49}$$

for any $l = 0, 1, 2, \dots$.

We write $\{\omega_h\}_{0<h<1} \in \{\Omega_{\rho,\delta}(\tau, h, \{\kappa_l\}_{l=0}^{\infty})\}_{0<h<1}$, or simply $\omega_h \in \Omega_{\rho,\delta}(\tau, h, \{\kappa_l\}_{l=0}^{\infty})$.

Proposition 4.5.

For any $\{\kappa_l\}_{l=0}^{\infty}(\kappa_0 = 1,\ \kappa_l \le \kappa_{l+1})$*, there exist a constant* $c_0 > 1$*(independent of* $\{\kappa_l\}_{l=0}^{\infty}$*) and a sequence of positive constants* $\{\kappa'_l\}_{l=0}^{\infty}(\kappa'_0 = 1,\ \kappa'_l \le \kappa'_{l+1},\ \kappa_l \le \kappa'_l)$ *(depending only on* $\kappa_1, \kappa_2, \dots, \kappa_l$*), satisfying the following:*

Let $\phi_{j,h} \in \mathbb{P}_{\rho,\delta}(\tau_j, h, \{\kappa_l\}_{l=0}^{\infty})$ *and* $\omega_{j,h} \in \Omega_{\rho,\delta}(\tau_j, h, \{\kappa_l\}_{l=0}^{\infty})$ $(j = 1, 2, \dots)$ *with* $\sum_{j=1}^{\infty} \tau_j \le 1/4$*.*

Then, for any $\nu = 1, 2, \dots,$ *it follows that*

$$
\tag{4.50}
\begin{cases}
\Phi_{\nu+1,h} \equiv (\phi_{1,h}\#\phi_{2,h}\#\cdots\#\phi_{\nu+1,h}) \in \mathbb{P}_{\rho,\delta}(c_0\bar{\tau}_{\nu+1}, h, \{\kappa'_l\}_{l=0}^{\infty}) \\
\sum_{j=1}^{\nu+1} \omega_{j,h}(X_{\nu,h}^{j-1}, \Xi_{\nu,h}^{j}) \in \Omega_{\rho,\delta}(c_0\bar{\tau}_{\nu+1}, h, \{\kappa'_l\}_{l=0}^{\infty})\,,
\end{cases}
$$

where $\bar{\tau}_{\nu+1} \equiv \tau_1 + \tau_2 + \cdots + \tau_{\nu+1}$*.*

Proof. By Proposition 2.4 and 2.5. □

Theorem 4.6 (Fujiwara's skip method).

For any $\{\kappa_l\}_{l=0}^{\infty}(\kappa_0 = 1,\ \kappa_l \le \kappa_{l+1})$*, there exist constants* $0 < \tau'_0 < 1/(4c_0)$ *(depending only on* $\kappa_1, \kappa_2, \dots, \kappa_{5n_0}$*) and* $C'_{3l+36n_0}, C''_{3l+36n_0}$*(depending only on* $\kappa_1, \kappa_2, \dots, \kappa_{3l+36n_0}$*) satisfying the following:*

For $\phi_{j,h} \in \mathbb{P}_{\rho,\delta}(\tau_j, h, \{\kappa_l\}_{l=0}^{\infty})$ *and* $\omega_{j,h} \in \Omega_{\rho,\delta}(\tau_j, h, \{\kappa_l\}_{l=0}^{\infty})(j = 1, 2, \dots)$ *with* $\sum_{j=1}^{\infty} \tau_j \le \tau'_0$*, set*

$$
\tag{4.51}
\begin{cases}
\phi_{1,2,\dots,\nu+1,h}(x,\xi) \equiv (\phi_{1,h}\#\phi_{2,h}\#\cdots\#\phi_{\nu+1,h})(x,\xi) \\
\omega_{1,2,\dots,\nu+1,h}(x,\xi) \equiv \sum_{j=1}^{\nu+1} \omega_{j,h}(X_{\nu,h}^{j-1}, \Xi_{\nu,h}^{j})\,,
\end{cases}
$$

and

$$
\tag{4.52}
\begin{cases}
p_{j,h}(x,\xi) \equiv \exp\Big(i\omega_{j,h}(x,\xi)\Big) \in \mathbb{B}^0_{\rho,\delta}(h) \\
p^{\#}_{1,2,\dots,\nu+1,h}(x,\xi) \equiv \exp\Big(i\omega_{1,2,\dots,\nu+1,h}(x,\xi)\Big) \in \mathbb{B}^0_{\rho,\delta}(h)\,.
\end{cases}
$$

Then, for any $\nu = 1, 2, \dots,$ *there exists* $\Upsilon_{1,2,\dots,\nu+1,h}(x,\xi) \in \mathbb{B}^0_{\rho,\delta}(h)$ *such that*

$$
\tag{4.53}
\begin{aligned}
&p_{1,h}(\phi_{1,h}; X, D_x)p_{2,h}(\phi_{2,h}; X, D_x)\cdots p_{\nu+1,h}(\phi_{\nu+1,h}; X, D_x) \\
&\quad = p^{\#}_{1,2,\dots,\nu+1,h}(\phi_{1,2,\dots,\nu+1,h}; X, D_x) \\
&\qquad + \Upsilon_{1,2,\dots,\nu+1,h}(\phi_{1,2,\dots,\nu+1,h}; X, D_x)\,,
\end{aligned}
$$

(4.54) $$|\Upsilon_{1,2,\dots,\nu+1,h}|_l^{(0)} \leq C'_{3l+36n_0}(\bar{\tau}_{\nu+1})^2\,,$$

and

(4.55) $$|(p^{\#}_{1,2,\dots,\nu+1,h} + \Upsilon_{1,2,\dots,\nu+1,h})|_l^{(0)} \leq C''_{3l+36n_0}\,.$$

PROOF.

(I) By Proposition 2.5 and Proposition 4.5, there exist a constant $c_0 > 1$ and a sequence of positive constants $\{\kappa'_l\}_{l=0}^{\infty}(\kappa'_0 = 1,\ \kappa'_l \leq \kappa'_{l+1},\ \kappa_l \leq \kappa'_l)$ (depending only on $\kappa_1, \kappa_2, \dots, \kappa_l$) such that

(4.56) $$\begin{cases} \phi_{1,2,\dots,\nu,h} \in \mathbb{P}_{\rho,\delta}(c_0\bar{\tau}_\nu, h, \{\kappa'_l\}_{l=0}^{\infty}), \\ \phi_{\nu+1,h} \in \mathbb{P}_{\rho,\delta}(c_0\tau_{\nu+1}, h, \{\kappa'_l\}_{l=0}^{\infty}) \\ \omega_{1,2,\dots,\nu,h} \in \Omega_{\rho,\delta}(c_0\bar{\tau}_\nu, h, \{\kappa'_l\}_{l=0}^{\infty}), \\ \ \omega_{\nu+1,h} \in \Omega_{\rho,\delta}(c_0\tau_{\nu+1}, h, \{\kappa'_l\}_{l=0}^{\infty})\,. \end{cases}$$

Hence, there exist a constant $C'''_l > 1$(depending only on $\kappa'_1, \kappa'_2, \dots, \kappa'_l$) such that

(4.57) $$\begin{cases} |p^{\#}_{1,2,\dots,\nu,h}|_l^{(0)} \leq C'''_l, \quad |p_{\nu+1,h}|_l^{(0)} \leq C'''_l \\ |\partial_{\xi_k} p^{\#}_{1,2,\dots,\nu,h}|_l^{(\rho)} \leq C'''_l \bar{\tau}_\nu, \quad |D_{x_k} p_{\nu+1,h}|_l^{(-\delta)} \leq C'''_l \tau_{\nu+1} \\ (\nu = 1, 2, \dots, \quad k = 1, 2, \dots, n, \quad p^{\#}_{j,h} \equiv p_{j,h}, \quad \bar{\tau}_\nu \equiv \sum_{j=1}^{\nu} \tau_j)\,. \end{cases}$$

By Theorem 4.1 and Proposition 2.5(2), there exist a constant $C''''_{3l+2n+3}$ (depending only on $\kappa'_1, \kappa'_2, \dots, \kappa'_{3l+2n+3}$) and $r_{1,2,\dots,\nu+1,h}(x,\xi) \in \mathbb{B}^0_{\rho,\delta}(h)$ such that

(4.58) $$\begin{aligned} &p^{\#}_{1,2,\dots,\nu,h}(\phi_{1,2,\dots,\nu,h}; X, D_x) p_{\nu+1,h}(\phi_{\nu+1,h}; X, D_x) \\ &\ = p^{\#}_{1,2,\dots,\nu+1,h}(\phi_{1,2,\dots,\nu+1,h}; X, D_x) \\ &\quad + r_{1,2,\dots,\nu+1,h}(\phi_{1,2,\dots,\nu+1,h}; X, D_x)\,, \end{aligned}$$

and

(4.59) $$|r_{1,2,\dots,\nu+1,h}|_l^{(0)} \leq C''''_{3l+2n+3}(\bar{\tau}_\nu)\tau_{\nu+1}\,.$$

For simplicity, set

$$
\left\{
\begin{array}{l}
P^{\#}_{1,2,\dots,\nu+1,h} \equiv p^{\#}_{1,2,\dots,\nu+1,h}(\phi_{1,2,\dots,\nu+1,h}; X, D_x) \\
R_{1,2,\dots,\nu+1,h} \equiv r_{1,2,\dots,\nu+1,h}(\phi_{1,2,\dots,\nu+1,h}; X, D_x)\,.
\end{array}
\right.
\tag{4.60}
$$

Furthermore, replacing $\{\phi_{1,h}, \phi_{2,h}, \dots, \phi_{\nu+1,h}\}$ and $\{\omega_{1,h}, \omega_{2,h}, \dots, \omega_{\nu+1,h}\}$ by $\{\phi_{j+1,h}, \phi_{j+2,h}, \dots, \phi_{j+\nu+1,h}\}$ and $\{\omega_{j+1,h}, \omega_{j+2,h}, \dots, \omega_{j+\nu+1,h}\}$, set $P^{\#}_{j+1,j+2,\dots,j+\nu+1,h}$ and $R_{j+1,j+2,\dots,j+\nu+1,h}$.

(II) We use Fujiwara's skip method: Using (4.58) inductively, we can get

$$
\begin{aligned}
&p_{1,h}(\phi_{1,h}; X, D_x) p_{2,h}(\phi_{2,h}; X, D_x) \cdots p_{\nu+1,h}(\phi_{\nu+1,h}; X, D_x) \\
&\quad = p^{\#}_{1,2,\dots,\nu+1,h}(\phi_{1,2,\dots,\nu+1,h}; X, D_x) \\
&\qquad + \Upsilon_{1,2,\dots,\nu+1,h}(\phi_{1,2,\dots,\nu+1,h}; X, D_x)\,,
\end{aligned}
\tag{4.61}
$$

where

$$
\begin{aligned}
&\Upsilon_{1,2,\dots,\nu+1,h}(\phi_{1,2,\dots,\nu+1,h}; X, D_x) \\
&\quad \equiv {\sum}' R_{j_0+1,j_0+2,\dots,j_1,h} R_{j_1+1,j_1+2,\dots,j_2,h} \\
&\qquad \cdots R_{j_{s-1}+1,j_{s-1}+2,\dots,j_s,h} P^{\#}_{j_s+1,j_s+2,\dots,\nu+1,h}\,,
\end{aligned}
\tag{4.62}
$$

${\sum}'$ stands for the summation with respect to the sequences of integers $(j_1, j_2, \dots, j_{s-1}, j_s)$ with the property

$$
0 \equiv j_0 < j_1 - 1 < j_1 < j_2 - 1 < \cdots < j_{s-1} < j_s - 1 < j_s \le \nu + 1\,,
\tag{4.63}
$$

and, in the special case of $j_s = \nu + 1$, we set $P^{\#}_{j_s+1,j_s+2,\dots,\nu+1,h} \equiv I$.

Now, let C_{3l+23n_0} is the same constant as that in Theorem 4.3.

By (4.57), (4.59), (4.62), (4.63) and Theorem 4.3, we have

$$
\begin{aligned}
&|\Upsilon_{1,2,\dots,\nu+1,h}|^{(0)}_{l} \\
&\quad \le {\sum}' (C_{3l+23n_0})^{s+1} C'''_{l+11n_0} \\
&\qquad \cdot \prod_{k=1}^{s} \Big(C''''_{3(l+11n_0)+2n+3} \big(\sum_{k'=j_{k-1}+1}^{j_k-1} \tau_{k'} \big) \tau_{j_k} \Big) \\
&\quad \le C_{3l+23n_0} C'''_{l+11n_0} \left(\prod_{j=1}^{\nu+1} \Big(1 + C_{3l+23n_0} C''''_{3l+36n_0}(\bar{\tau}_{\nu+1}) \tau_j \Big) - 1 \right) \\
&\quad \le C'_{3l+36n_0} (\bar{\tau}_{\nu+1})^2\,.
\end{aligned}
\tag{4.64}
$$

Here C'_{3l+36n_0} is a constant depending only on $\kappa_1, \kappa_2, \dots, \kappa_{3l+36n_0}$. $\square$

The following lemma is a revised version of that in Fujiwara [4].

LEMMA 4.7.
For any $\{\kappa_l\}_{l=0}^{\infty}(\kappa_0 = 1, \kappa_l \le \kappa_{l+1})$ *and* $\{\chi_l\}_{l=0}^{\infty}(0 \le \chi_l \le \chi_{l+1})$, *there exists a constant* $C_{3l+4n+5}$(*depending only on* $\kappa_1, \kappa_2, \dots, \kappa_{3l+4n+5}$ *and* $\chi_0, \chi_1, \dots, \chi_{3l+4n+5}$) *satisfying the following:*

Let $\phi_{j,h} \in \mathbb{P}_{\rho,\delta}(\tau_j, h, \{\kappa_l\}_{l=0}^{\infty})$, $p_{j,h}(x,\xi) \in \mathbb{B}^0_{\rho,\delta}(h)$ $(j = 1,3)$, $\phi_{2,\vartheta,h} \in \mathbb{P}_{\rho,\delta}(\tau_2, h, \{\kappa_l\}_{l=0}^{\infty})$, $p_{2,\vartheta,h}(x,\xi) \in \mathbb{B}^0_{\rho,\delta}(h)$ $(0 \le \vartheta \le 1)$ *and* $\sum_{j=1}^{3} \tau_j \le 1/(4c_0)$.

Furthermore, let $\partial_\vartheta p_{2,\vartheta,h}(x,\xi) \in \mathbb{B}^{0,1}_{\rho,\delta}(h)$ $(0 \le \vartheta \le 1)$ *and let* $\widetilde{\phi}_{2,\vartheta,h}$ *satisfy* (2.26).

Then, there exists $q_{\vartheta,h}(x,\xi) \in \mathbb{B}^0_{\rho,\delta}(h)$ *such that*

$$\begin{aligned} &q_{\vartheta,h}(\phi_{1,h}\#\phi_{2,\vartheta,h}\#\phi_{3,h}; X, D_x) \\ &\quad = p_{1,h}(\phi_{1,h}; X, D_x) p_{2,\vartheta,h}(\phi_{2,\vartheta,h}; X, D_x) p_{3,h}(\phi_{3,h}; X, D_x)\,, \end{aligned} \tag{4.65}$$

$$\partial_\vartheta q_{\vartheta,h}(x,\xi) \in \mathbb{B}^{0,1}_{\rho,\delta}(h)\,, \tag{4.66}$$

and

$$\begin{aligned} |\partial_\vartheta q_{\vartheta,h}|_l^{(0,1)} &\le C_{3l+4n+5} |p_{1,h}|^{(0)}_{3l+4n+5} |p_{3,h}|^{(0)}_{3l+4n+5} \\ &\quad \times \Big((\tau_2)^2 |p_{2,h}|^{(0)}_{3l+4n+5} + |\partial_\vartheta p_{2,\vartheta,h}|^{(0,1)}_{3l+4n+5} \Big) \\ &\qquad\qquad (0 \le \vartheta \le 1)\,. \end{aligned} \tag{4.67}$$

PROOF.
(I) For $u \in \mathcal{S}$, we can write

$$\begin{aligned} \widetilde{q}_{\vartheta,h}(x,\xi) = h^{-4n\sigma} \mathrm{Os}\text{-}\!\!\iiiint & e^{ih^{-2\sigma}\widetilde{\psi}_{\vartheta,h}(x,\xi^1,x^1,\xi^2,x^2,\xi)} \\ & \times \widetilde{p}_{1,h}(x,\xi^1) \widetilde{p}_{2,\vartheta,h}(x^1,\xi^2) \widetilde{p}_{3,h}(x^2,\xi) dx^1 đ\xi^1 dx^2 đ\xi^2\,, \end{aligned} \tag{4.68}$$

where

$$\widetilde{\psi}_{\vartheta,h}(x,\xi^1,x^1,\xi^2,x^2,\xi) \equiv \widetilde{\phi}_{1,h}(x,\xi^1) - x^1\cdot\xi^1 + \widetilde{\phi}_{2,\vartheta,h}(x^1,\xi^2) - x^2\cdot\xi^2 + \widetilde{\phi}_{3,h}(x^2,\xi) - (\widetilde{\phi}_{1,h}\#\widetilde{\phi}_{2,\vartheta,h}\#\widetilde{\phi}_{3,h})(x,\xi)\,, \tag{4.69}$$

and $\sigma \equiv (\rho-\delta)/2$.

Let $\{\widetilde{X}^j_{2,\vartheta,h}, \widetilde{\Xi}^j_{2,\vartheta,h}\}_{j=1}^2(x,\xi)$ be the solution of the equations (2.24).

By a change of variables:

$$\begin{cases} x^1 = \widetilde{X}^1_{2,\vartheta,h}(x,\xi) + h^\sigma y^1, & \xi^1 = \widetilde{\Xi}^1_{2,\vartheta,h}(x,\xi) + h^\sigma\eta^1 \\ x^2 = \widetilde{X}^2_{2,\vartheta,h}(x,\xi) + h^\sigma y^2, & \xi^2 = \widetilde{\Xi}^2_{2,\vartheta,h}(x,\xi) + h^\sigma\eta^2\,, \end{cases} \tag{4.70}$$

we can write

$$\widetilde{q}_{\vartheta,h}(x,\xi) = \mathrm{O_s}\text{—}\iiiint e^{i\widetilde{\varphi}_{\vartheta,h}(y^1,\eta^1,y^2,\eta^2;x,\xi)} \cdot \widetilde{a}_{\vartheta,h}(y^1,\eta^1,y^2,\eta^2;x,\xi)dy^1 đ\eta^1 dy^2 đ\eta^2\,, \tag{4.71}$$

where

$$\widetilde{a}_{\vartheta,h}(y^1,\eta^1,y^2,\eta^2;x,\xi) \equiv \widetilde{p}_{1,h}(x,\widetilde{\Xi}^1_{2,\vartheta,h} + h^\sigma\eta^1) \times \widetilde{p}_{2,\vartheta,h}(\widetilde{X}^1_{2,\vartheta,h} + h^\sigma y^1, \widetilde{\Xi}^2_{2,\vartheta,h} + h^\sigma\eta^2) \times \widetilde{p}_{3,h}(\widetilde{X}^2_{2,\vartheta,h} + h^\sigma y^2,\xi)\,, \tag{4.72}$$

and

$$\begin{aligned} \widetilde{\varphi}_{\vartheta,h}(y^1,\eta^1,y^2,\eta^2;x,\xi) &\equiv h^{-2\sigma}\widetilde{\psi}_{\vartheta,h}(x,\xi^1,x^1,\xi^2,x^2,\xi) \\ &= -y^1\cdot\eta^1 - y^2\cdot\eta^2 + y^1\cdot\eta^2 \\ &+ \int_0^1 (1-\theta)(\overrightarrow{\nabla}_\xi\nabla_\xi \widetilde{J}_{1,h})(x,\widetilde{\Xi}^1_{2,\vartheta,h} + \theta h^\sigma\eta^1)d\theta\eta^1\cdot\eta^1 \\ &+ \int_0^1 (1-\theta)(\overrightarrow{\nabla}_x\nabla_x \widetilde{J}_{2,\vartheta,h})(\widetilde{X}^1_{2,\vartheta,h} + \theta h^\sigma y^1, \widetilde{\Xi}^2_{2,\vartheta,h} + \theta h^\sigma\eta^2)d\theta y^1\cdot y^1 \\ &+ 2\int_0^1 (1-\theta)(\overrightarrow{\nabla}_x\nabla_\xi \widetilde{J}_{2,\vartheta,h})(\widetilde{X}^1_{2,\vartheta,h} + \theta h^\sigma y^1, \widetilde{\Xi}^2_{2,\vartheta,h} + \theta h^\sigma\eta^2)d\theta y^1\cdot\eta^2 \end{aligned} \tag{4.73}$$

$$+\int_0^1 (1-\theta)(\overrightarrow{\nabla}_\xi \nabla_\xi \widetilde{J}_{2,\vartheta,h})(\widetilde{X}^1_{2,\vartheta,h}+\theta h^\sigma y^1, \widetilde{\Xi}^2_{2,\vartheta,h}+\theta h^\sigma \eta^2)d\theta\eta^2\cdot\eta^2$$
$$+\int_0^1 (1-\theta)(\overrightarrow{\nabla}_x \nabla_x \widetilde{J}_{3,h})(\widetilde{X}^2_{2,\vartheta,h}+\theta h^\sigma y^2, \xi)d\theta y^2\cdot y^2\,.$$

Furthermore, we have

$$\tag{4.74} \begin{cases} \nabla_{\eta^1}\widetilde{\varphi}_{\vartheta,h} = -y^1 + \int_0^1 (\overrightarrow{\nabla}_\xi \nabla_\xi \widetilde{J}_{1,h})(x, \widetilde{\Xi}^1_{2,\vartheta,h}+\theta h^\sigma\eta^1)d\theta\eta^1 \\ \nabla_{y^1}\widetilde{\varphi}_{\vartheta,h} = -\eta^1+\eta^2 \\ \quad +\int_0^1 (\overrightarrow{\nabla}_x \nabla_x \widetilde{J}_{2,\vartheta,h})(\widetilde{X}^1_{2,\vartheta,h}+\theta h^\sigma y^1, \widetilde{\Xi}^2_{2,\vartheta,h}+\theta h^\sigma\eta^2)d\theta y^1 \\ \quad +\int_0^1 (\overrightarrow{\nabla}_\xi \nabla_x \widetilde{J}_{2,\vartheta,h})(\widetilde{X}^1_{2,\vartheta,h}+\theta h^\sigma y^1, \widetilde{\Xi}^2_{2,\vartheta,h}+\theta h^\sigma\eta^2)d\theta \eta^2 \\ \nabla_{\eta^2}\widetilde{\varphi}_{\vartheta,h} = y^1-y^2 \\ \quad +\int_0^1 (\overrightarrow{\nabla}_x \nabla_\xi \widetilde{J}_{2,\vartheta,h})(\widetilde{X}^1_{2,\vartheta,h}+\theta h^\sigma y^1, \widetilde{\Xi}^2_{2,\vartheta,h}+\theta h^\sigma\eta^2)d\theta y^1 \\ \quad +\int_0^1 (\overrightarrow{\nabla}_\xi \nabla_\xi \widetilde{J}_{2,\vartheta,h})(\widetilde{X}^1_{2,\vartheta,h}+\theta h^\sigma y^1, \widetilde{\Xi}^2_{2,\vartheta,h}+\theta h^\sigma\eta^2)d\theta \eta^2 \\ \nabla_{y^2}\widetilde{\varphi}_{\vartheta,h} = -\eta^2 + \int_0^1 (\overrightarrow{\nabla}_x \nabla_x \widetilde{J}_{3,h})(\widetilde{X}^2_{2,\vartheta,h}+\theta h^\sigma y^1, \xi)d\theta y^2\,. \end{cases}$$

(II) By (4.72), (4.73), (4.74) and Proposition 2.4, there exists a constant C (independent of $\{\kappa_l\}_{l=0}^\infty$) and C'_l(depending only on $\kappa_1,\kappa_2,\dots,\kappa_l$) such that

$$\tag{4.75} \begin{cases} C^{-1}\langle y^1;\eta^1;y^2;\eta^2\rangle \le \langle \nabla_{(y^1,\eta^1,y^2,\eta^2)}\widetilde{\varphi}_{\vartheta,h}\rangle \le C\langle y^1;\eta^1;y^2;\eta^2\rangle \\ |\partial_\xi^\alpha D_x^\beta \partial_{\eta^1}^{\alpha^1} D_{y^1}^{\beta^1} \partial_{\eta^2}^{\alpha^2} D_{y^2}^{\beta^2} \nabla_{(y^1,\eta^1,y^2,\eta^2)}\widetilde{\varphi}_{\vartheta,h}| \\ \qquad \le C'_{|\alpha+\beta+\alpha^1+\beta^1+\alpha^2+\beta^2|}\langle y^1;\eta^1;y^2;\eta^2\rangle \\ |\partial_\xi^\alpha D_x^\beta \partial_{\eta^1}^{\alpha^1} D_{y^1}^{\beta^1} \partial_{\eta^2}^{\alpha^2} D_{y^2}^{\beta^2} \widetilde{\varphi}_{\vartheta,h}| \\ \qquad \le C'_{|\alpha+\beta+\alpha^1+\beta^1+\alpha^2+\beta^2|}\langle y^1;\eta^1;y^2;\eta^2\rangle^2\,, \end{cases}$$

and

$$\tag{4.76} \begin{aligned} &|\partial_\xi^\alpha D_x^\beta \partial_{\eta^1}^{\alpha^1} D_{y^1}^{\beta^1} \partial_{\eta^2}^{\alpha^2} D_{y^2}^{\beta^2} \widetilde{a}_{\vartheta,h}| \\ &\quad \le C'_{|\alpha+\beta+\alpha^1+\beta^1+\alpha^2+\beta^2|-1} |\widetilde{p}_{1,h}|^{(0)}_{|\alpha+\beta+\alpha^1|} \\ &\quad \times |\widetilde{p}_{2,\vartheta,h}|^{(0)}_{|\alpha+\beta+\beta^1+\alpha^2|} |\widetilde{p}_{3,h}|^{(0)}_{|\alpha+\beta+\beta^2|} \quad (0\le\vartheta\le 1)\,, \end{aligned}$$

where $\langle y;\eta;y';\eta'\rangle \equiv (1+|y|^2+|\eta|^2+|y'|^2+|\eta'|^2)^{1/2}$.

Furthermore, by (4.72), (4.73), Lemma 2.6, 2.7, and Proposition 2.4, there exists a constant C_l''(depending only on $\kappa_1,\kappa_2,\dots,\kappa_l$ and $\chi_0,\chi_1,\dots,\chi_l$) such that

$$
\begin{aligned}
(4.77)\qquad &|\partial_\xi^\alpha D_x^\beta \partial_{\eta^1}^{\alpha^1} D_{y^1}^{\beta^1} \partial_{\eta^2}^{\alpha^2} D_{y^2}^{\beta^2} \partial_\vartheta \widetilde{\varphi}_{\vartheta,h}| \\
&\qquad \le C''_{|\alpha+\beta+\alpha^1+\beta^1+\alpha^2+\beta^2|+1} (\tau_2)^2 \langle y^1;\eta^1;y^2;\eta^2\rangle^3 \langle x;\xi\rangle\,,
\end{aligned}
$$

and

$$
\begin{aligned}
(4.78)\qquad &|\partial_\xi^\alpha D_x^\beta \partial_{\eta^1}^{\alpha^1} D_{y^1}^{\beta^1} \partial_{\eta^2}^{\alpha^2} D_{y^2}^{\beta^2} \partial_\vartheta \widetilde{a}_{\vartheta,h}| \\
&\le C''_{|\alpha+\beta+\alpha^1+\beta^1+\alpha^2+\beta^2|} \langle y^1;\eta^2\rangle \langle x;\xi\rangle \\
&\quad \cdot |\widetilde{p}_{1,h}|^{(0)}_{|\alpha+\beta+\alpha^1|+1} |\widetilde{p}_{3,h}|^{(0)}_{|\alpha+\beta+\beta^2|+1} \\
&\times \Big((\tau_2)^2 |\widetilde{p}_{2,\vartheta,h}|^{(0)}_{|\alpha+\beta+\beta^1+\alpha^2|+1} + |\partial_\vartheta \widetilde{p}_{2,\vartheta,h}|^{(0,1)}_{|\alpha+\beta+\beta^1+\alpha^2|} \Big) \\
&\qquad\qquad (0\le\vartheta\le 1)\,,
\end{aligned}
$$

Set

$$
\begin{aligned}
(4.79)\qquad &\widetilde{b}_{\vartheta,h}(y^1,\eta^1,y^2,\eta^2;x,\xi) \equiv \partial_\vartheta \widetilde{a}_{\vartheta,h}(y^1,\eta^1,y^2,\eta^2;x,\xi) \\
&\qquad + i\partial_\vartheta \widetilde{\varphi}_{\vartheta,h}(y^1,\eta^1,y^2,\eta^2;x,\xi) \widetilde{a}_{\vartheta,h}(y^1,\eta^1,y^2,\eta^2;x,\xi)\,.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
(4.80)\quad \partial_\vartheta \widetilde{q}_{\vartheta,h}(x,\xi) = \mathrm{O_s}\text{—}\iiiint & e^{i\widetilde{\varphi}_{\vartheta,h}(y^1,\eta^1,y^2,\eta^2;x,\xi)} \\
&\cdot \widetilde{b}_{\vartheta,h}(y^1,\eta^1,y^2,\eta^2;x,\xi)\, dy^1 đ\eta^1 dy^2 đ\eta^2\,.
\end{aligned}
$$

(III) Set

$$
\begin{aligned}
(4.81)\qquad & e^{i\widetilde{\varphi}_{\vartheta,h}(y^1,\eta^1,y^2,\eta^2;x,\xi)} \widetilde{b}_{\vartheta,h,\alpha,\beta}(y^1,\eta^1,y^2,\eta^2;x,\xi) \\
&\equiv \partial_\xi^\alpha D_x^\beta \Big(e^{i\widetilde{\varphi}_{\vartheta,h}(y^1,\eta^1,y^2,\eta^2;x,\xi)} \widetilde{b}_{\vartheta,h}(y^1,\eta^1,y^2,\eta^2;x,\xi) \Big).
\end{aligned}
$$

By (4.75)-(4.79), there exists a constant C'''_l(depending only on $\kappa_1,\kappa_2,\dots,\kappa_l$ and $\chi_0,\chi_1,\dots,\chi_l$) such that

$$
\begin{aligned}
(4.82)\quad & |\partial_\xi^\gamma D_x^\mu \partial_{\eta^1}^{\alpha^1} D_{y^1}^{\beta^1} \partial_{\eta^2}^{\alpha^2} D_{y^2}^{\beta^2} \widetilde{b}_{\vartheta,h,\alpha,\beta}(y^1,\eta^1,y^2,\eta^2;x,\xi)| \\
& \le C'''_{|\alpha+\beta+\alpha^1+\beta^1+\alpha^2+\beta^2+\gamma+\mu|+1} \langle y^1;\eta^1;y^2;\eta^2\rangle^{2|\alpha+\beta|+3}\langle x;\xi\rangle \\
& \times |\widetilde{p}_{1,h}|^{(0)}_{|\alpha+\beta+\alpha^1+\gamma+\mu|+1} |\widetilde{p}_{3,h}|^{(0)}_{|\alpha+\beta+\beta^2+\gamma+\mu|+1} \\
& \times \left((\tau_2)^2 |\widetilde{p}_{2,\vartheta,h}|^{(0)}_{|\alpha+\beta+\beta^1+\alpha^2+\gamma+\mu|+1} + |\partial_\vartheta \widetilde{p}_{2,\vartheta,h}|^{(0,1)}_{|\alpha+\beta+\beta^1+\alpha^2+\gamma+\mu|}\right) \\
& (0\le\vartheta\le 1)\,.
\end{aligned}
$$

Hence, for $l' = 2|\alpha+\beta|+4n+4$, we can write

$$
\begin{aligned}
(4.83)\quad & \partial_\xi^\alpha D_x^\beta \partial_\vartheta \widetilde{q}_{\vartheta,h}(x,\xi) \\
& = \mathrm{O_s}\text{—}\iiiint e^{i\widetilde{\varphi}_{\vartheta,h}(y^1,\eta^1,y^2,\eta^2;x,\xi)} \\
& \quad \cdot \widetilde{b}_{\vartheta,h,\alpha,\beta}(y^1,\eta^1,y^2,\eta^2;x,\xi)dy^1 đ\eta^1 dy^2 đ\eta^2 \\
& = \iiiint e^{i\widetilde{\varphi}_{\vartheta,h}} (1+i\nabla_{(y^1,\eta^1,y^2,\eta^2)}\cdot\nabla_{(y^1,\eta^1,y^2,\eta^2)}\widetilde{\varphi}_{\vartheta,h})^{l'} \\
& \quad \times \langle \nabla_{(y^1,\eta^1,y^2,\eta^2)}\widetilde{\varphi}_{\vartheta,h}\rangle^{-2l'} \widetilde{b}_{\vartheta,h,\alpha,\beta}(y^1,\eta^1,y^2,\eta^2;x,\xi)dy^1 đ\eta^1 dy^2 đ\eta^2
\end{aligned}
$$

and there exists a constant $C_{3l+4n+5}$(depending only on $\kappa_1,\kappa_2,\dots,\kappa_{3l+4n+5}$) such that

$$
\begin{aligned}
(4.84)\quad |\partial_\vartheta \widetilde{q}_{\vartheta,h}|^{(0,1)}_l & \le C_{3l+4n+5} |\widetilde{p}_{1,h}|^{(0)}_{3l+4n+5} |\widetilde{p}_{3,h}|^{(0)}_{3l+4n+5} \\
& \quad \times ((\tau_2)^2 |\widetilde{p}_{2,\vartheta,h}|^{(0)}_{3l+4n+5} + |\partial_\vartheta \widetilde{p}_{2,\vartheta,h}|^{(0,1)}_{3l+4n+5}) \\
& \qquad\qquad (0\le\vartheta\le 1)\quad .
\end{aligned}
$$

Therefore, we get (4.67). □

In Theorem 4.6, for $\nu = 1,2,\dots$ and $0\le\vartheta\le 1$, set

$$
\begin{aligned}
(4.85)\quad & p_{1,2,\dots,\nu+1,\vartheta,h}(x,\xi) \\
& \equiv \exp\left(i\sum_{j=1}^{\nu+1} \left(\vartheta\omega_{j,h}(X^{j-1}_{\nu,h},\varXi^j_{\nu,h}) + (1-\vartheta)\omega_{j,h}(x,\xi)\right)\right) \\
& \quad + \vartheta\varUpsilon_{1,2,\dots,\nu+1,h}(x,\xi)\,.
\end{aligned}
$$

Then we have

$$(4.86)\qquad \begin{cases} p_{1,2,\dots,\nu+1,1,h}(x,\xi) = (p^{\#}_{1,2,\dots,\nu+1,h} + \Upsilon_{1,2,\dots,\nu+1,h})(x,\xi) \\ p_{1,2,\dots,\nu+1,0,h}(x,\xi) = \exp\Big(i\sum_{j=1}^{\nu+1}\omega_{j,h}(x,\xi)\Big) . \end{cases}$$

Furthermore, we get the following lemma.

Lemma 4.8.

For any $\{\kappa_l\}_{l=0}^{\infty}(\kappa_0 = 1,\, \kappa_l \le \kappa_{l+1})$, *there exist constants* $0 < \tau_0' < 1/(4c_0)$ (*depending only on* $\kappa_1, \kappa_2, \dots, \kappa_{5n_0}$) *and* $C''_{3l+36n_0}, C'''_{3l+36n_0}$(*depending only on* $\kappa_1, \kappa_2, \dots, \kappa_{3l+36n_0}$) *satisfying the following:*

Let $\phi_{j,h} \in \mathbb{P}_{\rho,\delta}(\tau_j, h, \{\kappa_l\}_{l=0}^{\infty})$ *and* $\omega_{j,h} \in \Omega_{\rho,\delta}(\tau_j, h, \{\kappa_l\}_{l=0}^{\infty})(j = 1, 2, \dots)$ *with* $\sum_{j=1}^{\infty}\tau_j \le \tau_0'$.

Then, it follows that

$$(4.87)\qquad \begin{cases} p_{1,2,\dots,\nu+1,\vartheta,h}(x,\xi) \in \mathbb{B}^{0}_{\rho,\delta}(h) \\ \partial_\vartheta p_{1,2,\dots,\nu+1,\vartheta,h}(x,\xi) \in \mathbb{B}^{0,1}_{\rho,\delta}(h) \quad (0 \le \vartheta \le 1) , \end{cases}$$

and

$$(4.88)\qquad \begin{cases} |p_{1,2,\dots,\nu+1,\vartheta,h}|_l^{(0)} \le C''_{3l+36n_0} \\ |\partial_\vartheta p_{1,2,\dots,\nu+1,\vartheta,h}|_l^{(0,1)} \le C'''_{3l+36n_0}(\bar{\tau}_{\nu+1})^2 \quad (0 \le \vartheta \le 1) . \end{cases}$$

Proof. Note that

$$(4.89)\qquad \sum_{j=1}^{\nu+1}\Big(\widetilde{\omega}_{j,h}(\widetilde{X}^{j-1}_{\nu,h}, \widetilde{\Xi}^{j}_{\nu,h}) - \widetilde{\omega}_{j,h}(x,\xi)\Big)$$
$$= \sum_{j=1}^{\nu+1}\int_0^1 (\overrightarrow{\nabla}_{(x,\xi)}\widetilde{\omega}_{j,h})(x + \theta(\widetilde{X}^{j-1}_{\nu,h} - x), \xi + \theta(\widetilde{\Xi}^{j}_{\nu,h} - \xi))d\theta$$
$$\cdot \begin{pmatrix} \widetilde{X}^{j-1}_{\nu,h} - x \\ \widetilde{\Xi}^{j}_{\nu,h} - \xi \end{pmatrix} .$$

By Proposition 2.4, 4.5 and Theorem 4.6, we get (4.88). □

5. A Construction of the Fundamental Solution

In this section, using the theory developed in Section 1, 2, 3 and 4, we give a construction of the fundamental solution for the Cauchy problem of a pseudo-differential equation of Schrödinger's type. We regard this construction as a multi-product of Fourier integral operators, and investigate the pointwise convergence of the phase function and that of the symbol.

Let $H_h(t,x,\xi)$ be a real-valued function on $[0,T]\times\boldsymbol{R}^{2n}$ with a parameter $0<h<1$, which has a continuous derivative $\partial_\xi^\alpha D_x^\beta H_h(t,x,\xi)$ on $[0,T]\times\boldsymbol{R}^{2n}$ for any α,β, satisfying

$$(5.1)\qquad \begin{aligned}&|\partial_\xi^\alpha D_x^\beta\{h^{\rho-\delta}H_h(t,h^\delta x,h^{-\rho}\xi)\}|\\ &\qquad\le\begin{cases}C_{\alpha,\beta}\langle x;\xi\rangle^{2-|\alpha+\beta|} & (|\alpha+\beta|\le 1)\\ C_{\alpha,\beta} & (|\alpha+\beta|\ge 2)\,,\end{cases}\end{aligned}$$

on $[0,T]\times\boldsymbol{R}^{2n}$. Here $C_{\alpha,\beta}$ is a constant independent of a parameter $0<h<1$.

Let $W_h(t,x,\xi)$ be a complex-valued function on $[0,T]\times\boldsymbol{R}^{2n}$ with a parameter $0<h<1$, which has a continuous derivative $\partial_\xi^\alpha D_x^\beta W_h(t,x,\xi)$ on $[0,T]\times\boldsymbol{R}^{2n}$ for any α,β, satisfying

$$(5.2)\qquad \begin{cases}|\partial_\xi^\alpha D_x^\beta\{W_h(t,h^\delta x,h^{-\rho}\xi)\}|\le\begin{cases}C'_{0,0}\langle x;\xi\rangle & (\alpha=\beta=0)\\ C'_{\alpha,\beta} & (|\alpha+\beta|\ge 1)\end{cases}\\ Im\{W_h(t,h^\delta x,h^{-\rho}\xi)\}\ge -C'_{0,0}\,,\end{cases}$$

on $[0,T]\times\boldsymbol{R}^{2n}$. Here $C'_{\alpha,\beta}$ is a constant independent of a parameter $0<h<1$.

Set

$$(5.3)\qquad \begin{cases}K_h(t,x,\xi)\equiv H_h(t,x,\xi)+W_h(t,x,\xi)\\ L_h\equiv i\partial_t+K_h(t,X,D_x)\,.\end{cases}$$

For sufficiently small T_0 $(0<T_0\le T)$, we consider the fundamental solution $U_h(t,s)$ $(0\le s\le t\le T_0)$ for the operator L_h such that

$$(5.4)\qquad \begin{cases}L_hU_h(t,s)=0 \quad \text{on } [s,T_0]\\ \quad U_h(s,s)=I \qquad (0\le s\le T_0)\,.\end{cases}$$

Our purpose is to construct $U_h(t,s)$ by

$$U_h(t,s) = \lim_{|\varDelta_{t,s}|\to 0} e^{i\int_{t_1}^{t} K_h(\tau)d\tau}(X,D_x)e^{i\int_{t_2}^{t_1} K_h(\tau)d\tau}(X,D_x) \cdots e^{i\int_{s}^{t_\nu} K_h(\tau)d\tau}(X,D_x)\,, \tag{5.5}$$

where $\varDelta_{t,s} : (T_0 \geq)\, t \equiv t_0 \geq t_1 \geq \cdots \geq t_\nu \geq t_{\nu+1} \equiv s\, (\geq 0)$ is an arbitrary division of interval [s,t] into subintervals, $|\varDelta_{t,s}| \equiv \max_{1\leq j\leq \nu+1} |t_{j-1} - t_j|$, and

$$\begin{aligned} & e^{i\int_{t_j}^{t_{j-1}} K_h(\tau)d\tau}(X,D_x)u(x) \\ & \qquad \equiv \int e^{ix\cdot\xi} e^{i\int_{t_j}^{t_{j-1}} K_h(\tau,x,\xi)d\tau} \hat{u}(\xi) đ\xi \quad (u \in \mathcal{S})\,. \end{aligned} \tag{5.6}$$

Now, set

$$\begin{cases} \phi_h(t,s)(x,\xi) \equiv x\cdot\xi + \int_s^t H_h(\tau,x,\xi)d\tau \\ \omega_h(t,s)(x,\xi) \equiv \int_s^t W_h(\tau,x,\xi)d\tau \quad (0 \leq s \leq t \leq T)\,. \end{cases} \tag{5.7}$$

and

$$p_h(t,s)(x,\xi) \equiv \exp\Big(i\omega_h(t,s)(x,\xi)\Big) \quad (0 \leq s \leq t \leq T)\,. \tag{5.8}$$

Then we can write

$$e^{i\int_s^t K_h(\tau)d\tau}(X,D_x)u(x) \equiv p_h(t,s)(\phi_h(t,s);X,D_x)u(x)\,. \tag{5.9}$$

Therefore, we regard the operator $e^{i\int_s^t K_h(\tau)d\tau}(X,D_x)$ as the Fourier integral operator $p_h(t,s)(\phi_h(t,s);X,D_x)$ with the phase function $\phi_h(t,s)(x,\xi)$ and the symbol $p_h(t,s)(x,\xi)$, and investigate the properties of $e^{i\int_s^t K_h(\tau)d\tau}(X,D_x)$.

THEOREM 5.1.

Under the assumptions (5.1), (5.2) *and* (5.3), *there exist sufficiently small* T_0 $(0 < T_0 \leq T)$, *constants* $C_{1,l}$, c' *and a sequences of positive constants* $\{\kappa'_l\}_{l=0}^{\infty}$ $(\kappa'_0 = 1, \kappa'_l \leq \kappa'_{l+1})$, *satisfying the following:*

For any $\Delta_{t_0,t_{\nu+1}}$: $(T_0 \geq)\, t_0 \geq t_1 \geq \cdots \geq t_{\nu+1}\,(\geq 0)$, *there exist a phase function* $\phi_h(\Delta_{t_0,t_{\nu+1}})(x,\xi) \in \mathbb{P}_{\rho,\delta}(c'(t_0-t_{\nu+1}), h, \{\kappa'_l\}_{l=0}^{\infty})$ *and a symbol* $p_h(\Delta_{t_0,t_{\nu+1}})(x,\xi) \in \mathbb{B}^0_{\rho,\delta}(h)$ *such that*

$$
\tag{5.10}
\begin{aligned}
p_h(\Delta_{t_0,t_{\nu+1}})(\phi_h(\Delta_{t_0,t_{\nu+1}}); X, D_x) &= e^{i\int_{t_1}^{t_0} K_h(\tau)d\tau}(X, D_x) \\
&\cdot e^{i\int_{t_2}^{t_1} K_h(\tau)d\tau}(X, D_x) \cdots e^{i\int_{t_{\nu+1}}^{t_\nu} K_h(\tau)d\tau}(X, D_x)\,,
\end{aligned}
$$

and

$$
\tag{5.11}
|p_h(\Delta_{t_0,t_{\nu+1}})|_l^{(0)} \leq C_{1,l}\,.
$$

Furthermore, the following equation holds:

$$
\tag{5.12}
\begin{aligned}
& e^{i\phi_h(\Delta_{t_0,t_{\nu+1}})(x,\xi)-ix'\cdot\xi} p_h(\Delta_{t_0,t_{\nu+1}})(x,\xi) \\
&= \mathrm{O_s}\text{—}\iint \cdots \iint \exp\Bigg(\sum_{j=1}^{\nu+1}\Big(i(x^{j-1}-x^j)\cdot\xi^j \\
&\quad + i\int_{t_j}^{t_{j-1}} K_h(\tau, x^{j-1}, \xi^j)d\tau\Big)\Bigg) dx^1 đ\xi^1 \cdots dx^\nu đ\xi^\nu\,,
\end{aligned}
$$

where $x^0 \equiv x$, $x^{\nu+1} \equiv x'$ *and* $\xi^{\nu+1} \equiv \xi$.

PROOF.

(I) By (5.1), (5.2) and (5.7), there exist a constant $c > 0$ and a sequence of positive constants $\{\kappa_l\}_{l=0}^{\infty}(\kappa_0 = 1,\ \kappa_l \leq \kappa_{l+1})$ such that

$$
\tag{5.13}
\begin{cases}
\phi_h(t,s)(x,\xi) \in \mathbb{P}_{\rho,\delta}(c(t-s), h, \{\kappa_l\}_{l=0}^{\infty}) \\
\omega_h(t,s)(x,\xi) \in \Omega_{\rho,\delta}(c(t-s), h, \{\kappa_l\}_{l=0}^{\infty})\,.
\end{cases}
$$

Let $c_0 > 1$ be the same constant as that in Proposition 4.5, $T' \equiv \min\{T, 1/(4c_0c)\}$ and $c' \equiv c_0c$. For any $\Delta_{t_0,t_{\nu+1}}$: $(T' \geq)\, t_0 \geq t_1 \geq \cdots \geq t_{\nu+1}\,(\geq 0)$, let $\{X^j_{\nu,h}, \Xi^j_{\nu,h}\}_{j=1}^{\nu}(x,\xi)$ be the solution of the equations

$$
\tag{5.14}
\begin{cases}
X^j_{\nu,h} = \nabla_\xi\phi_h(t_{j-1},t_j)(X^{j-1}_{\nu,h}, \Xi^j_{\nu,h}) \\
\Xi^j_{\nu,h} = \nabla_x\phi_h(t_j,t_{j+1})(X^j_{\nu,h}, \Xi^{j+1}_{\nu,h}) \\
(j = 1,\ldots,\nu,\quad X^0_{\nu,h} \equiv x,\quad \Xi^{\nu+1}_{\nu,h} \equiv \xi)\,,
\end{cases}
$$

and set

$$
(5.15) \qquad \begin{cases} \phi_h(\varDelta_{t_0,t_{\nu+1}})(x,\xi) \\ \quad \equiv (\phi_h(t_0,t_1)\#\phi_h(t_1,t_2)\#\cdots\#\phi_h(t_\nu,t_{\nu+1}))(x,\xi) \\ \omega_h(\varDelta_{t_0,t_{\nu+1}})(x,\xi) \equiv \sum_{j=1}^{\nu+1} \omega_h(t_{j-1},t_j)(X_{\nu,h}^{j-1},\varXi_{\nu,h}^{j}) \,. \end{cases}
$$

Then, by Proposition 2.5 and 4.5, there exists a sequence of positive constants $\{\kappa_l'\}_{l=0}^{\infty}(\kappa_0'=1,\ \kappa_l'\le\kappa_{l+1}',\ \kappa_l\le\kappa_l')$ such that

$$
(5.16) \qquad \begin{cases} \phi_h(\varDelta_{t_0,t_{\nu+1}})(x,\xi) \in \mathbb{P}_{\rho,\delta}(c'(t-s),h,\{\kappa_l'\}_{l=0}^{\infty}) \\ \omega_h(\varDelta_{t_0,t_{\nu+1}})(x,\xi) \in \Omega_{\rho,\delta}(c'(t-s),h,\{\kappa_l'\}_{l=0}^{\infty}) \\ (\nu=0,1,2,\dots,\quad T'\ge t_0\ge t_1\ge\cdots\ge t_{\nu+1}\ge 0)\,. \end{cases}
$$

Furthermore, set

$$
(5.17) \qquad p_h^{\#}(\varDelta_{t_0,t_{\nu+1}})(x,\xi) \equiv \exp\Big(i\omega_h(\varDelta_{t_0,t_{\nu+1}})(x,\xi)\Big) \in \mathbb{B}^0_{\rho,\delta}(h)\,.
$$

Then, by (5.16), there exists a constant $C_{1,l}'$ such that

$$
(5.18) \qquad \begin{aligned} &|p_h^{\#}(\varDelta_{t_0,t_{\nu+1}})|_l^{(0)} \le C_{1,l}' \\ &(\nu=0,1,2,\dots,\quad T'\ge t_0\ge t_1\ge\cdots\ge t_{\nu+1}\ge 0)\,. \end{aligned}
$$

(II) We choose sufficiently small T_0 $(0<T_0<T')$ such that Theorem 4.6 holds. By Theorem 4.6, there exists a constant $C_{1,l}''$ satisfying the following:

For any $\varDelta_{t_0,t_{\nu+1}} : (T_0 \ge)\, t_0 \ge t_1 \ge \cdots \ge t_{\nu+1}\,(\ge 0)$, there exists $\varUpsilon_h(\varDelta_{t_0,t_{\nu+1}})(x,\xi) \in \mathbb{B}^0_{\rho,\delta}(h)$ such that

$$
(5.19) \qquad \begin{aligned} p_h(t_0,t_1)&(\phi_h(t_0,t_1);X,D_x)p_h(t_1,t_2)(\phi_h(t_1,t_2);X,D_x) \\ &\qquad\qquad \cdots p_h(t_\nu,t_{\nu+1})(\phi_h(t_\nu,t_{\nu+1});X,D_x) \\ &= p_h^{\#}(\varDelta_{t_0,t_{\nu+1}})(\phi_h(\varDelta_{t_0,t_{\nu+1}});X,D_x) \\ &\quad + \varUpsilon_h(\varDelta_{t_0,t_{\nu+1}})(\phi_h(\varDelta_{t_0,t_{\nu+1}});X,D_x)\,, \end{aligned}
$$

and

$$
(5.20) \qquad |\varUpsilon_h(\varDelta_{t_0,t_{\nu+1}})|_l^{(0)} \le C_{1,l}''(t_0-t_{\nu+1})^2\,.
$$

(III) Set

$$p_h(\Delta_{t_0,t_{\nu+1}}) = p_h^{\#}(\Delta_{t_0,t_{\nu+1}}) + \Upsilon_h(\Delta_{t_0,t_{\nu+1}})\,. \tag{5.21}$$

Then we get (5.11). □

Lemma 5.2.

(1) *There exists a sequence of positive constants* $\{\chi_l\}_{l=0}^{\infty}(0 \le \chi_l \le \chi_{l+1})$ *satisfying the following:*

For any $\Delta_{t_0,t_{\nu+1}} : (T_0 \ge)\, t_0 \ge t_1 \ge \cdots \ge t_{\nu+1}\, (\ge 0)$, *there exists* $\widetilde{\phi}_{\vartheta,h}(\Delta_{t_0,t_{\nu+1}}) \in \mathbb{P}_{0,0}(c'(t_0 - t_{\nu+1}), h, \{\kappa'_l\}_{l=0}^{\infty})$ $(0 \le \vartheta \le 1)$ *such that*

$$\begin{cases} \widetilde{\phi}_{1,h}(\Delta_{t_0,t_{\nu+1}})(x,\xi) = \widetilde{\phi}_h(\Delta_{t_0,t_{\nu+1}})(x,\xi) \\ \widetilde{\phi}_{0,h}(\Delta_{t_0,t_{\nu+1}})(x,\xi) = \widetilde{\phi}_h(t_0, t_{\nu+1})(x,\xi)\,, \end{cases} \tag{5.22}$$

and

$$\begin{cases} |\partial_\vartheta \widetilde{J}_{\vartheta,h}(\Delta_{t_0,t_{\nu+1}})(x,\xi)| \le \chi_0 (t_0 - t_{\nu+1})^2 \langle x;\xi\rangle^2 \\ |\partial_\xi^\alpha D_x^\beta \nabla_{(x,\xi)} \partial_\vartheta \widetilde{J}_{\vartheta,h}(\Delta_{t_0,t_{\nu+1}})(x,\xi)| \le \chi_{|\alpha+\beta|} (t_0 - t_{\nu+1})^2 \langle x;\xi\rangle \\ (0 < h < 1, \quad 0 \le \vartheta \le 1)\,, \end{cases} \tag{5.23}$$

where $\widetilde{J}_{\vartheta,h}(\Delta_{t_0,t_{\nu+1}})(x,\xi) \equiv \widetilde{\phi}_{\vartheta,h}(\Delta_{t_0,t_{\nu+1}})(x,\xi) - x\cdot\xi$.

(2) *There exist constants* $C_{1,l}$, $C_{2,l}$, *satisfying the following:*

For any $\Delta_{t_0,t_{\nu+1}} : (T_0 \ge)\, t_0 \ge t_1 \ge \cdots \ge t_{\nu+1}\, (\ge 0)$, *there exists* $p_{\vartheta,h}(\Delta_{t_0,t_{\nu+1}})(x,\xi) \in \mathbb{B}^0_{\rho,\delta}(h)$ $(0 \le \vartheta \le 1)$ *such that*

$$\begin{cases} p_{1,h}(\Delta_{t_0,t_{\nu+1}})(x,\xi) = p_h(\Delta_{t_0,t_{\nu+1}})(x,\xi) \\ p_{0,h}(\Delta_{t_0,t_{\nu+1}})(x,\xi) = p_h(t_0, t_{\nu+1})(x,\xi)\,, \end{cases} \tag{5.24}$$

$$\partial_\vartheta p_{\vartheta,h}(\Delta_{t_0,t_{\nu+1}})(x,\xi) \in \mathbb{B}^{0,1}_{\rho,\delta}(h) \quad (0 \le \vartheta \le 1)\,, \tag{5.25}$$

and

$$\begin{cases} |p_{\vartheta,h}(\Delta_{t_0,t_{\nu+1}})|_l^{(0)} \le C_{1,l} \\ |\partial_\vartheta p_{\vartheta,h}(\Delta_{t_0,t_{\nu+1}})|_l^{(0,1)} \le C_{2,l}(t_0 - t_{\nu+1})^2 \quad (0 \le \vartheta \le 1)\,. \end{cases} \tag{5.26}$$

Proof. Set

$$\widetilde{\phi}_{\vartheta,h}(\Delta_{t_0,t_{\nu+1}})(x,\xi) \equiv \vartheta\widetilde{\phi}_h(\Delta_{t_0,t_{\nu+1}})(x,\xi) + (1-\vartheta)\widetilde{\phi}_h(t_0,t_{\nu+1})(x,\xi)\,, \tag{5.27}$$

and

$$\begin{aligned} p_{\vartheta,h}(\Delta_{t_0,t_{\nu+1}})(x,\xi) \\ \equiv \exp\Big(i\Big(\vartheta\omega_h(\Delta_{t_0,t_{\nu+1}})(x,\xi) + (1-\vartheta)\omega_h(t_0,t_{\nu+1})(x,\xi)\Big)\Big) \\ + \vartheta\Upsilon_h(\Delta_{t_0,t_{\nu+1}})(x,\xi)\,. \end{aligned} \tag{5.28}$$

By Lemma 2.8 and Lemma 4.8, we get (1) and (2). □

Theorem 5.3.

(1) *There exists a constant $C_{3,l}$ satisfying the following:*

For any $(T_0 \geq)\, t \geq s\, (\geq 0)$, let $\Delta_{t,s}$ be an arbitrary division of interval $[s,t]$ into subintervals, and $\Delta'_{t,s}$ be an arbitrary refinement of $\Delta_{t,s}$.

Then it follows that

$$|\phi_h(\Delta_{t,s}) - \phi_h(\Delta'_{t,s})|_l^{(\delta-\rho,2)} \leq C_{3,l}|\Delta_{t,s}|(t-s)\,. \tag{5.29}$$

Furthermore, there exists a phase function $\phi_h^\star(t,s) \in \mathbb{P}_{\rho,\delta}(c'(t-s), h, \{\kappa'_l\}_{l=0}^\infty)$ such that

$$|\phi_h(\Delta_{t,s}) - \phi_h^\star(t,s)|_l^{(\delta-\rho,2)} \leq C_{3,l}|\Delta_{t,s}|(t-s)\,. \tag{5.30}$$

(2) *There exists a constant $C_{4,l}$ satisfying the following:*

For any $(T_0 \geq)\, t \geq s\, (\geq 0)$, let $\Delta_{t,s}$ be an arbitrary division of interval $[s,t]$ into subintervals, and $\Delta'_{t,s}$ be an arbitrary refinement of $\Delta_{t,s}$.

Then it follows that

$$|p_h(\Delta_{t,s}) - p_h(\Delta'_{t,s})|_l^{(0,1)} \leq C_{4,l}|\Delta_{t,s}|(t-s)\,. \tag{5.31}$$

Furthermore, there exists a symbol $p_h^\star(t,s) \in \mathbb{B}^0_{\rho,\delta}(h)$ *such that*

$$|p_h(\varDelta_{t,s}) - p_h^\star(t,s)|_l^{(0,1)} \le C_{4,l}|\varDelta_{t,s}|(t-s)\,. \tag{5.32}$$

and

$$|p_h^\star(t,s)|_l^{(0)} \le C_{1,l}\,. \tag{5.33}$$

(3) *Furthermore, the following equation holds:*

$$\begin{aligned} & e^{i\phi_h^\star(t,s)(x,\xi)-ix'\cdot\xi} p_h^\star(t,s)(x,\xi) \\ &= \lim_{|\varDelta_{t,s}|\to 0} \mathrm{O_s}\text{—}\iint\cdots\iint \exp\Bigg(\sum_{j=1}^{\nu+1}\Big(i(x^{j-1}-x^j)\cdot\xi^j \\ &\quad + i\int_{t_j}^{t_{j-1}} K_h(\tau,x^{j-1},\xi^j)d\tau\Big)\Bigg) dx^1 đ\xi^1\cdots dx^\nu đ\xi^\nu\,, \end{aligned} \tag{5.34}$$

where $\varDelta_{t,s} : (T_0 \ge)\, t \equiv t_0 \ge t_1 \ge \cdots \ge t_{\nu+1} \equiv s\,(\ge 0)$, $x^0 \equiv x$, $x^{\nu+1} \equiv x'$ *and* $\xi^{\nu+1} \equiv \xi$.

Proof.
(I) Set

$$\varDelta_{t,s} : (T_0 \ge)\, t \equiv t_0 \ge t_1 \ge \cdots \ge t_{\nu+1} \equiv s\,(\ge 0)\,, \tag{5.35}$$

and

$$\varDelta'_{t,s} : \begin{cases} t_0 \equiv t_{0,0} \ge t_{0,1} \ge \cdots \ge t_{0,k_0} \equiv t_1 \\ t_1 \equiv t_{1,0} \ge t_{1,1} \ge \cdots \ge t_{1,k_1} \equiv t_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ t_\nu \equiv t_{\nu,0} \ge t_{\nu,1} \ge \cdots \ge t_{\nu,k_\nu} \equiv t_{\nu+1} \\ (k_j > 0, \quad j = 0,1,2,\ldots,\nu) \quad . \end{cases} \tag{5.36}$$

Then we have

$$
\begin{aligned}
(5.37)\quad & \widetilde{\phi}_h(\Delta'_{t,s}) - \widetilde{\phi}_h(\Delta_{t,s}) \\
&= \sum_{j=1}^{\nu+1} \Big(\widetilde{\phi}_h(\Delta'_{t_0,t_{j-1}}) \# \widetilde{\phi}_h(\Delta'_{t_{j-1},t_j}) \# \widetilde{\phi}_h(\Delta_{t_j,t_{\nu+1}}) \\
&\qquad - \widetilde{\phi}_h(\Delta'_{t_0,t_{j-1}}) \# \widetilde{\phi}_h(t_{j-1},t_j) \# \widetilde{\phi}_h(\Delta_{t_j,t_{\nu+1}}) \Big) \\
&= \sum_{j=1}^{\nu+1} \int_0^1 \partial_\vartheta \Big(\widetilde{\phi}_h(\Delta'_{t_0,t_{j-1}}) \# \widetilde{\phi}_{\vartheta,h}(\Delta'_{t_{j-1},t_j}) \# \widetilde{\phi}_h(\Delta_{t_j,t_{\nu+1}}) \Big) d\vartheta \,.
\end{aligned}
$$

By Lemma 2.7, Lemma 5.2 and Theorem 5.1, there exists a constant χ'_l such that

$$
\begin{aligned}
(5.38)\qquad & |\partial_\xi^\alpha D_x^\beta \{ \widetilde{\phi}_h(\Delta'_{t,s}) - \widetilde{\phi}_h(\Delta_{t,s}) \}| \\
&\quad \le \sum_{j=1}^{\nu+1} \int_0^1 \chi'_{|\alpha+\beta|-1} (t_{j-1}-t_j)^2 \langle x;\xi\rangle^2 d\vartheta \\
&\quad \le \chi'_{|\alpha+\beta|-1} |\Delta_{t,s}| (t-s) \langle x;\xi\rangle^2 \,.
\end{aligned}
$$

Hence we get (1).

(II) Define $q_{j,\vartheta,h}(\Delta'_{t,s},\Delta_{t,s})(x,\xi) \in \mathbb{B}^0_{\rho,\delta}(h)$ by

$$
\begin{aligned}
(5.39)\quad & q_{j,\vartheta,h}(\Delta'_{t,s},\Delta_{t,s})(\phi_h(\Delta'_{t_0,t_{j-1}}) \# \phi_{\vartheta,h}(\Delta'_{t_{j-1},t_j}) \# \phi_h(\Delta_{t_j,t_{\nu+1}}); X, D_x) \\
&\equiv p_h(\Delta'_{t_0,t_{j-1}})(\phi_h(\Delta'_{t_0,t_{j-1}}); X, D_x) \\
&\qquad \circ p_{\vartheta,h}(\Delta'_{t_{j-1},t_j})(\phi_{\vartheta,h}(\Delta'_{t_{j-1},t_j}); X, D_x) \\
&\qquad \circ p_h(\Delta_{t_j,t_{\nu+1}})(\phi_h(\Delta_{t_j,t_{\nu+1}}); X, D_x) \,.
\end{aligned}
$$

Then we have

$$
(5.40)\qquad \begin{cases} q_{j,1,h}(\Delta'_{t,s},\Delta_{t,s})(x,\xi) = p_h(\Delta'_{t_0,t_j},\Delta_{t_{j+1},t_{\nu+1}})(x,\xi) \\ q_{j,0,h}(\Delta'_{t,s},\Delta_{t,s})(x,\xi) = p_h(\Delta'_{t_0,t_{j-1}},\Delta_{t_j,t_{\nu+1}})(x,\xi) \,. \end{cases}
$$

Hence, it follows that

$$
\begin{aligned}
(5.41)\qquad & p_h(\varDelta'_{t,s}) - p_h(\varDelta_{t,s}) \\
&= \sum_{j=1}^{\nu+1} \Big(p_h(\varDelta'_{t_0,t_j}, \varDelta_{t_{j+1},t_{\nu+1}}) - p_h(\varDelta'_{t_0,t_{j-1}}, \varDelta_{t_j,t_{\nu+1}}) \Big) \\
&= \sum_{j=1}^{\nu+1} \int_0^1 \partial_\vartheta \Big(q_{j,\vartheta,h}(\varDelta'_{t,s}, \varDelta_{t,s}) \Big) d\vartheta \,.
\end{aligned}
$$

By Lemma 4.7, Lemma 5.2 and Theorem 5.1, we get (2).

(III) From (1) and (2), (3) is clear. □

Theorem 5.4.

The Fourier integral operator $U_h(t,s) \equiv p^\star_h(t,s)(\phi^\star_h(t,s); X, D_x)$ with the phase function $\phi^\star_h(t,s)(x,\xi)$ and the symbol $p^\star_h(t,s)(x,\xi)$ in Theorem 5.3, satisfies the following:

(1) *The operators $U_h(t,s) : \boldsymbol{L}^2 \to \boldsymbol{L}^2$, $\boldsymbol{H}_{2,h} \to \boldsymbol{H}_{2,h}$ are uniformly bounded for $0 \le s \le t \le T_0$ and $0 < h < 1$.*

(2) *The operators $K_h(t, X, D_x)U_h(t,s)$, $\partial_t U_h(t,s) : \boldsymbol{H}_{2,h} \to \boldsymbol{H}_{0,h}$ are uniformly bounded for $0 \le s \le t \le T_0$ and $0 < h < 1$.*

(3) *The following relation holds:*

$$
\begin{aligned}
(5.42)\qquad U_h(t,s) = \lim_{|\varDelta_{t,s}|\to 0} & e^{i\int_{t_1}^{t} K_h(\tau)d\tau}(X, D_x) e^{i\int_{t_2}^{t_1} K_h(\tau)d\tau}(X, D_x) \\
& \cdots e^{i\int_{s}^{t_\nu} K_h(\tau)d\tau}(X, D_x) \,,
\end{aligned}
$$

where $\varDelta_{t,s} : (T_0 \ge) t \equiv t_0 \ge t_1 \ge \cdots \ge t_\nu \ge t_{\nu+1} \equiv s\,(\ge 0)$ is an arbitrary division of interval [s,t] into subintervals, and $|\varDelta_{t,s}| \equiv \max\limits_{1\le j\le \nu+1} |t_{j-1} - t_j|$.

(4) *$U_h(t,s)$ is the fundamental solution for the operator L_h such that*

$$
(5.43)\qquad \left\{
\begin{aligned}
L_h U_h(t,s) &= 0 \quad \textit{on } [s, T_0] \\
U_h(s,s) &= I \quad (0 \le s \le T_0) \,.
\end{aligned}
\right.
$$

Proof.

(0) By Proposition 3.7, Theorem 4.2 and Theorem 5.3, (1) and (2) are clear. We have only to check (3) and (4) for $u \in \mathcal{S}$.

(I) By Proposition 3.3 and Theorem 5.1, there exist a constant $M_{1,h,l}$ and an integer l_1 such that

$$
\begin{aligned}
&|p_h(\varDelta_{t_0,t_{\nu+1}})(\phi_h(\varDelta_{t_0,t_{\nu+1}}); X, D_x)u|_{l,\mathcal{S}} \leq M_{1,h,l}|u|_{l_1,\mathcal{S}} \\
&\Big(\varDelta_{t_0,t_{\nu+1}} : (T_0 \geq)\, t_0 \geq t_1 \geq \cdots \geq t_{\nu+1}\, (\geq 0)\Big)\,.
\end{aligned}
\tag{5.44}
$$

Furthermore, we can write

$$
\begin{aligned}
&\Big(p_h(\varDelta_{t_0,t_{\nu+1}})(\phi_h(\varDelta_{t_0,t_{\nu+1}}); X, D_x) \\
&\quad - p_h(t_0, t_{\nu+1})(\phi_h(t_0, t_{\nu+1}); X, D_x)\Big)u(x) \\
&= \int_0^1 \partial_\vartheta \Big(\int e^{i\phi_{\vartheta,h}(\varDelta_{t_0,t_{\nu+1}})(x,\xi)} p_{\vartheta,h}(\varDelta_{t_0,t_{\nu+1}})(x,\xi)\hat{u}(\xi) đ\xi \Big) d\vartheta \\
&= \int_0^1 q_{\vartheta,h}(\varDelta_{t_0,t_{\nu+1}})(\phi_{\vartheta,h}(\varDelta_{t_0,t_{\nu+1}}); X, D_x)u(x) d\vartheta\,,
\end{aligned}
\tag{5.45}
$$

where

$$
\begin{aligned}
q_{\vartheta,h}(\varDelta_{t_0,t_{\nu+1}})(x,\xi) \equiv\, &\partial_\vartheta p_{\vartheta,h}(\varDelta_{t_0,t_{\nu+1}})(x,\xi) \\
&+ i\partial_\vartheta \phi_{\vartheta,h}(\varDelta_{t_0,t_{\nu+1}})(x,\xi) p_{\vartheta,h}(\varDelta_{t_0,t_{\nu+1}})(x,\xi)\,.
\end{aligned}
\tag{5.46}
$$

Hence, by Proposition 3.3, Theorem 5.1 and Lemma 5.2, there exist a constant $M_{2,h,l}$ and an integer l_2 such that

$$
\begin{aligned}
&\Big|\Big(p_h(\varDelta_{t_0,t_{\nu+1}})(\phi_h(\varDelta_{t_0,t_{\nu+1}}); X, D_x) \\
&\quad - p_h(t_0, t_{\nu+1})(\phi_h(t_0, t_{\nu+1}); X, D_x)\Big)u\Big|_{l,\mathcal{S}} \\
&\quad \leq M_{2,h,l}(t_0 - t_{\nu+1})^2 |u|_{l_2,\mathcal{S}} \\
&\Big(\varDelta_{t_0,t_{\nu+1}} : (T_0 \geq)\, t_0 \geq t_1 \geq \cdots \geq t_{\nu+1}\, (\geq 0)\Big)\,.
\end{aligned}
\tag{5.47}
$$

Similarly, there exist a constant $M_{3,h,l}$ and an integer l_3 such that

$$\left|\Big(i\partial_t + K(t,X,D_x)\Big)p_h(t,t_1)(\phi_h(t,t_1);X,D_x)u\right|_{l,\mathcal{S}} \le M_{3,h,l}(t-t_1)|u|_{l_3,\mathcal{S}} \quad (T_0 \ge t \ge t_1 \ge 0)\,. \tag{5.48}$$

(II) Let $\Delta_{t,s} : (T_0 \ge)\, t \equiv t_0 \ge t_1 \ge \cdots \ge t_{\nu+1} \equiv s\,(\ge 0)$ and let $\Delta'_{t,s}$ be any refinement of $\Delta_{t,s}$. Then we can write

$$\begin{aligned}&\Big(p_h(\Delta'_{t,s})(\phi_h(\Delta'_{t,s});X,D_x) - p_h(\Delta_{t,s})(\phi_h(\Delta_{t,s});X,D_x)\Big)u \\ &= \sum_{j=1}^{\nu+1} p_h(\Delta'_{t_0,t_{j-1}})(\phi_h(\Delta'_{t_0,t_{j-1}});X,D_x) \\ &\circ \Big(p_h(\Delta'_{t_{j-1},t_j})(\phi_h(\Delta'_{t_{j-1},t_j});X,D_x) \\ &\quad - p_h(t_{j-1},t_j)(\phi_h(t_{j-1},t_j);X,D_x)\Big) \\ &\quad \circ p_h(\Delta_{t_j,t_{\nu+1}})(\phi_h(\Delta_{t_j,t_{\nu+1}});X,D_x)u\,.\end{aligned} \tag{5.49}$$

Hence, using (5.44) and (5.47), there exist a constant $M_{4,h,l}$ and an integer l_4 such that

$$\begin{aligned}&\left|\Big(p_h(\Delta'_{t,s})(\phi_h(\Delta'_{t,s});X,D_x) - p_h(\Delta_{t,s})(\phi_h(\Delta_{t,s});X,D_x)\Big)u\right|_{l,\mathcal{S}} \\ &\le M_{4,h,l}|\Delta_{t,s}|(t-s)|u|_{l_4,\mathcal{S}} \\ &\Big(\Delta_{t,s} : (T_0 \ge)\, t \equiv t_0 \ge t_1 \ge \cdots \ge t_{\nu+1} \equiv s\,(\ge 0)\Big)\,.\end{aligned} \tag{5.50}$$

Therefore, there exists $U_h(t,s)u \in \mathcal{S}$ such that

$$U_h(t,s)u = \lim_{|\Delta_{t,s}|\to 0} p_h(\Delta_{t,s})(\phi_h(\Delta_{t,s});X,D_x)u \quad (\text{ uniformly on } [s,T_0])\,. \tag{5.51}$$

Similarly, using (5.44), (5.47) and (5.48), there exist a constant $M_{5,h,l}$ and an integer l_5 such that

$$\begin{aligned}&\left|\Big(\partial_t p_h(\Delta'_{t,s})(\phi_h(\Delta'_{t,s});X,D_x) - \partial_t p_h(\Delta_{t,s})(\phi_h(\Delta_{t,s});X,D_x)\Big)u\right|_{l,\mathcal{S}} \\ &\le M_{5,h,l}|\Delta_{t,s}||u|_{l_5,\mathcal{S}} \\ &\Big(\Delta_{t,s} : (T_0 \ge)\, t \equiv t_0 \ge t_1 \ge \cdots \ge t_{\nu+1} \equiv s\,(\ge 0)\Big)\,.\end{aligned} \tag{5.52}$$

Therefore, there exists $V_h(t,s)u \in \mathcal{S}$ such that

$$V_h(t,s)u = \lim_{|\Delta_{t,s}|\to 0} \partial_t p_h(\Delta_{t,s})(\phi_h(\Delta_{t,s}); X, D_x)u \qquad \text{(uniformly on } [s, T_0]) \,. \tag{5.53}$$

Hence, $U_h(t,s)u$ is differentiable and $\partial_t U_h(t,s)u = V_h(t,s)u$.

Furthermore, using (5.44) and (5.48), there exist a constant $M_{6,h,l}$ and an integer l_6 such that

$$\begin{aligned} &\Big|\Big(i\partial_t + K_h(t, X, D_x)\Big)p_h(\Delta_{t,s})(\phi_h(\Delta_{t,s}); X, D_x)u\Big|_{l,\mathcal{S}} \\ &\quad \le M_{6,h,l}(t-t_1)|u|_{l_6,\mathcal{S}} \\ &\Big(\Delta_{t,s} : (T_0 \ge)\, t \equiv t_0 \ge t_1 \ge \cdots \ge t_{\nu+1} \equiv s\,(\ge 0)\Big)\,. \end{aligned} \tag{5.54}$$

Therefore, we have

$$\lim_{|\Delta_{t,s}|\to 0} \Big(i\partial_t + K_h(t, X, D_x)\Big)p_h(\Delta_{t,s})(\phi_h(\Delta_{t,s}); X, D_x)u = 0 \tag{5.55}$$

As a consequence, we get

$$\begin{cases} L_h U_h(t,s)u = 0 \quad \text{on } [s, T_0] \\ \quad U_h(s,s)u = u \quad (0 \le s \le T_0)\,. \end{cases} \tag{5.56}$$

(III) By Theorem 5.1 and Theorem 5.3, there exist a constant $M_{7,h,l}$ and an integer l_7 such that

$$\begin{aligned} &\Big|\Big(p_h(\Delta_{t,s})(\phi_h(\Delta_{t,s}); X, D_x) - p_h^\star(t,s)(\phi_h^\star(t,s); X, D_x)\Big)u\Big|_{l,\mathcal{S}} \\ &\quad \le M_{7,h,l}|\Delta_{t,s}|(t-s)|u|_{l_7,\mathcal{S}} \\ &\Big(\Delta_{t,s} : (T_0 \ge)\, t \equiv t_0 \ge t_1 \ge \cdots \ge t_{\nu+1} \equiv s\,(\ge 0)\Big)\,. \end{aligned} \tag{5.57}$$

Hence, we have

$$U_h(t,s) = p_h^\star(t,s)(\phi_h^\star(t,s); X, D_x)\,. \tag{5.58}$$

(VI) Finally, we prove the uniqueness of $U_h(t,s)$:

Let $u_h(t,s) \in \mathcal{C}^1([s,T_0];\mathcal{S})$ be a solution of the Cauchy problem such that

$$\left\{ \begin{array}{l} L_h u_h(t,s) = 0 \quad \text{on } [s,T_0] \\ \quad u_h(s,s) = 0 \quad . \end{array} \right. \tag{5.59}$$

Set

$$\left\{ \begin{array}{l} K'_h(t,\xi,x') \equiv \overline{K_h(t,x',\xi)} \\ L'_h \equiv i\partial_t + K'_h(t,D_x,X')\,. \end{array} \right. \tag{5.60}$$

Then there exists $W'_h(t,x,\xi)$ such that

$$L'_h = i\partial_t + H_h(t,X,D_x) + W'_h(t,X,D_x) \tag{5.61}$$

and

$$\left\{ \begin{array}{l} |\partial_\xi^\alpha D_x^\beta \{W'_h(t,h^\delta x, h^{-\rho}\xi)\}| \le \left\{ \begin{array}{ll} C''_{0,0}\langle x;\xi\rangle & (\alpha=\beta=0) \\ C''_{\alpha,\beta} & (|\alpha+\beta|\ge 1) \end{array} \right. \\ Im\{W'_h(t,h^\delta x,h^{-\rho}\xi)\} \le C''_{0,0}\,, \end{array} \right. \tag{5.62}$$

on $[0,T_0]\times \boldsymbol{R}^{2n}$. Here $C''_{\alpha,\beta}$ is a constant independent of $0<h<1$.

Hence, for any $0 \le \theta \le T_0$, there exists a fundamental solution $U'(t,\theta)$ $(0 \le t \le \theta)$ for the operator L'_h such that

$$\left\{ \begin{array}{l} L'_h U'_h(t,\theta) = 0 \quad \text{on } [0,\theta] \\ \quad U'_h(\theta,\theta) = I \quad . \end{array} \right. \tag{5.63}$$

For any $v(t) \in \mathcal{C}^0([s,T_0];\mathcal{S})$, we set

$$z_h(t) \equiv -i\int_{T_0}^t U'_h(t,\theta)v(\theta)d\theta \quad (s \le t \le T_0) \quad . \tag{5.64}$$

Then we have

$$\left\{ \begin{array}{l} L'_h z_h(t) = v(t) \quad \text{on } [s,T_0] \\ \quad z_h(T_0) = 0 \quad . \end{array} \right. \tag{5.65}$$

Hence, using (5.59) and (5.65), we have

$$
\begin{aligned}
(5.66) \quad & \int_s^{T_0} (u_h(t,s), v(t))dt \\
&= \int_s^{T_0} (u_h(t,s), L'_h z_h(t))dt \\
&= \Big[-i(u_h(t,s), z_h(t)) \Big]_{t=s}^{t=T_0} + \int_s^{T_0} (L_h u_h(t,s), z_h(t))dt \\
&= 0 \quad (v(t) \in \mathcal{C}^0([s,T_0];\mathcal{S})) \quad .
\end{aligned}
$$

Therefore, we get $u_h(t,s) = 0$ on $[s, T_0]$. □

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Graduate School of Mathematical Sciences
University of Tokyo
3-8-1, Komaba
Meguro-ku, Tokyo 153
Japan