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Topological entropy and topologically mixing property in symbolic dynamics

By Li SIMIN

Abstract. In this paper we construct an example to show usually positive topological entropy does not imply mixing property at all in symbolic dynamics.

In ergodic theory, it is well-known that a K- systems, i.e., a measure preserving transformation having positive metric entropy with respect to every nontrivial finite measurable partition, is strongly mixing. But F.Blandchard proved the topological setting of above is not true. He constructed a subshift which has completely positive topological entropy but not topologically mixing. It is not too difficult to check that for a subshift of finite type, it is topologically mixing if and only if it has completely positive topological entropy.

On the other hand, it is well-known [2] that if a continuous map $f : X \to X$ from an interval or a circle to itself has positive entropy, then some subsystems are topologically mixing, i.e., there exists a closed subset $\Lambda \subset X$ and an integer $m > 0$ such that $f^m(\Lambda) \subset \Lambda$ and $f^m|_\Lambda$ is topologically mixing. A. Katok [3] also got analogous results for $C^{1+\alpha}$ $(\alpha > 0)$ diffeomorphism of a two dimensional compact manifold.

In zero dimension case, it is easy to check it is also true for a subshift of finite type. However we will construct a subshift which has positive entropy but has no mixing subsystem.

In the construction we use the Denjoy C^1 diffeomorphism of the circle. We will state the construction of the Denjoy diffeomorphism briefly without proof. For the detail see for example [4].

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Consider a countable set $\{I_m = [0, l_m] : m \in \mathbb{Z}, l_m > 0\}$ of closed interval which satisfies $\sum_{m\in\mathbb{Z}} l_m = l < \infty$. Let α be an irrational number and define $\alpha_m, m \in \mathbb{Z}$, by $\alpha_m = m\alpha \pmod 1$, $0 \leq \alpha_m < 1$. We construct an interval of length of $1+l$ by inserting I_m at $\alpha_m, m \in \mathbb{Z}$, in the interval [0,1]. We obtain a circle by identifying the point 0 and 1, and denote it by S^1. Let α_m and β_m denote the left endpoint and the right endpoint of I_m, respectively.

Construct a C^1 map $f_m : I_m \to I_{m+1}, f_m(0) = 0, f_m(l_m) = l_{m+1}$ and some other condition (See [4]). Define a map $f : S^1 \to S^1$ by

$$f(x) = \begin{cases} f_m(x) \in I_{m+1} \subset S^1 & x \in I_m \\ x + \alpha(\text{mod } 1) \in [0,1] \subset S^1 & x \in [0,1] \end{cases}$$

It is easily seen f is well defined and it can be made to be a C^1 diffeomorphism. It is called a Denjoy C^1 diffeomorphism of the circle.

For $m \in \mathbb{Z}$, define a half open half closed interval $J_m \subset S^1$ by

$$J_m = \begin{cases} [\beta_{m-1}, \beta_m) \subset S^1 & \beta_{m-1} < \beta_m \\ [\beta_{m-1}, 1] \cup [0, \beta_m) \subset S^1 & \beta_{m-1} > \beta_m \end{cases}$$

Now define a map $\phi : S^1 \to \Sigma_2 = \{0,1\}^{\mathbb{Z}}$ by

$$\phi(x)(m) = \begin{cases} 0 & x \notin J_m \\ 1 & x \in J_m \end{cases}$$

for $m \in \mathbb{Z}$ and $x \in S^1$. Give S^1 the usual topology with metric d. Let $X = S^1 - \bigcup_{m\in\mathbb{Z}} \text{Int}(I_m) = [0,1] \cup \{\beta_m : m \in \mathbb{Z}\}$. X is a closed subset of S^1 and $f(X) = X$. we will show :

(1) $\phi|_X : X \to \Sigma_2$ is injective,
(2) $\phi|_X : X \to \Sigma_2$ is continuous.

Proof of (1). For $x, y \in [0,1], x < y$, since $\{\alpha_m : m \in \mathbb{Z}\}$ is dense in $[0,1]$, there exists an integer $m \in \mathbb{Z}$ and $x \leqslant \alpha_m < y$. Let $n = \inf\{k > m : a_k \geqslant y$ or $a_k < x\}$. Then $\phi(y)(n) = 1$ but $\phi(x)(n) = 0$. So $\phi(x) \neq \phi(y)$. Similarly $\phi(x) \neq \phi(\beta_m)$ for any $x \in [0,1]$ and any $m \in \mathbb{Z}$. consequently $\phi|_X$ is injective.

Proof of (2). For any $k > 0, k \in \mathbb{Z}$, let $\delta = \min\{l_m| - (k+1) \leq m \leq (k+1)\}$. Let $x, y \in X$ and $d(x,y) < \delta$. If for some $m, -k \leq m \leq k, x \in J_m$, but $y \notin_m$, then $\beta_{m-1} \leq x \leq \beta_m$, but $y > \beta_m$ or $y < \alpha_{m-1}$. So $d(x, y) > l_m$ or $d(x, y) > l_{m-1}$. This contradicts with $d(x, y) < \delta$.

Since $\phi|_X$ is an injective continuous map of a compact space to a Hausdorff space, $\phi(X) \subset \Sigma_2$ is closed and $\phi|_X : X \to \Lambda = \phi(X) \subset \Sigma_2$ is a homeomorphism. Obviously $\phi \circ f(x) = \sigma \circ \phi(x)$. We get a subshift $\sigma : \Lambda \to \Lambda$ which is conjugate to $f : X \to X$. Since f acts in X as an irrational rotation. $f|_X$ can not have mixing subsystems. Neither can $\sigma : \Lambda \to \Lambda$.

Let $\Sigma_2' = \{a, b\}^{\mathbb{Z}}$ and $\Sigma_4 = \{0a, 0b, 1a, 1b\}^{\mathbb{Z}}$. Define a map $\psi : \Lambda \times \Sigma_2' \to \Sigma_4$ by

$$\psi(x, y)(m) = x(m)y(m) \text{ for } m \in \mathbb{Z},\ x \in \Lambda,\ y \in \Sigma_2'$$

Obviously ψ is injective and continuous. Let $W = \psi(\Lambda \times \Sigma_2') \subset \Sigma_4$. $\sigma \times \sigma : \Lambda \times \Sigma_2' \to \Lambda \times \Sigma_2'$ is conjugate to $\sigma : W \to W$.

Since $\sigma : \Lambda \to \Lambda$ can not have mixing subsystems, neither can $\sigma \times \sigma : \Lambda \times \Sigma_2' \to \Lambda \times \Sigma_2'$. Consequently $\sigma : W \to W$ does not have mixing subsystems. But $h(\sigma|_W) = h(\sigma \times \sigma|_{\Lambda \times \Sigma_2'}) \geq h\,(\sigma|_{\Sigma_2}) = \log 2 > 0$

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Department of Mathematical Sciences
University of Tokyo

Hongo, Tokyo 113
JAPAN